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FINITENESS OF THE INTEGRAL CLOSURE OF A NOETHERIAN RING

by

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1. INTRODUCTION

All the rings considered will be commutative, with unit and noetherian. For a ring A, we denote by Q(A) the total quotient ring of A. For notations and definitions not explained here, one can see [4].

Recall the following definition.

(1.1) DEFINITION. A noetherian integral domain A is called japanese, if for any field L, finite exten$ion of the quotient field K of A, the integral closure of A in L is a finite A-algebra.

We will also need some definitions and results concerning various sets of asymptotic prime divisors (for more details see [5] and [2]).

(1.2) DEFINITION. Let A be a ring, I an ideal of A. The set $I_a = \{x \in A /$ there exists $c_1, \ldots, c_n \in A$, such that $c_i \in I^i$ for any i and $x^m + c_1 x^{m-1} + \ldots + c_m = 0\}$ is called the integral closure of I.

It is easily seen that I_a is an ideal of A.

(1.3) THEOREM ([5] , 1.5,3.9). Let I be an ideal of the noetherian ring A. Then the sets $Ass_A(A/I^n)_a)$ eventually stabilize for large n to the well-defined finite sets denoted by A(I), resp. $A_a(I)$.

(1.4) PROPOSITION ([9], 4.14). Let A be a noetherian ring, $x \in A$ a non zero-divisor. Then $A_a(xA) \subseteq Ass_A(A/xA)$.

(1.5) PROPOSITION ([5], 3.17). For any noetherian ring A and any ideal I, we have that $A_a(I) \subseteq A(I)$.

(1.6) PROPOSITION ([5], 3.5). Let $A \subseteq B$ be an integral extension of integral domains, where A is noetherian. Let I be an ideal of A, Q a prime ideal of B, minimal over IB. Then $Q \cap A \in A_a(I)$ (and also to A(I), by 1.5).

(1.7) PROPOSITION ([5], 3.18). Let A be a noetherian ring, I an ideal of A, P a prime ideal of A. Then $P \in A_a(I)$ if and only if there is a minimal prime ideal Q of A, such that $Q \subseteq P$ and $P/Q \in A_a(I + Q/Q)$.

(1.8) DEFINITION. Let A be a noetherian ring, I an ideal of A. We will denote $E(I) = \{P \in \mathrm{Spec}A /$ there is $Q \in Ass(R_p)^\wedge$ such that $I(R_p)^\wedge + Q$ is an $P(R_p)^\wedge$ -primary ideal$\}$. This set is called the set of essential prime divisors of I. We will also denote $U(I) = \{P \cap A / P \in E(uR(I))\}$, where $R(I) = A[IX, X^{-1}]$ is the extended Rees ring of A with respect to I and $u = X^{-1}$. U(I) is called the set of U-essential prime divisors of I.

(1.9) PROPOSITION ([2] 2.2.1, 2.3, 2.5.7, 2.5.2);

Let A be a noetherian ring, I an ideal of I. Then :

(a) $E(I) \cup U(I) \subseteq A(I)$;

(b) $\mathrm{Min}\,(I) \subseteq U(I)$;

(c) $A_a(I) \cup E(I) \subseteq U(I)$;

(d) $P \in U(I)$ if and only if there is $Q \in Ass\,(A)$ such that $P/Q \in U(I + Q/Q)$.

We must say that some ideas of the results which follow go back to J.Marot (3). Thanks are also due to Dorin Popescu for useful suggestions.

2. ON A THEOREM OF MAROT

In [3] , J.Marot proved the following result concerning the finiteness of the integral closure of a noetherian domain.

(2.1) THEOREM ([3], 1.2) Let A be a noetherian integral domain , K its field of quotients, L a field, finite extension of K, A' the integral closure of A in L. Let I be an ideal of A such that:

a) A is I-adically complete;

b) A/P is japanese, for any prime P containing I.

