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by

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§o. Introduction

Let X be a complex space and denote by $F_q(X)$ the set of all functions $f:X\to\mathbb{R}$ such that for any $x\in X$ there exists an open neighbourhood U of x and finitely many smooth strongly q-convex functions $f_1,\dots,f_s$ on U such that $f|_U=\max(f_1,\dots,f_s)$. These are the "strongly q-convex functions with corners" ([3], [4], [14]). The main result in [3], [4] shows that on a complex space X of dimension n for any $f\in F_q(X)$ and any continuous function $\eta>0$ on X there is a smooth strongly $\tilde{q}$-convex function $\tilde{f}$ on X, with $\tilde{q}=n-[\frac{n}{q}]+1$, such that $|f-\tilde{f}|<\eta$ on X.

In this paper we consider only the particular case q=n. Hence the above theorem of Diederich and Fornaess asserts that on an n-dimensional complex space any strongly n-convex function with corners can be approximated uniformly by smooth strongly n-convex functions. We get easily from this last statement that any irreducible n-dimensional complex space X is strongly n-concave and, if addition X is non-compact, then X is n-complete. The n-completeness is a well known result of Ohsawa [13] (see also [2], [8]).

§1. Preliminaries

We recall some classical definitions [1] :

A complex space X is called q-complete if there exists a smooth strongly q-convex function $\varphi:X\to\mathbb{R}$ such that $\{\varphi<c\}\subset\subset X$ for any $c\in\mathbb{R}$.

X is called strongly q-concave if there exists a smooth function $\varphi:X\to(0,\infty)$ which is strongly q-convex outside a compact set and such that $\{\varphi>\varepsilon\}\subset\subset X$ for any $\varepsilon>0$.

If X is a complex space we denote by $F_q(X)$ the strongly q-convex functions with corners on X ([3] ,[4], [14]) i.e. those continuous functions on X such that for any $x\in X$ there is an open neighbourhood $U=U(x)$ of x and finitely many smooth strongly q-convex functions $f_1,\dots,f_s$ on U such that $f|_U=\max(f_1,\dots,f_s)$. In [3] and [4] Diederich and Fornaess have proved the following :

Theorem 1. Let X be a complex space of dimension n, $f\in F_q(X)$, $1\leq q\leq n$, and $\eta>0$ a continuous function on X. Then there exists a smooth strongly $\tilde{q}$-convex function $\tilde{f}$ on X, where $\tilde{q}=n-\left[\frac{n}{q}\right]+1$, such that $|f-\tilde{f}|<\eta$ on X.

In fact in our proofs we shall need this theorem only in the particular case q=n, hence $\tilde{q}=n$.

Let us also recall some results concerning polydiscs in complex manifolds (see [6] , [7]). An open subset of a complex manifold X of dimension n is called polydisc if it is biholomorphically equivalent to the unit polydisc $U^n\subset\mathbb{C}^n$. Fornaess and Stout [7] have shown :

Theorem 2. Any connected complex manifold has a finite covering with polydiscs.

If $W\subset X$ is a polydisc in the complex manifold X, the subset $W_1\subset\subset W$ is said to be a concentric polydisc if for some biholomorphic map $\psi: U^n\to W$ there is $r\in(0,1)$ such that if $rU^n=\{rz \mid z\in U^n\}$, then $W_1=\psi(rU^n)$. In ([6] ,Lemma 3) it is proved: if Δ_1 and Δ_2 are disjoint polydiscs in the complex connected manifold X and if $\Delta_1'\subset\subset\Delta_1$, $\Delta_2'\subset\subset\Delta_2$ are concentric polydiscs, then there exists a polydisc $\Delta\subset X$ containing Δ_1' and Δ_2'. From this statement it follows easily :

Corollary 1. If X is a connected complex manifold and $A\subset X$ a finite set then there is a polydisc $W\subset X$ containing A.

§2. The concave case

In this section we obtain, as a consequence of the approximation theorem of Diederich and Fornaess (Theorem 1), the following:

Theorem 3. Any n-dimensional irreducible complex space X is strongly n-concave.

In fact we establish a stronger result, which clearly implies the above statement, namely:

Theorem 3'. Let X be an irreducible n-dimensional complex space, $z_o \in X$ any point and $V=V(z_o) \subset\subset X$ any neighbourhood. Then there exists a smooth function $\varphi : X \to (o,\infty)$ which is strongly n-convex outside $\overline{V}$ and such that $\{\varphi > \varepsilon\} \subset\subset X$ for any $\varepsilon > o$.

