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COMPATIBLE PREORDERS AND LINEAR OPERATORS ON R^n

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ABSTRACT

We characterize the total preorders on R^n which are compatible with the vector space structure, in terms of linear operators and the lexicographical order. In particular, we obtain that the lexicographical order is, up to a linear isometry, the unique compatible total order on R^n. We also study compatible total extensions of compatible preorders and give some Szpilrajn type results for compatible preorders and compatible orders.

§ 0. INTRODUCTION

In some previous papers [9-12] (see also [7, 8, 15]), we have studied the lexicographical order and linear operators on R^n, and some of their applications, e.g. to separation of convex sets, vector optimization, hemi-spaces (i.e., convex sets with convex complements), etc. In the present paper, continuing these investigations, we shall give some applications of the lexicographical order and linear operators to the study of compatible total preorders (and, in particular, compatible total orders) on R^n and to compatible total extensions of compatible preorders.

We recall that a preorder (i.e., a reflexive and transitive binary relation) ρ on R^n is said to be <u>compatible with the vector space structure of</u> R^n, or, briefly, <u>compatible</u>, if

$$y_1 \rho y'_1 , \; y_2 \rho y'_2 \Rightarrow y_1 + y_2 \rho \, y'_1 + y'_2 , \tag{0.1}$$

$$y \rho y' , \; \lambda \geq 0 \Rightarrow \lambda y \rho \lambda y' . \tag{0.2}$$

The natural partial order $\leq$ and the lexicographical order $\leq_L$ are well-known examples of compatible orders (i.e., compatible anti-symmetric preorders) on R^n. Let us recall that the natural partial order $\leq$ on R^n is defined componentwise, i.e., denoting the elements of R^n by column vectors and the transpose of a row vector by T,

$x = (\xi_1, \ldots, \xi_n)^T \in R^n$ is said to be "less than or equal to" $y = (\eta_1, \ldots, \eta_n)^T \in R^n$ (in symbols, $x \leq y$) if $\xi_i \leq \eta_i$ $(i = 1, \ldots, n)$. We write $x < y$ if $x \leq y$ and $x \neq y$. Furthermore, $x = (\xi_1, \ldots, \xi_n)^T \in R^n$ is said to be "lexicographically less than" $y = (\eta_1, \ldots, \eta_n)^T \in R^n$ (in symbols, $x <_L y$) if $x \neq y$ and if for $k = \min\{i \in \{1, \ldots, n\} \mid \xi_i \neq \eta_i\}$ we have $\xi_k < \eta_k$. We write $x \leq_L y$ if $x <_L y$ or $x = y$. The notations $y >_L x$ and $y \geq_L x$, respectively, will be also used.

We recall that a preorder ρ on R^n is said to be total, if for any $y, y' \in R^n$ we have either $y \rho y'$ or $y' \rho y$. A well-known example of a total order is the lexicographical order $\leq_L$ on R^n.

In § 1 we shall characterize the compatible total preorders on R^n, in terms of linear operators and the lexicographical order. In particular, for orders, we shall show that the lexicographical order is, up to a linear isometry, the unique compatible total order on R^n. We shall denote by $\mathcal{L}(R^n, R^r)$, $\mathcal{U}(R^n)$ and $\mathcal{O}(R^n)$ the families of all linear operators $u: R^n \to R^r$, all isomorphisms $v: R^n \to R^n$, and all linear isometries $v: R^n \to R^n$ (for the euclidean norm on R^n), respectively. We shall consider on $\mathcal{L}(R^n, R^r)$ the lexicographical order $u \geq_L 0$ in the sense of [8], defined columnwise, i.e., $u \geq_L 0$ if and only if all columns of the $r \times n$ matrix of u (with respect to the unit vector bases of R^n and R^r) are $\geq_L 0$. We shall denote by u^* the adjoint of the operator $u \in \mathcal{L}(R^n, R^r)$ and by I the identity operator on R^r.

We recall that a preorder ρ_2 on R^n is said to be an extension of a preorder ρ_1 on R^n, and we write $\rho_1 \precsim \rho_2$, if

$$y \rho_2 y' \qquad (y, y' \in R^n,\ y \rho_1 y'); \tag{0.3}$$

when ρ_2 is total, we shall say that ρ_2 is a total extension of ρ_1. Let us also recall that a preorder σ on R^n is said to be the intersection of the family of preorders $\{\rho_j\}_{j \in J}$ on R^n, in symbols

$$\sigma = \bigcap_{j \in J} \rho_j , \tag{0.4}$$

provided that for any $y, y' \in R^n$ we have the equivalence

$$y \sigma y' \Leftrightarrow y \rho_j y' \qquad (j \in J); \tag{0.5}$$

hence, in this case, $\sigma \preccurlyeq \rho_j (j \in J)$. By a classical theorem of E. Szpilrajn [16], any partial order is the intersection of its total extensions, and many authors have investigated the problem whether this result remains valid for orders having some prescribed additional property (for a survey, see [1]).

In § 2 we shall characterize the compatible total preorders which are extensions of the natural partial order $\leq$ on R^n, in terms of linear operators and the lexicographical order, and we shall give some Szpilrajn type results for compatible preorders and compatible orders.

The tools which we shall use in the sequel (the correspondence between compatible preorders and convex cones, some concepts and results of [11] on hemi-spaces, etc.) will be recalled in § 1 and § 2.

§ 1. COMPATIBLE TOTAL PREORDERS. UNIQUENESS OF THE LEXICOGRAPHICAL ORDER

It is well-known (see e.g. [14], p.3) that there exists a canonical one-to-one correspondence between the collections of all compatible preorders on R^n and all convex cones (containing 0 as a vertex) in R^n, i.e., all subsets C of R^n satisfying

$$C + C \subseteq C, \tag{1.1}$$

$$\lambda C \subseteq C \qquad (\lambda \geq 0); \tag{1.2}$$

namely, to a preorder ρ there corresponds the convex cone

$$C_\rho = \{y \in R^n \mid y \rho 0\}, \tag{1.3}$$

and, conversely, to a convex cone $C \subseteq R^n$ there corresponds the preorder ρ on R^n, defined by

$$y \rho y' \Leftrightarrow y - y' \in C. \tag{1.4}$$

In particular, this induces a canonical one-to-one correspondence between the collections of all compatible orders on R^n and all "pointed" convex cones in R^n, i.e., all convex cones $C \subseteq R^n$ satisfying

$$C \cap (-C) = \{0\} \tag{1.5}$$

(or, equivalently, containing no line through 0).

We shall use the notation

$$\mathcal{P} = \text{the set of all compatible total preorders on } R^n. \tag{1.6}$$

Theorem 1.1. For any compatible total preorder ρ on R^n there exist a unique $r \in \{0, 1, \dots, n\}$ and a unique $u \in \mathcal{L}(R^n, R^r)$ with $uu^* = I$ (hence $u(R^n) = R^r$), such that

$$y \rho y' \Leftrightarrow u(y) \leq_L u(y') \qquad (y, y' \in R^n). \tag{1.7}$$

Conversely, given any r and $u \in \mathcal{L}(R^n, R^r)$ with $uu^* = I$, if we define $\rho = \rho_u$ by (1.7), then ρ_u is a compatible total preorder on R^n. Consequently, the mapping

$$\delta : u \to \rho_u \tag{1.8}$$

is a bijection of $\bigcup_{r=0}^{n} \{u \in \mathcal{L}(R^n, R^r) \mid uu^* = I\}$ onto $\mathcal{P}$.

Proof. Let $\rho \in \mathcal{P}$. Then, by (1.3) and since ρ is total, we have

$$R^n \setminus C_\rho = \{y \in R^n \mid 0 \rho y\}. \tag{1.9}$$

Therefore, by (0.1) and (0.2), the sets C_ρ (of (1.3)) and $R^n \setminus C_\rho$ are convex, so C_ρ is a hemi-space. Hence, by [11], theorem 2.1, there exist unique $r \in \{0, 1, \ldots, n\}$, $\tau \in \{<_L, \leq_L\}$, $x \in R^r$ and $u \in \mathcal{L}(R^n, R^r)$ with $uu^* = I$, such that

$$C_\rho = \{y \in R^n \mid u(y) \tau x\} . \quad (1.10)$$

We claim that $x = 0$ and τ is $\leq_L$, i.e.,

$$C_\rho = \{y \in R^n \mid u(y) \leq_L 0\} . \quad (1.11)$$

Indeed, by (1.3) and the reflexivity of ρ, we have $0 \in C_\rho$, whence, by (1.10), $0 = u(0) \tau x$. Assume now, a contrario, that $x \neq 0$. Then, by $0 \tau x$ and $\tau \in \{<_L, \leq_L\}$, we have $0 <_L x$. By $uu^* = I$, we have $\operatorname{rank} u = r$, so there exists $y_o \in R^n$ such that $u(y_o) = x$. Then $u(\frac{1}{2} y_o) = \frac{1}{2} x <_L x$, whence $\frac{1}{2} y_o \in C_\rho$ and hence, by (0.2) (with $y' = 0$) and (1.3), $\lambda y_o \in C_\rho$ for all $\lambda > 0$. But then, by (1.10) and $\tau \in \{<_L, \leq_L\}$, we obtain $\lambda x = u(\lambda y_o) \leq_L x$ for all $\lambda > 0$, whence $x \leq_L 0$, a contradiction. Thus, $x = 0$, whence, by (1.10), $\tau \in \{<_L, \leq_L\}$ and $0 \in C_\rho$, it follows that τ is $\leq_L$, which proves the claim (1.11). Hence,

$$y \rho y' \Leftrightarrow y - y' \rho 0 \Leftrightarrow y - y' \in C_\rho \Leftrightarrow u(y - y') \leq_L 0 \Leftrightarrow u(y) \leq_L u(y'), \quad (1.12)$$

which proves (1.7). Also, since for any u satisfying (1.7) we have (1.11), from the uniqueness of r and $u \in \mathcal{L}(R^n, R^r)$ with $uu^* = I$, satisfying (1.11), there follows the uniqueness of r and $u \in \mathcal{L}(R^n, R^r)$ with $uu^* = I$, satisfying (1.7).

Finally, the converse part follows from the properties of the lexicographical order $\leq_L$ and the linearity of u. ☐

Remark 1.1. a) As shown by the above proof, the converse part in theorem 1.1 remains valid for any $u \in \mathcal{L}(R^n, R^r)$ (which need not satisfy $uu^* = I$, nor even $\operatorname{rank} u = r$).

b) By (1.7), u may be regarded as a "vector-valued (namely, R^r-valued) linear lexicographical utility function (for $r = 1$, it becomes a linear utility function, in the usual sense) representing the preorder ρ". Then, for example, remark 2.2 e) of [11] may be regarded as a linear version, for this case, of the well-known condition for two utility functions to represent the same preorder (see e.g. [2], p.97).

For any compatible preorder ρ on R^n, we shall denote by D_ρ the "indifference set" of ρ, i.e.,

$$D_\rho = C_\rho \cap (-C_\rho) = \{y \in R^n \mid y \rho 0, 0 \rho y\} . \qquad (1.13)$$

By (1.2), we have $\lambda D_\rho \subseteq D_\rho$ ($\lambda \in R$), and hence, by (1.1), D_ρ is a linear subspace of R^n (the "indifference subspace" of ρ). We recall that, following [11], a hemi-space H in R^n, represented (uniquely) in the form $H = \{y \in R^n \mid u(y) \tau x\}$, where $r \in \{0, 1, \dots, n\}$, $\tau \in \{<_L, \leq_L\}$, $x \in R^r$ and $u \in \mathcal{L}(R^n, R^r)$, $uu^* = I$, is said to be of type $<_L$ (respectively, of type $\leq_L$), if τ is $<_L$ (respectively, $\leq_L$), and (in either case) the set $M = M(H) = \{y \in R^n \mid u(y) = x\}$ is called "the linear manifold associated to the hemi-space H"(by "linear manifold" we mean a translate of a linear subspace).

Corollary 1.1. For any compatible total preorder ρ on R^n, the cone C_ρ (of (1.3)) is a hemi-space of type $\leq_L$, and the associated linear manifold to C_ρ is the linear subspace

$$M(C_\rho) = D_\rho . \qquad (1.14)$$

Conversely, for any hemi-space H of type $\leq_L$, for which the associated linear manifold M(H) is a linear subspace of R^n, there exists a unique compatible total preorder $\rho = \rho_H$ on R^n, such that

$$C_\rho = H ; \qquad (1.15)$$

moreover, this ρ satisfies $D_\rho = M(H)$.

Proof. The first part follows from the above proof of theorem 1.1, observing that, by (1.11), we have

$$M(C_\rho) = \{y \in R^n \mid u(y) = 0\} = C_\rho \cap (-C_\rho) = D_\rho . \tag{1.16}$$

Conversely, let H be a hemi-space of type $\leq_L$, such that the associated linear manifold M(H) is a linear subspace of R^n. Then, by [11], theorem 2.1, there exist unique $r \in \{0, 1, \dots, n\}$, $x \in R^r$ and $u \in \mathcal{L}(R^n, R^r)$ with $uu^* = I$, such that

$$H = \{y \in R^n \mid u(y) \leq_L x\} . \tag{1.17}$$

But, since $M(H) = \{y \in R^n \mid u(y) = x\}$ is a linear subspace of R^n, we must have $x = 0$, and hence (1.17) becomes

$$H = \{y \in R^n \mid u(y) \leq_L 0\} . \tag{1.18}$$

Therefore, by theorem 1.1, for $\rho = \rho_u$ defined by (1.7) we have $\rho_u \in \mathcal{P}$ and (taking $y' = 0$ in (1.7))

$$H = \{y \in R^n \mid y\rho 0\} = C_\rho , \quad M(H) = \{y \in R^n \mid y \rho 0,\ 0 \rho y\} = D_\rho .$$

Finally, the uniqueness of $\rho \in \mathcal{P}$ satisfying (1.15) follows from the fact that C_ρ determines ρ uniquely. ∎

Remark 1.2. For related results on relations between hemi-spaces and compatible total preorders see also corollaries 1.2, 1.4 below and [6], proposition 2.2 and proposition 2.3, equivalence (2)⇔(7) (in the framework of linear spaces over an ordered field).

