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COEFFICIENT RINGS IN THE NONSEPARABLE CASE

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PREPRINT SERIES IN MATHEMATICS

No.73/1990

BUCURESTI

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November 1990

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The structure theory of noncommutative algebras of Wedderburn and the structure theory of commutative complete local rings do not look to have very much in common. Meanwhile both theories are looking for nice coefficient rings. The best thing that might happen is that the coefficient ring is isomorphic to the residual algebra, this happens in the case of separable algebras and in the commutative case. This is no more true in the nonseparable case, however under certain assumptions one can find coefficient rings for the "separable part" and for "the pure inseparable part" of the residual algebra and describe the coefficient ring as the compositum of these two rings.

We shall denote by $J = J(A)$ the Jacobson radical of A by $Z = \mathbb{Z}(A)$ the center of A. All algebras we consider are finite dimensional algebras over a field k of positive characteristic p. Denote by $\bar{x}$ the image of $x \in A$ in the residual algebra A/J.

We say that a subring $B \subset A$ is a coefficient ring if B is minimal with the property $B/J \cap B = A/J$. By Zorn's lemma such minimal rings always exists.

Let us look now at two counterexamples which prove that without the separability assumption the conclusion of Wedderburn Principal Theorem is no more true.

Example 1 [De] p.25. Let $k = F_2(t)$, $B = k[b]/(b^2)$, $A = B[\sqrt{b+t}]$.

Then $J(A) = Ab = kb + kc$ and $\bar{A} = A/J \simeq k(\sqrt{t})$ because if $a = \sqrt{b+t}$ in A/J, $\bar{a}^2 = \bar{t}$. But A contains no k subalgebra isomorphic to A because then A would contain an $a' \in A$ such that $a'^2 = t$ and $a' \in a + J$ which is impossible because for any $a' = a + ub + vc$ we have $a'^2 = a^2 = t + b$ as char $k = 2$.

As A contains $F_2(t+b)$ a coefficient ring in A is $F_2(t+b)[\sqrt{t+b}]$ which is no more a k subalgebra. As A is a commutative ring Cohen's theory works here.

Example 2 [RV] . Let $R = F_p\langle X,Y\rangle$ with noncommuting variables X and Y such that $T = [X,Y] = XY - YX$ and $[X,T] = [X,T] = T^2 = 0$.

Then the set S of non-zero divisors in R satisfy the Ore condition and $A = S^{-1}R$ the ring of quotients of R relative to S has an unique maximal ideal $\underline{m} = AT = TA$ and residual field $A/\underline{m} = F_p(\bar{X},\bar{Y})$ where $\bar{X}$ and $\bar{Y}$ commute.

A contains no subring isomorphic to A/J because since A/J is commutative and $\underline{m}$ is included in the center it would follow that A is commutative, a contradiction with $T \neq 0$.

In what follows as the problem of rising idempotents and matrix units has a positive answer in our context we shall consider the problem of finding coefficient rings for the case when A is a completely primary K algebra i.e. A/J is a division algebra.

Let us supose that $Z(A/J) = K$ is not a separable extension of k. Then we denote by L the separable closure of k in K and K is a purely inseparable extension of L, $K = L(\theta_1,\ldots,\theta_t)$ with $\theta_i^{p^{n_i}} = a_i \in L \setminus L^p$.

We have now to extend the result in [HP1] as follows.

Theorem 1. Let e be the natural number such that $J^e = 0$ and let P_i be the irreducible polynomials of $a_i = \theta_i^{p^{n_i}} \in L \setminus L^p$ over k. In $k\langle X_1,\ldots,X_t\rangle$ set m the bilateral ideal generated by the commuters $[X_i,X_j]$ and $P_i(X_i^{p^{n_i}})$. Then A contains a subring R which is a homomorphic image of $k\langle X_1,\ldots,X_t\rangle/\underline{m}^e$ which is a coefficient ring of representatives for K.

The proof is similar to that given in [HP1]. Now we need the following

Theorem (Hochcshild). For any extension $L \subset K$ such that $K^{p^n} \subset L$ for an integer n and such that K is a projective L module of finite type the homomorphism between Brauer groups $Br(L) \to B(K)$ is onto.

See [H] th.5 and [KO] th. 6.1.

Theorem 2. Let A be completely primary k algebra with residual division algebra $D = A/J$ of center K a purely inseparable extension of L the separable closure of k in K. Then if $[D] = [\mathcal{D} \otimes_L K]$ in the Brauer group Br(K) and if the Schur index Ind $\mathcal{D}$ is prime to p the characteristic of A, than A contains a separable subalgebra isomorphic to $\mathcal{D}$ and an ring R as described in theorem 1 with residual field K.

The subring generated in A by $\mathcal{D}$ and R is a coefficient ring for A.

Proof. If ind $\mathcal{D}$ is prime to p then $\mathcal{D} \otimes_L K$ is a division algebra so in $\mathcal{D} \otimes_L K = D_n$ given by Hochschild theorem we can take $n = 1$. If $\varphi : A \to A/J$ is the canonical mapping and $B = \varphi^{-1}(\mathcal{D})$ as $\mathcal{D} \subset \mathcal{D} \otimes_L K = D$ we can use Wedderburn Principal Theorem and obtain an k subalgebra of B isomorphic to $\mathcal{D}$, which we shall denote again by $\mathcal{D}$. Using theorem 1 the rest of the proof is clear. The coefficient ring is an homomorphic image of the coproduct of R and $\mathcal{D}$.

Remark. An analogous result can be deduced for the unequal characteristic case, see [HP2] and [Az].

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