We first verify that Theorem 3' holds when X is the unit polydisc U^n.

Lemma 1. Theorem 3' holds if $X=U^n$.

Proof

By means of an automorphism of U^n we may assume z_o=o=the origin. We make also the following remark: Let $\Omega \subset \mathbb{C}$ be an open set, $F=F(u,t): \Omega \times [-1,1]^n \to \mathbb{R}$ a continuous function and define $F_t(u)=F(u,t)$ $f(u)= \int_{[-1,1]^n} F(u,t)dt$. We assume that F_t is subharmonic on Ω for any t. Then : a) f is subharmonic and continuous on Ω

b) if moreover F_t is strongly subharmonic for t in a set of positive Lebesgue measure then f is strongly subharmonic and continuous on Ω

The proof of this statement is straightforward and is ommited.

Let $\varepsilon(z) > o$ $z \in U^n \setminus \{o\}$ be C^∞ and "small" (vanishing rapidly at the boundary $\partial(U^n \setminus \{o\})$) and $\lambda \in C^\infty(\mathbb{R}), \lambda \geq o$ with supp$\lambda \subset [-1,1]$ and $\int_{\mathbb{R}} \lambda(t)dt=1$. We define

$$\varphi(z)= \int_{[-1,1]^n} \frac{\lambda(t_1)\cdots\lambda(t_n)dt_1\cdots dt_n}{\max(|z_1|^2-\varepsilon(z)t_1,\ldots,|z_n|^2-\varepsilon(z)t_n)} - 1 \qquad z \in U^n \setminus \{o\}$$

If $\varepsilon(z)>0$ is small enough (vanishing rapidly at the boundary of $U^n\setminus\{0\}$) φ is smooth, $\varphi>0$, $\varphi\to 0$ at ∂U^n and $\varphi\to\infty$ at 0.
We show that, if in addition the first and second derivatives of ε are small enough, φ is strongly n-convex on $U^n\setminus\{0\}$ (by modifying φ near 0 we will get then the required function).

Let $z^0\in U^n\setminus\{0\}$ be any point and $L(z^0)$ the complex line passing through 0 and z^0. Near z^0, on $L(z^0)$, we have $z=uz^0$ with a complex parameter u in a neighbourhood of $1\in\mathbb{C}$.

We consider first the case $|z_1^0|=\ldots=|z_n^0|=\alpha>0$. Then the restriction of φ on $L(z^0)$, as a function of u, has the following form :

$$\varphi(u)=\int_{[-1,1]^n}\frac{\lambda(t_1)\cdots\lambda(t_n)\,dt_1\ldots dt_n}{\max(\alpha^2|u|^2-\varepsilon(uz^0)t_1,\ldots,\alpha^2|u|^2-\varepsilon(uz^0)t_n)}-1$$

If ε and its first and second derivatives at z^0 are small enough then the function $u\to\frac{1}{\alpha^2|u|^2-\varepsilon(uz^0)\mu}$ is strongly subharmonic in a fixed neighbourhood of $1\in\mathbb{C}$ for any $-1\le\mu\le 1$. For $t=(t_1,\ldots,t_n)\in[-1,1]^n$ consider the function $\varphi_t(u)=\frac{\lambda(t_1)\cdots\lambda(t_n)}{\alpha^2|u|^2-\varepsilon(uz^0)\min(t_i)}$. For any t φ_t is subharmonic and for t in a set of positive Lebesgue measure φ_t is strongly subharmonic. By the remark made at the beginning of the proof $\varphi|_{L(z^0)}$ is strongly subharmonic in a neighbourhood of z^0, hence, being smooth, it is also strongly subharmonic along the lines parallel to $L(z^0)$ (in a neighbourhood of z^0).

