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ISSN 0250 3638

REMARKS ON OPERATORS IN NORMED ALMOST LINEAR SPACES

by

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PREPRINT SERIES IN MATHEMATICS

No. 69/1990

BUCURESTI

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November 1990

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INTRODUCTION

The normed almost linear spaces were introduced in [5] as a generalization of normed linear spaces. An example of a normed almost linear space is the collection of all nonempty, bounded and convex subsets of a normed linear space (see [5]).

The almost linear spaces, which generalize the linear spaces were introduced by Mayer ([9], who called them quasilinear spaces as an abstraction of the algebraic structure of the class of all closed intervals of R. These spaces have been subsequently studie by Kracht and Schröder ([8]) and Ratschek and Schröder ([10]).

A much more studied notion is that of a convex cone [2] (or in other terminology a semilinear space [1], or simply cone [12], [13]), which is a set X satisfying all the axioms of an almost linear space (see the definition in Section 1 below) replacing everywhere R by R_+. (For some more general notions see [3], [14]). It is easy to see that a convex cone X can be organized as an almost linear space if we define $\lambda \circ x = |\lambda| \circ x$ for $\lambda < 0$ and $x \in X$.

In [9] Mayer considered non-negative norms on an almost linear space but, as he observed there, such norms are not convenient. To compensate the weakening of the axioms of a linea space, we defined ([5]) a norm on an almost linear space by adding to the usual axioms of a norm on a linear space an additional one which makes the framework productive. In [4]-[7] we began to develop a theory for the normed almost linear spaces, similar with that of the normed linear spaces. Thus, we defined the dual space of a normed almost linear space (where the functionals are no longer linear, but almost linear), the bounded, linear and almost linear operators between two such spaces and we obtained in this more general framework basic results from the theory of normed linear spaces.

The main tool for the theory of normed almost linear spaces was given in Theorem 3.2 of [7], where we proved that any normed almost linear space can be embedded into a normed linear space, allowing us the use of the techniques of the normed linear spaces. As we have shown in [6], the space of bounded almost linear operators between two normed almost linear spaces can be organized as an almost linear space and we can endow it with a "norm" which does not always satisfy the additional axiom requierd in the definition of a norm on an almost linear space. In [6] we proved the embedding theorem of [7] for the space of bounded, almost linear operators under a condition which implies that this space is a normed almost linear space. In Theorem 2.9 of Section 2, we embed the space of bounded almost linear operators into a normed linear space, even when it is not a normed almost linear space and when it is, all conditions in Theorem 3.2 of [7] are satisfied. Besides this result, in this paper we also give conditions in order that this space belong to the simplest classes of (normed) almost linear spaces.

Finally, the author thanks to Dr. K.D. Schmidt for drawing attention on the papers [8]-[10] and for providing copies of [11] - [13].

1. PRELIMINARIES

For an easy understanding of this paper, in this section besides notation, we recall some definitions and results on normed almost linear spaces. We assume that <u>all spaces are over the real field</u> R and we denote by R_+ the set $\{\lambda \in R : \lambda \geq 0\}$.

A commutative semigroup X with zero 0 is called an <u>almost linear space</u> ([9]) if there is also given a mapping $(\lambda, x) \longrightarrow \lambda \circ x$ of $R \times X$ into X satisfying (i)-(v) below. Let $x, y \in X$ and $\lambda, \mu \in R$:

(i) $1 \circ x = x$; (ii) $0 \circ x = 0$; (iii) $\lambda \circ (x+y) = \lambda \circ x + \lambda \circ y$;

(iv) $\lambda \circ (\mu \circ x) = (\lambda \mu) \circ x$; (v) $(\lambda + \mu) \circ x = \lambda \circ x + \mu \circ x$ for $\lambda, \mu \in R_+$.

We set off the following two subsets of X ([5]):

$$V_X = \{ x \in X : x + (-1 \circ x) = 0 \}$$
$$W_X = \{ x \in X : x = -1 \circ x \}$$

These are <u>almost linear subspaces of</u> X (i.e., closed under additio and multiplication by reals) and V_X is a linear space. Clearly, an almost linear space X is a linear space iff $X = V_X$, iff $W_X = \{0$

In an almost linear space X we use the notation $\lambda \circ x$ for the multiplication of $\lambda \in R$ by $x \in X$, the notation λx being used only in a linear space.

A <u>normed almost linear space</u> ([5]) is an almost linear spac X together with a <u>norm</u> $|||\cdot||| : X \longrightarrow R$ satisfying (N_1)-(N_4) below.

(N_1) $|||x+y||| \leq |||x||| + |||y|||$ $\quad (x, y \in X)$

(N_2) $|||x||| = 0$ iff $x = 0$ $\quad (x \in X)$

(N_3) $|||\lambda \circ x||| = |\lambda| \, |||x|||$ $\quad (\lambda \in R,\ x \in X)$

(N_4) $|||x||| \leq |||x+w|||$ $\quad (x \in X,\ w \in W_X)$

Note that $|||x||| \geq 0$ for each $x \in X$. We also have ([4]):

$$|||w||| \leq |||x+w||| \quad . \quad (x \in X,\ w \in W_X) \tag{1.1}$$

We draw attention that in [4] and [5] we have worked with an equivalent definition of the norm and in [5] the last axiom of the norm is superfluous.

1.1. LEMMA ([4]). Let X be a normed almost linear space and let $x,y \in X$, $w_i \in W_X$, $v_i \in V_X$, $i=1,2$.

(i) If $x+y \in V_X$ then $x,y \in V_X$.

(ii) If $w_1+v_1 = w_2+v_2$ then $w_1=w_2$ and $v_1=v_2$.