Corollary 1.2. <u>For a compatible preorder</u> ρ <u>on</u> R^n, <u>the following statements are equivalent:</u>

1°. ρ <u>is total.</u>

2°. C_ρ <u>(of</u> (1.3)) <u>is a hemi-space of type</u> $\leq_L$, <u>and the associated linear manifold</u>

$M(C_\rho)$ is a linear subspace of R^n.

3°. There exists a unique $r \in \{0, 1, \dots, n\}$ and a unique $u \in \mathcal{L}(R^n, R^r)$ with $uu^* = I$, such that we have (1.11). ∎

Definition 1.1. For any $\rho \in \mathcal{P}$, we define the rank of ρ by

$$r(\rho) = \text{the unique } r \in \{0, 1, \dots, n\} \text{ of corollary 1.2 .} \tag{1.19}$$

Remark 1.3. Definition 1.1 yields a classification of all compatible total preorders ρ on R^n (into those with $r(\rho) = 0$, those with $r(\rho) = 1$, etc.). This is quite natural, by [11], theorem 2.2, according to which the rank and type of hemi-spaces give a (metric-affine) classification of hemi-spaces, since now $r(\rho)$ coincides with the rank of C_ρ and the type of ρ is fixed (namely, it is $\leq_L$, by corollary 1.2).

b) By [11], theorem 2.1, for the class of compatible preorders on R^n, of a given rank r, one can choose a "canonical representative" ρ^r, defined by

$$(\eta_1, \dots, \eta_n)^T \rho^r (\eta'_1, \dots, \eta'_n)^T \Leftrightarrow (\eta_1, \dots, \eta_r)^T \leq_L (\eta'_1, \dots, \eta'_r)^T. \tag{1.20}$$

Corollary 1.3. For $\rho_1, \rho_2 \in \mathcal{P}$, the following statements are equivalent:

1°. We have $r(\rho_1) = r(\rho_2)$.

2°. We have (with D_ρ of (1.13))

$$\dim D_{\rho_1} = \dim D_{\rho_2} . \tag{1.21}$$

3°-4°. There exists $v \in \mathcal{U}(R^n)$ (respectively, $v \in \mathcal{O}(R^n)$), such that

$$y \rho_1 y' \Leftrightarrow v(y) \rho_2 v(y') \qquad (y, y' \in R^n) \tag{1.22}$$

(in other words, such that $v : (R^n, \rho_1) \to (R^n, \rho_2)$ is an isomorphism of preordered sets).

Proof. 1°⇔2°, by $uu^* = I$ and $D_\rho = \operatorname{Ker} u$, for u of corollary 1.2. Finally,

$1° \Leftrightarrow 3° \Leftrightarrow 4°$, by (1.11) and [11], theorem 2.2. ■

Let us denote

$$\mathcal{T} = \text{the set of all compatible total orders on } R^n. \tag{1.23}$$

Theorem 1.2. For any compatible total order ρ on R^n there exists a unique $u \in \mathcal{O}(R^n)$ such that we have (1.7).

Conversely, given any $u \in \mathcal{O}(R^n)$, if we define $\rho = \rho_u$ by (1.7), then ρ_u is a compatible total order on R^n. Consequently, the restriction of the bijection ρ of (1.8), to $\mathcal{O}(R^n) = \{u \in \mathcal{L}(R^n, R^n) \mid uu^* = I\}$, maps $\mathcal{O}(R^n)$ onto $\mathcal{T}$.

Proof. Let $\rho \in \mathcal{T}$ and, by theorem 1.1, let $u = \delta^{-1}(\rho) \in \mathcal{L}(R^n, R^r)$, with $r \in \{0, 1, \ldots, n\}$. Then, for any $y \in \operatorname{Ker} u$, by $u(y) = 0 = u(0)$ and (1.7), we have $y \rho 0$ and $0 \rho y$. Hence, since $\rho \in \mathcal{T}$ is anti-symmetric, we obtain $y = 0$. Thus, $\operatorname{Ker} u = \{0\}$, which, by $r \leq n$ and $u \in \mathcal{L}(R^n, R^r)$, implies $r = n$. Hence, by $uu^* = I$, we obtain $u \in \mathcal{O}(R^n)$.

Conversely, let $u \in \mathcal{O}(R^n)$ (hence $uu^* = I$), and let $\rho_u = \delta(u) \in \mathcal{P}$. Then, for any $y, y' \in R^n$ with $y \rho_u y'$ and $y' \rho_u y$ we have, by (1.7), $u(y) \leq_L u(y')$ and $u(y') \leq_L u(y)$, i.e., $u(y) = u(y')$, whence, by $u \in \mathcal{O}(R^n)$, we obtain $y = y'$. Thus, $\rho_u \in \mathcal{P}$ is anti-symmetric, i.e., $\delta(u) = \rho_u \in \mathcal{T}$. ■

Remark 1.4. a) As shown by theorem 1.2, the lexicographical order $\leq_L$ on R^n is, up to a (unique) $u \in \mathcal{O}(R^n)$, the unique compatible total order on R^n.

b) The theorem on "lexicographical separation" of a convex set from an outside point ([17], p.258; see also [5,9]) can be restated, using theorem 1.2, in the following equivalent form: A set $G \subsetneq R^n$ is convex if and only if for each $y_o \in R^n \setminus G$ there exists $\rho = \rho(y_o) \in \mathcal{T}$ such that

$$g \rho y_o \qquad (g \in G). \tag{1.24}$$

Note that this is a characterization of convexity in R^n in terms of the compatible total orders ρ on R^n, which permits to generalize the concept of convexity (see [13]).

c) The theorem on "lexicographical separation" of two sets in R^n ([9], theorem 2.1, equivalence (1)$\Leftrightarrow$(4)) can be restated, using theorem 1.2, in the following equivalent form: For two sets $G_1, G_2 \subset R^n$ we have $\mathrm{co}\, G_1 \cap \mathrm{co}\, G_2 = \emptyset$ if and only if there exists $\rho = \rho(G_1, G_2) \in \mathcal{T}$ such that

$$g_1 \rho g_2 \qquad (g_1 \in G_1, g_2 \in G_2); \tag{1.25}$$

this latter result has been obtained, with different methods, by Coquet and Dupin ([3], theorem 7, equivalence (1)$\Leftrightarrow$(2)), in arbitrary real linear spaces.

Corollary 1.4. For a compatible order ρ on R^n, the following statements are equivalent:

1°. ρ is total.

2°. C_ρ is the complement of a semi-space at 0 (i.e., $C_\rho = R^n \setminus S$, where S is a maximal convex subset of R^n such that $0 \notin S$).

3°. $r(\rho) = n$.

Proof. $1^\circ \Leftrightarrow 2^\circ$. By theorem 1.2, we have 1° if and only if there exists (a unique) $u \in \mathcal{U}(R^n)$ such that we have (1.7). But, (1.7) holds if and only if it holds for $y' = 0$, i.e.,

$$C_\rho = \{y \in R^n \mid u(y) \leq_L 0\} = R^n \setminus \{y \in R^n \mid u(y) >_L 0\}; \tag{1.26}$$

also, by [15], lemma 1.1, a set $S \subset R^n$ is a semi-space at 0 if and only if $S = \{y \in R^n \mid u(y) >_L 0\}$ for some $u \in \mathcal{U}(R^n)$.

$2^\circ \Leftrightarrow 3^\circ$. By definition 1.1, $r(\rho)$ is the rank [11] of the hemi-space C_ρ, or, equivalently, of the hemi-space $S = R^n \setminus C_\rho$. But, by [11], remark 2.3 d), S is a semi-space if and only if it is of rank n. ∎

Remark 1.5. The equivalence $1^\circ \Leftrightarrow 2^\circ$ is due, essentially, to P.C. Hammer ([14], theorem 1.4).

§ 2. COMPATIBLE TOTAL EXTENSIONS OF COMPATIBLE PREORDERS

Note that, if ρ_1 and ρ_2 are compatible preorders on R^n, we have (with the notations $\preccurlyeq$ and C_ρ of § 0 and (1.3)) the equivalence

$$\rho_1 \preccurlyeq \rho_2 \Leftrightarrow C_{\rho_1} \subseteq C_{\rho_2}. \tag{2.1}$$

Let us first consider compatible total extensions $\rho \in \mathcal{P}$ of the natural partial order $\leq$ on R^n. Since (by theorem 1.1 and remark 1.1a)) the compatible total preorders $\rho \in \mathcal{P}$ are precisely the preorders $\rho = \rho_u$ defined by (1.7), where $u \in \mathcal{L}(R^n, R^r)$, we shall state the next result in terms of ρ_u.

Theorem 2.1. For a compatible total preorder $\rho = \rho_u \in \mathcal{P}$ defined by (1.7), where $u \in \mathcal{L}(R^n, R^r)$, the following statements are equivalent:

1°. ρ_u is an extension of the natural partial order $\leq$ on R^n.

2°. $u \geq_L 0$.

Proof. By (0.3), (1.7) and [8], corollary 2.3, we have the equivalences

$1° \Leftrightarrow y \rho_u y' \ (y \leq y') \Leftrightarrow u(y) \leq_L u(y') \ (y \leq y') \Leftrightarrow u(y) \geq_L 0 \ (y \geq 0) \Leftrightarrow 2°.$ ∎

Corollary 2.1. For a compatible total preorder $\rho \in \mathcal{P}$, the following statements are equivalent:

1°. ρ is an extension of $\leq$.

2°. $\delta^{-1}(\rho) \geq_L 0$ (where δ is the bijection (1.8)). ∎

Remark 2.1. a) As shown by simple examples, for $n > 1$ there exist total extensions of $\leq$ which are not compatible with the vector space structure of R^n.

b) By the equivalence (2.1) for compatible preorders, theorem 2.1 means, geometrically, that for $\rho_u \in \mathcal{P}$ we have $C_{\leq} = \{y \in R^n \mid y \leq 0\} \subseteq C_{\rho_u}$ if and only if $u \geq_L 0$.

Now we shall give some Szpilrajn type results for compatible preorders and compatible orders. Note that, if σ and ρ_j ($j \in J$) are compatible preorders on R^n, we have (with the notations $\bigcap_{j\in J} \rho_j$ of §0 and (1.3)) the equivalence

$$\sigma = \bigcap_{j\in J} \rho_j \Leftrightarrow C_\sigma = \bigcap_{j\in J} C_{\rho_j} . \tag{2.2}$$

Let us first prove that following proposition, which we shall need in the sequel.

Proposition 2.1. Let σ be a compatible preorder on R^n and let $\bar{y} \notin -C_\sigma$. Then there exists a compatible total extension ρ of σ, such that

$$D_\rho = D_\sigma , \tag{2.3}$$

$$\bar{y} \in C_\rho . \tag{2.4}$$

Proof. We claim that

$$\text{co}\,((C_\sigma \setminus D_\sigma) \cup \{\bar{y}\}) \cap D_\sigma = \emptyset . \tag{2.5}$$

Indeed, assume, a contrario, that there exist $y \in C_\sigma \setminus D_\sigma$ and $0 < \lambda < 1$ such that $(1-\lambda)y + \lambda\bar{y} \in D_\sigma$. Then, since D_σ is a linear subspace, $-y - (\lambda/1-\lambda)\bar{y} \in D_\sigma$, whence, since $y \in C_\sigma$ and C_σ is a convex cone, we obtain $-(\lambda/1-\lambda)\bar{y} = y + (-y - (\lambda/1-\lambda)\bar{y}) \in C_\sigma$. Hence, $-\bar{y} \in C_\sigma$, in contradiction with our assumption. This proves the claim (2.5).

Now, by (2.5) and Zorn's lemma, there exists a maximal convex subset H of R^n (with respect to inclusion), such that

$$\text{co}\,((C_\sigma \setminus D_\sigma) \cup \{\bar{y}\}) \subseteq H, \qquad H \cap D_\sigma = \emptyset . \tag{2.6}$$

But, then, H is also a maximal convex subset of R^n such that $H \cap D_\sigma = \emptyset$; indeed, if not, i.e., if there exists a convex subset C of R^n such that $H \subset C$, $H \neq C$ and $C \cap D_\sigma = \emptyset$, then $co((C_\sigma \setminus D_\sigma) \cup \{\bar{y}\}) \subseteq H \subset C$, $H \neq C$ and $C \cap D_\sigma = \emptyset$, in contradiction with the maximality of H with property (2.6). Hence, since D_σ is a linear subspace of R^n, from [11], theorem 3.2 it follows that H is a hemi-space of type $<_L$, with associated linear manifold $M(H) = D_\sigma$, so $H \cup D_\sigma$ is a hemi-space of type $\leq_L$, with $M(H \cup D_\sigma) = D_\sigma$. Therefore, by corollary 1.1, there exists a unique compatible total preorder ρ on R^n, such that

$$C_\rho = H \cup D_\sigma, \; D_\rho = M(H \cup D_\sigma) = D_\sigma. \tag{2.7}$$

Then, since $C_\sigma \setminus D_\sigma \subseteq H$, we have $C_\sigma \subseteq H \cup D_\sigma = C_\rho$, so ρ is an extension of σ. Finally, $\bar{y} \in H \subset H \cup D_\sigma = C_\rho$. ∎

Remark 2.2. Proposition 2.1 corresponds to a lemma of Szpilrajn ([16], p.387) on general posets.