Consider now the general case. We may assume that $|z_1^0|=\ldots=|z_k^0|=\alpha>|z_{k+1}^0|\geq\ldots\geq|z_n^0|$. If ε is sufficiently small at z^0, then on $L(z^0)$ (near z^0) $\max(|z_1|^2-\varepsilon(z)t_1,\ldots,|z_n|^2-\varepsilon(z)t_n)=\max(\alpha^2|u|^2-\varepsilon(uz^0)t_1,\ldots,\alpha^2|u|^2-\varepsilon(uz^0)t_k)$ where $(t_1,\ldots,t_n)\in[-1,1]^n$ and u is in a fixed neighbourhood of $1\in\mathbb{C}$. Since $\int_{\mathbb{R}}\lambda(t)dt=1$ it follows that on $L(z^0)$ (near z^0), as a function of u, φ has the form

$$\varphi(u)=\int_{[-1,1]^k}\frac{\lambda(t_1)\cdots\lambda(t_k)\,dt_1\cdots dt_k}{\max(\alpha^2|u|^2-\varepsilon(uz^0)t_1,\ldots,\alpha^2|u|^2-\varepsilon(uz^0)t_k)}-1$$

The same arguments as before show that $\varphi|_{L(z^0)}$ is strongly subharmonic near z^0 if ε and its first and second derivatives at z^0

are small enough. By an exhaustion argument the conclusion of Lemma 1 follows easily.

Lemma 2. Theorem 3' holds for nonsingular X.

Proof

Let $W_o,\dots,W_p$ be polydiscs in X such that $X=W_o\cup\dots\cup W_p$. We may assume that $z^o\in W_o$ and let $z_1,\dots,z_p$ be points with $z_i\neq z_j$ for any $i\neq j$ $i,j\in\{o,\dots,p\}$. By Corollary 1 there is a polydisc W_{p+1} containing $z_o,\dots,z_p$. Choose open subsets $V_o,\dots,V_p$ such that : $z_i\in V_i$, $V_i\subset\subset W_i\cap W_{p+1}$, $V_o\subset\subset V(z_o)$ and $\overline{V}_i\cap\overline{V}_j=\emptyset$ for $i\neq j$. For any $o\leq i\leq p$ let $\varphi_i:W_i\to(o,\infty)$ be smooth functions which are strongly n-convex outside $\overline{V}_i$ and $\{\varphi_i>\varepsilon\}\subset\subset W_i$ for any $\varepsilon>o$ (which exist by Lemma 1). Let $\varphi_{p+1}:W_{p+1}\to(o,\infty)$ be smooth, strongly n-convex outside $\overline{V}_o$, $\{\varphi_{p+1}>\varepsilon\}\subset\subset W_{p+1}$ for any $\varepsilon>o$ and $\varphi_{p+1}|\overline{V}_i>\varphi_i|\overline{V}_i$ $o\leq i\leq p$.

If we define $\varphi(x)=\max\{\varphi_i(x)\mid x\in W_i\}$ then $\varphi|_{X\setminus\overline{V}_o}\in F_n(X\setminus\overline{V}_o)$ and $\{\varphi>\varepsilon\}\subset\subset X$ for any $\varepsilon>o$. Lemma 2 follows now from the approximation theorem of Diederich and Fornaess.

Lemma3. Let X be a complex space, $A\subset X$ a closed analytic subset, $f\in F_q(A)$ and $\eta>o$ a continuous function on A. Then there exists an open neighbourhood V of A and $\tilde{f}\in F_q(V)$ such that $|\tilde{f}-f|<\eta$ on A.

Proof

By a perturbation argument we see that there is a locally finite open covering of A $(U_i)_{i\in N}$, $U_i\subset\subset A$, and smooth strongly q-convex functions ρ_i near $\overline{U}_i$ (which admit smooth strongly q-convex extensions to open subsets $\tilde{U}_i$ of X, $\overline{U}_i\subset\tilde{U}_i$) such that :

a) $\rho=\max\{\rho_i\}$ satisfies $|f-\rho|<\eta$

b) $\rho_i|\partial U_i<\rho|\partial U_i$

It follows easily from b) that ρ can be extended to a function $\tilde{f}\in F_q(V)$ if V is a sufficiently small open neighbourhood of A.

Lemma 4. Let X be a complex space of dimension n. Then there exists a smooth strongly (n+1)-convex function $\varphi:X\to(o,\infty)$ such that $\{\varphi>\varepsilon\}\subset\subset X$ for any $\varepsilon>o$.