Let X,Y be two (normed) almost linear spaces. For a mapping $T:X \longrightarrow Y$ the definition of a linear operator (an isometry) is similar with that from the linear case. We draw attention that a linear isometry is not always one-to-one. For $A \subset X$ we denote by $T(A)$ the set $\{T(a): a \in A\}$.

The following result is the main tool for the theory of normed almost linear spaces. In Section 2, we will make repeated use of this result.

1.2. THEOREM. ([7], Theorem 3.2). For any normed almost linear space $(X, |||\cdot|||)$ there exist a normed linear space $(E_X, \|\cdot\|_{E_X})$ and a mapping $\omega_X : X \longrightarrow E_X$ with the following properties:

(i) $E_X = \omega_X(X) - \omega_X(X)$ and $\omega_X(X)$ can be organized as an almost linear space where the addition and the multiplication by non-negative reals are the same as in E_X.

(ii) For $z \in E_X$ we have:

$$\|z\|_{E_X} = \inf\{ |||x||| + |||y||| : x,y \in X,\ z = \omega_X(x) - \omega_X(y)\}$$

and $(\omega_X(X), \|\cdot\|_{E_X})$ is a normed almost linear space.

(iii) ω_X is a linear isometry of $(X, |||\cdot|||)$ onto $(\omega_X(X), \|\cdot\|_{E_X})$.

We shall sometimes denote $\|\cdot\|_{E_X}$ by $\|\cdot\|$ when this will not lead to misunderstanding.

The proof of the following lemma is contained in the proof of ([7], Theorem 3.2 (iv), fact I).

1.3. LEMMA. Let $(X,|||\cdot|||)$ be a normed almost linear space and let $x,y \in X$. If $\omega_X(x) = \omega_X(y)$ then for each $\varepsilon > 0$ there exist $x_\varepsilon, y_\varepsilon, u_\varepsilon \in X$ such that $|||x_\varepsilon||| = |||y_\varepsilon||| < \varepsilon$ and $x+y_\varepsilon+u_\varepsilon = y+x_\varepsilon+u_\varepsilon$.

A subset $C \subset X$ is called a cone if the relations $c \in C$, $\lambda \in R_+$ imply that $\lambda \circ c \in C$. The definition of a convex set of X is similar with that from the linear case.

Let X and Y be two normed almost linear spaces and C a conve cone of Y. A mapping $T: X \longrightarrow Y$ is called an almost linear operator with respect to C ([6]) if T is additive, positively homogeneous and $T(W_X) \subset C$. The set $\mathcal{L}(X,(Y,C))$ of all such operators can be organized as an almost linear space if we define the addition and 0 as in the linear case, while for $\lambda \in R$ and $T \in \mathcal{L}(X,(Y,C))$ we define $(\lambda \circ T)(x) = T(\lambda \circ x)$, $x \in X$. For $T \in \mathcal{L}(X,(Y,C))$ define

$$(1.2) \quad |||T||| = \sup \{ |||T(x)|||;\ x \in X,\ |||x||| \leq 1 \}$$

and let $L(X,(Y,C)) = \{ T \in \mathcal{L}(X,(Y,C)) : |||T||| < \infty \}$. Since $|||\cdot|||$ defined by (1.2) satisfies (N_1)-(N_3), $L(X,(Y,C))$ is an almost linear space. It is not always a normed almost linear space for arbitrary convex cones $C \subset Y$ (see Theorem 1.5 below). Though we shall avoid the word "norm" when (N_4) does not hold for $|||\cdot|||$ given by (1.2), in the sequel we shall always consider the almost linea space $L(X,(Y,C))$ equipped with the $|||\cdot|||$ defined by (1.2).

Let us note that when $X = V_X$, $Y = V_Y$, then C is superfluou and $L(X,(Y,C)) = L(X,Y)$ is the usual normed linear space of all bounded, linear operators $T: X \longrightarrow Y$.

Let X be a normed almost linear space and C a convex cone of X.

1.4. DEFINITION. ([6], Definition 3.1). The convex cone $C \subset X$ has *property* (P) *in* X if the relations $x,y \in X$, $x+y \in C$ and $c \in C$ imply that

$$(1.3) \qquad \max\{|||x|||, |||y|||\} \leq \max\{|||x+c|||, |||y+c|||\}$$

The existence of convex cones $C \neq \{0\}$ having property (P) in $X(\neq \{0\})$ is guaranteed by Proposition 3.2 of [6] .

1.5. THEOREM. ([6], Theorem 4.15). *Let* C *be a convex cone of the normed almost linear space* Y. *Then* $L(X,(Y,C))$ *is a normed almost linear space for each normed almost linear space* X *iff* C *has property* (P) *in* Y.

1.6. PROPOSITION. ([6], Proposition 4.14). *If* $L(X,(Y,C))$ *is a normed almost linear space then* $|||c_1||| \leq |||c_1+c_2|||$, $c_i \in C, i=1,2$.

We recall that X satisfies the *law of cancellation* if the relations $x,y,z \in X$, $x+y=x+z$ imply that $y=z$.

1.7. LEMMA. ([6], Lemma 3.5 (i)). *Let* X *be a normed almost linear space satisfying the law of cancellation and let* $C \subset X$ *be a convex cone having property* (P) *in* X *and such that* $W_X \subset C$. *Then* $\omega_X(C)$ *is a convex cone having property* (P) *in* $\omega_X(X)$.

Let us note that when $Y = R$ and $C = R_+$ then $L(X,(R,R_+))$ is the dual space X^* of X ([5]), where the functionals are no longer "linear" but "almost linear". Since R_+ has property (P) in R, X^* is a normed almost linear space.

1.8. PROPOSITION. ([6], Corollary 2.9). If X is a normed almost linear space such that $X \neq V_X$ then $W_{X^*} \neq \{0\}$.