Now we can prove the following Szpilrajn type result.

Theorem 2.2. Any compatible preorder σ on R^n is the intersection of its compatible total extensions ρ satisfying (2.3).

Proof. By the equivalences (2.1) and (2.2) for compatible preorders, theorem 2.2 means that

$$C_\sigma = \bigcap_{\substack{\rho \in \mathcal{P} \\ C_\sigma \subseteq C_\rho, D_\rho = D_\sigma}} C_\rho . \tag{2.8}$$

The inclusion $\subseteq$ in (2.8) is obvious, recalling that $\bigcap_{\rho \in \emptyset} C_\rho = R^n$ (but, actually, $\{\rho \in \mathcal{P} \mid C_\sigma \subseteq C_\rho, D_\rho = D_\sigma\} \neq \emptyset$, by corollary 2.2 below).

In order to prove the opposite inclusion in (2.8), assume, a contrario, that there exists an element

$$y \in \Big(\bigcap_{\substack{\rho \in \mathcal{P} \\ C_\sigma \subseteq C_\rho,\, D_\rho = D_\sigma}} C_\rho\Big) \setminus C_\sigma . \tag{2.9}$$

Then, since $y \notin C_\sigma$, by proposition 2.1 (applied to $\bar{y} = -y$) there exists $\rho_0 \in \mathcal{P}$ with $C_\sigma \subseteq C_{\rho_0}$, $D_{\rho_0} = D_\sigma$, such that $-y \in C_{\rho_0}$, whence, by (2.9), $y \in C_{\rho_0} \cap (-C_{\rho_0}) = C_\sigma \cap (-C_\sigma)$, in contradiction with $y \notin C_\sigma$. ■

One can define a trivial compatible total preorder ρ_0 on R^n, by

$$y \rho_0 y' \qquad (y, y' \in R^n); \tag{2.10}$$

then, by the equivalence (2.1), ρ_0 is the largest compatible preorder on R^n. We shall call the compatible preorders $\sigma \neq \rho_0$ non-trivial.

Remark 2.3. a) By theorem 2.2, every non-trivial maximal compatible preorder σ on R^n (i.e., admitting no proper extension to a non-trivial compatible preorder on R^n) is total. However, the converse is not true, as shown e.g. by the lexicographical order $\leq_L$ on R^n, which is compatible and total, but not a maximal compatible preorder on R^n; indeed, it admits, for example, the proper extension $\rho \in \mathcal{P}$ defined by

$$(\eta_1, \ldots, \eta_n)^T \rho (\eta'_1, \ldots, \eta'_n)^T \Leftrightarrow \eta_1 \leq \eta'_1 . \tag{2.11}$$

b) By the equivalence (2.1) and by (1.3), a non-trivial compatible preorder ρ on R^n is maximal if and only if C_ρ is a closed half-space (since the closed half-spaces are the maximal non-trivial convex cones).

Corollary 2.2. A compatible preorder σ on R^n is total if and only if it is maximal among those having the same indifference subspace $D_\sigma = C_\sigma \cap (-C_\sigma)$.

Proof. If $\sigma \in \mathcal{P}$ is not total, then, by theorem 2.2, it admits a proper total extension $\rho \in \mathcal{P}$ with $D_\rho = D_\sigma$, so σ is not maximal (among the compatible preorders $\rho \in \mathcal{P}$ with $D_\rho = D_\sigma$).

Conversely, assume now that σ is total and let $\rho \in \mathcal{P}$ be any compatible preorder such that $\sigma \preccurlyeq \rho$, $D_\rho = D_\sigma$. We shall show that $\rho \preccurlyeq \sigma$, whence $\rho = \sigma$, so σ is maximal (among the $\rho \in \mathcal{P}$ with $D_\rho = D_\sigma$). Assume, a contrario, that there exist $y_1, y_2 \in R^n$ such that $y_1 \rho y_2$, $y_1 \bar{\sigma} y_2$. Then, since σ is total, we have $y_2 \sigma y_1$, whence, by $\sigma \preccurlyeq \rho$, we obtain $y_2 \rho y_1$. Therefore, $y_1 - y_2 \in D_\rho = D_\sigma$, which contradicts $y_1 \bar{\sigma} y_2$. ∎

Let us consider now compatible orders on R^n.

Theorem 2.3. Any compatible order σ on R^n is the intersection of its extensions to compatible total orders on R^n.

Proof. This follows immediately from theorem 2.2, since a compatible preorder ρ is an order if and only if $D_\rho = \{0\}$. ∎

Note that, since the preorder ρ_o of (2.10) is not an order, every compatible order is non-trivial.

In contrast with remark 2.3 a), we have now

Corollary 2.3. A compatible order σ on R^n is total if and only if it is maximal (i.e., admits no proper extension to a compatible order ρ on R^n).

Proof. This follows from corollary 2.2, applied to $D_\sigma = \{0\}$. ∎

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MULTI-ORDER CONVEXITY

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Dedicated to Professor V. Klee in honor of his 65^{th} birthday

Abstract. We introduce and study the concepts of segmential convexity and separational convexity in a multi-ordered set (i.e., a set endowed with a non-empty family of partial orders). We prove that, for a suitable family $\mathcal{T}$ of total orders on R^n, the segmentially convex sets and the separationally convex sets in the multi-ordered set $(R^n,\mathcal{T})$ coincide with the usual (vector) convex sets in R^n. Furthermore, we study two concepts of discrete convexity, namely, segmential and separational convexity in $(Z^n,\mathcal{T}')$, where $\mathcal{T}'$ is the family of restrictions to Z^n of the orders belonging to $\mathcal{T}$. Also, we prove that the usual order convexity in a poset coincides with the segmential convexity and the separational convexity in some associated multi-ordered sets.

§0. INTRODUCTION

The aim of the present paper is to introduce two natural general concepts of convexity of subsets G of a set S, in terms of (partial) orders on S, encompassing, as particular cases, various known notions of (continuous and discrete) convexity. Our first concept is defined with the "segmential" (or "inner") approach, defining first the notion of a "segment" $<x,y>$ in S and then calling a set G "convex", if the relations $x,y\in G$ imply $<x,y>\subseteq G$. The second one of our concepts is defined with the "separational" (or "outer") approach, calling a set $G\subseteq S$ "convex", if every $x\in S\setminus G$ can be "separated" from G, in a certain sense (defined in terms of partial orders on S).

Concerning the well-known concept of "order convexity" of a subset G of a poset $(S,\leqq)$, we shall see in §2 that there exists no order relation $\leqq$ on R^n $(n\geqq 2)$ for which the order convex subsets of $(R^n,\leqq)$ are the usual (vector) convex subsets of R^n. Therefore we shall use, instead of posets, the following natural framework.

Definition 0.1. We call multi-ordered set an ordered pair $(S,\mathcal{O})$, where S is a set and $\mathcal{O}$ is a non-empty family of partial orders (i.e., reflexive anti-symmetric transitive binary relations) on S.

Remark 0.1. a) In the particular case when $\mathcal{O}$ is a singleton, say, $\mathcal{O}=\{\leqq\}$, the multi-ordered set $(S,\mathcal{O})=(S,\{\leqq\})$ can be identified with the poset $(S,\leqq)$.

b) Although not in the above generality, multi-ordered sets have been used e.g. in [4] (in linear spaces) and [13], [14]; moreover, in [14] there has been introduced a concept of convexity in multi-ordered topological spaces, different from those introduced in the present paper. For other concepts of a set endowed with a family of binary relations, see also [1] and the references therein.

In §1 of the present paper we introduce and study $\mathcal{O}$-segments and $\mathcal{O}$-segmentially convex (or, briefly, $\mathcal{O}_{seg}$-convex) sets in a multi-ordered set $(S,\mathcal{O})$. We also consider the concept of an $\mathcal{O}_{seg}$-semi-space in $(S,\mathcal{O})$, generalizing the notion of a semi-space in a linear space, introduced by P.C. Hammer [5]; the first fundamental results about semi-spaces in linear spaces have been obtained by P.C. Hammer [5] and V. Klee [9].

In §2 we study segmential convexity in the multi-ordered set $(R^n,\mathcal{T})$, where $\mathcal{T}$ is the set of all total orders on R^n which are compatible with the vector space structure of R^n. We prove that the $\mathcal{T}$-segments of R^n coincide with the usual segments of R^n and, hence, $\mathcal{T}_{seg}$-convexity in R^n coincides with the usual convexity in R^n.

In §3 we show that segmential convexity in $(Z^n,\mathcal{T}')$, where $Z=\{\dots,-2,-1,0,1,2,\dots\}$ and $\mathcal{T}'$ is the family of restrictions to the subset Z^n of R^n of the order relations belonging to $\mathcal{T}$ of §2, is equivalent to a concept of discrete convexity considered by L. Lupşa ([11], [12]).

In §4 we prove that the usual order convex sets of a poset $(S,\leq)$ coincide with the segmentially convex sets of some associated multi-ordered sets, having the same "ground set" S.

In §5 we introduce and study $\mathcal{O}$-separationally convex (or, briefly, $\mathcal{O}_{sep}$-convex) sets in a multi-ordered set $(S,\mathcal{O})$, defined by means of a separation property by elements of $\mathcal{O}$. We show that every $\mathcal{O}_{sep}$-convex set is $\mathcal{O}_{seg}$-convex, but the converse is not true (in general). We also prove that, for any multi-ordered set $(S,\mathcal{O})$, the family of $\mathcal{O}_{sep}$-convex sets coincides with that of $\mathscr{T}_{sep}$-convex sets in an associated multi-ordered set $(S,\mathscr{T})$ (with the same S), such that $\mathscr{T}$ consists of total orders.

In §6 we prove that in $(R^n,\mathcal{T})$ (with $\mathcal{T}$ of §2), $\mathcal{T}_{sep}$-convexity, too, coincides with the usual convexity in R^n.

In §7 we give some characterizations of separational convexity in $(Z^n,\mathcal{T}')$ (with $\mathcal{T}'$ of §3), one of which shows that separational convexity in $(Z^n,\mathcal{T}')$ is equivalent to another concept of discrete convexity considered by L. Lupşa ([11], [12]). Hence, we infer that there exist $\mathcal{T}'_{seg}$-convex sets which are not $\mathcal{T}'_{sep}$-convex.

Finally, in §8, we prove that the usual order convexity in a poset $(S,\leq)$ coincides with $\mathcal{O}_{sep}$-convexity in an associated multi-ordered set, having the same "ground set" S.

We emphasize that the concepts of segmential and separational convexity in multi-ordered sets, introduced in this paper, are motivated by the above mentioned results of §2 and §6, according to which the $\mathcal{T}_{seg}$-convexity and $\mathcal{T}_{sep}$-convexity in $(R^n,\mathcal{T})$ coincide with the usual convexity in R^n. The proofs of both results are based on a lexicographical separation theorem ([15], p. 258; see also [16], theorem 2.1), closely related to a lexicographical separation theorem of V. Klee ([10], §2.4). Since the lexicographical order on R^n will play an important role in the sequel, we recall that $x=\{\xi_i\}_1^n\in R^n$ is said to be "lexicographically less than" $y=\{\eta_i\}_1^n \in R^n$ (in symbols, $x<_L y$) if $x\neq y$ and if for $k=\min\{i\in\{1,\dots,n\} \mid \xi_i\neq\eta_i\}$ we have $\xi_k<\eta_k$. We write $x\leq_L y$ if $x<_L y$ or $x=y$. We denote by $(R^n)^*$ the conjugate space of R^n, identified with R^n in the usual way, and by $O(R^n)$ the family of all linear isometries $v:R^n\to R^n$ (for the euclidean norm $\|\cdot\|$ on R^n).

§1. SEGMENTIAL MULTI-ORDER CONVEXITY

Definition 1.1. Let $(S,\mathcal{O})$ be a multi-ordered set.

a) Given $x,y\in S$, we define the $\mathcal{O}$-segment $\langle x,y\rangle=\langle x,y\rangle_{\mathcal{O}}$ by

$$\langle x,y\rangle=\langle x,y\rangle_{\mathcal{O}}=\bigcap_{\rho\in\mathcal{O}}\langle x,y\rangle_\rho \quad , \tag{1.1}$$

where

$$\langle x,y\rangle_\rho=\{z\in S \mid \text{either } x\rho z\rho y \text{ or } y\rho z\rho x\} \qquad (\rho\in\mathcal{O}). \tag{1.2}$$

b) We call a set $G\subseteq S$, "$\mathcal{O}$-segmentially convex", or, briefly, $\mathcal{O}_{seg}$-convex, if

$$\langle x,y\rangle\subseteq G \qquad (x,y\in G). \tag{1.3}$$

In the sequel we shall assume, without any special mention, that $(S,\mathcal{O})$ is a multi-ordered set.

Proposition 1.1. For any $x,y\in S$, the following statements are equivalent:

1°. $\langle x,y\rangle\neq\emptyset$.

2°. $x,y\in\langle x,y\rangle$.

