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A GENERAL RESULT ON ABSTRACT FLOWCHART SCHEMES WITH APPLICATIONS TO THE STUDY OF ACCESSIBILITY, REDUCTION AND MINIMIZATION

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Abstract. An abstract flowchart scheme [CS87b] differs from a usual flowchart scheme by the fact that the set of arrows which connect the atomic elements is replaced with an element from an adequate abstract structure (called support theory). Deterministic schemes, nondeterministic schemes or other kind of digraph-like models are instances of abstract schemes obtained by using particular support theories.

Such an abstract scheme is obtained from atomic schemes (variables) and trivial schemes (elements of the underlying support theory) by using three operations: sum, composition and feedback.

The aim of this paper is to present a general result on abstract flowchart schemes and to apply it to the study of accessibility, reduction and minimization.

Contents

0. Introduction

There is an increasing need to find some basic algebraic structures for theoretical computer science. The present paper deals with the algebraization of the theory of the flowchart schemes, as was initiated by Elgot in 1970.

Elgot was interested in getting an axiomatization for the input (step-by-step) behaviour of the deterministic flowchart schemes [El75, El76a]. Roughly speaking, two deterministic flowchart schemes have the same input behaviour if and only if they unfold into the same (regular) tree [EBT78]. An equivalent characterization is the property that by deletion of the inaccessible vertices and by identification of the vertices with the same behaviour both schemes reduce to the same minimal one [El77, St87a].

In this setting two algebraic structures have been proposed, namely <u>iteration theories</u>, defined in [BEW80] and axiomatized in [Es80], and <u>strong iteration theories</u> [St87a]. Iteration theories are weaker than strong iteration theories in the sense that the implication scheme used in strong iteration theories (i.e. functorial implication for functions) is replaced by an equation scheme. So iteration theories are defined using only equations. Iteration theories have been obtained from the analysis of regular trees (the first charaterization of the input behaviour given above), while strong iteration theories have appeared from the analysis of minimization (the second characterization of the input behaviour).

In the nondeterministic case the problem of axiomatizing nondeterministic flowchart schemes is closely related to the old problem of axiomatizing the automata behaviour (i.e. the algebra of regular events). Elgot was aware about this. Indeed, the first algebra for flowchart schemes proposed by Elgot in [El75], i.e. iterative theory, uses an implication scheme (unique solution of the equation $x = f\langle x, I_p\rangle$ for each ideal morphism f) which may be viewed as a variant of the implication scheme used in Salomaa's axiomatization (unique solution of the equation $X = AX + B$ for A satisfying the empty word property). So it is natural to develop an algebra for nondeterministic flowchart schemes using Kleene's operations: union, composition and repetition. Such an algebra is proposed in [St87b].

In 1986 one of us introduced a new looping operation called feedback [St86a, St86b] and in a series of papers [CS87b, CS89a, CS87a, CS88b, CS88a, CS89b] we have tryed to develop the theory of flowchart schmes and their behaviours in the new setting: sum-composition-feedback. The framework of this theory is presented in [CS87b].

The basic algebra, called biflow, gives a complete axiomatization of flowchart schemes [St86b, Ba87, CS87a, CS88b]. A biflow is a symmetric strict monoidal category [ML71] endowed with a feedback operation fulfilling some natural axioms.

The aim of the present paper is to extend the above result in order to obtain a complete axiomatization for accessible, reduced and minimal schemes (with respect to the input-behaviour), using the general result on abstract flowchart schemes obtained in the first part of the paper.

The notion of abstract flowchart scheme we use was introduced by Căzănescu and Ungureanu in [CU82] (and developed in [CG84, St87a, St87b, CS87b]), where we replaced the set of arrows which connect the vertices by an element from an adequate algebraic structure.

Beside the simplicity of the biflow structure one further benefit of the sum-composition-feedback setting is the simplification of the study of minimization with respect to the input behaviour. More precisely, this setting allows to separate the study of accessibility from the study of reduction (i.e. identification of the vertices with the same behaviour). Moreover, it turns out that accessibility and reduction are dual phenomena and both follow from a common general study presented here. This duality is analogous to the well-known duality between "reachability" and "observability" which has been noticed in system theory (see, for example [AM75]).

Finally we mention that the general study reported here may be applied (with slight variations) to other classes of flowchart schemes, e.g. to input-output minimal schemes.

1. Preiminaries

To make the reading easier we recall some things from our previous papers.

The objects of the categories we use form a monoid that we denote by (Ob(B), + ,e) for each category B. If $a \in Ob(B)$ then I_a denote the identity morphism of a. The composite of $f \in B(a,b)$ and $g \in B(b,c)$ is denoted by $f \cdot g$ or by fg.

The additional operations we use are:

a) sum $\_+\_ : B(a,b) \times B(c,d) \to B(a + c, b + d)$,

b) block traspositions ${}^aX^b \in B(a + b, b + a)$,

c) right feedback $\_\uparrow^a : B(b + a, c + a) \to B(b,c)$.

The axioms for this operation are given in Table 1. B1-2 are the usual axioms for the categories.

When only the sum is used as an additional operation and the axioms B1-6 hold then the algebraic structure is called <u>strict monoidal category</u> (smc, for short) [ML71, Ma76]. The <u>nonpermutable smc</u> (nsmc, for short) [CS89a] is a weaker concept as the axiom B6 is required to hold only if g <u>or</u> u is an identity. The magmoïds [AD78] are smc having the additive monoid of nonnegative integer as monoid of objects.

(B1)	$(fg)h = f(gh)$	(B2)	$I_a f = f = f I_b$
(B3)	$(f + g) + h = f + (g + h)$	(B4)	$I_e + f = f = f + I_e$
(B5)	$I_a + I_b = I_{a+b}$	(B6)	$(f + g)(u + v) = fu + gv$
(B7)	${}^aX^{b+c} = ({}^aX^b + I_c)(I_b + {}^aX^c)$	(B8)	${}^aX^e = I_a$
(B9)	${}^cX^a(u + g){}^bX^d = g + u$ for $u : a \to b$ and $g : c \to d$		
(B10)	$f(g\uparrow^a)h = ((f + I_a)g(h + I_a))\uparrow^a$	(B11)	$f + g\uparrow^a = (f + g)\uparrow^a$
(B12)	$(f(I_d + g))\uparrow^a = ((I_c + g)f)\uparrow^b$ for $f : c + a \to d + b$ and $g : b \to a$		
(B13)	$f\uparrow^{a+b} = f\uparrow^b\uparrow^a$ (B14) $I_a\uparrow^a = I_e$	(B15)	${}^aX^a\uparrow^a = I_a$

Table 1. These axioms define a biflow.

When the sum and the block transpositions are used as additional operations and axioms B1-9 hold then the algebraic structure is called symmetric smc (ssmc, for short) [ML71, Ma76]. The symmetric nsmc (snsmc, for short) [CS89a] is a weaker concept as axioms B6 and B9 are required to hold only if g or u is a block transposition (by B8 the identities are block transpositions). In an snsmc an aα-morphisms is a composite of morphisms of the type $I_a + {}^bX^c + I_d$. In an snsmc if u or g are aα-morphisms then B6 and B9 hold.

The ssmc concept is the basic algebraic structure to study acyclic flowchart schemes. To study flowchart schemes we use feedback to model loops.

A flow [CS87a] is an snsmc having a feedback satisfying axioms B10-11, B13-15 and axiom B12 whenever g is a block transposition. A biflow [CS88b] is a flow over an ssmc. In a biflow B12 holds. The biflow concept is our basic algebraic structure to study flowchart schemes.