Since the next section is concerned with some properties of $L(X,(Y,C))$, the question wether this space is not $\{0\}$ must be setteled. Due to certain inaccuracies in [6], Remark 4.12, we take this oportunity to correct them.

1.9. REMARK. ([6], Remark 4.12). If $C \neq \{0\}$ and $X \neq V_X$. then $L(X,(Y,C)) \neq \{0\}$. Indeed, let $c \in C \setminus \{0\}$ and let $f \in W_{X^*} \setminus \{0\}$ (which exists by Proposition 1.8). Then $f(x) \geq 0$ for each $x \in X$ since by $f \in W_{X^*}$ we have $f = -1 \circ f$, i.e., $f(x) = f(-1 \circ x)$ for each $x \in X$ and so $0 \leq f(x+(-1 \circ x)) = 2f(x)$. Define $T(x)=f(x) \circ c$, $x \in X$. Then $T \in L(X,(Y,C)) \setminus \{0\}$. As we have observed in [6], when $C = \{0\}$ then $L(X,(Y,C))$ may be or may be not $\{0\}$. When $X = V_X \neq \{0\}$ then $L(X,(Y,C))$ may be or may be not $\{0\}$. Indeed, when $X = V_X$ $(\neq \{0\})$ and $Y = V_Y (\neq \{0\})$ then $L(X,Y) \neq \{0\}$ and when $X = V_X$ and $V_Y = \{0\}$ then $L(X,(Y,C)) = \{0\}$ (this is a consequence of the next remark). An example of a normed almost linear space $X \neq \{0\}$ with $V_X = \{0\}$ can be found in Example 2.5, Section 2.

1.10. REMARK. ([6], Remark 4.10). We have $T(V_X) \subset V_Y$ for each $T \in L(X,(Y,C))$.

2. THE SPACE OF BOUNDED ALMOST LINEAR OPERATORS

Throughout this section X and Y are normed almost linear spaces and C is a convex cone of Y. As we have observed in Section 1, $|||\cdot|||$ given by (1.2) does not always satisfy (N_4), i.e., the almost linear space L(X,(Y,C)) is not always a normed almost linear space. In the sequel, if otherwise not stated, L(X,(Y,C)) equipped with $|||\cdot|||$ given by (1.2) is not necessarily a normed almost linear space.

Among the simplest classes of (normed) almost linear spaces X are those of the form $X = V_X$ (when we recover the class of (normed) linear spaces), $X = W_X$ and $X = W_X + V_X$. In Proposition 2.1 (see also Remark 2.2 (i)) and Proposition 2.3 (see also Corollary 3.4) we give sufficient or necessary conditions in order that $L(X,(Y,C)) = W_{L(X,(Y,C))} + V_{L(X,(Y,C))}$. Let us note that by Remark 1.10, when $X = V_X$ then L(X,(Y,C)) is the usual normed linear space $L(V_X, V_Y)$, i.e., $L(X,(Y,C)) = V_{L(X,(Y,C))}$.

2.1. PROPOSITION. If $Y = V_Y$ or $X = W_X + V_X$ then $L(X,(Y,C)) = W_{L(X,(Y,C))} + V_{L(X,(Y,C))}$.

PROOF. Suppose $Y = V_Y$ and let $T \in L(X,(Y,C))$. For i=1,2 define $T_i : X \longrightarrow Y$ in the following way:

$$T_1(x) = \frac{T(x)+T(-1\circ x)}{2} \qquad (x \in X)$$

$$T_2(x) = \frac{T(x)+(-1\circ T(-1\circ x))}{2} \qquad (x \in X)$$

It is easy to show that $T_i \in L(X,(Y,C))$, i=1,2 and since $Y = V_Y$ we get $T = T_1 + T_2$. For each $x \in X$ we have $(-1\circ T_1)(x) = T_1(-1\circ x) =$

$= T_1(x)$, i.e., $T_1 \in W_{L(X,(Y,C))}$ and $(T_2+(-1\circ T_2))(x) = T_2(x)+T_2(-1\circ x)$ $= \frac{T(x)+(-1\circ T(-1\circ x))}{2} + \frac{T(-1\circ x)+(-1\circ T(x))}{2} = 0$ (since $Y = V_Y$), i.e., $T_2 \in V_{L(X,(Y,C))}$.

Suppose now $X = W_X+V_X$ and let $T \in L(X,(Y,C))$. By Lemma 1.1(ii) for each $x \in X$ there exist unique $w_x \in W_X$ and $v_x \in V_X$ such that $x = w_x+v_x$. Define $T_i: X \longrightarrow Y$, i=1,2 in the following way:

$$(2.1) \qquad T_1(x) = T(w_x) \qquad (x \in X)$$

$$(2.2) \qquad T_2(x) = T(v_x) \qquad (x \in X)$$

For i=1,2 it is easy to show that T_i is additive and positively homogeneous. By (2.1) and since $T(W_X) \subset C$, it follows that $T_1(x) \in C$ for each $x \in X$ and by (2.2) it follows that $T_2(w) = 0 \in C$ for each $w \in W_X$ (since $v_w = 0$), i.e., $T_i \in \mathcal{L}(X,(Y,C))$, i=1,2. Using (1.1) we get for each $x \in X$ that $|||T_1(x)||| = |||T(w_x)||| \leq |||T|||\,|||w_x||| \leq$ $\leq |||T|||\,|||w_x+v_x||| = |||T|||\,|||x|||$, i.e., $T_1 \in L(X,(Y,C))$. Similarly, using (N_4) instead of (1.1) we obtain that $T_2 \in L(X,(Y,C))$. Since for each $x \in X$, $x = w_x+v_x$ we have $-1\circ x = w_x+(-1\circ v_x)$ we get $(-1\circ T_1)(x) = T_1(-1\circ x) = T(w_x) = T_1(x)$, i.e., $T_1 \in W_{L(X,(Y,C))}$ and $(T_2+(-1\circ T_2))(x) = T_2(x)+T_2(-1\circ x) = T(v_x)+T(-1\circ v_x) = T(v_x+(-1\circ v_x))$ $= T(0) = 0$, i.e., $T_2 \in V_{L(X,(Y,C))}$. Clearly, by (2.1) and (2.2) we get $T = T_1+T_2$, which completes the proof.