3°. For each $\rho\in\mathcal{O}$, either $x\rho y$ or $y\rho x$.

Proof. 1°⇒3°, by the transitivity of each $\rho\in\mathcal{O}$. Also, 3°⇒2°, by the reflexivity of each $\rho\in\mathcal{O}$. Finally, the implication 2°⇒1° is obvious.

We recall that an order ρ on S is said to be total, if for any $x,y\in S$ we have either $x\rho y$ or $y\rho x$.

Corollary 1.1. The following statements are equivalent:

1°. $\langle x,y\rangle_{\mathcal{O}}\neq\emptyset$ $(x,y\in S)$.

2°. $x,y\in\langle x,y\rangle_{\mathcal{O}}$ $(x,y\in S)$.

3°. All $\rho\in\mathcal{O}$ are total.

Remark 1.1. a) By corollary 1.1, implication 1°⇒3°, if there exists $\rho\in\mathcal{O}$ which is not total, then there exist $x,y\in S$ such that $\langle x,y\rangle_{\mathcal{O}}=\emptyset$; thus, in this case, the axiom J1 for "join geometries" (see [19], p. 209) is not satisfied. Moreover, let us also note that, by proposition 1.1, equivalence 1°⇔3°, for $x,y\in S$ we have $\langle x,y\rangle_{\mathcal{O}}=\emptyset$ if and only if there exists $\rho=\rho(x,y)\in\mathcal{O}$ such that $x\bar\rho y$ and $y\bar\rho x$, where $\bar\rho$ denotes the negation of ρ.

b) One may (and, sometimes, we shall) assume, without loss of generality, that for (some, or all) $\rho\in\mathcal{O}$ we have $\rho^{-1}\in\mathcal{O}$ (since the segments $\langle x,y\rangle$ remain the same), where ρ^{-1} is the "reverse order" to ρ, defined by

$$x\rho^{-1}y\Leftrightarrow y\rho x \qquad (x,y\in S). \tag{1.4}$$

c) We have

$$\langle x,y\rangle=\langle y,x\rangle \qquad (x,y\in S). \tag{1.5}$$

Clearly, the whole set S, the empty set $\emptyset$, and every singleton $\{x\}$, $(=\langle x,x\rangle)$, where $x\in S$, are $\mathcal{O}_{seg}$-convex. Furthermore, we have

Proposition 1.2. For any $x,y\in S$, *the segment* $\langle x,y\rangle$ *is* $\mathcal{O}_{seg}$*-convex.*

Proof. Let $z,t\in\langle x,y\rangle$, $u\in\langle z,t\rangle$ and $\rho\in\mathcal{O}$. Then, there are eight cases:

i) If $z\rho u\rho t$, $x\rho z\rho y$, $x\rho t\rho y$, then $x\rho z\rho u\rho t\rho y$, whence $x\rho u\rho y$, so $u\in\langle x,y\rangle_\rho$.

ii) If $z\rho u\rho t$, $x\rho z\rho y$, $y\rho t\rho x$, then $z\rho u\rho t\rho x\rho z\rho y\rho t$, whence $z=u=t=x=y$, so $u\in\langle x,y\rangle_\rho$.

iii) If $z\rho u\rho t$, $y\rho z\rho x$, $x\rho t\rho y$, then $z\rho u\rho t\rho y\rho z\rho x\rho t$, whence $z=u=t=y=x$, so $u\in\langle x,y\rangle_\rho$.

iv) If $z\rho u\rho t$, $y\rho z\rho x$, $y\rho t\rho x$, then $y\rho z\rho u\rho t\rho x$, whence $y\rho u\rho x$, so $u\in\langle x,y\rangle_\rho$.

Finally, the other four cases are obtained from i)-iv) by interchanging the roles of z and t.

Proposition 1.3. a) *For any* $\rho\in\mathcal{O}$, *each set* $M\subseteq S$ *such that*

$$x\in M,\quad u\in S,\quad x\rho u \Rightarrow u\in M, \tag{1.6}$$

is $\mathcal{O}_{seg}$*-convex.*

b) *For any* $\rho\in\mathcal{O}$, *each set* $N\subseteq S$ *such that*

$$x\in N,\quad u\in S,\quad u\rho x \Rightarrow u\in N, \tag{1.7}$$

is $\mathcal{O}_{seg}$*-convex.*

Proof. a) Assume that $\rho\in\mathcal{O}$ and $M\subseteq S$ satisfy (1.6), and let $z,t\in M$ and $u\in\langle z,t\rangle$. Then, for each $\rho'\in\mathcal{O}$, we have either $z\rho'u\rho't$ or $t\rho'u\rho'z$. Let $\rho'=\rho$. If $z\rho u\rho t$, then, by $z\in M$ and (1.6), we have $u\in M$. On the other hand, if $t\rho u\rho z$, then by $t\in M$ and (1.6) we have, again, $u\in M$. Thus, M is $\mathcal{O}_{seg}$-convex.

b) The proof is similar to that of part a). Alternatively, assuming that $\rho^{-1}\in\mathcal{O}$ (see remark 1.1 b)), part b) follows from part a) applied to ρ^{-1}.

For each $a\in S$ and $\rho\in\mathcal{O}$, let us denote

$$M_{a\rho}=\{x\in S\,|\,x\bar\rho a\},\qquad M'_{a\rho}=\{x\in S\,|\,a\bar\rho x\}, \tag{1.8}$$

$$C_{a\rho}=\{x\in S\,|\,a\rho x\},\qquad C'_{a\rho}=\{x\in S\,|\,x\rho a\}; \tag{1.9}$$

clearly, $C_{a\rho}\subseteq M_{a\rho}\cup\{a\}$ and $C'_{a\rho}\subseteq M'_{a\rho}\cup\{a\}$. The sets $C_{a\rho}$ have been called "upper cones" in [20], and the sets $C'_{a\rho}$ may be called "lower cones", in the poset (S,ρ). The sets $M_{a\rho}$ and $M'_{a\rho}$ are useful in vector optimization.

Definition 1.2. We call $\mathcal{O}_{seg}$*-hemi-space*, any set $M\subseteq S$ such that both M and $S\setminus M$ are $\mathcal{O}_{seg}$-convex.

From proposition 1.2 we obtain

Corollary 1.2. For any $a\in S$ *and* $\rho\in\mathcal{O}$, *we have the pairs of complementary* $\mathcal{O}_{seg}$*-hemi-spaces*

$$(M_{a\rho},C'_{a\rho}),\ (M_{a\rho}\cup\{a\},\ C'_{a\rho}\setminus\{a\}),\ (M'_{a\rho},C_{a\rho}),\ (M'_{a\rho}\cup\{a\},\ C_{a\rho}\setminus\{a\}). \tag{1.10}$$

Proof. If $a\in S$, $\rho\in\mathcal{O}$, $x\in M_{a\rho}$ and $x\rho u$, then $u\in M_{a\rho}$ (since otherwise $x\rho u\rho a$, in contradiction with $x\in M_{a\rho}$), so $M_{a\rho}$ satisfies (1.6). Furthermore, let $x=a$ and $x\rho u$. If $u=a$, then $u\in M_{a\rho}\cup\{a\}$; if $u\neq a$, then $u\bar\rho a$, whence $u\in M_{a\rho}\subseteq M_{a\rho}\cup\{a\}$. Thus, $M_{a\rho}\cup\{a\}$ satisfies (1.6). Finally, if $a\in S$, $\rho\in\mathcal{O}$, $x\in C_{a\rho}$ and $x\rho u$, then $u\in C_{a\rho}$ (since $a\rho x\rho u$) and, if also $x\neq a$, then $u\neq a$ (by $a\rho x\rho u$); thus, $C_{a\rho}$ and $C_{a\rho}\cup\{a\}$ satisfy (1.6). With a similar argument (or, alternatively, assuming that $\rho^{-1}\in\mathcal{O}$ and applying the above to ρ^{-1}), it follows that $M'_{a\rho}$, $M'_{a\rho}\cup\{a\}$, $C'_{a\rho}$ and $C'_{a\rho}\cup\{a\}$ satisfy (1.7). Hence, by proposition 1.3, all eight sets in (1.10) are $\mathcal{O}_{seg}$-convex.

Definition 1.3. For $a\in S$, we call $\mathcal{O}_{seg}$*-semi-space at a* (or, $\mathcal{O}_{seg}$*-copoint at a*), any maximal (in the sense of inclusion) $\mathcal{O}_{seg}$-convex subset of $S\setminus\{a\}$.

Corollary 1.3. If $a\in S$ is a ρ-minimal or a ρ-maximal element for some $\rho\in\mathcal{O}$, then $S\setminus\{a\}$ is the only $\mathcal{O}_{seg}$-semi-space at a.

Proof. If $a\in S$ is ρ-minimal, i.e., if

$$M_{a\rho}=\{x\in S\,|\,x\bar{\rho}a\}=S\setminus\{a\}, \tag{1.11}$$

then, by corollary 1.2, $S\setminus\{a\}$ is $\mathcal{O}_{seg}$-convex, and hence, clearly, a maximal $\mathcal{O}_{seg}$--convex subset of $S\setminus\{a\}$.

If $a\in S$ is ρ-maximal, i.e., if

$$M'_{a\rho}=\{x\in S\,|\,a\bar{\rho}x\}=S\setminus\{a\}, \tag{1.12}$$

the argument is similar.

Proposition 1.4. For any $\rho\in\mathcal{O}$, the family of all subsets M *(respectively,* N*) of* S *which satisfy* (1.6) *(respectively,* (1.7)*) is closed under union and intersection. Hence, the union and the intersection of any family of subsets of* S *whose all members satisfy* (1.6), *or* (1.7), *is $\mathcal{O}_{seg}$-convex.*

Proof. Let $\{M_i\}_{i\in I}\subseteq S$ be such that each M_i satisfies (1.6), and let $x\in\bigcup_{i\in I}M_i$, $u\in S$, $x\rho u$. Then $x\in M_{i_o}$ for some $i_o\in I$, whence, by our assumption, $u\in M_{i_o}\subseteq\bigcup_{i\in I}M_i$. Thus, $\bigcup_{i\in I}M_i$ satisfies (1.6) and hence, by proposition 1.3, it is $\mathcal{O}_{seg}$-convex. For $\bigcap_{i\in I}M_i$ the argument is similar (see also proposition 1.6 below). For (1.7) the proof is similar.

Corollary 1.4. For any set $A\subseteq S$ and any $\rho\in\mathcal{O}$, we have the pairs of complementary $\mathcal{O}_{seg}$-hemi-spaces

$$\Big(\bigcup_{a\in A}M_{a\rho},\bigcap_{a\in A}C'_{a\rho}\Big),\quad \Big(\Big(\bigcup_{a\in A}M_{a\rho}\Big)\cup A,\ \bigcap_{a\in A}(C'_{a\rho}\setminus\{a\})\Big), \tag{1.13}$$

$$\Big(\bigcup_{a\in A}M'_{a\rho},\bigcap_{a\in A}C_{a\rho}\Big),\quad \Big(\Big(\bigcup_{a\in A}M'_{a\rho}\Big)\cup A,\ \bigcap_{a\in A}(C_{a\rho}\setminus\{a\})\Big), \tag{1.14}$$

$$\Big(\bigcap_{a\in A}M_{a\rho},\bigcup_{a\in A}C'_{a\rho}\Big),\quad \Big(\bigcap_{a\in A}M'_{a\rho},\ \bigcup_{a\in A}C_{a\rho}\Big). \tag{1.15}$$

Proof. This follows from proposition 1.4 and corollary 1.2, observing also that the pairs in (1.13)-(1.15) are complementary. Indeed, e.g., for (1.13), we have

$$S\setminus\Big(\bigcup_{a\in A}M_{a\rho}\Big)=\bigcap_{a\in A}(S\setminus M_{a\rho})=\bigcap_{a\in A}C'_{a\rho},$$

$$S\setminus\Big(\Big(\bigcup_{a\in A}M_{a\rho}\Big)\cup A\Big)=S\setminus\Big(\bigcup_{a\in A}(M_{a\rho}\cup\{a\})\Big)=\bigcap_{a\in A}(C'_{a\rho}\setminus\{a\}).$$

Corollary 1.5. If $A\subseteq S$ is an antichain (i.e., a set of pairwise incomparable elements) for some $\rho\in\mathcal{O}$, then the sets

$$\Big(\bigcap_{a\in A}M_{a\rho}\Big)\cup A,\quad \Big(\bigcap_{a\in A}M'_{a\rho}\Big)\cup A, \tag{1.16}$$

are $\mathcal{O}_{seg}$-hemi-spaces.