As sometimes we prefer to use in a flow the left feedback $\uparrow^a\_ : B(a + b, a + c) \longrightarrow B(b,c)$ instead of the right feedback, we recall that

$$\uparrow^a f = ({}^bX^a f\, {}^aX^c)\uparrow^a \quad \text{for} \quad f : a + b \longrightarrow a + c.$$

The above algebraic structures form categories where the morphisms are functors that are monoid morphisms on objects and that preserve the additional algebraic structure. Sometimes we are interested in certain subcategories, namely where the monoid M of objects is kept fixed in the above algebraic structures (call them : M-smc, M-nsmc, M-ssmc, M-snsmc, M-flow, M-biflow) and where the morphisms are object preserving functors (call them : M-smc morphism,... ...,M-biflow morphism). These subcategories are varieties in the sense of the many-sorted universal algebra.

For a nonnegative integer n we use the notation $[n] = \{1, 2, \ldots, n\}$.

The biflow $\mathbf{Rel}_S$ of the finite S-sorted relations is used to build nondeterministic flowchart schemes. A word $a \in S^*$ is written as $a = a_1 + a_2 + \ldots + a_{|a|}$ where $|a|$ is its length and a_i are its letters. For $a,b \in S^*$ by definition

$$\mathbf{Rel}_S(a,b) = \{ f \subset [|a|] \times [|b|] \mid (i,j) \in f \text{ implies } a_i = b_j \}.$$

The operations in $\mathbf{Rel}_S$ are:

$$fg = \{(i,k) \mid (\exists j)[(i,j) \in f \text{ and } (j,k) \in g]\},$$

$$I_a = \{(i,i) \mid i \in [|a|]\},$$

$$f + g = f \cup \{(|a| + i, |b| + j) \mid (i,j) \in g\} \text{ where } f : a \to b,$$

$${}^a X^b = \{(i, |b| + i) \mid i \in [|a|]\} \cup \{(|a| + i, i) \mid i \in [|b|]\},$$

for $s \in S$ and $f \in \mathbf{Rel}_S(a + s, b + s)$

$$f\uparrow^s = \{ (i,j) \mid (i,j) \in f \text{ or } [(i,|b| + 1) \in f \text{ and } (|a| + 1,j) \in f]\}.$$

In this case $\uparrow^a$ is defined by induction using $f\uparrow^\lambda = f$ (where λ is the empty word) and B13 in Table 1. Other notation is

$$T_a = \emptyset \in \mathbf{Rel}_S(\lambda,a), \quad V_a = I_a \cup \{(|a| + i,i) \mid i \in [|a|]\} \in \mathbf{Rel}_S(a + a,a),$$

$$\perp^a = \emptyset \in \mathbf{Rel}_S(a,\lambda), \quad \wedge^a = I_a \cup \{(i,|a| + i) \mid i \in [|a|]\} \in \mathbf{Rel}_S(a,a + a).$$

There are some interesting subbiflows of $\mathbf{Rel}_S$. The biflow $\mathbf{Pfn}_S$ of the finite S-sorted partial functions is used to build (deterministic) flowchart schemes. $\mathbf{Bi}_S$ is the biflow of the finite S-sorted bijections and $\mathbf{In}_S$ is the biflow of the finite S-sorted injections. The ssmc $\mathbf{Sur}_S$ of the finite S-sorted surjections and the ssmc $\mathbf{Fn}_S$ of the finite S-sorted functions are not subbiflows as they are not closed under feedback. When S is a singleton, we drop the subscript S and we identify S^* to the additive monoid of the nonnegative integers.

Passing to flowchart schemes, we explain once again our viewpoint. As atomic flowchart schemes we use a set Σ of statements. Two functions $i : \Sigma \to N$ and $o : \Sigma \to N$ show for each statement x, the number $i(x)$ of its inputs and the number $o(x)$ of its output. A (partial) finite function $f : [n] \to [m]$ is thought as a very simple flowchart scheme having n inputs and m outputs, without statements and such that the flow control go from the input j to the output k if and only if $(j,k) \in f$.

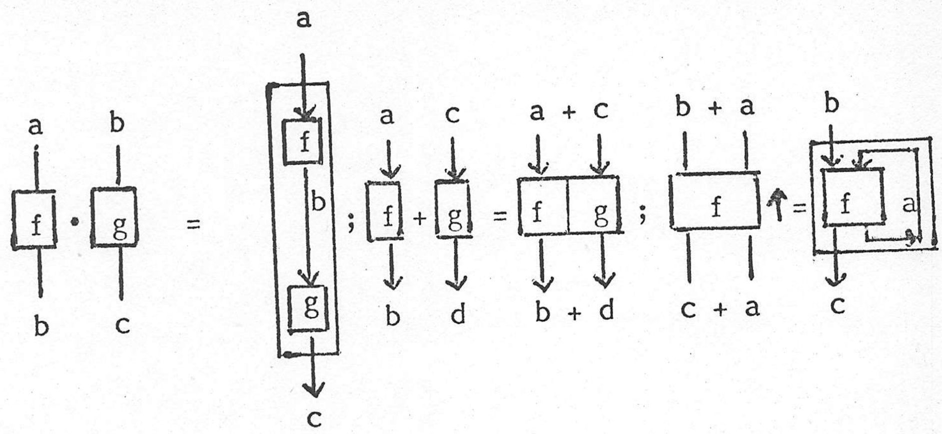

Figure 1.

The operation used to build flowchart schemes are composition, sum and feedback (see Figure 1). Every flowchart scheme is isomorphic to a scheme in a normal form (see Figure 2) where $x \in \Sigma^*$ is thought as the sum of its letters and f is a (partial) function. Therefore a scheme with n inputs and m outputs may be represented as a pair (x,f), where $x \in \Sigma^*$ and $f : [n + o(x)] \longrightarrow [m + i(x)]$. Here i and o are the unique monoid morphisms $i,o : \Sigma^* \longrightarrow (N,+,0)$ which extend the given functions i and o. The normal form of a flowchart scheme is not unique as the letters of x may be permuted.

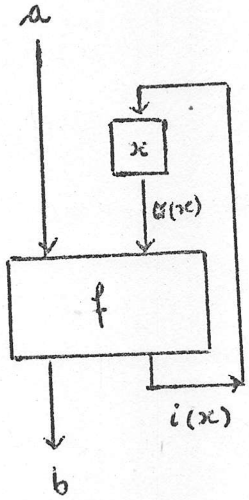

Figure 2.

To define the operations for scheme representations we use the formulas (1.1), (1.2) and (1.3) below. To obtain the right hand sides we put in a normal form the result of the operations from Figure 1 made using the schemes representated by the pairs in the left hand sides. Look more careful to (1.3). The right hand side may have no sense if f is a function. This formula has sense if f is a partial function. Working with partial functions instead of functions we pass from schemes to partial schemes. We think it is better to work with partial schemes instead of schemes having a loop vertex $\perp$ as for example in [BE85, Ba87]. The same idea was used to replace the total trees using a distinguished nullary operation by the partial trees. Remark that the partial functions as well as the partial flowchart schemes form a biflow [CS88b].

Passing to nondeterministic flowchart schemes, remark that they may be represented by pairs, too: the pair (x,f) represents a nondeterministic flowchart scheme with n inputs and m outputs if and only if $x \in \Sigma^*$ and $f \in \mathbf{Rel}(n + o(x), m + i(x))$. The formulas used to define operations (1.1), (1.2) and (1.3) are the same as in the deterministic case.