2.2. REMARKS. (i) An inspection of the above proof shows that $\mathcal{L}(X,(Y,C)) = W_{\mathcal{L}(X,(Y,C))}+V_{\mathcal{L}(X,(Y,C))}$ when $Y = V_Y$ or $X = W_X+V_X$ and $\mathcal{L}(X,(Y,C)) = W_{\mathcal{L}(X,(Y,C))}$ when $X = W_X$. Consequently, when $X = W_X$ then $L(X,(Y,C)) = W_{L(X,(Y,C))}$.

(ii) When $Y = R$ and $C = R_+$, by Proposition 2.1 we obtain that $X^* = W_{X^*}+V_{X^*}$.

We give now a necessary condition in order that $L(X,(Y,C)) = {}^{W}L(X,(Y,C))+{}^{V}L(X,(Y,C))$:

2.3. PROPOSITION. *Let* $T\in L(X,(Y,C))$. *If* $T = T_1+T_2$, $T_1\in {}^{W}L(X,(Y,C))$, $T_2\in {}^{V}L(X,(Y,C))$ *then* $T(X)\subset C+V_Y$. *Consequently, if* $L(X,(Y,C)) = {}^{W}L(X,(Y,C))+{}^{V}L(X,(Y,C))$ *then* $T(X)\subset C+V_Y$ *for each* $T\in L(X,(Y,C))$.

PROOF. Let $T\in L(X,(Y,C))$, $T = T_1+T_2$, $T_1\in {}^{W}L(X,(Y,C))$, $T_2\in {}^{V}L(X,(Y,C))$ and let $x\in X$. We have :

$$T(x+(-1\circ x)) = T_1(x+(-1\circ x))+T_2(x+(-1\circ x)) \tag{2.3}$$

Since $T_1\in {}^{W}L(X,(Y,C))$, we get

$$T_1(x) = (-1\circ T_1)(x) = T_1(-1\circ x) \tag{2.4}$$

Since $T_2\in {}^{V}L(X,(Y,C))$, we get

$$0 = (T_2+(-1\circ T_2))(x) = T_2(x)+T_2(-1\circ x) = T_2(x+(-1\circ x)) \tag{2.5}$$

Using (2.4) and (2.5) in (2.3) we get

$$T_1(x) = \frac{T(x+(-1\circ x))}{2}$$

Then $T_1(x)\in C$, since $T\in L(X,(Y,C))$ and $x+(-1\circ x)\in W_X$. By (2.5) and Lemma 1.1 (i) we get $T_2(x)\in V_Y$. Consequently, $T(x)=T_1(x)+T_2(x)\in C+V_Y$, which completes the proof.

2.4. COROLLARY. *If* $Y\neq C+V_Y$ *and* $W_Y\subset C$ *then* $L(Y,(Y,C))\neq$

$\neq W_{L(Y,(Y,C))}+V_{L(Y,(Y,C))}$. In particular, if $X \neq W_X+V_X$ then $L(X,(X,W_X)) \neq W_{L(X,(X,W_X))}+V_{L(X,(X,W_X))}$.

PROOF. Let $T:Y \to Y$ be defined by $T(y) = y$, $y \in Y$. Then (since $W_Y \subset C$) we get $T \in L(Y,(Y,C))$. Since $T(Y) = Y$ and $Y \neq C+V_Y$, by the above proposition $T \notin W_{L(Y,(Y,C))}+V_{L(Y,(Y,C))}$.

The next example shows that the condition given in Proposition 2.3 is not sufficient for $L(X,(Y,C)) = W_{L(X,(Y,C))}+V_{L(X,(Y,C))}$. This example also shows that the assumption $Y = W_Y+V_Y$ does not always imply that $L(X,(Y,C)) = W_{L(X,(Y,C))}+V_{L(X,(Y,C))}$, even when $Y = W_Y = C$.

2.5. EXAMPLE. Let $X = Y = \{(\alpha,\beta) \in R^2 : \beta \geq |\alpha|\}$.

We organize X as an almost linear space where the addition is as in R^2 and for $\lambda \in R$ and $x = (\alpha,\beta) \in X$ we define $\lambda \circ x = (\lambda\alpha, |\lambda|\beta)$. We have $W_X = \{(0,\lambda) : \lambda \geq 0\}$ and $V_X = \{0\}$, i.e., $X \neq W_X+V_X$. For $(\alpha,\beta) \in X$ define $|||(\alpha,\beta)||| = |\alpha| + \beta$. Then X is a normed almost linear space.

We organize Y as an almost linear space where the addition is as in R^2 and for $\lambda \in R$ and $y = (\alpha,\beta) \in Y$ we define $\lambda \circ y = (|\lambda|\alpha, |\lambda|\beta)$. Then $Y = W_Y$ and $V_Y = \{0\}$, i.e., $Y = W_Y+V_Y$. For $y = (\alpha;\beta) \in Y$ define $|||(\alpha,\beta)||| = |\alpha| + \beta$. Then Y is a normed almost linear space.