Proof. Let us first show that $\big(\bigcap_{a\in A}M_{a\rho}\big)\cup A$ satisfies (1.6). If $x\in\bigcap_{a\in A}M_{a\rho}$, $u\in S$, $x\rho u$, then, by the proof of corollary 1.2, we have $u\in\bigcap_{a\in A}M_{a\rho}\subseteq\big(\bigcap_{a\in A}M_{a\rho}\big)\cup A$. Assume now that $x\in A$, $u\in S$, $x\rho u$, and $u\notin\bigcap_{a\in A}M_{a\rho}$, so there exists $a'\in A$ such that $u\rho a'$. Then $x\rho u\rho a'$, whence, since A is an antichain, we obtain $x=a'$. Hence, again by $x\rho u\rho a'$, it follows that

$u=a'\in A\subseteq(\bigcap_{a\in A} M_{a\rho})\cup A$. Let us show now that $S\setminus((\bigcap_{a\in A} M_{a\rho})\cup A)$ satisfies (1.7). If $x\in S\setminus$ $\setminus((\bigcap_{a\in A} M_{a\rho})\cup A)$, $u\in S$, $u\rho x$, then $x\notin\bigcap_{a\in A} M_{a\rho}$ and $x\notin A$. Let $a'\in A$ be such that $x\notin M_{a'\rho}$, i.e., $x\rho a'$. Then $u\rho x\rho a'$, so $u\notin\bigcap_{a\in A} M_{a\rho}$. We claim that $u\notin A$. Indeed, if $u\in A$, then, by $u\rho a'$ and since A is an antichain, we obtain $u=a'$. Hence, by $u\rho x\rho a'$, it follows that $x=a'\in A$, in contradiction with our assumption that $x\notin A$. This proves the claim that $u\notin A$, and thus $u\in S\setminus((\bigcap_{a\in A} M_{a\rho})\cup A)$. Hence, by proposition 1.3, $(\bigcap_{a\in A} M_{a\rho})\cup A$ is an $\mathcal{O}_{seg}$-hemi-space.

Finally, the result for $(\bigcap_{a\in A} M'_{a\rho})\cup A$ follows by assuming that $\rho^{-1}\in\mathcal{O}$ (see remark 1.1b)) and applying the above to ρ^{-1}.

Proposition 1.5. Assume that each $\rho\in\mathcal{O}$ is a total order on S and, for each $a,x\in S$ with $x\neq a$, let

$$\mathcal{P}_{a,x}=\{\rho\in\mathcal{O}\mid x\rho a\},\quad \mathcal{P}'_{a,x}=\{\rho\in\mathcal{O}\mid a\rho x\}=\mathcal{O}\setminus\mathcal{P}_{a,x}\,. \tag{1.17}$$

Then, for any $a,x\in S$ with $x\neq a$, the sets

$$D_{a,x}=\{y\in S\setminus\{a\}\mid \mathcal{P}_{a,y}=\mathcal{P}_{a,x}\},\quad D'_{a,x}=\{y\in S\setminus\{a\}\mid \mathcal{P}_{a,y}=\mathcal{P}'_{a,x}\} \tag{1.18}$$

and $D_{a,x}\cup\{a\}$, $D'_{a,x}\cup\{a\}$, are $\mathcal{O}_{seg}$-convex.

Proof. Fix $z,t\in D_{a,x}$ and $u\in\langle z,t\rangle$. Let $\rho\in\mathcal{P}_{a,u}$. If $z\rho u\rho t$, then $z\rho u\rho a$, so $\rho\in\mathcal{P}_{a,z}$; on the other hand, if $t\rho u\rho z$, then $t\rho u\rho a$, so $\rho\in\mathcal{P}_{a,t}$. Hence, in either case, by $z,t\in$ $\in D_{a,x}$ we get $\rho\in\mathcal{P}_{a,x}$. Conversely, if $\rho\in\mathcal{P}_{a,x}$, then, by $z,t\in D_{a,x}$, we have $\rho\in\mathcal{P}_{a,z}=\mathcal{P}_{a,t}$ and hence $\rho\in\mathcal{P}_{a,u}$ (indeed, if $\rho\notin\mathcal{P}_{a,u}$, then, since ρ is total, we have $a\rho u\rho t$ when $z\rho u\rho t$, and $a\rho u\rho z$ when $t\rho u\rho z$, whence, by $\rho\in\mathcal{P}_{a,z}=\mathcal{P}_{a,t}$, we get either $a=t$ or $a=z$, contradicting $z,t\in D_{a,x}$). Thus, $\mathcal{P}_{a,u}=\mathcal{P}_{a,x}$. Also, $u\neq a$; indeed, if $u=a$, then $a\in\langle z,t\rangle$, so either $z\rho a\rho t$, whence $\rho\in\mathcal{P}_{a,z}\cap\mathcal{P}'_{a,t}=\mathcal{P}_{a,t}\cap\mathcal{P}'_{a,t}$, whence $a=t$, contradicting $t\in D_{a,x}$, or $t\rho a\rho z$, whence $\rho\in\mathcal{P}_{a,t}\cap\mathcal{P}'_{a,z}=\mathcal{P}_{a,z}\cap\mathcal{P}'_{a,z}$, whence $a=z$, contradicting $z\in D_{a,x}$. Thus, $u\in D_{a,x}$, which proves that $D_{a,x}$ is $\mathcal{O}_{seg}$-convex.

Now, let $z\in D_{a,x}$ and $u\in\langle z,a\rangle$, $u\neq a$. If $\rho\in\mathcal{P}_{a,u}$, then, by $u\in\langle z,a\rangle$, we have $z\rho u\rho a$ (the case $a\rho u\rho z$ cannot occur, since $u\rho a$, $u\neq a$), whence $\rho\in\mathcal{P}_{a,z}$. Conversely, if $\rho\in\mathcal{P}_{a,z}$, then, by $u\in\langle z,a\rangle$, we have, again, $z\rho u\rho a$ (the case $a\rho u\rho z$ cannot occur, since $z\rho a$, $z\neq a$, by $u\in\langle z,a\rangle$, $u\neq a$), whence $\rho\in\mathcal{P}_{a,u}$. Thus, $\mathcal{P}_{a,z}=\mathcal{P}_{a,u}$, and, $\mathcal{P}_{a,z}=\mathcal{P}_{a,x}$ (by $z\in D_{a,x}$), whence $\mathcal{P}_{a,u}=\mathcal{P}_{a,x}$, i.e., $u\in D_{a,x}\subset D_{a,x}\cup\{a\}$. This proves that $D_{a,x}\cup\{a\}$ is $\mathcal{O}_{seg}$-convex (since we have proved already that $D_{a,x}$ is $\mathcal{O}_{seg}$-convex).

Finally, the proofs for the sets $D'_{a,x}$ and $D'_{a,x}\cup\{a\}$ are similar.

Proposition 1.6. The pair $(S,\mathcal{C}_{seg})$, where $\mathcal{C}_{seg}$ is the family of all $\mathcal{O}_{seg}$-convex subsets of S, is an aligned space in the sense of [8], i.e.:

A.1. $\emptyset$ and S are $\mathcal{O}_{seg}$-convex.

A.2. An arbitrary intersection of $\mathcal{O}_{seg}$-convex sets is $\mathcal{O}_{seg}$-convex.

A.3. The union of any family of $\mathcal{O}_{seg}$-convex sets totally ordered by inclusion, is $\mathcal{O}_{seg}$-convex.

Proof. A1 has been observed before proposition 1.2. A2 and A3 hold for any "segmential convexity" (i.e., defined by (1.3), for some concept of "segment" $\langle x,y\rangle$).

By proposition 1.6, one can apply to the pair $(S,\mathcal{C}_{seg})$ the theory of aligned spaces (see [8]). In particular, for any set $G\subseteq S$, the $\mathcal{O}_{seg}$-convex hull $co_{\mathcal{O}_{seg}} G$ of G is defined as the smallest $\mathcal{O}_{seg}$-convex set containing G.

Proposition 1.7. We have

$$co_{\mathcal{O}_{seg}}\{x,y\}=\langle x,y\rangle \qquad (x,y\in S). \qquad (1.19)$$

Proof. By proposition 1.2, we have the inclusion $\subseteq$ in (1.19). The opposite inclusion $\supseteq$ follows from (1.3).

Corollary 1.6. If there exists $\rho\in\mathcal{O}$ which is not total, then there exists no family $\mathcal{T}$ of total orders on S, such that $\mathcal{T}_{seg}$-convexity in S coincides with $\mathcal{O}_{seg}$-convexity in S.

Proof. If $\mathcal{T}$ is any family of orders on S, such that $\mathcal{T}_{seg}$-convexity in S coincides with $\mathcal{O}_{seg}$-convexity in S, then, for any subset G of S, the $\mathcal{T}_{seg}$-convex hull of G coincides with the $\mathcal{O}_{seg}$-convex hull of G, and hence, in particular, by (1.19),

$$\langle x,y\rangle_{\mathcal{T}}=co_{\mathcal{T}_{seg}}\{x,y\}=co_{\mathcal{O}_{seg}}\{x,y\}=\langle x,y\rangle_{\mathcal{O}} \qquad (x,y\in S). \qquad (1.20)$$

But, since there exists an order $\rho\in\mathcal{O}$ which is not total, from corollary 1.1 it follows that there exists no family $\mathcal{T}$ of total orders on S, satisfying (1.20).

§2. SEGMENTIALLY CONVEX SETS IN $(R^n,\mathcal{T})$

Let us first show that the concept of order convexity is not sufficient to encompass, as a particular case, the usual (vector) convexity in R^n.

Proposition 2.1. There exists no order relation $\leqq$ on R^n $(n\geqq 2)$ for which the order convex sets are the usual convex sets.

Proof. We recall (see e.g. [2]) that, for any poset $(S,\leqq)$, the order segments of $(S,\leqq)$ are defined by

$$[x,y]=\{z\in S \mid x\leqq z\leqq y\} \qquad (x,y\in S), \qquad (2.1)$$

and that a subset G of S is said to be order convex, if

$$[x,y]\subseteq G \qquad (x,y\in G). \qquad (2.2)$$

Clearly, for any $x\neq y$ in S we have either $[x,y]=\emptyset$ or $[y,x]=\emptyset$, and hence $[x,y]\cup[y,x]$ is an order convex set containing x and y; on the other hand, by (2.2), for any order convex set G containing x and y, we have $[x,y]\cup[y,x]\subseteq G$. Thus,

$$co_{\leqq}\{x,y\}=[x,y]\cup[y,x] \qquad (x,y\in S), \qquad (2.3)$$

where $co_{\leqq}\{x,y\}$ denotes the order convex hull of $\{x,y\}$.

Assume now, a contrario, that there exists an order $\leqq$ on R^n such that the order convex sets are the usual convex sets. Then, by (2.3) we obtain, as in the above proof of corollary 1.6, that

$$[x,y]\cup[y,x]=co_{\leqq}\{x,y\}=co_{R^n}\{x,y\} \qquad (x,y\in R^n), \qquad (2.4)$$

where co_{R^n} denotes the usual convex hull in R^n. Let $x,y,z,t\in R^n$ be any four distinct points with $z\in \operatorname{co}_{R^n}\{x,y\}$. We may assume that $\operatorname{co}_{R^n}\{x,y\}=[x,y]$, whence $z\in[x,y]$, i.e., $x\leqq z\leqq y$. Since $[z,t]\cup[t,z]=\operatorname{co}_{R^n}\{z,t\}\neq\emptyset$, we have either $z\leqq t$ or $t\leqq z$, whence either $x\leqq z\leqq t$, so $z\in[x,t]=\operatorname{co}_{R^n}\{x,t\}$, or $t\leqq z\leqq y$, so $z\in[t,y]=\operatorname{co}_{R^n}\{t,y\}$. Thus, in either case, t belongs to the line determined by x,y and z, in contradiction with the arbitrariness of $t\in R^n\setminus\{x,y,z\}$.

We recall that an order ρ on R^n is said to be <u>compatible with the vector space structure of</u> R^n, or, briefly, <u>compatible</u>, if

$$y_1\rho y_1', \quad y_2\rho y_2' \Rightarrow y_1+y_2\rho y_1'+y_2', \tag{2.5}$$

$$y\rho y', \quad \lambda\geqq 0 \Rightarrow \lambda y\rho\lambda y'. \tag{2.6}$$

We shall consider the multi-ordered set $(R^n,\mathcal{T})$, where

$$\mathcal{T}=\text{the set of all compatible total orders on } R^n. \tag{2.7}$$

Let us recall the following result of [18].

<u>Theorem 2.1.</u> ([18], theorem 1.2). <u>For any</u> $\rho\in\mathcal{T}$ <u>there exists an unique</u> $v\in O(R^n)$ <u>such that</u>

$$y\rho y' \Leftrightarrow v(y)\leqq_L v(y') \qquad (y,y'\in R^n). \tag{2.8}$$

<u>Conversely, given any</u> $v\in O(R^n)$, <u>if we define</u> $\rho=\rho_v$ <u>by</u> (2.8), <u>then</u> $\rho_v\in\mathcal{T}$. <u>Consequently, the mapping</u>

$$\delta: v\to\rho_v \tag{2.9}$$

<u>is a bijection of</u> $O(R^n)$ <u>onto</u> $\mathcal{T}$.

Now we can prove the main result of this Section.

<u>Theorem 2.2.</u> <u>In</u> $(R^n,\mathcal{T})$, <u>we have</u>

$$\langle x,y\rangle=\{(1-\lambda)x+\lambda y\,|\,\lambda\in[0,1]\} \qquad (x,y\in R^n), \tag{2.10}$$

<u>i.e., the</u> $\mathcal{T}$<u>-segments of</u> R^n <u>coincide with the usual segments of</u> R^n. <u>Hence,</u> $\mathcal{T}_{seg}$<u>-convexity in</u> R^n <u>coincides with the usual convexity in</u> R^n.

