INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

# HOMOLOGY AND COHOMOLOGY OF WEIGHTED COMPLETE INTERSECTIONS

by

Louis FUNAR

PREPRINT SERIES IN MATHEMATICS

No. 75/1990

BUCURESTI

HOMOLOGY AND COHOMOLOGY OF WEIGHTED

COMPLETE INTERSECTIONS

by

Louis FUNAR*)

December 1990

*) Institute of Mathematics, Bd. Pacii 220, 79622 Bucharest, Romania, and

Universite de Paris-Sud, Mathematiques, Bat 425, Orsay 91405, France.

## HOMOLOGY AND COHOMOLOGY OF WEIGHTED COMPLETE INTERSECTIONS

by Louis FUNAR :Institute of Mathematics,Bucharest,Romania
and Universite de Paris-Sud,Mathematiques,Bat 425
Orsay 91405 ,France

The first part of this paper is devoted to the computation of the integer homology and cohomology of weighted projective spaces $P(a_0,a_1,...,a_n)$.These spaces are defined as follows:denote by $\zeta_l$ the l-th root of unity $\exp(2\pi i/l)$.Let the group $G=Z/a_0Z \oplus Z/a_1Z \oplus ...\oplus Z/a_nZ$ acts on $CP^n$ by

$$(k_0,k_1,...,k_n)(z_0,z_1,...,z_n)=(\zeta_{a_0}^{k_0}z_0,\zeta_{a_1}^{k_1}z_1,...,\zeta_{a_n}^{k_n}z_n)$$

Then $P(a)$ is the quotient $CP^n/G$ where a denotes the n-tuple $(a_0,a_1,...,a_n)$.An entirely elementary computation of their integral homology was carried out for n=2 in [4].

Next we shall consider a quasi-smooth complete intersection $Y \subseteq P(a)$ coming from an isolated singularity.We shall obtain some informations about its cohomology Z-algebra from which we can get a topological invariant of our singularity computable in terms of weights only.

AKNOWLEDGEMENTS : I express my gratitude to A.DIMCA for introducing me in this field,to P.IONESCU and S.PAPADIMA for many and helpful discussions during this paper was written and to A.NEMETHI for his careful reading of the manuscript .

### 1.Weighted projective spaces

Our first result is a generalization of those proved in [4]:

**THEOREM 1.1: The integral homology and cohomology groups of P(a) are torsion free.In fact there are isomorphic as graded abelian groups to those of $CP^n$.**

Proof: We can assume $\gcd(a_0,a_1,\dots,a_n)=1$ without loss of generality since the obvious projection $P(a) \quad P(ca)$ is an homeomorphism (see [3] ).We shall use induction over n.The begin n=0 is trivial because any weighted projective space is then a point.Consider now the following push-out:

$$\begin{tikzcd} C^n \supset S^{2n-1} \arrow[r, "\phi"] \arrow[d, hook, "i"'] & CP^{n-1} \arrow[d, hook, "i"] \\ C^n \supset D^{2n} \arrow[r, "\Psi"] & CP^n \end{tikzcd}$$

where $\phi(z_0,z_1,\dots,z_{n-1})=(z_0,z_1,\dots,z_{n-1})$ and

$$\Psi(z_0,z_1,\dots,z_{n-1})=(z_0,z_1,\dots,z_{n-1},1-(\Sigma_{0\leq i\leq n-1} \mid z_i\mid^2)^{1/2})$$

and i denotes the inclusion.Let G acts on $D^{2n}\subset C^n$ by

$(k)(z)=(\zeta_{a_n}^{-k_n}\zeta_{a_0}^{-k_0}z_0,\zeta_{a_n}^{-k_n}\zeta_{a_1}^{-k_1}z_1,\dots,\zeta_{a_n}^{-k_n}\zeta_{a_{n-1}}^{-k_{n-1}}z_{n-1})$

and by restriction on $S^{2n-1}$.Then $(\phi,\psi)$ becomes a G-equivariant push-out.This implies