Remark that in the definition of the operations for scheme representations we use only biflow operations. All these remarks lead to a natural idea: replace f in a scheme representation (x,f) by a morphism from an N-biflow. Using this natural idea we unify the study of the deterministic flowchart schemes and the study of the nondeterministic flowchart schemes.

We prefer to work more abstract as you may see in the following definitions.

Assume B is an ssmc and $(X, +, \varepsilon)$ is a monoid. Let $i : X \to Ob(B)$ and $o : X \to Ob(B)$ be two monoid morphisms.

For $a,b \in Ob(B)$ we say the pair (x,f) represents a flowchart scheme (see Figure 2) with input a and output b if $x \in X$ and $f \in B(a + o(x), b + i(x))$. The morphism f which in the usual cases gives all the arrows of the scheme will be called connection. Let $\mathbf{Fl}_{X,B}(a,b)$ be the set of all flowchart scheme representations with input a and output b. If there is no danger of confusion we omit the subscripts X and B in $\mathbf{Fl}_{X,B}$. The operations in $\mathbf{Fl}$ are defined as follows.

If $(x,f) \in \mathbf{Fl}(a,b)$ and $(y,g) \in \mathbf{Fl}(b,c)$ then

(1.1) $$(x,f)(y,g) = (x + y, (f + I_{o(y)})(I_b + {}^{i(x)}X^{o(y)})(g + I_{i(x)})(I_c + {}^{i(y)}X^{i(x)})).$$

$I_a = (\varepsilon, I_a)$ for every object a of B.

If $(x,f) \in \mathbf{Fl}(a,b)$ and $(y,g) \in \mathbf{Fl}(c,d)$ then

(1.2) $$(x,f) + (y,g) = (x + y, (I_a + {}^{c}X^{o(x)} + I_{o(y)})(f + g)(I_b + {}^{i(x)}X^{d} + I_{i(y)})).$$

$^{a}X^{b} = (\varepsilon, {}^{a}X^{b})$ for every a and b objects of B.

Endowed with the above operations **Fl** becomes an snsmc.

In an nsmc C, the set of its morphisms Mor(C) is a monoid having the sum as operation. To embed X in **Fl** we define the monoid morphism $E_X : X \longrightarrow \mathrm{Mor}(\mathbf{Fl})$ by $E_X(x) = (x, {}^{i(x)}X^{o(x)})$. To embed B in **Fl** we define the Ob(B)-snsmc morphism $E_B : B \longrightarrow \mathbf{Fl}$ by $E_B(f) = (\varepsilon, f)$ for every morphism f of B.

Using these embedings we may identify X and B with subsets of **Fl**.

If B is a biflow we define the feedback for $(x,f) \in \mathbf{Fl}(b + a, c + a)$ by

$$(1.3) \qquad (x,f)\uparrow^{a} = (x,[(I_b + {}^{o(x)}X^{a})\, f\, (I_c + {}^{a}X^{i(x)})]\uparrow^{a}).$$

Therefore **Fl** becomes a flow and E_B an Ob(B)-flow morphism.

In [CS88b] we have shown that the biflow of the flowchart schemes with statements from the monoid X and connections from the biflow B denoted by $FS_{X,B}$ may be obtained by the factorization of **Fl** to the least flow congruence relation containing all the pairs

$$(X\,X) \quad ((x + y)^{o(x)}X^{o(y)},\ {}^{i(x)}X^{i(y)}(y + x)) \text{ where } x,y \in X.$$

From the computer science viewpoint our generalization (connections from an arbitrary biflow instead of **Pfn** or **Rel**) has another signifiquence beside the unification of the determinism and of the nondeterminism in the study of flowcharts, namely <u>the unification of the syntax and of the semantics.</u> This affirmation is motivated as follows: in [CS87a, CS88b] we have shown that the basic semantic model in the deterministic case and the basic semantic model in the nondeterministic case [CS87b, section 2] are biflows.

From an algebraic viewpoint our generalization has another signifiquence : the algebra of the flowchart schemes may be developed in the same way as the algebra of the polynomials. The theorems in [CS88b] have been made having in mind this idea.

2. The role of functoriality rule in flowchart scheme theories

Let us consider $AFS_{\Sigma,Pfn}$ (resp. $FS_{\Sigma,Pfn}$), the theory of deterministic acyclic (resp. cyclic) Σ-schemes over **Pfn**; these schemes are precisely those built up from atomic schemes in the double ranked set Σ and trivial schemes in **Pfn** using the operations of separated sum "+" and composition "$\cdot$" (resp. separated sum, composition and feedback "$\uparrow$"). Let us consider the following rules of identification:

$$(TX)\quad T_m \cdot x = T_n, \quad \text{for } x \in \Sigma(m,n);$$

$$(VX)\quad V_m \cdot x = (x+x)\cdot V_n, \quad \text{for } x \in \Sigma(m,n);$$

$$(\ X)\quad x \cdot \perp^n = \perp^m, \quad \text{for } x \in \Sigma(m,n).$$

In $AFS_{\Sigma,Pfn}$ some natural equivelence relations are precisely captured by the least congruence relations generated by subsets of rules in $\{TX, VX, \perp X\}$. For example, the least congruence relation $\equiv_{a\beta}$ generated by the identifications (TX) precisely capture accessibility, i.e. two acyclic schemes F and F' are $\equiv_{a\beta}$-equivalent iff F and F' have the same accessible part. Do analogous results works for cyclic schemes? The answer is "not". The following example may help the reader to understand why.

Example 2.1 (schemes over **In** and one biscalar variable). Suppose $\Theta(1,1) = \{x\}$ and $\Theta(m,n) = \emptyset$ othervise.

Every scheme in $AFS_{\Theta,In}(m,n)$ may be represented as

$$\Big(\sum_{i=1}^{m} x^{k_i} + T_p \cdot \sum_{i=1}^{p} x^{r_i}\Big)\cdot c, \quad \text{where } c \in \mathbf{In}(m+p,n),\ k_i \geq 0 \text{ and } r_i \geq 1$$

(by convention $x^0 = I_1$), hence it is uniquely determinated by the injection c and the pair $(k_1,\ldots,k_m; r_i,\ldots,r_p)$ of sequences of natural numbers. If $\equiv_{a\beta}$ denotes the least congruence relation (with respect to sum and composition) generated by the identifications (TX), then two schemes F and F' represented by c and $(k_1,\dots,k_m; r_1,\dots,r_p)$, and by c' and $(k'_1,\dots,k'_m; r'_1,\dots,r'_{p'})$, respectively are $\equiv_{a\beta}$-equivalent iff $k_i = k'_i$, $\forall i \in [m]$ and $(I_m+T_p)\cdot c = (I_m+T_{p'})\cdot c'$.

That is F and F' are $\equiv_{a\beta}$-equivalent iff F and F' have the same accessible part.

Every scheme in $\mathbf{FS}_{\theta,\mathbf{In}}(m,n)$ may be represented as

$$(\sum_{i=1}^{m} x^{k_i} + T_p \cdot \sum_{i=1}^{p} x^{r_i})\cdot c + \sum_{i=1}^{q} (x^{s_i}) \quad , \text{ where } c \in \mathbf{In}(m+p,n),\ k_i \geq 0,\ r_i \geq 1,\ s_i \geq 1,$$

hence it is uniquely determinated by the injection c and the triple $(k_1,\dots,k_m; r_1,\dots,r_p; s_1,\dots,s_q)$ of sequences of natural numbers. If $\approx_{a\beta}$ denoted the least congruence relation (with respect to sum, composition and feedback) generated by the identifications (TX), then

two schemes F and F' represented by c and $(k_1,\dots,k_m; r_1,\dots,r_p; s_1,\dots,s_q)$, and by c' and $(k'_1,\dots,k'_m; r'_1,\dots,r'_{p'}; s'_1,\dots,s'_{q'})$, respectively are $\approx_{a\beta}$-equivalent iff $k_i = k'_i$, $\forall i \in [m]$, $(I_m+T_p)\cdot c = (I_m+T_{p'})\cdot c'$, $q = q'$ and there exists a bijection $b \in \mathbf{Bi}(q,q')$ such that $s_i = s'_{b(i)}$, $\forall i \in [q]$.