<u>Proof.</u> Let $z\in\langle x,y\rangle$ and assume, a contrario, that $z\notin\{(1-\lambda)x+\lambda y\,|\,\lambda\in[0,1]\}$. Then, by the lexicographical separation theorem ([15], p. 258), there exists $v\in O(R^n)$ such that

$$v(z)<_L v((1-\lambda)x+\lambda y) \qquad (\lambda\in[0,1]), \tag{2.11}$$

whence, in particular (for $\lambda=0$ and $\lambda=1$), $v(z)<_L v(x)$ and $v(z)<_L v(y)$, that is, $z\rho_v x$ and $z\rho_v y$, for $\rho_v\in\mathcal{T}$ of theorem 2.1. On the other hand, since $z\in\langle x,y\rangle$, we have either $x\rho_v z\rho_v y$ or $y\rho_v z\rho_v x$, whence, by the antisymmetry of ρ_v, either $z=x$ or $z=y$, in contradiction with $z\notin\{(1-\lambda)x+\lambda y\,|\,\lambda\in[0,1]\}$. Thus, we have the inclusion $\subseteq$ in (2.10).

Conversely, let $z=(1-\lambda)x+\lambda y$, where $\lambda\in[0,1]$, and, for any $\rho\in\mathcal{T}$, let $v=\delta^{-1}(\rho)\in O(R^n)$ of theorem 2.1. Then, since $\leqq_L$ belongs to $\mathcal{T}$, we have either $v(x)\leqq_L v(y)$, or $v(y)\leqq_L v(x)$. Hence, again since $\leqq_L$ is in $\mathcal{T}$, if $v(x)\leqq_L v(y)$, then

$$v(x)\leqq_L v((1-\lambda)x+\lambda y)=v(z)\leqq_L v(y), \tag{2.12}$$

while if $v(y)\leqq_L v(x)$, then

$$v(y) \leqq_L v((1-\lambda)x+\lambda y) = v(z) \leqq_L v(x). \tag{2.13}$$

Therefore (by (2.8)), we have either $x\rho z\rho y$ or $y\rho z\rho x$, whence, since $\rho\in\mathcal{T}$ has been arbitrary, $z\in<x,y>$. Thus, we also have the inclusion $\supseteq$ in (2.10), and hence the equality.

Remark 2.1. Since the hemi-spaces and semi-spaces in R^n are known (see [17] and [5]), so are, by theorem 2.2, the $\mathcal{T}_{seg}$-hemi-spaces and $\mathcal{T}_{seg}$-semi-spaces. In the converse direction, from corollary 1.2 and theorem 2.1 it follows that for each $v\in O(R^n)$ and $a\in R^n$, the sets

$$M'_{a\rho_v} = \{x\in R^n \mid a\bar{\rho}_v x\} = \{x\in R^n \mid v(x) <_L v(a)\}, \tag{2.14}$$

$$C'_{a\rho_v} = \{x\in R^n \mid x\rho_v a\} = \{x\in R^n \mid v(x) \leqq_L v(a)\}, \tag{2.15}$$

are hemi-spaces in R^n, and, from [17], theorem 2.1, it follows that these exhaust all hemi-spaces "of rank n", i.e., all semi-spaces and all complements of semi-spaces in R^n. However, there are also other hemi-spaces in R^n, which might perhaps be encompassed by a more general theory of "multiply preordered sets". Note also that, since the other sets occurring in corollaries 1.2 and 1.4 for $\rho=\rho_v$ $(v\in O(R^n))$, are again of the form (2.14) or (2.15), and since no $a\in R^n$ is minimal or maximal (for any $\rho\in\mathcal{T}$) and every non-empty antichain in R^n (for any $\rho\in\mathcal{T}$) is a singleton, corollaries 1.2-1.4 and proposition 1.5 do not yield any further results in $(R^n,\mathcal{T})$.

Proposition 2.2. In $(R^n,\mathcal{T})$, for any $a\in R^n$ and $x\neq a$, the sets $D_{a,x}$ and $D'_{a,x}$ of (1.18) are the open half-lines

$$D_{a,x} = \{a+\lambda(x-a) \mid \lambda>0\}, \quad D'_{a,x} = \{a+\lambda(x-a) \mid \lambda<0\}. \tag{2.16}$$

Proof. It is sufficient to consider the case when $a=0$, i.e., to show that, for any $x\neq 0$, we have

$$D_{0,x} = \{\lambda x \mid \lambda>0\}, \quad D'_{0,x} = \{\lambda x \mid \lambda<0\}. \tag{2.17}$$

Indeed, for any $a,y\in R^n$ we have

$$\mathcal{P}_{a,y} = \{\rho\in\mathcal{T} \mid y\rho a\} = \{\rho\in\mathcal{T} \mid y-a\rho 0\} = \mathcal{P}_{0,y-a}, \tag{2.18}$$

and hence, if (2.17) holds for all $x\neq 0$, then, for all $x\neq a$,

$$D_{a,x} = \{y\in R^n\setminus\{a\} \mid \mathcal{P}_{a,y} = \mathcal{P}_{a,x}\} = \{y\in R^n \mid \mathcal{P}_{0,y-a} = \mathcal{P}_{0,x-a}\} =$$
$$= a+\{y'\in R^n\setminus\{0\} \mid \mathcal{P}_{0,y'} = \mathcal{P}_{0,x-a}\} = a+D_{0,x-a} =$$
$$= a+\{\lambda(x-a) \mid \lambda>0\} = \{a+\lambda(x-a) \mid \lambda>0\},$$

and the proof for $D'_{a,x}$ is similar.

Thus, let $x\neq 0$. Then,

$$\mathcal{P}_{0,\lambda x} = \{\rho\in\mathcal{T} \mid \lambda x\rho 0\} = \{\rho\in\mathcal{T} \mid x\rho 0\} = \mathcal{P}_{0,x} \qquad (\lambda>0), \tag{2.19}$$

so $\{\lambda x \mid \lambda>0\}\subseteq D_{0,x}$.

Conversely, let $y\notin\{\lambda x \mid \lambda>0\}$. If $y=0$, then $\mathcal{P}_{0,y} = \mathcal{P}_{0,0} = \mathcal{T}$. Thus, since $x\neq 0$, taking any $v\in O(R^n)$ with $v(x) >_L 0$, we obtain $\rho_v\in\mathcal{T}\setminus\mathcal{P}_{0,x} = \mathcal{P}_{0,y}\setminus\mathcal{P}_{0,x}$, whence $y\notin D_{0,x}$. On the other hand, if $y\neq 0$, then, since $x\neq 0$, we have $y\notin\{\lambda x \mid \lambda\geqq 0\}$, which is a closed convex set in R^n. Hence, by the classical strict separation theorem, there exists a functional $\Phi\in(R^n)^*$ with $||\Phi||=1$, such that

$$\Phi(y)>\Phi(\lambda x)=\lambda\Phi(x) \qquad (\lambda\geq 0). \qquad (2.20)$$

Then, in particular, we obtain $\Phi(x)<0$ (by taking $\lambda\to+\infty$) and $\Phi(y)\geq 0$ (by taking $\lambda=0$). If $\Phi(y)=0$ (whence $n\geq 2$, since otherwise we would obtain $y=0$, a contradiction), take $\psi\in(R^n)^*$ with $||\psi||=1$, such that Φ and ψ are orthogonal, and $\psi(y)>0$; if $\Phi(y)>0$, take $\psi\in(R^n)^*$ satisfying the same conditions, except the last one. Furthermore, take $\Phi_3,\dots,\Phi_n\in(R^n)^*$, such that for $v:R^n\to R^n$ defined by

$$v(x)=(\Phi(x),\psi(x),\Phi_3(x),\dots,\Phi_n(x)) \qquad (x\in R^n) \qquad (2.21)$$

we have $v\in O(R^n)$. Then, $v(x)<_L 0$ (since $\Phi(x)<0$) and $v(y)>_L 0$ (since either $\Phi(y)>0$ or, if $\Phi(y)=0$, then $\psi(y)>0$). Thus, for $\rho_v\in\mathcal{T}$ of theorem 2.1, we have $\rho_v\in\mathcal{P}_{0,x}\setminus\mathcal{P}_{0,y}$, whence $y\notin D_{0,x}$. Thus, $D_{0,x}\subseteq\{\lambda x|\lambda>0\}$, and hence $D_{0,x}=\{\lambda x|\lambda>0\}$.

Finally, the proof of $D'_{0,x}=\{\lambda x|\lambda<0\}$ is similar.

§3. SEGMENTIALLY CONVEX SETS IN $(Z^n,\mathcal{T}')$

Now we shall consider the multi-ordered set $(Z^n,\mathcal{T}')$, where

$$\mathcal{T}'=\mathcal{T}|_{Z^n}=\{\rho|_{Z^n}|\rho\in\mathcal{T}\}, \qquad (3.1)$$

with $\mathcal{T}$ of (2.7). By (3.1), the $\mathcal{T}'$-segments in Z^n are

$$\langle x,y\rangle_{\mathcal{T}'}=Z^n\cap\langle x,y\rangle_{\mathcal{T}} \qquad (x,y\in Z^n). \qquad (3.2)$$

Therefore, from theorem 2.2 we obtain

Theorem 3.1. In $(Z^n,\mathcal{T}')$ we have

$$\langle x,y\rangle_{\mathcal{T}'}=Z^n\cap\{(1-\lambda)x+\lambda y|\lambda\in[0,1]\} \qquad (x,y\in Z^n). \qquad (3.3)$$

Hence, a subset G of Z^n is $\mathcal{T}'_{seg}$-convex if and only if we have the implication

$$x,y\in G,\quad \lambda\in[0,1],\ (1-\lambda)x+\lambda y\in Z^n \Rightarrow (1-\lambda)x+\lambda y\in G. \qquad (3.4)$$

Remark 3.1. The sets $G\subseteq Z^n$ with property (3.4) have been called, in [12], 2-convex sets.

§4. SEGMENTIALLY CONVEX SETS IN MULTI-ORDERED SETS ASSOCIATED TO POSETS

For any poset $(S,\leq)$, it is natural to consider the multiordered sets $(S,\{\leq\})$ and $(S,\{\leq,\geq\})$; note that $\geq$ is nothing else than $\leq^{-1}$, in the sense of (1.4), so remark 1.1 b) applies.

Proposition 4.1. For a poset $(S,\leq)$, the multi-ordered sets $(S,\{\leq\})$ and $(S,\{\leq,\geq\})$, and any $x,y\in S$, we have (with $[x,y]$ of (2.1))

$$\langle x,y\rangle_{\{\leq\}}=[x,y]\cup[y,x]=\begin{cases}[x,y], & \text{if } x\leq y\\ [y,x], & \text{if } y\leq x\end{cases}=\langle x,y\rangle_{\{\leq,\geq\}}. \qquad (4.1)$$

Hence, for a subset G of S the following statements are equivalent:

1°. G is $\{\leq\}_{seg}$-convex.

2°. G is $\{\leq,\geq\}_{seg}$-convex.

3°. G <u>is order convex</u>.

<u>Proof</u>. Obvious from the definitions.

Some further remarks on posets and related multi-ordered sets are collected in

<u>Remark 4.1</u>. a) Given any multi-ordered set $(S,\mathcal{T})$, where $\mathcal{T}$ is a family of total orders on S, one can define an order $\leqq_{\mathcal{T}}$ on S, by

$$x \leqq_{\mathcal{T}} y \Longleftrightarrow x\rho y \qquad (x,y\in S, \rho\in\mathcal{T}). \tag{4.2}$$

However, then segmential convexity in $(S,\mathcal{T})$ need not coincide with order convexity in $(S,\leqq_{\mathcal{T}})$ and, in fact, $\leqq_{\mathcal{T}}$ may be the "trivial order" on S, in which

$$x \leqq_{\mathcal{T}} y \Longleftrightarrow x=y \qquad (x,y\in S); \tag{4.3}$$

indeed, in particular, if $\mathcal{T}$ contains ρ and ρ^{-1} (for example, if $(S,\mathcal{T})=(R^n,\mathcal{C})$, with $\mathcal{C}$ of (2.7)), then we have (4.3), and hence every set in $(S,\leqq_{\mathcal{T}})$ is order convex.

b) R. Jamison-Waldner [7] has observed that in a poset $(S,\leqq)$, at each $a\in S$ there exist at most two semi-spaces; namely, if $a\in S$ is not maximal and not minimal, then $M_{a\leqq}$ and $M'_{a\leqq}$ of (1.8) above (with ρ being $\leqq$) are the only semi-spaces at a (while if $a\in S$ is minimal or maximal, then corollary 1.3 above applies). It is an unsolved problem to find the semi-spaces for segmential convexity in multi-ordered sets $(S,\mathcal{O})$.

c) An aligned space $(S,\mathcal{L})$ (see proposition 1.6 above) is said to be <u>determined by an order</u> [6], if there exists an order $\leqq$ on S such that $\mathcal{L}$ coincides with the family of all order convex sets in $(S,\leqq)$. In [6], R. Jamison-Waldner has given necessary and sufficient conditions for an aligned space $(S,\mathcal{L})$ to be determined by an order (in terms of "natural" properties of aligned spaces). It is an unsolved problem to extend this result to an aligned space $(S,\mathcal{L})$ "determined by a family of orders" (replacing order convex sets by $\mathcal{O}_{seg}$-convex sets, in the above definition).

In connection with remark 4.1 b), let us give the following result on posets, which may have some interest for applications.