Then the following are equivalent:

1) A' is a finite A-algebra;

2) The radical of IA' is a finitely generated ideal of A' and IA' has only a finite number of minimal prime ideals.

We will prove a stronger version of this result. For that we will need the following:

(2.2) PROPOSITION (see also [5] , 1.9). Let A be a noetherian integral domain, K its field of quotients, L a field, finite extension of K, A' the integral closure of A in L. Let J be a finitely generated ideal of A'. Then there are only finitely many prime ideals of A', minimal over J.

Proof Let B be a finite A-algebra with field of quotients L.Then B is a noetherian integral domain and its integral closure is A'. Now we can apply ([5], 1.9) to get the conclusion.

(2.3) PROPOSITION . Let A be a noetherian integral domain, K its field of quotients, L a field, finite extention of K, A' the integral closure of A in L. Let I be an ideal of A such that:

a) A is I-adically complete;

b) A/P is japanese for any prime ideal $P \in A_a(I)$.

Then the following are equivalent:

1) A' is a finite A-algebra;

2) the radical of IA' is finitely generated.

Proof. Obviously 1) $\Rightarrow$ 2). For the proof of 2) $\Rightarrow$ 1) let J be the radical of IA'. Then by (2.2) $J = Q_1 \cap \ldots \cap Q_n$, where $Q_1, \ldots, Q_n$ are the minimal prime over-ideals of IA'.

Let $P_i = Q_i \cap A$, for $i = 1, \ldots, n$. Then $P_i \in A_a(I)$ by (1.6) and $[k(Q_i) : k(P_i)] < \infty$. Since A/P_i is japanese it follows that A'/Q_i is an A/P_i - module of finite type. In particular, A'/Q_i is noetherian. We have also that A'/J is a finite A/I - algebra. From the exact sequence

$$0 \longrightarrow J^n/J^{n+1} \longrightarrow A'/J^{n+1} \longrightarrow A'/J^n \longrightarrow 0$$

we get by induction that A'/J^n is a finite A-algebra, for any natural number n, since J is finitely generated. But there is a natural number n, such that $J^n \subseteq IA'$, so A'/IA' is a finite A-algebra. By [3], (1.1) A' is separated in the I-adic topology, so by a well-known lemma (see for instance [4], 8.4), we obtain that A' is a finite A-algebra.

The next step is to see what's happening when A is no more a domain.

(2.4) PROPOSITION. Let A be a noetherian ring, I an ideal of A, Q a minimal prime ideal of A. Suppose that:

a) A is I-adically complete;

b) A/P is japanese for any prime $P \in A_a(I)$, which contains Q.

Let L a field, finite extension of $k(Q)$, A' the integral closure of A/Q in L. Then the following are equivalent:

1) A' is a finite A-algebra;

2) the radical of IA' is finitely generated.

Proof. We have only to prove 2) $\Longrightarrow$ 1). Let $P' \in A_a(I+Q/Q)$. Then $P' = P/Q$, where $P \in A_a(I)$ by (1.7). Now we can apply (2.3) to the integral domain A/Q to get the conclusion.

Finally we try to drop out the completeness assumption on A.

(2.5) THEOREM. Let A be a noetherian reduced ring, I an ideal contained in the Jacobson radical of A. Let Q be a minimal prime ideal of A, L a field, finite extension of $k(Q)$, A' the integralclosure of A in L. Suppose that:

a) the fibre in Q of the morphism $A \longrightarrow \hat{A} = (A,I)^\wedge$ is geometrically reduced;

b) A/P is japanese for any prime $P \in A_a(I)$ which contains Q.

Then the following are equivalent:

1) A' is a finite A-algebra;

2) the radical of IA' is finitely generated.