That is F and F' are $\approx_{a\beta}$-equivalent iff F and F' have the same accessible part <u>and</u> the same (inaccessible) cycles. Hence $\approx_{a\beta}$ does not capture accessibility. □

The reason for the answer "not" above is the imposibility of using the identifications given by (TX), (VX), and ($\perp$X) in cycles. Consequently the least congruence relation generated by certain such identifications is too strong, i.e. it identifies too few schemes.

To overcome this difficulty we combine the identifications (TX), (VX), and ($\perp$X) with an additional identification rule, called functoriality rule, which allows us to use these identifications in cycles. The rule is defined as follows. We say a relation $\equiv$ on a biflow B fulfills (func : y) for a $y \in B(p,q)$ or $y \in B(p,q)$ is $\equiv$-functorial if

(func : y) $\qquad f\cdot(I_n+y) \equiv (I_m+y)\cdot g \implies f\uparrow^p = g\uparrow^q$ for all $f{:}m+p \to n+p$ and $g{:}m+q \to n+q$

holds. If E is a subset of morphisms of a biflow B, we say $\equiv$ fulfills (func : E) if $\equiv$

fulfills (func : y) for all y in E. Finally, we say a biflow B satisfies the functoriality axiom (func : E) if the equality relation on B fulfills (func : E).

In $\mathbf{FS}_{\Sigma,\mathbf{Pfn}}$ some natural equivalence relations, corresponding to those for acyclic schemes, are precisely captured by the least congruence relations genearted by subsets of $\{TX, VX, \perp X\}$ in the class of congruence relations satisfying (func : E) for an adequate E included in **Pfn.** (Since the class of congruence relations satisfying (func : E) is nonempty and closed with respect to intersection such a congruence relation does exists, namely it is the intersection of all relations in this class.) For example, the least congruence relation $\sim_{a\beta}$ generated by the identifications (TX) in the class of congruence relations satisfying (func : **In**) precisely captures accessibility, i.e. two cyclic schemes F and F' are $\sim_{a\beta}$-equivalent iff F and F' have the same accessible part.

In conclusion using the functoriality rule to restrict the class of congruence relations used for generating we get weaker congruence relations (i.e. they identify more schemes) which correspond to some naturally introducing ones.

Example 2.1 (continued). Consider three schemes F, F' and F" in $\mathbf{FS}_{\mathcal{C},\mathbf{In}}(m,n)$ represented by c and $(k_1,\ldots,k_m;\ r_1,\ldots,r_p;\ s_1,\ldots,s_q)$, by $(I_m+T_p)\cdot c$ and $(k_1,\ldots,k_m;r;\ s_1,\ldots,s_q)$, and by $(I_m+T_p)\cdot c$ and $(k_1,\ldots,k_m;r;r)$, respectively, where r is the empty sequence. Clearly $F \approx_{a\beta} F'$. Since $F''(I_n+T_q) \approx_{a\beta} (I_m+T_q)G$ where $G \in \mathbf{FS}_{\mathcal{C},\mathbf{In}}(m+q,n+q)$ is the scheme represented by $(I_m+T_p)c + I_q$ and $(k_1,\ldots,k_m,s_1,\ldots,s_q;r;r)$, by the functoriality rule (func : **In**) we get $F''\uparrow^0 \sim_{a\beta} G\uparrow^q = F'$, hence $F \sim_{a\beta} F''$. Consequently the difficulty is overcame: a scheme is $\sim_{a\beta}$-equivalent to its accessible part. □

3. Enriched symmetric strict monoidal categories

In a previous paper [CS89a] we have given characterizations for certain classes of finite relations as initial abstract data types. These classes, denoted xy-**Rel** for

$x \in \{a,b,c,d\}$ and $y \in \{\alpha,\beta,\gamma,\delta\}$ correspond to some natural classes of relations, e.g. $a\alpha$-**Rel** = bijections, $a\beta$-**Rel** = injections, $a\gamma$-**Rel** = surjections, $a\delta$-**Rel** = functions, $b\beta$-**Rel** = partially defined injections, $b\delta$-**Rel** = partially defined functions, etc. (see Table 6 in Section 11). The characterization involves the concept of an xy-ssmc, defined below.

Suppose we are given an ssmc $(B,\cdot,I,+,X)$, where the monoid of the objects of the underlying category is $(Ob(B),+,e)$. We enrich the ssmc-structure with some constants (zero-ary operations)

$$T_a \in B(e,a) \qquad\qquad \perp^a \in B(a,e)$$

$$V_a \in B(a+a,a) \qquad\qquad \wedge^a \in B(a,a+a)$$

for $a \in Ob(B)$. Now we define an xy-ssmc, for $x \in \{a,b,c,d\}$ and $y \in \{\alpha,\beta,\gamma,\delta\}$ as an ssmc enriched with the constants corresponding to xy specified in Table 2 and fulfilling all the axioms in Table 3 in which these and only these constants appear. For example, a $c\delta$-ssmc is defined as an ssmc enriched with the constants $\wedge^a$, T_a, and V_a and fulfilling the axioms A, A°, B, B°, C, D°, F, G, SV1-4, and SV3°-4°.

The acyclic algebra SC_0 of Bloom and Esik in [BE85] (completed with the axiomatization [CS89a] of finite, partially defined functions) is equivalent with a $b\delta$-ssmc.

A morphism of an xy-ssmc B is called an <u>xy-base morphism</u> (or shortly, <u>xy-morphism</u>) if it is the evaluation in B of a term written with "+", "·", I, X, and the constants in $T,V,\perp,\wedge$ corresponding to xy.

The xy-base morphisms of an xy-ssmc B form the least sub-xy-ssmc of B which we denote in the sequel by B_{xy}. Due to the axioms that define ssmc-ies we get the following equivalent characterization.