$$H_*(P(a),P(a^\wedge))= H_*(D^{2n}/G,S^{2n-1}/G)$$

where $a^\wedge$ is the (n-1)-tuple which omits $a_n$.But $D^{2n}/G$ is contractible hence we have:

$$H_*(D^{2n}/G,S^{2n-1}/G)=H_{*-1}(S^{2n-1}/G)$$

Now the G-action on $S^{2n-1}$ extends to a G-action on $C^{n*}$.Consider also the following $Z/a_nZ$-action on $C^{n*}$ which invaries $S^{2n-1}$:

$$k\,(z)=(\zeta_{a_n}^{-ka_0}z_0,\zeta_{a_n}^{-ka_1}z_1,\dots,\zeta_{a_n}^{-ka_{n-1}}z_{n-1})$$

Then the map $\tau:C^{n*}\longrightarrow C^{n*}$ given by $\tau(z)=(z_0^{a_0},z_1^{a_1},\dots,z_{n-1}^{a_{n-1}})$ induces an homeomorphism $C^{n*}/G \longrightarrow C^{n*}/(Z/a_n)$.On the other hand observe that we have the retractions

$$C^{n*}/G \longrightarrow S^{2n-1}/G\ ,\quad C^{n*}/(Z/a_nZ)\longrightarrow S^{2n-1}/(Z/a_nZ)$$

Since the $Z/a_nZ$-action on $C^{n*}$ extends to a $S^1$-action the $Z/a_nZ$-action induced in homology is trivial.Now the usual properties of the transfer map (see [1]) imply that the natural maps

$$H_*(S^{2n-1}) \simeq H_*(S^{2n-1})^{Z/a_nZ} \underset{r}{\overset{p}{\rightleftarrows}} H_*(S^{2n-1}/(Z/a_nZ))$$

satify:

$$p \circ r = a_n \cdot 1$$

$$r \circ p = \Sigma_{g \in Z/a_n Z} (\ )^g = a_n \cdot 1$$

Therefore

$$H_i(S^{2n-1}/(Z/a_n Z)) \otimes Z[1/a_n] = \begin{cases} 0 & \text{, for } i \neq o,2n-1 \\ Z[1/a_n] & \text{, for } i=0,2n-1 \end{cases}$$

We derive from inspecting the long homology sequence of the pair $(P(a),P(a\hat{}))$ and the induction hypothesis the following exact sequence

$$0 \longrightarrow H_i(P(a\hat{})) \longrightarrow H_i(P(a)) \longrightarrow H_{i-1}(S^{2n-1}/(Z/a_n Z)) \longrightarrow 0$$

for $0 < i < 2n$ and the isomorphism

$$H_{2n}(P(a)) = H_{2n-1}(S^{2n-1}/(Z/a_n Z))$$

This implies that $H_*(P(a))$ has no torsion if we can invert $a_n$.But this is true for all $a_i$'s.Since $\gcd(a_0,a_1,...,a_n)=1$ holds we have shown that $H_*(P(a))$ contains no torsion.Now from [3] we have

$$H_*(P(a),Q)=H_*(CP^n,Q)$$

thus the theorem follows.□

Fix now a prime p,and write $a_i=p^{r_i} c_i$ with $\gcd(c_i,p)=1$.Choose a permutati o n $\sigma$ of $\{0,1,2,...,n\}$ such that

$$r_{\sigma(1)} \geq r_{\sigma(2)} \geq ... \geq r_{\sigma(n)} \geq r_{\sigma(0)}$$

Define then

$$b_i(p)= \Pi_{0 \leq j \leq i} \ p^{r_{\sigma(j)}}$$

and

$$b_i = \Pi_{p \text{ prime}} \ b_i(p)$$

Set $g_i$ for the generator of $H^{2i}(P(a))$.We may state:

**THEOREM 1.2 :The cohomology Z-algebra of P(a) is determined by the following relations:**

$$g_i \cup g_j = (b_i b_j/b_{i+j}) g_{i+j} \text{ , for all i,j with } i+j \leq n$$