<u>Proposition 4.2. In any poset</u> $(S,\leqq)$, <u>the family</u>

$$\mathcal{M}=\{M_{a\leqq}|a\in S\}\cup\{M'_{a\leqq}|a\in S\}\cup S \tag{4.4}$$

<u>is an intersectional basis for the family of all order convex sets, i.e., for every order convex set</u> $G\subseteq S$ <u>we have</u>

$$G=\bigcap_{\substack{M\in\mathcal{M}\\ G\subseteq M}} M . \tag{4.5}$$

<u>Proof</u>. Let $G\subseteq S$ be order convex. Since the inclusion $\subseteq$ in (4.5) is obvious, it will be enough to show that for each $a\notin G$ there exists $M\in\mathcal{M}$, namely, either $M=M_{a\leqq}$ or $M=M'_{a\leqq}$, such that $G\subseteq M$ (indeed, then, by $a\notin M$, a does not belong to the right hand side of (4.5)). If, a contrario, $G\not\subseteq M_{a\leqq}$ and $G\not\subseteq M'_{a\leqq}$, then there exist $g_1,g_2\in G$ with $g_1\leqq a\leqq g_2$, whence, since G is order convex, it follows that $a\in G$, a contradiction.

<u>Remark 4.2</u>. It is an unsolved problem to find an intersectional basis for the family of all segmentially convex sets in a multi-ordered set.

Finally, let us give a characterization of the family of all order segments in a poset, which will imply again proposition 2.1, and may have some other applications.

Proposition 4.3. Let S *be a set and let* $\{\langle a,b\rangle\}_{a,b\in S}$ *be a family of subsets of* S. *The following conditions are equivalent:*

1°. *There exists an order relation* $\leqq$ *on* S *such that*

$$[x,y]=\langle x,y\rangle \qquad (x,y\in S), \tag{4.6}$$

where $[x,y]$ *is the order segment* (2.1) *in* $(S,\leqq)$.

2°. *We have*

$$\langle x,x\rangle=\{x\} \qquad (x\in S), \tag{4.7}$$

$$\langle x,y\rangle=\{z\in S \mid \langle x,z\rangle\neq\emptyset,\ \langle z,y\rangle\neq\emptyset\} \qquad (x,y\in S). \tag{4.8}$$

Moreover, in this case the order $\leqq$ *of* 1° *is unique, namely, it is the order defined, for any* $x,y\in S$, *by*

$$x\leqq y \Leftrightarrow \langle x,y\rangle\neq\emptyset. \tag{4.9}$$

Proof. 1° $\Rightarrow$ 2°. Assume 1°. Then, by (4.6), for any $x\in S$ we have

$$z\in\langle x,x\rangle \Leftrightarrow x\leqq z\leqq x \Leftrightarrow z=x \Leftrightarrow z\in\{x\},$$

which proves (4.7). Now, let $x,y\in S$. Then, by (4.6),

$$z\in\langle x,y\rangle \Rightarrow x\leqq z\leqq y \Rightarrow x,z\in\langle x,z\rangle,\ z,y\in\langle z,y\rangle,$$

which proves the inclusion $\subseteq$ in (4.8). Conversely, let $z\in S$ be such that $\langle x,z\rangle\neq\emptyset$, $\langle z,y\rangle\neq\emptyset$, and take any $t\in\langle x,z\rangle$, $s\in\langle z,y\rangle$. Then, by (4.6), we have $x\leqq t\leqq z\leqq s\leqq y$, whence, again by (4.6), $z\in\langle x,y\rangle$. This proves the inclusion $\supseteq$ in (4.8), and hence the equality.

2° $\Rightarrow$ 1°. Assume 2° and define a binary relation $\leqq$ on S, by (4.9). Then, by (4.7), we have $x\in\{x\}=\langle x,x\rangle$ $(x\in S)$, whence, by (4.9), $x\leqq x$ $(x\in S)$. Furthermore, if $x\leqq y$ and $y\leqq x$, then, by (4.9), we have $\langle x,y\rangle\neq\emptyset$ and $\langle y,x\rangle\neq\emptyset$, whence, by (4.8) and (4.7) we obtain $y\in\langle x,x\rangle=\{x\}$, i.e., $y=x$; thus, $\leqq$ is anti-symmetric. Also, if $x\leqq z\leqq y$, then, by (4.9), we have $\langle x,z\rangle\neq\emptyset$ and $\langle z,y\rangle\neq\emptyset$, whence, by (4.8), we obtain $z\in\langle x,y\rangle\neq\emptyset$, and hence, again by (4.9), $x\leqq y$; thus, $\leqq$ is transitive, which proves that $\leqq$ is an order on S. Finally, for any $x,y,z\in S$ we have, by (4.9) and (4.8),

$$z\in[x,y] \Leftrightarrow x\leqq z\leqq y \Leftrightarrow \langle x,z\rangle\neq\emptyset,\ \langle z,y\rangle\neq\emptyset \Leftrightarrow z\in\langle x,y\rangle,$$

which proves (4.6).

Finally, in order to prove the uniqueness statement, let $\leqq$ be any order on S satisfying (4.6). Then, for any $x,y\in S$, we have

$$x\leqq y \Rightarrow x,y\in[x,y]=\langle x,y\rangle \Rightarrow \langle x,y\rangle\neq\emptyset,$$

and, conversely,

$$\langle x,y\rangle\neq\emptyset \Leftrightarrow \exists\, z\in\langle x,y\rangle=[x,y] \Rightarrow x\leqq y,$$

which proves (4.9).

Remark 4.3. a) Proposition 4.3 implies again proposition 2.1, by (2.4) and since the family of all usual segments $\mathrm{co}_{R^n}\{x,y\}$ in R^n does not satisfy (4.8).

b) It is an unsolved problem to find a corresponding characterization of the family of all $\mathcal{O}$-segments in a multi-ordered set $(S,\mathcal{O})$.

§5. SEPARATIONAL MULTI-ORDER CONVEXITY

Definition 5.1. Let $(S,\mathcal{O})$ be a multi-ordered set. We call a set $G\subseteq S$, "*$\mathcal{O}$-separationally convex*", or, briefly, *$\mathcal{O}_{sep}$-convex*, if for each $x_o\in S\setminus G$ there exists an order

$\rho=\rho(x_o)\in\mathcal{O}$ such that (where $\bar{\rho}$ denotes the negation of ρ)

$$x_o\bar{\rho}g \qquad (g\in G). \qquad (5.1)$$

Remark 5.1. a) In other words, G is $\mathcal{O}_{sep}$-convex, if every outside point x_o can be "separated from G by some order $\rho=\rho(x_o)\in\mathcal{O}$"; this motivates our terminology. Note also that this definition is of different type from that of $\mathcal{O}_{seg}$-convexity, which has involved all $\rho\in\mathcal{O}$; also, $\mathcal{O}_{sep}$-convexity is a "one-sided" concept (see (5.1)), while $\mathcal{O}_{seg}$-convexity is a "symmetric" concept (see (1.5)).

b) When all $\rho\in\mathcal{O}$ are total, a set $G\subseteq S$ is $\mathcal{O}_{sep}$-convex if and only if for each $x_o\in S\setminus G$ there exists $\rho=\rho(x_o)\in\mathcal{O}$ such that

$$g\rho x_o \qquad (g\in G). \qquad (5.2)$$

c) In contrast with the case of $\mathcal{O}_{seg}$-convexity (see remark 1.1 b)), if $\mathcal{O}'=\mathcal{O}\cup\{\rho^{-1}\}$, for some $\rho\in\mathcal{O}$, then $\mathcal{O}'_{sep}$-convexity need not coincide with $\mathcal{O}_{sep}$-convexity (see remark 5.2 and proposition 5.2).

d) In contrast with the case of $\mathcal{O}_{seg}$-convexity, a singleton $\{x\}$, where $x\in S$, need not be $\mathcal{O}_{sep}$-convex. Indeed, for example, when $\mathcal{O}=\{\rho\}$ (a singleton) and $x_o\rho x$, $x_o\neq x$, then $\{x\}$ is not $\mathcal{O}_{sep}$-convex. Also, an $\mathcal{O}$-segment $\langle x,y\rangle$ need not be $\mathcal{O}_{sep}$-convex. Indeed, for example, if $\mathcal{O}=\{\rho\}$ and $y\rho x$, $y\neq x$, then $\langle x,y\rangle=\{z\in S \mid y\rho z\rho x\}$. Hence, if $x_o\rho y$, $x_o\neq y$, then $x_o\notin\langle x,y\rangle$, $x\in\langle x,y\rangle$ and $x_o\rho y\rho x$, so (5.1) (with $G=\langle x,y\rangle$) is not satisfied, and therefore $\langle x,y\rangle$ is not $\mathcal{O}_{sep}$-convex. Thus, $\mathcal{O}_{seg}$-convexity does not imply $\mathcal{O}_{sep}$-convexity.

e) Clearly, the whole set S and the empty set $\emptyset$ are $\mathcal{O}_{sep}$-convex.

Proposition 5.1. For any $\rho\in\mathcal{O}$, each set $N\subseteq S$ satisfying (1.7) is $\mathcal{O}_{sep}$-convex.

Proof. Let $x_o\notin N$. If (5.1) (with $G=N$) does not hold for the given ρ, then there exists $x'\in N$ such that $x_o\rho x'$, whence, by (1.7), we obtain $x_o\in N$, in contradiction with our assumption. Thus, for any $x_o\notin N$ and for the given ρ (which does not depend on x_o), we have (5.1) (with $G=N$).

Corollary 5.1. For any set $A\subseteq S$ and any $\rho\in\mathcal{O}$, the sets $\bigcap_{a\in A}C'_{a\rho}$, $\bigcap_{a\in A}(C'_{a\rho}\setminus\{a\})$, $\bigcup_{a\in A}M'_{a\rho}$, $(\bigcup_{a\in A}M'_{a\rho})\cup A$, $\bigcup_{a\in A}C'_{a\rho}$ and $\bigcap_{a\in A}M'_{a\rho}$, of (1.13)-(1.15), are $\mathcal{O}_{sep}$-convex.

Proof. By the arguments of §1, these sets satisfy (1.7), and hence, by proposition 5.1, they are $\mathcal{O}_{sep}$-convex.

Remark 5.2. The results corresponding to proposition 1.3 a) and its consequences given in §1, do not hold, in general, for $\mathcal{O}_{sep}$-convexity. Moreover, let us show that if $\mathcal{O}=\{\rho\}$, where ρ is total, then no set M (different from $\emptyset$ and from S) satisfying (1.6) is $\mathcal{O}_{sep}$-convex. Indeed, if $x_o\notin M$ and (5.1) holds (with $G=M$), then, since ρ is total, we have $x\rho x_o$ for all $x\in M$, whence, by (1.6), we obtain $x_o\in M$, in contradiction with our assumption. Thus, M is not $\mathcal{O}_{sep}$-convex. However, for any multi-ordered set $(S,\mathcal{O})$, we have

Proposition 5.2. If for some $\rho\in\mathcal{O}$ we have $\rho^{-1}\in\mathcal{O}$, then each set $M\subseteq S$ satisfying (1.6) is $\mathcal{O}_{sep}$-convex.

Proof. Let $x_0 \notin M$. We shall show that

$$x_0 \overline{\rho^{-1}} x \qquad (x \in M), \tag{5.3}$$

which, since $\rho^{-1} \in \mathcal{O}$, will prove that M is $\mathcal{O}_{sep}$-convex. If (5.3) does not hold, then there exists $x' \in M$ such that $x_0 \rho^{-1} x'$, or, equivalently $x' \rho x_0$. Hence, by (1.6), we obtain $x_0 \in M$, in contradiction with our assumption. Thus, M is $\mathcal{O}_{sep}$-convex.

Definition 5.2. We call $\mathcal{O}_{sep}$-hemi-space, any set $M \subseteq S$ such that both M and $S \setminus M$ are $\mathcal{O}_{sep}$-convex.

Corollary 5.2. If for some $\rho \in \mathcal{O}$ we have $\rho^{-1} \in \mathcal{O}$, then, for any set $A \subseteq S$, (1.13)-(1.15) are pairs of complementary $\mathcal{O}_{sep}$-hemi-spaces. If, in addition, A is an antichain for ρ, then the sets (1.16) are $\mathcal{O}_{sep}$-hemi-spaces.

Proof. By the above proofs of corollaries 1.4 and 1.5, these sets satisfy (1.6) or (1.7), so the result follows from propositions 5.1 and 5.2.

Proposition 5.3. An arbitrary intersection of $\mathcal{O}_{sep}$-convex sets in $\mathcal{O}_{sep}$-convex.

Proof. Let $\{G_i\}_{i \in I}$ be a family of $\mathcal{O}_{sep}$-convex subsets of S and let $x_0 \notin \bigcap_{i \in I} G_i$. Then there exists $i_0 \in I$ such that $x_0 \notin G_{i_0}$, whence, since G_{i_0} is $\mathcal{O}_{sep}$-convex, there exists $\rho = \rho_{i_0}(x_0) \in \mathcal{O}$ such that $x_0 \bar{\rho} g$ for all $g \in G_{i_0}$, and hence, in particular, also for all $g \in \bigcap_{i \in I} G_i \subseteq G_{i_0}$.

Remark 5.3. We do not know under what conditions on $\mathcal{O}$ is the union of any family of $\mathcal{O}_{sep}$-convex sets, totally ordered by inclusion, again $\mathcal{O}_{sep}$-convex, i.e., under what conditions is the pair $(S, \mathcal{C}_{sep})$ an aligned space, where $\mathcal{C}_{sep}$ is the family of all $\mathcal{O}_{sep}$-convex subsets of S (cp. proposition 1.6). Nevertheless, proposition 5.3 permits us to define the $\mathcal{O}_{sep}$-convex hull $\mathrm{co}_{\mathcal{O}_{sep}} G$ of a set $G \subseteq S$ as the smallest $\mathcal{O}_{sep}$-convex set containing G.