Observation 3.1. A morphism is xy-base if and only if it is a composite of morphisms of type I_a+g+I_b, where g is ${}^cX^d$ or a constant in $\{T_c,V_c,\perp^c,\wedge^c\}$ corresponding to xy (acorrding to Table 2). □

The motivation we have given in Section 2 shows that we have to consider the

x	operations	y	operations
a	nothing	α	nothing
b	$\perp^a$	β	T_a
c	Λ^a	γ	V_a
d	$\perp^a$ and Λ^a	δ	T_a and V_a

Table 2. Operations for xy-ssmc

A) $(V_a + I_a)V_a = (I_a + V_a)V_a$

A°) $\Lambda^a(\Lambda^a + I_a) = \Lambda^a(I_a + \Lambda^a)$

B) ${}^aX^a \cdot V_a = V_a$

B°) $\Lambda^a \cdot {}^aX^a = \Lambda^a$

C) $(T_a + I_a)V_a = I_a$

C°) $\Lambda^a(\perp^a + I_a) = I_a$

D) $V_a \cdot \perp^a = \perp^a + \perp^a$

D°) $T_a \cdot \Lambda^a = T_a + T_a$

E) $T_a \cdot \perp^a = I_e$

F) $V_a \cdot \Lambda^a = (\Lambda^a + \Lambda^a)(I_a + {}^aX^a + I_a)(V_a + V_a)$

G) $\Lambda^a \cdot V_a = I_a$

- -

SV1) $T_e = I_e$

SV1°) $\perp^e = I_e$

SV2) $T_{a+b} = T_a + T_b$

SV2°) $\perp^{a+b} = \perp^a + \perp^b$

SV3) $V_e = I_e$

SV3°) $\Lambda^e = I_e$

SV4) $V_{a+b} = (I_a + {}^bX^a + I_b)(V_a + V_b)$

SV4°) $\Lambda^{a+b} = (\Lambda^a + \Lambda^b)(I_a + {}^aX^b + I_b)$

Table 3. Axioms for xy-ssmc

ST) $T_a f = T_b$

S$\perp$) $f \perp^b = \perp^a$

SV) $(f + f)V_b = V_a f$

SΛ) $\Lambda^a(f + f) = f\Lambda^b$

Table 4. Axioms for strong xy-ssmc ($f : a \to b$)

stronger axioms in Table 4. They are stronger in the following sense: in an arbitrary xy-ssmc only their restrictions to the case when f is an xy-morphism hold.

Let us consider the order $<_L$ on $\{a,b,c,d\}$ given by $a < b < d$, $a < c < d$, $\neg(b < c)$ and $\neg(c < b)$, and similarly $<_G$ for Greek letters in $\{\alpha,\beta,\gamma,\delta\}$. We define an x'y'-strong xy-ssmc, for $x' \leq_L x$ and $y' \leq_G y$, as an xy-ssmc in which all the axioms in Table 4 corresponding to x'y' hold. A strong xy-ssmc is by definiton an xy-strong xy-ssmc. For example, in order to define a cγ-strong cδ-ssmc one have to add the axioms (SV) and (S$\wedge$) to the axioms that define a cδ-ssmc.

There are very important instances of strong xy-ssmc-ies. The concept of a strong aδ-ssmc coincides with the concept of an algebraic theory -- in the sense of Lawvere -- used by Elgot, ADJ-group, etc (see [BTW85], for example). The concept of a strong dδ-ssmc coincides with the concept of an idempotent matrix theory introduced by Elgot [El76b]; if the monoid of the objects is equal to the additive monoid of the nonnegative integers this concept is equivalent with the concept of a theory of matrices over an idempotent semiring.

To be more exact, the above coincidences is with the extensions of the usual concept of algebraic theory and (idempotent) matrix theory, respectively to the case when the objects of the underlying category form an arbitrary monoid. This extension is defined in [CS89a]. Note that the concept of a matrix theory is equivalent to an ssmc which is a strong aδ-ssmc and a strong dα-ssmc, too. Since D, D°, E, and F follow from ST, SV, S$\perp$, and S$\wedge$ it follows that in a matrix theory all the axioms in Tables 3 and 4 hold, with only one possible exception: the axiom G. The axiom G holds iff the matrix theory is idempotent.

$H(T_a) = T_{H(a)}$	$H(\perp^a) = \perp^{H(a)}$
$H(V_a) = V_{H(a)}$	$H(\wedge^a) = \wedge^{H(a)}$

Table 5. Conditions for xy-ssmc morphisms.

An xy-ssmc morphism is an ssmc-morphism fulfilling all the conditions in Table

5 corresponding to the restriction xy. Note that an xy-ssmc morphism maps an xy-morphism to an xy-morphism.

Sometimes we are interested to keep fixed the monoid of the objects of the underlying categories of xy-ssmc-ies. Let M be a monoid. An M-xy-ssmc is an xy-ssmc B such that Ob(B) = M. An M-xy-ssmc morphism H is an xy-ssmc morphism that preserves the objects, i.e. H(a) = a, for every $a \in M$.

Proposition 3.2. If $H : B \longrightarrow B'$ is an M-xy-ssmc morphism then for every xy-morphism f' in B' there exists an xy-morphism f in B such that H(f) = f'. □

We don't know even if the restriction of H on objects is surjective if this proposition is valid when H is only an xy-ssmc morphism. Perhaps adding some hypotheses such a result may be obtained.

Proposition 3.3. Suppose $x \in \{b,d\}$ and B is an xy-ssmc. All the morphisms $f \in B(a,b)$ satisfying $f \perp^b = \perp^a$ form a sub-xy-ssmc of B.

Proposition 3.4. Suppose $x \in \{c,d\}$ and B is an xy-ssmc. All the morphisms $f \in B(a,b)$ satisfying $f \wedge^b = \wedge^a(f + f)$ form a sub-xy-ssmc of B.

Proposition 3.5. Suppose $y \in \{\beta,\delta\}$ and B is an xy-ssmc. All the morphisms $f \in B(a,b)$ satisfying $T_a f = T_b$ form a sub-xy-ssmc of B.

Proposition 3.6. Suppose $y \in \{\gamma,\delta\}$ and B is an xy-ssmc. All the morphisms $f \in B(a,b)$ satisfying $V_a f = (f + f)V_b$ form a sub-xy-ssmc of B.

Theorem 3.7. If B is an xy-ssmc then the category of its xy-morphisms B_{xy} is a strong xy-ssmc.

4. Simulation

In Section 2 we have shown that the local conditions (TX), (VX), or ($\perp$X) are not enough to generate useful equivalence relations. In order to do so one have to use also certain global rules, for example functoriality. The combination of functoriality with the above local conditions leads to certain equivalence relations which may, perhaps more directly, be introduced by using simulaton.

The using of simulation by bijective, injective, or surjective functions had become a standard way to define morphisms of automata, or graphs; see [Ho69], [Go74], [TWW79], for example. In the theory of multi-input/multi-output flowchart schemes the simulation by functions was used by Elgot in [El77] to study the complete minimization. In our theory of flowchart schemes we have defined and studied simulation by bijective functions in [CG84], by surjective functions in [St87a, version 1984], by injective functions in [St87a version 1985], and by arbitrary relations in [St87b version 1985]; see also [St86a], [St86b], [St87a], [St87b], [CS87b], [CS88b]. This study of simulation has led to an abstract setting for the definition of simulaton, introduced in [CS88b]. Namely, since an ssmc structure may naturally be defined on the relations used to define simulaton, we may be more abstract and define simulation via morphisms in an arbitrary ssmc.

Definition 4.1. Let Y, B be two ssmc-ies, $i,o : Y \to B$ two ssmc morphisms, and (x,f), (y,g) two pairs in $\mathbf{Fl}_{Ob(Y),B}(a,b)$. We say (x,f) and (y,g) are <u>similar via $u \in Y(x,y)$</u>, and write $(x,f) \to_u (y,g)$, if

(s) $$f\cdot(I_b+i(u)) = (I_a+o(u))\cdot g.$$

The relation $(x,f) \to_Y (y,g)$ means $(x,f) \to_j (y,g)$ for some $j \in Y(x,y)$. The relation $\to_Y$ is called <u>simulation via Y-morphisms.</u> □

Example 4.2. Let us consider the partial schemes obtained using atomic schemes in a double ranked set Σ, i.e. the schemes represented by pairs in $\mathbf{Fl}_{\Sigma,\mathbf{Pfn}}$. Suppose also the functions $i,o : \Sigma \to N$, specifying the input and the output number, respectively are given.