Let $T:X \to Y$ be defined by

$$T((\alpha,\beta)) = (\alpha,\beta) \in Y \qquad ((\alpha,\beta) \in X) \tag{2.6}$$

Then clearly, $T \in L(X,(Y,Y))$. Suppose $T = T_1+T_2$, $T_1 \in W_{L(X,(Y,Y))}$ $T_2 \in V_{L(X,(Y,Y))}$. Similar with the proof of Proposition 2.3 we obtain that

$$T_1(x) = \frac{T(x)+T(-1\circ x)}{2} \qquad (x \in X)$$

Consequently, for $x = (\alpha, \beta) \in X$ we have $T_1((\alpha, \beta)) = (0, \beta)$ and $T_2((\alpha, \beta)) = (\gamma, \delta)$, $\delta \geq |\gamma|$. Since by our assumption $T = T_1 + T_2$, we get $(\alpha, \beta) = (0, \beta) + (\gamma, \delta)$, which is impossible for $(\alpha, \beta) \neq (0,$

Let us note that for T defined by (2.6) we have $T(X) = Y$ (= $= C + V_Y$).

When $L(X,(Y,C))$ is a normed almost linear space, it is of interest to determine $(E_{L(X,(Y,C))}, \|\cdot\|_{E_{L(X,(Y,C))}})$ and $\omega_{L(X,(Y,C)}$ given by Theorem 1.3. These were done in [6], Theorem 5.6 under the additional assumption that $L(\omega_X(X),(\omega_Y(Y),\omega_Y(C)))$ is a normed almost linear space. As we have observed in [6], this assumption implies that $L(X,(Y,C))$ is a normed almost linear space and we have no counterexample to show that the converse does not hold. In the next proposition we collect three simple conditions when the converse holds. We observe that in (i) and (iii) the assumption that $L(X,(Y,C))$ be a normed almost linear space is superfluous.

2.6. PROPOSITION. <u>Suppose</u> $L(X,(Y,C))$ <u>is a normed almost linear space. Then each of the conditions</u> (i)-(iii) <u>is sufficient for</u> $L(\omega_X(X),(\omega_Y(Y),\omega_Y(C)))$ <u>be a normed almost linear space</u>:

(i) $C = \{0\}$.

(ii) $C = Y$.

(iii) C <u>has property</u> (P) <u>in</u> Y, $W_Y \subset C$ <u>and</u> Y <u>satisfies the law of cancellation</u>.

PROOF. (i) If $C = \{0\}$ then $\omega_Y(C) = \{0\}$. Since $\{0\}$ has property (P) in $\omega_Y(Y)$, by Theorem 1.5 it follows that $L(\omega_X(X),(\omega_Y(Y),\omega_Y(C)))$ is a normed almost linear space.

(ii) Suppose $C = Y$ and that $|||\cdot|||$ given by (1.2) for $L(\omega_X(X),(\omega_Y(Y),\omega_Y(C)))$ does not satisfy (N_4). Then there exist $T \in L(\omega_X(X),(\omega_Y(Y),\omega_Y(C)))$ and $T_1 \in W_{L(\omega_X(X),(\omega_Y(Y),\omega_Y(C)))}$ such that

$$|||T+T_1||| < |||T|||$$

Consequently, there exists $x_o \in X$, $|||x_o||| \leq 1$ such that $|||T+T_1||| < \| T(\omega_X(x_o))\|$. Then

$$(2.7) \qquad \| T(\omega_X(x_o))+T_1(\omega_X(x_o))\| < \| T(\omega_X(x_o))\|$$

Let $y, y_1 \in Y$ be such that $\omega_Y(y) = T(\omega_X(x_o))$ and $\omega_Y(y_1) = T_1(\omega_X(x_o))$. Hence, by (2.7) we get $|||y+y_1||| = \|\omega_Y(y+y_1)\| = \| T(\omega_X(x_o))+T_1(\omega_X(x_o))\| < \| T(\omega_X(x_o))\| = \|\omega_Y(y)\| = |||y|||$, which contradicts Proposition 1.6, since $L(X,(Y,C))$ is a normed almost linear space and $Y = C$. Consequently, $L(\omega_X(X),(\omega_Y(Y),\omega_Y(C)))$ is a normed almost linear space.

(iii) If C has property (P) in Y, $W_Y \subset C$ and Y satisfies the law of cancellation, then by Lemma 1.7 $\omega_Y(C)$ has property (P) in $\omega_Y(Y)$, whence by Theorem 1.5 it follows that $L(\omega_X(X),(\omega_Y(Y),\omega_Y(C)))$ is a normed almost linear space, which completes the proof.

The next example shows that the assumption $Y = C$ does not imply that C has property (P) in Y. Hence by Theorem 1.5, there exists a normed almost linear space X such that $L(X,(Y,C))$ is not a normed almost linear space and so the assumption in Proposition 2.6 (ii) that $L(X,(Y,C))$ be a normed almost linear space is not superfluous.

2.7. EXAMPLE. Let $X = \{(\alpha,\beta) \in R^2 ; \beta \geq 0\}$. Define the addition as in R^2 and for $\lambda \in R$ and $(\alpha,\beta) \in X$ define $\lambda \circ (\alpha,\beta) = (\lambda\alpha, |\lambda|\beta)$. Then X is an almost linear space. We have $W_X = \{(0,\beta) ; \beta \geq 0\}$, $V_X = \{(\alpha,0) ; \alpha \in R\}$ and $X = W_X + V_X$. For $x = (\alpha,\beta) \in X$ define $|||x||| = |\alpha| + \beta$. Then X is a normed almost linear space. The cone $C = X$ has not property (P) in X. Indeed, let $x = (1,0) \in C$ and $y = (1/2,1/2) \in C$ and $c = (-1/2,0) \in C$. We have $\max\{|||x|||, |||y|||\} = 1$ and $\max\{|||x+c|||, |||y+c|||\} = 1/2$, i.e., $C = X$ has not property (P) in X.