Proof

It is clear that for smooth X it is nothing to prove. For singular X the proof is by induction on dimX. Let n=dimX and assume that the lemma holds for all complex spaces of dimension $\leq n-1$. If we set Y=Sing(X) then there exists $\varphi_1 : Y \to (o,\infty)$ which is a smooth strongly n-convex function and $\{\varphi_1 > \varepsilon\} \subset\subset Y$ for any $\varepsilon > o$. By Lemma 3 there is a neighbourhood V of Y and $\tilde{\varphi}_1 \in F_n(V)$ which approximates φ_1 on Y and from the approximation theorem of Diederich and Fornaess we may assume that $\tilde{\varphi}_1$ is smooth strongly n-convex. Let $\varphi : X \to (o,\infty)$ be a smooth function such that $\{\varphi > \varepsilon\} \subset\subset X$ for any $\varepsilon > o$ and $\varphi = \tilde{\varphi}_1$ near A. Clearly φ satisfies the required properties.

Lemma 5. Let $u : (o,a] \to (o,\infty)$ be any continuous function. Then there exists a smooth strictly increasing convex function $v : (o,\infty) \to (o,\infty)$ such that $v < u$ on $(o,a]$ and $\lim_{t \to o} v(t) = o$.

The proof is elementary and is ommited.

Proof of Theorem 3'

Let Y=Sing(X). We may assume that $z_o \in Reg(X)$ and $V = V(z_o) \subset\subset Reg(X)$. Since Reg(X) is connected it follows from Lemma 2 that there is a smooth function $\varphi_1 : Reg(X) \to (o,\infty)$ which is strongly n-convex outside $\overline{V}$ and $\{\varphi_1 > \varepsilon\} \subset\subset Reg(X)$ for any $\varepsilon > o$. From Lemma 4, Lemma 3 and the approximation theorem of Diederich and Fornaess there is an open neighbourhood U of Y, $U \cap \overline{V} = \phi$, and a smooth strongly n-convex function $\tilde{\varphi}_2 : U \to (o,\infty)$ such that $\{\tilde{\varphi}_2 > \varepsilon\} \subset\subset X$ for any $\varepsilon > o$. By shrinking U we may assume also that $\tilde{\varphi}_2$ is defined near $\overline{U}$. By Lemma 5 there is a smooth strictly increasing convex function $v : (o,\infty) \to (o,\infty)$ such that $v \circ \tilde{\varphi}_2 < \varphi_1$ on ∂U. If we set $V_1 = Reg(X)$, $V_2 = U$, $\varphi_2 = v \circ \tilde{\varphi}_2$ then $\varphi(x) = \max \{\varphi_i(x) \mid x \in V_i\}$ is in $F_n(X \setminus \overline{V})$ and $\{\varphi > \varepsilon\} \subset\subset X$ for any $\varepsilon > o$. Theorem 3' follows now from the approximation result of Diederich and Fornaess.

§3. The convex case

We give now a short proof of Ohsawa's theorem [13] using convex functions with corners.

Theorem 4. Let X be an irreducible n-dimensional noncompact complex space. Then X is n-complete.

Proof

Let $\pi:\tilde{X}\to X$ be a resolution of singularities [9], $Y=\mathrm{Sing}(X)$ and $\tilde{Y}=\pi^{-1}(Y)$. By Greene and Wu result [8] (i.e. Theorem 4 holds for nonsingular spaces) there is a smooth strongly n-convex exhaustion function $\tilde{\varphi}:\tilde{X}\to\mathbb{R}$.

Also it is easy to verify that any complex space Z of dimension k is (k+1)-complete (the proof is identical with the proof of Lemma 4). Hence we deduce from Lemma 3 and Theorem 1 (since $\dim Y< n$) that there is an open neighbourhood V of Y and a smooth strongly n-convex function τ on V such that $\tau|_Y$ is an exhaustion function. Shrinking V we may assume that τ is defined near $\bar{V}$ and $\tau|_{\bar{V}}$ is proper. We choose a function $g:X\to[-\infty,\infty)$, $Y=\{g=-\infty\}$, with g smooth outside Y and such that g is locally the sum of a plurisubharmonic function and a smooth function (the existence of such a function is proved by M.Peternell in [15]). We may assume also that $g=0$ outside V. Let $\alpha\in C^{\infty}(\mathbb{R})$ be a strictly increasing convex function such that $\alpha\circ\tilde{\varphi}>\tau\circ\pi$ on $\pi^{-1}(\partial V)$ and $\alpha\circ\tilde{\varphi}+g\circ\pi$ is strongly n-convex outside $\tilde{Y}$. We set $V_1=X\setminus Y$, $V_2=V$, $\varphi_1=g+\alpha\circ\tilde{\varphi}\circ\pi^{-1}$ $\varphi_1:V_1\to\mathbb{R}$ and $\varphi_2=\tau$ $\varphi_2:V_2\to\mathbb{R}$. We define $\varphi(x)=\max\{\varphi_i(x)\mid x\in V_i\}$. Then φ is an exhaustion function and $\varphi\in F_n(X)$. From the approximation result of Diederich and Fornaess X is n-complete.