Proof:The projection $CP^n \longrightarrow P(a)$ induces the morphisms $p_i$ between their homology groups of dimension 2i.An alternative way to describe them is to consider the transfer maps

$$H_{2i}(CP^n) \cong H_{2i}(CP^n)^G \underset{r}{\overset{p}{\rightleftarrows}} H_{2i}(P(a))$$

where the isomorphism is a consequence of the fact that the G-action may be extended to a $T^n$-action ,$T^n$ being the torus of dimension n.Then we have :

$$r \circ p = |G| \cdot 1$$
$$p \circ r = |G| \cdot 1$$

Thus there is some positive integer $d_i$ for which $p_i$ sends the generator of $H_{2i}(CP^n)$ to $d_i$-times the generator of $H_{2i}(P(a))$.Notice that this would imply that $g_i \cup g_j = (d_i d_j/d_{i+j})g_{i+j}$.So it remains to check that $d_i$ equals $b_i$.We can assume without loss of generality that $gcd(a_0,a_1,...,a_n)=1$ hence $r_{\sigma(0)}=0$ holds.Now observe that $d_0=1$ and $d_n=a_0a_1...a_n$.In fact the projection $S^{2n-1} \longrightarrow S^{2n-1}/G$ is around $(1/n^{1/2},...,1/n^{1/2})$ a nonramified covering of degree $|G|$ since G acts orientation preserving on $S^{2n-1}$and the considered value is regular with distinct $|G|$ preimages.Now the whole projection map which is a ramified covering will have degree $d_n=|G|$ hence $d_n=b_n$.On the other hand the projection $CP^n \longrightarrow P(a)$ factorizes into the composition of projections $CP^n \longrightarrow P(p^{r_{\sigma(0)}},...,p^{r_{\sigma(n)}}) \longrightarrow P(a)$.Notice that the second projection is the quotient map of an action of a group whose order is prime to p.Hence from transfer arguments it induces an isomorphism in homology if we localize at p.Therefore it suffice to prove the theorem under the assumption that the $c_i$'s are all 1.We may also suppose that $a_i= p^{r_i}$ for i=0,1,...,n , $a_0=1$ and $r_i \geq r_{i+1}$ for i= 1,2,...,n-1 holds.We shall use induction over n.For n=0 the claim is trivial.Now since $a_n$ divide $a_i$ for all $i \geq 1$ and $a_0=1$ the operation of $Z/a_nZ$ on $S^{2n-1}$ is given by the following formula:

$$k\ (z) = (\zeta_{a_n}{}^k z_0, z_1,...,z_{n-1})$$

This can be extended to a $Z/a_nZ$-action on $C^{n*}$.The map $v:C^{n*} \longrightarrow C^{n*}$ given by $v(z)=(z_0{}^{a_n},z_1,...,z_n)$ induces an homeomorphism $C^{n*} \longrightarrow C^{n*}/(Z/a_nZ)$.The exact sequence used in the proof of theorem 1.1

give us the following isomorphisms induced by inclusion

$$H_{2i}(P(a^\wedge)) \xrightarrow{\sim} H_{2i}(P(a)) \qquad \text{for } i \leq 2n-2$$

The argument above given for $d_n$ and the induction hypothesis applied to $P(a^\wedge)$ establishes our claim.□

Consider now the following arrow associated to the $a_i$'s : in the first column we put all the primes p dividing some $a_i$ in increasing order; after that we put on the row begining to p the exponents which appears in the $a_i$'s in decreasing order i.e.the $r_i$'s.Denote this arrow by R(a).