Simulation via bijections. It is proved in [CS89a] that there is a unique ssmc morphism i: $\mathbf{Bi}_\Sigma \to \mathbf{Pfn}$ (resp. o: $\mathbf{Bi}_\Sigma \to \mathbf{Pfn}$) which acts on Σ as the given function i (resp. o). Given two pairs (x,f) and (y,g) in $\mathbf{Fl}_{\Sigma,\mathbf{Pfn}}(a,b)$ it is shown in [CS88b] that

(x,f) and (y,g) represent isomorphic flowchart schemes iff there exists $u \in \mathbf{Bi}_\Sigma(x,y)$ such that $f(I_b+i(u)) = (I_a+o(u))g$. In this case we say that (x,f) and (y,g) are similar via the bijection u. One may easily see that this definition of simulation via a bijection is a particular case of Definition 4.1, namely when $Y = \mathbf{Bi}_\Sigma$.

Simulation via injections. It is proved in [CS89a] that there is a unique aβ-ssmc morphism $i: \mathbf{In}_\Sigma \to \mathbf{Pfn}$ (resp. $o: \mathbf{In}_\Sigma \to \mathbf{Pfn}$) which acts on Σ as the given function i (resp. o). Given two pairs (x,f) and (y,g) in $\mathbf{Fl}_{\Sigma,\mathbf{Pfn}}(a,b)$ it is proved in Section 13 that the scheme represented by (x,f) is isomorphic to a subscheme of the scheme represented by (y,g) iff there exists $u \in \mathbf{In}_\Sigma(x,y)$ such that $f(I_b+i(u)) = (I_a+o(u))g$. [Here by "subscheme" we understand that there is no arrow from an input or from a vertex in the subscheme to a statement which is not in the subscheme.] In this case we say that (x,f) and (y,g) are similar via the injection u. One may easily see that this definition of simulation via an injections is a particular case of Definition 4.1, namely when $Y = \mathbf{In}_\Sigma$. $\square$

Let us turn to the abstract setting. Suppose $i,o:Y \to B$ are two ssmc morphisms. We denote by ${}_Y\!\leftarrow$ the converse of $\rightarrow_Y$ and by $\sim_Y$ the least equivalence relation including $\rightarrow_Y$. Note that $\sim_Y$ is the transitive closure of $\rightarrow_Y \cup {}_Y\!\leftarrow$, i.e. $\sim_Y = (\rightarrow_Y \cup {}_Y\!\leftarrow)^+$.

To simplify the notation we denote the monoid Ob(Y) by X and we shall sometime write $\mathbf{Fl}$ (resp. $\rightarrow$, resp. $\sim$) instead of $\mathbf{Fl}_{X,B}$ (resp. $\rightarrow_Y$, resp. $\sim_Y$).

The following two results are proved in [CS88b].

Lemma 4.3. The simulation relation $\rightarrow_Y$ is a preorder which is compatible to summation and composition. The generated equivalence $\sim_Y$ may be written as $\sim_Y = (\rightarrow_Y \circ {}_Y\!\leftarrow)^+ = ({}_Y\!\leftarrow \circ \rightarrow_Y)^+$. Finally, the relation $\sim_Y$ is also the least congruence relation including $\rightarrow_Y$, i.e. it is compatible to summation and composition. $\square$

Consequently, summation and composition make sense in $\mathbf{Fl}/\sim_Y$, the quotient of

Fl by $\sim_Y$. Let $F_B: B \to Fl/\sim_Y$ be the composite of the embeddeing of B in **Fl**, i.e. $E_B: B \to Fl$, with the factorization morphism from **Fl** to $Fl/\sim_Y$.

Proposition 4.4. The quotient $Fl/\sim_Y$ is an ssmc and F_B is an ssmc morphism. $\square$

We try to find an algebraic structure such that:

(i) it have sufficient properties (including the validity of the strong axioms in Table 4 and functoriality) in order to make possible the study of the classes of flowchart schemes we are interested in;

(ii) the structure is preserved by passing from B to $Fl/\sim$.

The strong axioms extend simply to $Fl/\sim$, but for the extension of functoriality from B to $Fl/\sim$ we need some technical conditions. The additional conditions are chosen in such a way to be preserved by the passing from B to $Fl/\sim$, too.

Proposition 4.5. If B is an xy-ssmc, then $Fl/\sim_Y$ is an xy-ssmc and F_B is an xy-ssmc morphism.

Proof. It is enough to see that:

- if certain operations from T_a, V_a, $\perp^a$, or $\wedge^a$ are in B, then by E_B they are embedded in **Fl**;
- if certain axioms in Table 3 are satisfied in B, then they hold in **Fl**, too. $\square$

Proposition 4.6. Let $i,o: Y \to B$ be two xy-ssmc morphisms. If B is a strong xy-ssmc, then $Fl/\sim_Y$ is a strong xy-ssmc.

Proof. (a) Axiom (ST) is preserved (case $y \in \{\beta,\delta\}$): First note that $B(e,a) = \{T_a\}$. Indeed, if $f \in B(e,a)$, then $f = I_e \cdot f = T_e \cdot f = T_a$.

If $(x,f) \in Fl(e,a)$, then $(\varepsilon, T_a) \to_{T_x} (x,f)$. Indeed, $T_a(I_a + i(T_x)) = T_{a+i(x)} = T_{o(x)} \cdot f = (I_e + o(T_x)) \cdot f$. Consequently, $(x,f) \sim (\varepsilon, T_a)$ for every $(x,f) \in Fl(e,a)$, hence axiom (ST) holds in $\mathbf{Fl}/\sim$.

(b) Axiom ($S\perp$) is preserved (case $x \in \{b,d\}$): Dual to (a).

(c) Axiom (SV) is preserved (case $y \in \{\gamma,\delta\}$): Suppose $(x,f) \in \mathbf{Fl}(a,b)$. Note that $((x,f) + (x,f))\cdot V_b = (x+x, g)$, where $g = (I_a + {}^aX^{o(x)} + I_{o(x)})(f+f)(I_b + {}^{i(x)}X^b + I_{i(x)})(V_b + I_{i(x+x)})$.

We show that $((x,f) + (x,f))\cdot V_b \rightarrow_{V_x} V_a\cdot(x,f)$ holds. Indeed,

$g(I_b+i(V_x)) = (I_a + {}^aX^{o(x)} + I_{o(x)})(f+f)V_{b+i(x)} = (I_a + {}^aX^{o(x)} + I_{o(x)})V_{a+o(x)}f$

$= (I_{a+a}+o(V_x))[(V_a+I_{o(x)})f]$.

(d) Axiom ($S\wedge$) is preserved (case $x \in \{c,d\}$): Dual to (c). $\square$

5. Extending functoriality from connections to schemes

In this section we suppose moreover B is a biflow. We recall some results from [CS87a, CS88b]. The simulation relation $\longrightarrow_Y$ is compatible to the feedback, therefore $\sim_Y$ is the least flow congruence relation including $\longrightarrow_Y$.

Proposition 5.1. The quotient $\mathbf{Fl}/\sim_Y$ is an biflow and F_B is a biflow morphism. $\square$

We give in this section some conditions which assure the extension of functoriality from B to $\mathbf{Fl}_{X,B}/\sim_Y$ where $X = Ob(Y)$.

As we already defined, a morphism $j:a \rightarrow b$ in a flow B is called functorial if

$$f\cdot(I_d+j) = (I_c+j)\cdot g \implies f\uparrow^a = g\uparrow^b$$

for every $f \in B(c+a,d+a)$ and $g \in B(c+b,d+b)$.