Now we show that we can embed $L(X,(Y,C))$ into a linear subspace of $L(E_X,E_Y)$ without the additional assumption that $L(\omega_X(X),(\omega_Y(Y),\omega_Y(C)))$ is a normed almost linear space. We shall not assume even that $|||\cdot|||$ given by (1.2) for $L(X,(Y,C))$ satisfies (N_4) but when $L(X,(Y,C))$ is a normed almost linear space, then all the conditions in Theorem 1.3 will be satisfied. First we prove a lemma.

2.8. LEMMA. <u>For</u> $T \in L(X,(Y,C))$ <u>let</u> $\tilde{T}: E_X \to E_Y$ <u>be defined for</u> $z = \omega_X(x_1) - \omega_X(x_2) \in E_X$, $x_1, x_2 \in X$, <u>by</u>

$$\tilde{T}(z) = \omega_Y(T(x_1)) - \omega_Y(T(x_2)) \tag{2.8}$$

<u>Then</u> $\tilde{T}$ <u>is well defined and</u> $\tilde{T} \in L(E_X,E_Y)$. <u>We also have</u> $|||T||| = \|\tilde{T}\|$.

PROOF. Suppose $z = \omega_X(x_1) - \omega_X(x_2) = \omega_X(x_3) - \omega_X(x_4)$, $x_i \in X$,

$1 \leq i \leq 4$. Then $\omega_X(x_1+x_4) = \omega_X(x_2+x_3)$, whence by Lemma 1.3 for each $\varepsilon > 0$ there exist $x_\varepsilon, x'_\varepsilon, u_\varepsilon \in X$ such that (2.9) and (2.10) below hold:

$$(2.9) \qquad x_1+x_4+x_\varepsilon+u_\varepsilon = x_2+x_3+x'_\varepsilon+u_\varepsilon$$

$$(2.10) \qquad |||x_\varepsilon||| = |||x'_\varepsilon||| < \varepsilon$$

By (2.9) we get $T(x_1)+T(x_4)+T(x_\varepsilon)+T(u_\varepsilon) = T(x_2)+T(x_3)+T(x'_\varepsilon)+T(u_\varepsilon)$ and so $\omega_Y(T(x_1))+\omega_Y(T(x_4))+\omega_Y(T(x_\varepsilon))+\omega_Y(T(u_\varepsilon)) = \omega_Y(T(x_2))+$ $+\omega_Y(T(x_3))+\omega_Y(T(x'_\varepsilon))+\omega_Y(T(u_\varepsilon))$. Consequently, we have
$\|(\omega_Y(T(x_1))-\omega_Y(T(x_2)))-(\omega_Y(T(x_3))-\omega_Y(T(x_4)))\| =$
$= \|\omega_Y(T(x'_\varepsilon))-\omega_Y(T(x_\varepsilon))\| \leq \|\omega_Y(T(x'_\varepsilon))\| + \|\omega_Y(T(x_\varepsilon))\| =$
$= |||T(x'_\varepsilon)||| + |||T(x_\varepsilon)||| \leq |||T|||(|||x'_\varepsilon||| + |||x_\varepsilon|||) < 2\varepsilon|||T|||$. As $\varepsilon \to 0$ we get $\omega_Y(T(x_1))-\omega_Y(T(x_2)) = \omega_Y(T(x_3))-\omega_Y(T(x_4))$, i.e., $\tilde{T}$ is well defined.

We show now that $\tilde{T} \in \mathcal{L}(E_X, E_Y)$. Let $z, z_1 \in E_X$, say, $z = \omega_X(x_1)-\omega_X(x_2)$, $z_1 = \omega_X(x_3)-\omega_X(x_4)$, $x_i \in X$, $1 \leq i \leq 4$. Then $z+z_1 = \omega_X(x_1+x_3)-\omega_X(x_2+x_4)$ and so $\tilde{T}(z+z_1) =$
$= \omega_Y(T(x_1+x_3))-\omega_Y(T(x_2+x_4)) = \omega_Y(T(x_1))-\omega_Y(T(x_2))+$
$+\omega_Y(T(x_3))-\omega_Y(T(x_4)) = \tilde{T}(z)+\tilde{T}(z_1)$. Let now $\lambda \in R$ and $z = \omega_X(x_1)-\omega_X(x_2) \in E_X$, $x_1, x_2 \in X$. If $\lambda \geq 0$ then $\lambda z =$
$= \lambda\omega_X(x_1)-\lambda\omega_X(x_2) = \lambda\circ\omega_X(x_1)-\lambda\circ\omega_X(x_2) = \omega_X(\lambda\circ x_1)-\omega_X(\lambda\circ x_2)$
and so $\tilde{T}(\lambda z) = \omega_Y(T(\lambda\circ x_1))-\omega_Y(T(\lambda\circ x_2)) = \lambda\circ\omega_Y(T(x_1))-\lambda\circ\omega_Y(T(x_2))$
$= \lambda\omega_Y(T(x_1))-\lambda\omega_Y(T(x_2)) = \lambda(\omega_Y(T(x_1))-\omega_Y(T(x_2))) = \lambda\tilde{T}(z)$.
If $\lambda < 0$ then $\lambda z = |\lambda|\omega_X(x_2)-|\lambda|\omega_X(x_1) = |\lambda|\circ\omega_X(x_2)-|\lambda|\circ\omega_X(x_1) =$
$= \omega_X(|\lambda|\circ x_2)-\omega_X(|\lambda|\circ x_1)$ and as above we get $\tilde{T}(\lambda z) =$
$= \omega_Y(T(|\lambda|\circ x_2))-\omega_Y(T(|\lambda|\circ x_1)) = |\lambda|(\omega_Y(T(x_2))-\omega_Y(T(x_1))) =$
$= \lambda(\omega_Y(T(x_1))-\omega_Y(T(x_2))) = \lambda\tilde{T}(z)$, i.e., $\tilde{T} \in \mathcal{L}(E_X, E_Y)$.