**COROLLARY 1.3:If P(a) and P(c) have the same homotopy type then R(a)=R(c) .For n=2 the converse is also valid.**

Proof:The comparison of their Z-cohomology algebras and some arithmetical considerations would yield the first claim.In case when n=2 we may apply Whitehead's theorem about the classification of CW-complexes since $\pi_1(P(a)=0$.□

**REMARK 1.4:** Let us call a n-tuple (a) to be k-prime if every k elements of it have their greatest common divisor 1.A simple computation give us

$$b_j=\Pi_{0\leq i\leq j}\, a_i \quad \text{for } j\geq k-1$$

**REMARK 1.5:** Consider a linear $C^*$-action on $C^{n+1}$ such that $S^{2n+1}$ is $S^1$ invariant which diagonalizes hence takes the form

$$t \longrightarrow \mathrm{diag}(t^{a_0},\ldots,t^{a_n})$$

with integer $a_i$'s.With no essential changes in the above proofs it can be obtained the isomorphism

$$H^*(S^{2n+1}/S^1) = H^*(P(|a_0|,|a_1|,\ldots,|a_n|))$$

These spaces correspond to quotients (by the usual weighted G-action) of some homotopy projective spaces $CP^{n,k}$ which can be defined as the quotients $S^{2n+1}/S^1$ under the following $S^1$-action:

$$t \longrightarrow \mathrm{diag}(\underbrace{t^{-1},t^{-1},\ldots,t^{-1}}_{k},t,\ldots,t)$$

Then a spectral sequence argument shows that $CP^{n,k}$ has the same Z-cohomology algebra as $CP^n$,and because it is a 1-connected compact manifold Whitehead's theroem implies that it would have the same homotopy type.If $0<k<n+1$ then the above action may be extended to a stable $C^*$-action on $C^{n+1}$ and therefore $P(C^{n+1})/C^* = CP^{k-1}\times CP^{n-k}$.

Indeed it is easy to see that $C[z_0,\ldots,z_n]^{C^*}$ is the ring of $CP^{k-1}\times CP^{n-k}$ under the Segre embedding.Therefore $CP^{n,k}$ are not pairwise homeomorphic if 2k<n+1.On the other hand for n=2 all of them are diffeomorphic to

$CP^2$ as result from the theorem of Freedmann and the uniqueness of differentiable structures on $CP^2$.

## 2.Weighted intersections and singularities

Let now (V,0) be an isolated singularity of complete intersection in $C^{m+1}$ defined by the weighted homogeneous polynomials $f_i$ of degree $d_i$ with respect to the positive integer weights $wt(z_j)=a_j$ (j=0,m) for i=1,p.There are two spaces naturally associated to the singularity (V,0):the link $K=V\cap S^{2m+1}$ and the quasi-smooth weighted complete intersection $Y_\infty$ defined by the polynomials $f_i$ in P(a).Set n=m-p.Notice that K is a smooth compact oriented (2n+1)-dimensional manifold which is (n-1)-connected(see [ 5 ] ).The middle Betti numbers of K could be computed in terms of the $a_i$'s and $d_j$'s as Dimca shown ( [ 2 ] ).A more delicate question is the determination of the torsion subgroup of $H_n(K)$ (which can be identified with the torsion subgroup of coker$(1-m_*)$, $m_*$ being the monodromy in case when p equals 1).Our aim is to say something about the Z-cohomology algebra of $Y_\infty$.

Consider $F_i(z)=f_i(z_0^{a_0},\dots,z_m^{a_m})$, i=1,p and set $Z_\infty$ for the complete intersection defined by the polynomials $F_i$ in $CP^m$.Next observe that the G-action on $CP^m$ invaries $Z_\infty$ and we have $Z_\infty/G=Y_\infty$.

**PROPOSITION 2.1:Let G be a finite group ,A and B be G-spaces and $f:A\to B$ be a G-equivariant map.Suppose that for p prime and P a maximal p-group in G the induced map $f^*:H^*(B^P,Z/pZ)\longrightarrow H^*(A^P,Z/pZ)$ satifies the condition:(*) it is an isomorphism in rank less than q and a monomorphism in rank q .Then the map $f/G:H^*(B/G,Z/pZ)\longrightarrow H^*(A/G,Z/pZ)$ satisfies (*).Furthermore if this holds for any p prime and also the map $f:H^*(B)\longrightarrow H^*(A)$ satifies (*) then the map $f/G$: $H^*(B/G)\longrightarrow H^*(A/G)$ satifies (*) .**

The proof may be found in [1 ] .