Note that a morphism $j:a \rightarrow b$ in a flow B is functorial iff

$$f\cdot(j+I_d) = (j+I_c)\cdot g \implies {\uparrow}^a f = {\uparrow}^b g$$

for every $f \in B(a+c,a+d)$ and $g \in B(b+c,b+d)$.

Lemma 5.2 ($\rightarrow$ preserve functoriallity). If $j:a \rightarrow b$ is functorial in B, then j is

$\rightarrow$-functorial in $\mathbf{Fl}_{X,B}$; that is

(a) $F\cdot(j+I_d) \rightarrow_u (j+I_c)\cdot G \implies \uparrow^a F \rightarrow_u \uparrow^b G$ and

(b) $(j+I_c)\cdot G \rightarrow_v F\cdot(j+I_d) \implies \uparrow^b G \rightarrow_v \uparrow^a F$

for every $F \in \mathbf{Fl}_{X,B}(a+c,a+d)$ and $G \in \mathbf{Fl}_{X,B}(b+c,b+d)$.

Proof. a) Suppose $F = (x,f)$ and $G = (y,g)$. The simulation shows that $[f(j+I_d+I_{i(x)})](I_{b+d}+i(u)) = (I_{a+c}+o(u))[(j+I_c+I_{o(y)})g]$. Consequently, $[f(I_{a+d}+i(u))](j+I_{d+i(y)}) = (j+I_{c+o(x)})[(I_{b+c}+o(u))g]$. Since j is functorial in B it follows that $\uparrow^a(f(I_{a+d}+i(u))) = \uparrow^b((I_{b+c}+o(u))g)$, hence $(\uparrow^a f)(I_d+i(u)) = (I_c+o(u))(\uparrow^b g)$. This means $\uparrow^a F \rightarrow_u \uparrow^b G$.

b) Similar. □

Theorem 5.3. Suppose $j \in B(a,b)$. The implication

j functorial in B $\implies$ j functorial in $\mathbf{Fl}_{X,B}/\sim_Y$

is valid provided that the following two conditions are fulfilled:

(C1) $\sim_Y = {}_Y\!\leftarrow\circ\rightarrow_Y$;

(C2) $F \rightarrow_u G(j+I_d) \implies (\exists H)$ such that $F = H(j+I_d)$ and $H \rightarrow_u G$

for all c,d objects in B, $F: a+c \rightarrow b+d$, $G: a+c \rightarrow a+d$ morphisms in $\mathbf{Fl}_{X,B}$ and u morphism in Y.

Proof. Assume $F \in \mathbf{Fl}_{X,B}(a+c,a+d)$, $G \in \mathbf{Fl}_{X,B}(b+c,b+d)$ and $F(j+I_d) \sim (j+I_c)G$. By (C1) this means $F(j+I_d) \leftarrow H \rightarrow (j+I_c)G$ for a certain pair H. Applying (C2) to the left simulation we get a pair H' such that $H = H'(j+I_d)$ and $F \rightarrow H'$, hence $\uparrow^a F \rightarrow \uparrow^a H'$. The right simulation may be written as $H'(j+I_d) \rightarrow (j+I_c)G$, hence by Lemma 5.1.a we get $\uparrow^a H' \rightarrow \uparrow^b G$. It follows that $\uparrow^a F \leftarrow\circ\rightarrow \uparrow^b G$, hence $\uparrow^a F \sim \uparrow^b G$. □

This easily proved theorem leads to the following problem: For a given Y find "reasonable" conditions on B with respect to Y such that the conditions in this theorem hold.

6. Technical conditions

In this section we give certain conditions on B with respect to Y such that conditions (C1) and (C2) in Theorem 5.2 hold.

We say a pair (j',k') of morphisms $j' \in B(a',a_1)$ and $k' \in B(a',a_2)$ of a category B is a <u>weak pullback</u> of the pair (j,k) of morphisms $j \in B(a_1,a)$ and $k \in B(a_2,a)$, and write

$$(j',k') \text{ Wpb } (j,k),$$

if $j'j = k'k$ and if for every object $b \in Ob(B)$ and morphisms $f \in B(b,a_1)$ and $g \in B(b,a_2)$ such that $fj = gk$ there exists a morphism $h \in B(b,a')$ such that $hj' = f$ and $hk' = g$.

The adjective "weak" referes to the fact that we do not require uniqueness of h as in the analogous definition of pullbacks.

Analysis of Condition (C1) in Theorem 5.2. Suppose $i:Y \to B$ is an ssmc morphism. We say B fulfills the <u>wpb-condition</u> (weak pullback condition) with respect to Y and i if for every morphisms $u \in Y(x_1,x)$ and $v \in Y(x_2,x)$ there exist an object $x' \in Ob(Y)$ and two morphisms $u' \in Y(x',x_1)$ and $v' \in Y(x',x_2)$ such that

(b1) $u'u = v'v$ and

(b2) $(I_a+i(u'), I_a+i(v')) \text{ Wpb } (I_a+i(u), I_a+i(v))$, for every object $a \in Ob(B)$.

This wpb-condition rests on the following three conditions:

(wpb_1) Y has weak pullbacks;

(wpb_2) the functor i preserves weak pullbacks;

(wpb_3) addition of an object $a \in Ob(B)$ preserves weak pullbacks.

Clearly, the conditions wpb_{1-3} imply wpb-condition. Due to some technical reasons we prefer to work with this global wpb-condition.

The utility of this wpb-condition come from the following proposition.

Proposition 6.1. If B fulfills the wpb-condition with respect to Y and i, then Condition (C1) in Theorem 5.2 holds, i.e. $\sim_Y = {}_Y\!\!\leftarrow \circ \rightarrow_Y$.

Proof. Since by Lemma 4.3 $\sim = (\leftarrow \circ \rightarrow)^+$ it suffices to show $\leftarrow \circ \rightarrow$ is transitive. Consequently, it is enough to prove that $\rightarrow \circ \leftarrow \subset \leftarrow \circ \rightarrow$.

Suppose we are given three pairs in $\mathbf{Fl}_{X,B}(b,a)$ such that

$$(x_1,f_1) \rightarrow_u (x,f) \; {}_v\!\!\leftarrow (x_2,f_2)$$

for some morphisms $u \in Y(x_1,x)$ and $v \in Y(x_2,x)$. Since B fulfills the wpb-condition with respect to Y and i, there exists an object x' of Y and two morphisms $u' \in Y(x',x_1)$ and $v' \in Y(x',x_2)$ fulfilling (b1) and (b2) in the definition of the wpb-condition. Since

$$(I_b+o(u'))f_1(I_a+i(u)) = (I_b+o(u'u))f = (I_b+o(v'v))f = (I_b+o(v'))f_2(I_a+i(v))$$

by (b2) we get a morphism $f' \in B(b+o(x'), a+i(x'))$ such that

$$f'(I_a+i(u')) = (I_b+o(u'))f_1 \quad \text{and} \quad f'(I_a+i(v')) = (I_b+o(v'))f_2,$$

i.e. such that $(x_1,f_1) \; {}_{u'}\!\!\leftarrow (x',f') \rightarrow_{v'} (x_2,f_2)$. Hence we have proved that $\rightarrow \circ \leftarrow \subset \leftarrow \circ \rightarrow$ and the result follows. $\square$

Analysis of Condition (C2) in Theorem 5.2. We say two morphisms $j:a \rightarrow b$ and $k:c \rightarrow d$ in an ssmc B are <u>wc-connected</u> (weak cartesianly connected), and we write

$$j \;\mathbf{Wc}\; k \quad \text{if} \quad (j+I_c, I_a+k) \;\mathbf{Wpb}\; (I_b+k, j+I_d).$$

We also use the notation

$I + A$	for the set $\{I_a + j \mid a \in Ob(B) \text{ and } j \in A\}$ ($I + j$ means $I + \{j\}$),
$A + I$	for the set $\{j+I_a \mid j \in A \text{ and } a \in Ob(B)\}$ ($j + I$ means $\{j\} + I$) and
$A \;\mathbf{Wc}\; A'$	for $(\forall j \in A)(\forall k \in A')(j \;\mathbf{Wc}\; k)$,

where A and A' are sets of morphisms in B.