Proof. As in (2.3) it follows that A'/IA' is finite over A/I and so $\hat{A} \otimes_A A'/I(\hat{A} \otimes_A A')$ is finite over $\hat{A}$. Let A'' be the integral closure of $\hat{A}$ in $Q(\hat{A} \otimes_A L)$. Then $A'' = A''_1 \times \dots \times A''_n$, where A''_i is the integral closure of $\hat{A}$ in $k(Q_i)$, $i = 1,\dots,n$, $Q_1,\dots,Q_n$ being the minimal prime ideals of the reduced ring $\hat{A} \otimes_A L$. Each A''_i is I-adically separated, so A'' is I-adically separated. On the other hand we have inclusions $\hat{A} \hookrightarrow \hat{A} \otimes_A A' \hookrightarrow A''$ $\longleftrightarrow$ so $\hat{A} \otimes_A A'$ is I-adically separated. Again by ([4] , 8.4) it follows that $\hat{A} \otimes_A A'$ is finite over $\hat{A}$ and by faithful flatness A' is finite over A.

(2.6) COROLLARY. Let A be a noetherian reduced ring, I an ideal contained in the Jacobson radical of A. Suppose that:

a) the generic fibres of the morphism $A \longrightarrow (A,I)^\wedge$ are reduced;

b) A/P is japanese for any prime $P \in A_a(I)$.

Let A' be the integral closure of A in its total quotient ring. Then the following are equivalent:

1) A' is a finite A-algebra;

2) the radical of IA' is finitely generated.

Proof. Let $Q_1,\ldots,Q_n$ be the minimal prime ideals of A, $A_i = A/Q_i$, $K_i = k(Q_i)$ for $i = 1,\ldots,n$. Then $A' = A'_1 \times \ldots \times A'_n$, where A'_i is the integral closure of A in K_i. So A' is finite over A if and only if A'_i is finite over A for any i.
Let $P' \in A_a(I+Q_i/Q_i)$ for some i. Then $P' = P/Q_i$, where P is a prime ideal of A, and by (1.7) $P \in A_a(I)$ so A_i/P is japanese. Now we get the conclusion as in (2.5).

3. ON TATE'S THEOREM

Recall that J. Tate proved the following useful theorem about a lifting property for japanese rings.

(3.1) THEOREM (see [1]): Let A be a noetherian normal domain, x a non-zero element of A such that:

a) xA is a prime ideal;

b) A is xA-adically complete;

c) A/xA is japanese.

Then A is japanese.

The first generalization was given by Seydi ([8]) which dropped the assumption that A is normal. Then Marot ([3])

dropped the assumption that xA is a prime ideal, changed the condition c) in "A/P is japanese for any $P \in \mathrm{Ass}_A(A/xA)$" and put supplementary conditions on the integral closure of A. Lastly Chiriacescu gave the following general form:

(3.2) THEOREM ([1]). Let A be a noetherian integral domain, x a non-zero element of A such that:

a) A is xA-adically complete;

b) A/P is japanese for any $P \in \mathrm{Ass}_A(A/xA)$.

Then A is japanese.

We want to weaken condition b) and then we will try to generalize the result in the non-complete and non-domain case.

(3.3) PROPOSITION. Let A be a noetherian integral domain, x a non-zero element of A. Let S be a set of prime ideals of A with the following property:

"for any domain B, integral extension of A, and any prime ideal Q of B, minimal over xB, we have that $Q \cap A \in S$".

Suppose that:

1) A is xA-adically complete;

2) A/P is japanese for any $P \in S$;

Then A is japanese.

Proof. Let K the field of quotients of A, L a field, finite extension of K, A' the integral closure of A in L. Let $Q_1, \ldots, Q_n$ be the minimal prime over-ideals of xA' and $P_i = Q_i \cap A$ for $i = 1, \ldots, n$. Then $P_i \in S$ and $[k(Q_i) : k(P_i)] < \infty$ for any $i = 1, \ldots, n$. As by b) A'/Q_i is noetherian for any i, by a result of Nishimura ([7], see [1], 1.1) it follows that A'/xA' is noetherian.