Now we shall show that, as concerns total orders, the situation for $\mathcal{O}_{sep}$-convexity is better than the one for $\mathcal{O}_{seg}$-convexity, described in corollary 1.6. To this end, we need some preparation.

Let us recall that an order ρ_2 on a set S is said to be an extension of an order ρ_1 on S, if

$$y \rho_2 y' \qquad (y, y' \in S,\ y \rho_1 y'), \tag{5.4}$$

and that, by a classical theorem of Szpilrajn [21], every order relation ρ on a set S admits a total extension τ on S.

Lemma 5.1. Let ρ be an order relation on a set S and let $G \subseteq S$ and $x_0 \in S \setminus G$ satisfy (5.1). Then there exists a total extension τ of ρ on S, such that

$$g \tau x_0 \qquad (g \in G). \tag{5.5}$$

Proof. Let

$$[G] = \{x \in S \mid \exists g \in G,\ x \rho g\}. \tag{5.6}$$

Then, by (5.1), $x_o \in S\setminus[G]$. By Szpilrajn's theorem, there exist total order relations τ_1, τ_2 on $[G]$ and $S\setminus[G]$, respectively, extending the restrictions of ρ to these sets. Let us define a binary relation τ on S, by saying that $x\tau y$, if one of the following conditions holds:

a) $x,y\in[G]$ and $x\tau_1 y$;

b) $x,y\in S\setminus[G]$ and $x\tau_2 y$;

c) $x\in[G]$ and $y\in S\setminus[G]$.

Clearly, τ is a total order on S. Furthermore, if $x,y\in S$ are such that $x\rho y$, then we cannot have $x\in S\setminus[G]$ and $y\in[G]$, since otherwise, by $y\in[G]$, there would exist $g\in G$ such that $y\rho g$, whence $x\rho y\rho g$, contradicting $x\in S\setminus[G]$. Therefore, if $x\rho y$, then we have the following three possibilities:

1) $x,y\in[G]$; in this case, $x\tau_1 y$ (since τ_1 extends the restriction of ρ to $[G]$, whence $x\tau y$.

2) $x,y\in S\setminus[G]$; in this case, $x\tau_2 y$ (since τ_2 extends the restriction of ρ to $S\setminus[G]$), whence $x\tau y$.

3) $x\in[G]$, $y\in S\setminus[G]$, whence, again, $x\tau y$.

This proves that τ is an extension of ρ. Finally, by $G\subseteq[G]$ and $x_o\in S\setminus[G]$, we have (5.5).

Now we can prove the result on total orders, announced above. Namely, in contrast with corollary 1.6, we have

Theorem 5.1. For each multi-ordered set $(S,\mathcal{O})$ there exists a family $\mathcal{T}$ of total orders on S, such that $\mathcal{T}_{sep}$-convexity in S coincides with $\mathcal{O}_{sep}$-convexity in S.

Proof. The family $\mathcal{T}$ defined by

$$\mathcal{T}=\bigcup_{\rho\in\mathcal{O}}\mathcal{T}_\rho , \tag{5.7}$$

where, for each $\rho\in\mathcal{O}$,

$$\mathcal{T}_\rho = \text{the family of all total extensions of } \rho, \tag{5.8}$$

has the required property. Indeed, each $\mathcal{T}_{sep}$-convex set G is $\mathcal{O}_{sep}$-convex, since if $x_o\notin G$ and $\tau\in\mathcal{T}$ satisfy $x_o\bar{\tau}g$ $(g\in G)$, then for $\rho\in\mathcal{O}$ such that $\tau\in\mathcal{T}_\rho$, we have (5.1). Conversely, if G is an $\mathcal{O}_{sep}$-convex set and if $x_o\notin G$ and $\rho\in\mathcal{O}$ satisfy (5.1), then, by lemma 5.1, there exists $\tau\in\mathcal{T}_\rho\subseteq\mathcal{T}$ satisfying (5.5), and hence, by remark 5.1 b), G is $\mathcal{T}_{sep}$-convex.

Remark 5.4. By theorem 5.1, in the study of $\mathcal{O}_{sep}$-convexity it is no restriction of the generality to assume that each $\rho\in\mathcal{O}$ is total.

Theorem 5.2. $\mathcal{O}_{sep}$-convexity implies $\mathcal{O}_{seg}$-convexity.

Proof. Let G be an $\mathcal{O}_{sep}$-convex subset of S, and assume, a contrario, that G is not $\mathcal{O}_{seg}$-convex, i.e., that there exist $x,y\in G$ and $x_o\in\langle x,y\rangle$, such that $x_o\notin G$. Then, since G is $\mathcal{O}_{sep}$-convex, there exists $\rho=\rho(x_o)\in\mathcal{O}$ satisfying (5.1), whence, in particular, $x_o\bar{\rho}x$ and $x_o\bar{\rho}y$ (since $x,y\in G$). But, by $x_o\in\langle x,y\rangle$, for this ρ we must have either $x\rho x_o\rho y$, which contradicts $x_o\rho y$, or $y\rho x_o\rho x$, which contradicts $x_o\rho x$.

Remark 5.5. The converse implication need not hold, even when each $\rho\in\mathcal{O}$ is total, as shown by remark 5.1 d). However, theorems 2.2, 6.1 and, respectively, proposition 4.1 and theorem 8.1, show two important cases when $\mathcal{O}_{sep}$-convexity coincides with

$\mathcal{O}_{seg}$-convexity. In the general case, we do not know under what conditions on $\mathcal{O}$ is every $\mathcal{O}_{seg}$-convex set $\mathcal{O}_{sep}$-convex.

§6. SEPARATIONALLY CONVEX SETS IN $(R^n, \mathcal{T})$

Let us consider the multi-ordered set $(R^n, \mathcal{T})$, with $\mathcal{T}$ of (2.7) (so every ρ is total).

Theorem 6.1. $\mathcal{O}_{sep}$-convexity in R^n coincides with the usual convexity in R^n.

Proof. If $G \subseteq R^n$ is $\mathcal{O}_{sep}$-convex, then, by theorem 5.1, G is $\mathcal{O}_{seg}$-convex, and hence, by theorem 2.2, G is convex.

Conversely assume now that $G \subset R^n$ is convex, and let $x_o \in R^n \setminus G$. Then, by the lexicographical separation theorem ([15], p. 256), there exists $v \in O(R^n)$ such that

$$v(g) <_L v(x_o) \qquad (g \in G). \tag{6.1}$$

Let $\rho = \rho_v \in \mathcal{T}$ of theorem 2.1. Then, by (6.1) and (2.8), we have $g \rho_v x_o$ for all $g \in G$, and therefore, by remark 5.1 b), G is $\mathcal{O}_{sep}$-convex.

Remark 6.1. Actually, theorem 6.1 is equivalent to the lexicographical separation theorem.

§7. SEPARATIONALLY CONVEX SETS IN $(Z^n, \mathcal{T}')$

Let us consider now the multi-ordered set $(Z^n, \mathcal{T}')$ of §3 (so every $\rho \in \mathcal{T}'$ is total).

Theorem 7.1. For a subset G of Z^n, the following statements are equivalent:

1°. G is $\mathcal{T}'_{sep}$-convex.

2°. There exists a convex subset C of R^n, such that

$$G = Z^n \cap C. \tag{7.1}$$

3°. We have

$$G = Z^n \cap \mathrm{co}_{R^n} G, \tag{7.2}$$

where co_{R^n} denotes the (usual) convex hull in R^n.

4°. For each $x_o \in Z^n \setminus G$ there exists $v = v_{x_o} \in O(R^n)$ such that

$$v(g) <_L v(x_o) \qquad (g \in G). \tag{7.3}$$

Proof. 1° ⇒ 4°. If 1° holds and $x_o \in Z^n \setminus G$, then, by remark 5.1 b), there exists $\rho = \rho(x_o) \in \mathcal{T}'$ such that we have (5.2). By (3.1), let $\rho_o \in \mathcal{T}$ be such that $\rho = \rho_o|_{Z^n}$ and, by theorem 2.1, let $v = \delta^{-1}(\rho_o) \in O(R^n)$. Then, by (5.2), (2.8) and $x_o \notin G$, we have (7.3).

4° ⇒ 3°. Since the inclusion $\subseteq$ in (7.2) is obvious, it is enough to show that 4° implies the inclusion $\supseteq$ in (7.2). Assume 4° and let $x_o \in Z^n \cap \mathrm{co}_{R^n} G$. If $x_o \notin G$, then by (7.3), we obtain $x_o \notin \mathrm{co}_{R^n} G$, a contradiction.

The implication 3° ⇒ 2° is obvious.

$2° \Rightarrow 1°$. Assume 2° and let $x_o \in Z^n \setminus G$. Then, by (7.1), we have $x_o \notin C$, and, by theorem 6.1, C is $\mathcal{T}_{sep}$-convex. Hence, by remark 5.1 b), there exists $\rho_o = \rho_o(x_o) \in \mathcal{T}$ such that

$$x \rho_o x_o \qquad (x \in C). \tag{7.4}$$

Consequently, for $\rho = \rho_o|_{Z^n} \in \mathcal{T}'$ we have (5.2) (since $G \subset C$ by (7.1)), and thus, by remark 5.1 b), G is $\mathcal{T}'_{sep}$-convex.

<u>Remark 7.1</u>. a) The sets $G \subseteq Z^n$ satisfying (7.2) have been called, in [12], <u>strong convex sets</u>. Moreover, by [12], proposition 3.1, condition 2° is equivalent to

5°. G <u>is</u> <u>p-convex, where</u> p=dim G+1, <u>i.e., we have the implication</u>

$$x_1,\dots,x_p \in G, \lambda_1,\dots,\lambda_p \geq 0, \sum_{i=1}^{p} \lambda_i = 1, \sum_{i=1}^{p} \lambda_i x_i \in Z^n \Rightarrow \sum_{i=1}^{p} \lambda_i x_i \in G. \tag{7.5}$$

Also, by [12], example 1.3, the subset $G=\{(1,3), (0,2), (2,1)\}$ of Z^2 is 2-convex, but not 3-convex; hence, by the above, G <u>is</u> $\mathcal{T}'_{seg}$<u>-convex, but not</u> $\mathcal{T}'_{sep}$<u>-convex, in</u> Z^2. For other related results, in a more general framework, see [12].

b) The equivalence $1° \Leftrightarrow 4°$ means that a subset G of Z^n is $\mathcal{T}'_{sep}$-convex if and only if each outside point $x_o \in Z^n \setminus G$ "can be separated from G by a linear isometry $v \in O(R^n)$". Let us recall the similar concept of "W-convexity" (due to Ky Fan [3]), in which one separates (strictly) x_o and G by real-valued functions (instead of operators $v: R^n \to R^n$): If $W \subseteq R(R^n)$ (where $R(R^n)$ is the family of all functions $\phi: R^n \to R$), a subset G of Z^n is said to be <u>W-convex</u>, if for each $x_o \in Z^n \setminus G$ there exists a function $w = w_{x_o} \in W$ such that

$$\sup w(G) < w(x_o). \tag{7.6}$$

In particular, for $W = (R^n)^*$ (the dual of R^n) one can show, similarly to the above, that $G \subseteq Z^n$ is $(R^n)^*$-convex if and only if

$$G = Z^n \cap \overline{co}_{R^n} G, \tag{7.7}$$

where $\overline{co}_{R^n}$ denotes the closed convex hull in R^n. Hence, since $\overline{co}_{R^n} G$ is convex, from theorem 7.1 (implication $2° \Rightarrow 1°$) it follows that <u>every</u> $(R^n)^*$<u>-convex subset</u> G <u>of</u> Z^n <u>is</u> $\mathcal{T}'_{sep}$<u>-convex</u>. However, the converse is not true, as shown by simple examples (of infinite sets).

§8. SEPARATIONALLY CONVEX SETS IN MULTI-ORDERED SETS ASSOCIATED TO POSETS

For a poset $(S, \leq)$, let us consider again the multi-ordered sets $(S, \{\leq\})$ and $(S, \{\leq, \geq\})$ (see §4).

<u>Theorem 8.1</u>. <u>Let</u> $(S, \leq)$ <u>be a poset. Then, a subset</u> G <u>of</u> S <u>is</u> $\{\leq, \geq\}_{sep}$<u>-convex if and only if it is order convex</u>.

<u>Proof</u>. Assume that G is $\{\leq, \geq\}$-convex, but not order convex, so there exist $g_1, g_2 \in G$ and $x_o \in S \setminus G$, with $g_1 \leq x_o \leq g_2$. Then, since G is $\{\leq, \geq\}$-convex, we have (5.1) ei-

ther for ρ being $\leqq$, or for ρ being $\geqq$, i.e., either $x_o \nleqq g$ $(g \in G)$, which contradicts $x_o \leqq g_2$, or $x_o \ngeqq g$ $(g \in G)$, which contradicts $g_1 \leqq x_o$.

Conversely, assume now that G is order convex, but not $\{\leqq,\geqq\}$-convex, so there exists $x_o \in S\setminus G$ such that neither $x_o \nleqq g$ $(g\in G)$, nor $x_o \ngeqq g$ $(g \in G)$. Then, there exist $g_1, g_2 \in G$ such that $g_1 \leqq x_o \leqq g_2$. Hence, since G is order convex, we obtain $x_o \in G$, which contradicts our assumption on x_o.

Remark 8.1. By theorem 5.1 and proposition 4.1, every $\{\leqq\}_{sep}$-convex set is order convex. However, the converse is not true, as shown e.g. by remark 5.1 d).

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