**THEOREM 2.2:Suppose that (a) is k-prime.Then the following**

$$H^i(P(a),Y_\infty)=0$$

**holds for $i\leq n-k+1$.**

Proof: $P\subseteq G$ is a p-group hence $P=Z/p^{\alpha_0}Z\oplus\dots\oplus Z/p^{\alpha_m}Z$ where $p^{\alpha_i}$ divides $a_i$.Therefore

$$(CP^m)^P=\left\{\alpha_i z_i=0 \text{ for all } i\right\}$$

$$(Z_\infty)^P = Z_\infty \cap \{\alpha_i z_i=0 \text{ for all } i\}$$

Since $Z_\infty$ is a complete intersection $(Z_\infty)^P$ will be too.Also the number of non-zero $\alpha_i$'s cannot exceed (k-1) because (a) is k-prime.Then Lefschetz's theorem for complete intersections give us:

$$\pi_i((CP^m)^P,(Z_\infty)^P)=0 \text{ for } i\leq n-k+1$$

and all p-groups $P\subseteq G$ so proposition 2.1 applies.□

**COROLLARY 2.3: Suppose that (a) is k-prime .Then the set of integers**

$$R_{ij}=b_i b_j/b_{i+j} \quad \text{with} \quad 0\leq i,j\leq i+j\leq (n-k+1)/2$$

**is a topological invariant of the isolated singularity (V,0).**

Set now L (respectively $\mathcal{L},\mathcal{K}$) for the link of F (respectively for the links defined by the p-tuples of functions $\mathcal{F}_i= F_i-z_{m+1}^{d_i}$ i=1,p and $f_i=f_i-z_{m+1}^{d_i}$).Consider Z the fibre of F over 1 (the global Milnor fibre) and $\mathcal{Z}$ its projective closure.Observe that $\mathcal{Z}$ is in fact the quasi smooth weighted intersection associated to $\mathcal{F}$ and $Z_\infty$ may be identified with $\mathcal{Z}$-Z.Denote by $(a^*)$ the (m+1)-tuple $(a_0,a_1,\dots,a_m,1)$.The G-action on $CP^{m+1}$ which leads us to $P(a^*)$ is compatible with the usual weighted G-actions on $C^{m+1}$ and $CP^m$.On the other hand $C^{m+1}/G$ is biholomorphic to $C^{m+1}$ (see 6 ) such that $P(a^*)$ may be viewed as the compactification of $C^{m+1}$ whose locus at infinity is precisely P(a).Set $\mathcal{Y}$ for the quasi-smooth weighted intersection associated to $f$ .In the same vein the global Milnor fibre of f ,Y can be identified with $\mathcal{Y}$-$Y_\infty$ .We have in fact

$$(\mathcal{Z},Z,Z_\infty)/G = (\mathcal{Y},Y,Y_\infty)$$

But the projective objects may be obtained also from the links as follows : consider the $S^1$-action on $(S^{2m+3},S^{2m+1})$ given by

$$\rho\,(z) = (\rho^{a_0} z_0,\dots,\rho^{a_m} z_m,z_{m+1})$$

Therefore $(\mathcal{K},K)$ is $S^1$-invariant and it can be checked that $(\mathcal{K},K)/S^1=(\mathcal{Y},Y_\infty)$.$Y_\infty$ is called strongly smooth (see [2]) if the $S^1$-action on K is semi-free.