Let $J = Q_1 \cap \ldots \cap Q_n$ be the radical of xA'. As in (2.3) we obtain that A'/J is finite over A/xA and that A'/J^n is finite A-algebra for any natural n. As A'/xA' is noetherian there is a natural number n, such that $J^n \subseteq xA'$. As in (2.3) it follows that A' is a finite A-algebra.

(3.4) <u>COROLLARY</u> : Let A be a noetherian integral domain , x a non-zero element of A. Suppose that:

a) A is xA-adically complete;

b) A/P is japanese, for any $P \in A_a(xA)$.

Then A is japanese.

<u>Proof</u>. Obvious by (3.3) and (1.6).

(3.5) <u>REMARKS</u> : a) Taking $S = \mathrm{Ass}_A(A/xA)$ in (3.3) we obtain (3.2).

b) We can take also $S = A(xA)$ in (3.3). But in this case $A(xA) = \mathrm{Ass}_A(A/xA)$ because x is a non-zero divisor. On the other hand (3.4) is a generalization of (3.2) since $A_a(xA) \subseteq \mathrm{Ass}_A(A/xA)$ (1.4).

Also, in ([6], Example 1) is constructed a noetherian ring A and an element x of A such that $A_a(xA) \neq \mathrm{Ass}_A(A/xA)$.

Now we try to generalize further (3.4).

(3.6) <u>PROPOSITION</u> : Let A be a noetherian ring, x a non-zero element of A. Suppose that:

a) A is xA - adically complete;

b) A/P is japanese for any $P \in A_a(xA)$.

Then A/Q is japanese for any minimal prime ideal which contains Q and by (1.7) $P \in A_a(xA)$ so that by b) B/P is japanese. Now we can apply (3.4) to the integral domain B.

(3.7) <u>THEOREM</u>. Let A be a noetherian reduced ring, x a non-zero element contained in the Jacobson radical of A.

Suppose that:

a) the generic fibres of the morphism $A \longrightarrow \hat{A} = (A, xA)^{\wedge}$ are reduced;

b) A/P is japanese for any prime $P \in A_a(xA)$.

Then if A' is the integral closure of A in its total quotient ring, A' is a finite A-algebra.

Proof : Let $Q(A) = K_1 \times \ldots \times K_n$, where $K_i = k(Q_i)$, $Q_1, \ldots, Q_n$ being the minimal primes of A. Then A' is the direct product of the integral closures of A in K_i, so it is sufficient to prove that if A is an integral domain such that the generic fibre of the completion morphism $A \longrightarrow \hat{A}$ is reduced, then A' is a finite A-algebra. By (2.6) it is sufficient to prove that the radical of xA' is finitely generated. Let Q be a minimal prime over-ideal of xA'.

Then $Q \cap A \in A_a(xA)$ by (1.6) so $A/Q \cap A$ is japanese. It follows that A'/Q is finite over $A/Q \cap A$, so it is noetherian. Now by ([1], (1.1)) A'/xA' is noetherian. The conclusion follows.

The same method of proof shows that we have also:

(3.8) THEOREM. Let A be a noetherian ring, x a non-zero elementar contained in the Jacobson radical of A. Suppose that:

a) the generic fibres of the morphism $A \longrightarrow (A, xA)^{\wedge}$ are geometrically reduced;

b) A/P is japanese, for any $P \in A_a(xA)$.

Then A/Q is japanese for any minimal prime Q of A.

Proof. Let Q be a minimal prime of A, $B = A/Q$, $P' \in A_a(xB)$. Then by (1.7) $P' = P/Q$ where $P \in A_a(xA)$ so B/P' is japanese. By (2.5) we have only to show that if L is a field, finite extension of $Q(B)$, and B' the integral closure of B in L, the radical of xB' is finitely generated. But this follows exactly as in (3.6).