Let now $z \in E_X$, $\|z\| < 1$. Then there exist $x_1, x_2 \in X$ such that

$z = \omega_X(x_1) - \omega_X(x_2)$ and $|||x_1||| + |||x_2||| \leq 1$. We have $\|\tilde{T}(z)\| =$
$= \|\omega_Y(T(x_1)) - \omega_Y(T(x_2))\| \leq \|\omega_Y(T(x_1))\| + \|\omega_Y(T(x_2))\| =$
$= |||T(x_1)||| + |||T(x_2)||| \leq |||T|||(|||x_1||| + |||x_2|||) \leq |||T|||$ and so $\|\tilde{T}\| \leq |||T|||$, i.e., $\tilde{T} \in L(E_X, E_Y)$. Finally, we show that $|||T||| \leq \|\tilde{T}\|$. Let $\varepsilon > 0$ and let $x \in X$, $|||x||| \leq 1$ such that $|||T||| \leq |||T(x)||| + \varepsilon$. Then $\|\tilde{T}(\omega_X(x))\| = \|\omega_Y(T(x))\| = |||T(x)||| \geq |||T||| - \varepsilon$. As $\varepsilon \to 0$ we get $|||T||| \leq \|\tilde{T}(\omega_X(x))\| \leq \|\tilde{T}\| \|\omega_X(x)\| = \|\tilde{T}\| |||x||| \leq \|\tilde{T}\|$. Consequently $\|\tilde{T}\| = |||T|||$, which completes the proof.

Let $\omega_L : L(X,(Y,C)) \to L(E_X, E_Y)$ be the mapping defined by

$$(2.11) \qquad \omega_L(T) = \tilde{T} \qquad (T \in L(X,(Y,C)))$$

where $\tilde{T}$ is given by Lemma 2.8 and let $K \subset L(E_X, E_Y)$ be the set

$$(2.12) \qquad K = \omega_L(L(X,(Y,C)))$$

As in Lemma 2.8, for $\tilde{T} \in L(E_X, E_Y)$ we denote by $\|\tilde{T}\|$ the usual norm on $L(E_X, E_Y)$. Now, we can formulate and prove the new version of Theorem 5.6 of [6].

2.9. THEOREM. <u>The set</u> K <u>defined by</u> (2.12) <u>is a convex cone of</u> $L(E_X, E_Y)$ <u>which can be organized as an almost linear space where the addition and the multiplication by non-negative reals are the same as in</u> $L(E_X, E_Y)$ <u>and</u> $\omega_L : L(X,(Y,C)) \to K$ <u>is a linear operator. The linear subspace</u> $E_L = K - K$ <u>of</u> $L(E_X, E_Y)$ <u>equipped with the norm</u> $\|\cdot\|_{E_L}$ <u>defined for</u> $\tilde{T} \in E_L$ <u>by</u>

$$(2.13) \qquad \|\tilde{T}\|_{E_L} = \inf\{\|\tilde{T}_1\| + \|\tilde{T}_2\| : \tilde{T}_1, \tilde{T}_2 \in K,\ \tilde{T} = \tilde{T}_1 - \tilde{T}_2\}$$

and the mapping $\omega_L : L(X,(Y,C)) \longrightarrow E_L$ given by (2.11) satisfy (i) in Theorem 1.2 and $\| \omega_L(T) \|_{E_L} = |||T|||$, $T \in L(X,(Y,C))$. When $L(X,(Y,C))$ is a normed almost linear space then (i)-(iii) in Theorem 1.2 are satisfied.

PROOF. We first show that K is a convex cone of $L(E_X, E_Y)$. Let $\tilde{T} \in K$ and $\lambda \geq 0$. There exists $T \in L(X,(Y,C))$ such that $\omega_L(T) = \tilde{T}$. Let $\tilde{T}_1 = \omega_L(\lambda \circ T) \in K$. For $z = \omega_X(x_1) - \omega_X(x_2) \in E_X$, $x_1, x_2 \in X$, we have (using (2.8)) that $\tilde{T}_1(z) = \omega_Y((\lambda \circ T)(x_1)) - \omega_Y((\lambda \circ T)(x_2)) =$
$= \omega_Y(\lambda \circ (T(x_1))) - \omega_Y(\lambda \circ (T(x_2))) = \lambda \circ \omega_Y(T(x_1)) - \lambda \circ \omega_Y(T(x_2)) =$
$= \lambda \omega_Y(T(x_1)) - \lambda \omega_Y(T(x_2)) = \lambda(\omega_Y(T(x_1)) - \omega_Y(T(x_2))) = \lambda \tilde{T}(z)$, i.e.,

$$(2.14) \qquad \tilde{T}_1 = \omega_L(\lambda \circ T) = \lambda \tilde{T} = \lambda \omega_L(T) \in K$$