**THEOREM 2.4: Suppose that $Y_\infty$ is strongly smooth.Then $H_*(K)$ is torsion free and the Milnor lattice of f is equivalent to the cup product pairing $H^{n+1}(\mathcal{K},K)\otimes H^{n+1}(\mathcal{K},K) \longrightarrow H^{2n+2}(\mathcal{K},K) = Z$.Moreover if p equals 1 then this may be expressed also as the cup product pairing $H^m(S^{2m+1},K)\otimes H^m(S^{2m+1},K) \longrightarrow H^{2m}(S^{2m+1},K) = Z$ ;also the integral monodromy operator satisfies $m_*^{\ d} = 1$.**

Proof:The Smith-Gysin sequence associated to the $S^1$- action on K give us

$$H_n(Y_\infty)=H_n(K)\oplus H_n(CP^n)$$
$$H_j(Y_\infty)=H_j(CP^n) \quad \text{for } j\neq n$$
$$H_n(K) \text{ is torsion free}$$

Observe that $Y_\infty$ is strongly smooth if and only if $\mathcal{Y}$ is stronly smooth and therefore the $S^1$-action on $(\mathcal{K},K)$ will be semi-free.From the long sequence associated to the pair $(\mathcal{K},K)$ it can be derived

$$H^j(\mathcal{K},K)=0 \quad \text{for } j\neq n+1,n+2,2n+2,2n+3$$
$$H^{2n+2}(\mathcal{K},K)=Z$$
$$H^{2n+3}(\mathcal{K},K)=Z$$

Since Y has the homotopy type of a bouquet of (n+1)-spheres (see[6] ) we have

$$H^j(\mathcal{Y},Y_\infty)=0 \quad \text{for } j\neq n+1$$

from Lefschetz duality.Then the Smith-Gysin sequence associated to the $S^1$-action on $(\mathcal{K},K)$ yields:

$$0=H^{2n}(\mathcal{Y},Y_\infty)=\ker p^*:H^{2n+2}(\mathcal{Y},Y_\infty) \qquad H^{2n+2}(\mathcal{K},K)$$
$$0=H^{2n+1}(\mathcal{Y},Y_\infty)=\operatorname{coker} p^*$$
$$H^{n+2}(\mathcal{K},K)=H^{n+1}(\mathcal{Y},Y_\infty)$$
$$p^*:H^{n+1}(\mathcal{Y},Y)\longrightarrow H^{n+1}(\mathcal{K},K) \quad \text{is an isomorphism}$$

Using the functoriality of Lefschetz duality the first part of the theorem follows.If p equals 1 then $\mathcal{K}$-K is a non-ramified Z/dZ-covering of $S^{2m+1}$-K and the Alexander duality gives the second claim.Otherwise $\mathcal{K}$-K is the total space of a fibre bundle over $S^1$ with fibre Y and whose characteristic map is $m^d$ (where m is the geometric monodromy [7] ).The Wang sequence of this fibration leads us to

$$\ker(1-m_*^{\,d})=H^m(Y)$$
$$\operatorname{coker}(1-m_*^{\,d})=H^m(Y)$$

hence $m_*^{\,d}=1$.□

**REMARK 2.5:** If we consider the Wang sequence with Q-coefficients we obtain that $m_{*Q}^{\,d}=1$ for arbitrary quasi-homogeneous f with isolated singularity i.e.the result of Dimca [2].

REFERENCES


1.BREDON,G.:Introduction to compact transformation groups,Academic Press,New-York-London 1972

2.DIMCA,A.:Monodromy and Betti numbers of weighted complete intersections,Topology 24(1985), 369-374

3.DOLGACHEV,I.:Weighted projective varieties,in "Group actions and vector fields " Lecture Notes in Math.956, Springer Verlag 1982

4.FUNAR,L.:Homology of $P(w_0,w_1,w_2)$, An.Univ.A.I.Cuza,Jassy,1990(in print)

5.HAMM,H.:Locale topologische Eigenschaften Komplexer Raume, Math.Ann. 191(1971),235-252

6.LOOIJENGA,E.:Singular points of complete intersections,London Math.Soc.,London-New York 77 (1984)

7.NEMETHI,A.:The link of the sum of two functions,Preprint INCREST 40(1986)