(3.9) COROLLARY: Let A be a noetherian ring, x a non-zero element contained in the Jacobson radical of A. Suppose that:

a) the generic fibres of the morphism $A \longrightarrow (A,xA)^{\wedge}$ are geometrically reduced;

b) A/P is japanese, for any $P \in Ass_A(A/xA)$.

Then A/Q is japanese, for any minimal prime ideal of A.

Proof. Obvious by (3.8) and (1.4).

(3.10) REMARK. (3.9) is clearly a generalization of (3.2). The use of asymptotic prime divisors enables us to obtain (3.9). One cannot hope to generalize (3.2) in this way, without using the set $A_a(xA)$, because for $Ass_A(A/xA)$ we have not a result similar to (1.7).

4. APPLICATION TO JAPANESE RINGS

In this section we will apply some of these results to give an answer to the following question ([1]) :

QUESTION: Let A be a noetherian domain, I an ideal of A. Suppose that:

a) A is I-adically complete;

b) A/P is japanese, for any $P \in Ass(A/I)$.

Does it follows that A is japanese?

We will begin by listing a trivial consequence of (3.4).

(4.1) COROLLARY: Let A be a noetherian domain, x a non-zero element of A. Suppose that:

a) A is xA-adically complete;

b) A/P is japanese, for any $P \in U(xA)$.

Then A is japanese.

Proof: Trivial by (3.4) and (1.9);

(4.2) PROPOSITION. Let A be a noetherian ring, x an element of A such that:

a) A is xA-adically complete;

b) A/P is japanese, for any $P \in U(xA)$.

Then A/Q is japanese for any $Q \in Ass(A)$.

Proof: Let $Q \in Ass(A)$ and $P' \in U(xA + Q/Q)$. Then $P' = P/Q$, where P is a prime ideal which contains Q. By (1.9) it follows that $P \in U(xA)$. Let B = A/Q. It follows that B/P' is japanese.
From (4.1) we have that B is japanese.

(4.3) LEMMA Let A be a noetherian ring, x a non-zero divizor in A. Suppose that:

a) A is xA-adically complete;

b) A/P is japanese, for any $P \in Ass(A/xA)$.

Then A/Q is japanese, for any $Q \in Ass(A)$.

Proof. As x is a non-zero divizor, it follows that $Ass(A/xA) = A(xA) \supseteq U(xA)$. Now we apply (4.2)
Now we can give the promised result.

(4.4) THEOREM: Let A be a noetherian domain, I an ideal generated by an A-regular sequence. Suppose that:

a) A is I-adically complete;

b) A/P is japanese, for any $P \in Ass(A/I)$.

Then A is japanese.

Proof: Let $I = (x_1,\dots,x_n)$. We will use induction on n. For n=1 we can apply (3.2). Let $J = (x_1,\dots,x_{n-1})$, $A' = A/J$. Let $Q \in Ass(A'/x_nA')$. So A'/Q is japanese. As I is generated by a regular sequence, x_n is not a zero-divizor in A', so by (4.3) it follows that A'/P is japanese, for any $P \in Ass(A')$.
By induction it follows that A is japanese.

(4.6) COROLLARY: Let A be a noetherian domain, I an ideal generated by a regular sequence. Suppose that:

a) A is I-adically complete;

b) A/P is japanese, for any $P \in \mathrm{Ass}(A/I)$.

Then the restricted power series ring $A_I\{X\}$ is japanese.

Proof: Let $B = A_I\{X\}$. Then B is IB-adically complete and $B/IB \simeq A/I[X]$. Let $Q \in \mathrm{Ass}\,(B/IB)$. It follows that $Q \cap A/I = P$ where $P \in \mathrm{Ass}(A/I)$. So A/P is japanese and $Q = PA_{/I}[X]$, so that $A/I\,[X]_{/Q} = A/P[X]$ is a japanese ring.

(4.7) REMARK As we saw in (3.10), using asymptotic prime divisors is also essential in proving (4.6). The good property we used, was (1.9), d).

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