To show that K is a convex cone, let $\tilde{T}_1, \tilde{T}_2 \in K$ and let $T_1, T_2 \in L(X,(Y,C))$ be such that $\tilde{T}_i = \omega_L(T_i)$, $i=1,2$. Let $T = T_1 + T_2$ and $\tilde{T} = \omega_L(T) \in K$. For $z = \omega_X(x_1) - \omega_X(x_2) \in E_X$, $x_1, x_2 \in X$, we have
$\tilde{T}(z) = \omega_Y(T(x_1)) - \omega_Y(T(x_2)) = \omega_Y((T_1+T_2)(x_1)) - \omega_Y((T_1+T_2)(x_2)) =$
$= (\omega_Y(T_1(x_1)) - \omega_Y(T_1(x_2))) + (\omega_Y(T_2(x_1)) - \omega_Y(T_2(x_2))) =$
$= \tilde{T}_1(z) + \tilde{T}_2(z)$, i.e.,

$$(2.15) \qquad \tilde{T} = \omega_L(T) = \omega_L(T_1+T_2) = \tilde{T}_1 + \tilde{T}_2 = \omega_L(T_1) + \omega_L(T_2) \in K.$$

We organize K as an almost linear space defining the addition and the multiplication by non-negative reals as in $L(E_X, E_Y)$ and for $\lambda < 0$ and $\tilde{T} \in K$, say, $\tilde{T} = \omega_L(T)$, $T \in L(X,(Y,C))$ we define

$$(2.16) \qquad \lambda \circ \tilde{T} = \omega_L(\lambda \circ T)$$

To show that (2.16) is well defined, let $T_1, T_2 \in L(X,(Y,C))$ be such that $\omega_L(T_1) = \omega_L(T_2) = \tilde{T}$ and we must show that $\omega_L(\lambda\circ T_1) = \omega_L(\lambda\circ T_2)$. For $z = \omega_X(x_1)-\omega_X(x_2)\in E_X$, $x_1,x_2\in X$, let $z_1 = \omega_X(-1\circ x_1)-\omega_X(-1\circ x_2)\in E_X$. We have

$$\begin{aligned}(\omega_L(\lambda\circ T_1))(z) &= \omega_Y((\lambda\circ T_1)(x_1))-\omega_Y((\lambda\circ T_1)(x_2))\\ &= \omega_Y(T_1(\lambda\circ x_1))-\omega_Y(T_1(\lambda\circ x_2))\\ &= |\lambda|(\omega_Y(T_1(-1\circ x_1))-\omega_Y(T_1(-1\circ x_2)))\\ &= |\lambda|(\omega_L(T_1)(z_1))\\ &= |\lambda|\tilde{T}(z_1)\end{aligned}$$

Similarly, $\omega_L(\lambda\circ T_2)(z) = |\lambda|\tilde{T}(z_1)$, i.e., $\omega_L(\lambda\circ T_1)=\omega_L(\lambda\circ T_2)$ which proves that (2.16) is well defined. It is easy to show that K is an almost linear space. Using (2.14)-(2.16) we get that $\omega_L: L(X,(Y,C)) \longrightarrow K$ is a linear operator.

Since K is a convex cone, the set $E_L = K-K$ is a linear subspace of $L(E_X,E_Y)$ and $\|\cdot\|_{E_L}$ defined by (2.13) is a norm on E_L. Let now $T\in L(X,(Y,C))$ and let $\tilde{T} = \omega_L(T)\in K$ be given by (2.11). Then by Lemma 2.8 we have that $\|\tilde{T}\| = |||T|||$. Now since $\tilde{T} = \tilde{T}-0$ we get $\|\tilde{T}\|_{E_L} \leq \|\tilde{T}\|$ and if $\tilde{T} = \tilde{T}_1-\tilde{T}_2$, $\tilde{T}_1,\tilde{T}_2\in K$ then $\|\tilde{T}\|\leq\|\tilde{T}_1\|+\|\tilde{T}_2\|$, whence by (2.13), $\|\tilde{T}\|\leq\|\tilde{T}\|_{E_L}$. Consequently $\|\tilde{T}\|_{E_L} = \|\tilde{T}\| = |||T|||$, i.e., $\|\omega_L(T)\|_{E_L} = |||T|||$.

Finally, suppose that $L(X,(Y,C))$ is a normed almost linear space. We show that $(K,\|\cdot\|_{E_L})$ is a normed almost linear space, i.e., that $\|\cdot\|_{E_L}$ on K satisfies (N_4). Let $\tilde{T}\in K$, $\tilde{T}_1\in W_K$ and let $T,T'\in L(X,(Y,C))$ be such that $\omega_L(T) = \tilde{T}$ and $\omega_L(T') = \tilde{T}_1$. Let $T_1 = (T'+(-1\circ T'))/2\in W_{L(X,(Y,C))}$. Since ω_L is a linear operator and $\tilde{T}_1 = -1\circ\tilde{T}_1$, we get $\omega_L(T') = -1\circ\omega_L(T') = \omega_L(-1\circ T')$ and $\omega_L(T_1) = (\omega_L(T')+\omega_L(-1\circ T'))/2 = \omega_L(T') = \tilde{T}_1$. Then by the above we obtain $\|\tilde{T}\|_{E_L} = \|\omega_L(T)\|_{E_L} = |||T|||\leq |||T+T_1||| =$

=

$= \| \omega_L(T+T_1) \|_{E_L} = \| \omega_L(T) + \omega_L(T_1) \|_{E_L} = \| \tilde{T} + \tilde{T}_1 \|_{E_L}$ which shows that $(K, \|\cdot\|_{E_L})$ is a normed almost linear space and completes the proof.

2.10. REMARK. Let us note that the cone K defined by (2.12) satisfies the following condition:

$$(2.17) \qquad K \subset \{ \tilde{T} \in L(E_X, E_Y) : \tilde{T}(\omega_X(X)) \subset \omega_Y(Y),\ \tilde{T}(\omega_X(W_X)) \subset \omega_Y(C) \}$$

When the equality sign holds in (2.17), then by [6], Lemma 5.3(iv) it follows that $L(\omega_X(X), (\omega_Y(Y), \omega_Y(C)))$ is a normed almost linear space.

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