Remark

Let X be a complex space and WP(X) the weakly plurisubharmonic functions on X, i.e. those upper semi-continuous functions $\varphi:X\to[-\infty,\infty)$ such that for any holomorphic map $u:D\to X$ ($D\subset\mathbb{C}$ is the unit disc) it follows that $\varphi\circ u$ is subharmonic on D. We denote also by WSP(X) those $\varphi\in WP(X)$ with the following property: for any $\theta\in C_0^{\infty}(X)$ there is $\varepsilon_0>0$ such that $\varphi+\varepsilon\theta\in WP(X)$ if $|\varepsilon|\leq\varepsilon_0$.

Norguet and Siu [12] have proved the following result (see also [5]) :

Theorem 5. Let X be a complex space and assume that there exists $\varphi\in WSP(X)\cap C(X)$ such that $\{\varphi<c\}\subset\subset X$ for any $c\in\mathbb{R}$. Then X is Stein.

We show now that the method used to prove Theorem 4 of Ohsawa gives also a short proof for the theorem of Norguet and Siu.

Proof of the theorem of Norguet and Siu

Replacing X by its irreducible components [10] we may assume that X has finite dimension and let $n=\dim X$. By induction on $\dim X$ we may assume that $Y=\mathrm{Sing}(X)$ is Stein and by Richberg theorem [16] we may also assume that there is an open neighbourhood V of Y and a continuous strongly plurisubharmonic function τ on V (in the usual sense with local extensions [11]) with $\{x\in V\mid\tau(x)<c\}\subset\subset X$ for any $c\in\mathbb{R}$. Shrinking V we may suppose also that τ is defined near $\overline{V}$. Let $g:X\to[-\infty,\infty)$ be a function with $\exp(g)$ continuous, $Y=\{g=-\infty\}$ and g is locally the sum of a smooth function and a plurisubharmonic function [15]. The function g may be easily obtained as follows : we choose $D_i''\subset\subset D_i'\subset\subset D_i\subset\subset X$ Stein open subsets and $\varphi_i:D_i\to[-\infty,\infty)$ plurisubharmonic functions with $A\cap D_i=\{\varphi_i=-\infty\}$, $\exp(\varphi_i)$ continuous and such that a) $\{D_i\}_{i\in\mathbb{N}}$ is locally finite and $X=\bigcup_{i\in\mathbb{N}}D_i''$ b) $\varphi_i-\varphi_j$ is bounded on $D_i'\cap D_j'\setminus A$ (e.g. we may define $\varphi_i=\log\sum_k|f_{k,i}|^2$ where $\{f_{k,i}\}_k$ if a finite set of generators of $\mathcal{J}_A$ on D_i). We choose also functions $p_i\in C_0^\infty(X)$, $p_i\geq 0$, supp $p_i\subset D_i'$ and $\varphi_i+p_i<\varphi_j+p_j$ on $\partial D_i'\cap D_j''\setminus A$. If we define $g(x)=\max\{\varphi_i(x)+p_i(x)\mid x\in D_i'\}$ then g satisfies the required conditions. It is clear that we also may assume $g\geq 0$ on $\complement V$.

We choose a strictly increasing convex function α which is piecewise linear with $\alpha\circ\varphi>\tau$ on ∂V and $\alpha\circ\varphi+g\in WSP(X)$, so $\alpha\circ\varphi+g$ is strongly plurisubharmonic on $X\setminus Y$ because $X\setminus Y$ is smooth. Then $\max(\alpha\circ\varphi+g,\tau)$ defines a continuous strongly plurisubharmonic exhaustion function on X, hence X is Stein [11].

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