Lemma 6.2. If $F \to_u Gj$ and j **Wc** $i(u)$, then there exists a pair H such that $F = Hj$ and $H \to_u G$.

Proof. Suppose $F = (x,f) : a \to c$, $G = (y,g) : a \to b$ and $j \in B(b,c)$. Then $f(I_c+i(u)) = (I_a+o(u))g(j+I_{i(y)})$. Since j **Wc** $i(u)$ in B there exists $h \in B(a+o(x), b+i(x))$ such that $h(j+I_{i(x)}) = f$ and $h(I_b+i(u)) = (I_a+o(u))g$. Consequently, $H = (x,h)$ obeys $Hj = F$ and $H \to_u G$. ☐

Corollary 6.3. If $j+I$ **Wc** $i(Y)$, then Condition (C2) in Theorem 5.3 holds. ☐

Lemma 6.4. The implication

$$(j',k')\,\mathbf{Wpb}\,(j,k) \implies (pj'q,pk'w)\,\mathbf{Wpb}\,(q^{-1}jt,w^{-1}kt)$$

is valid provided that p,q w and t are isomorphisms. ☐

Lemma 6.5. 1) j **Wc** k $\implies$ k **Wc** j;

2) j isomorphism $\implies$ j **Wc** f, for all f;

3) j **Wc** f and j' **Wc** f $\implies$ $j \cdot j'$ **Wc** f. ☐

Proposition 6.6. If $I+j$ **Wc** $i(Y)$, then Condition (C2) in Theorem 5.3 holds.

Proof. By Corollary 6.3 and Lemma 6.5. ☐

7. Extending technical conditions from connections to schemes

In this section we try to answer the question asked after Proposition 4.4. To this aim we study the preservation of some properties by the passing from B to **Fl**/$\sim$.

Lemma 7.1. The implication

$$(j',k')\,\mathbf{Wpb}\,(j,k) \text{ in } B \implies (j',k')\,\mathbf{Wpb}\,(j,k) \text{ in } \mathbf{Fl}/\!\sim$$

is valid provided that the following three conditions are fulfilled.

(1) $\sim_Y \;=\; {}_Y\!\longleftarrow\circ\longrightarrow_Y$.

(2) $\{j,k\}\,\mathbf{Wc}\;i(Y)$.

(3) $(j' + I_{i(z)},\, k' + I_{i(z)})\,\mathbf{Wpb}\,(j + I_{i(z)},\, k + I_{i(z)})$ in B, for every $z \in X$.

[The premise of the implication is a particular case of (3)].

Proof. Suppose $j' \in B(a',a_1)$, $j \in B(a_1,a)$, $k' \in B(a',a_2)$ and $k \in B(a_2,a)$. Suppose moreover $F = (x,f) \in \mathbf{Fl}(b,a_1)$ and $G = (y,g) \in \mathbf{Fl}(b,a_2)$ satisfy

$$Fj \sim Gk.$$

By (1) there exists $H = (z,h) \in \mathbf{Fl}(b,a)$ and two morphisms u and v in Y such that

$$Fj \;{}_u\!\longleftarrow H \longrightarrow_v Gk.$$

By (2) $j\,\mathbf{Wc}\,i(u)$, hence by Lemma 6.2 there exists a pair $H_1 = (z,h_1) \in \mathbf{Fl}(b,a_1)$ such that

$$F \;{}_u\!\longleftarrow H_1 \quad\text{and}\quad H_1 j = H$$

and similarly from $k\,\mathbf{Wc}\,i(v)$ we deduce that there exists a pair $H_2 = (z,h_2) \in \mathbf{Fl}(b,a_2)$ such that

$$H = H_2 k \quad\text{and}\quad H_2 \longrightarrow_v G.$$

This means $h_1(j + I_{i(z)}) = h = h_2(k + I_{i(z)})$. By (3) there exists $h' \in B(b + o(z),\, a' + i(z))$ such that

$$h'(j' + I_{i(z)}) = h_1 \quad\text{and}\quad h'(k' + I_{i(z)}) = h_2.$$

For $H' = (z,h') \in \mathbf{Fl}(b,a')$ we deduce

$$H'j' = H_1 \longrightarrow_u F \quad\text{and}\quad H'k' = H_2 \;{}_v\!\longleftarrow G,$$

hence $H'j' \sim F$ and $H'k' \sim G$. Hence $(j',k')\,\mathbf{Wpb}\,(j,k)$ in $\mathbf{Fl}/\!\sim$.

Proposition 7.2. The implication

"B fulfills the wpb-condition with respect to Y and i ==>

$Fl/\sim$ fulfills the wpb-condition with respect to Y and iF_B"

is valid provided that $I + i(Y)\, \mathrm{Wc}\, i(Y)$.

Proof. Let $u \in Y(x_1,x)$ and $v \in Y(x_2,x)$. As B fulfills the wpb-condition with respect to Y and i there exists $u' \in Y(x',x_1)$ and $v' \in Y(x',x_2)$ fulfilling (b1) and (b2) in the definition of the wpb-condition. To prove (b2) in $Fl/\sim$ we apply Lemma 7.1. The conditions (1) and (2) in Lemma 7.1 follow from Proposition 6.1 and the hypothesis $I + i(Y)\, \mathrm{Wc}\, i(Y)$. Therefore we only have to show condition (3) in Lemma 7.1 holds.

Let $z \in Ob(Y)$. From (b2) in B we deduce

$$(I_{a+i(z)} + i(u'),\ I_{a+i(z)} + i(v'))\ \mathbf{Wpb}\ (I_{a+i(z)} + i(u),\ I_{a+i(z)} + i(v)).$$

Using Lemma 6.4 for $p = I_a + {}^{i(x')}X^{i(z)}$, $q = I_a + {}^{i(z)}X^{i(x_1)}$, $w = I_a + {}^{i(z)}X^{i(x_2)}$ and $t = I_a + {}^{i(z)}X^{i(x)}$ we deduce

$$(I_a + i(u') + I_{i(z)},\ I_a + i(v') + I_{i(z)})\ \mathrm{Wpb}\ (I_a + i(u) + I_{i(z)},\ I_a + i(v) + I_{i(z)})$$

hence condition (3) in Lemma 7.1 is valid. $\square$

Lemma 7.3. Let $j \in B(a,b)$ and $k \in B(c,d)$. The implication

$$j\ \mathrm{Wc}\ k \text{ in } B \implies j\ \mathrm{Wc}\ k \text{ in } Fl/\sim .$$

is valid provided that the following three conditions are fulfilled.

(i) $\sim_Y = {}_Y\!\longleftarrow \circ \longrightarrow_Y$

(ii) $\{I_b + k,\ j + I_d\}\ \mathrm{Wc}\ i(Y)$

(iii) $j\ \mathrm{Wc}\ k + I_{i(z)}$ in B, for every $z \in Ob(Y)$.

[The premise of the implication is a particular case of (iii)].

Proof. We apply Lemma 7.1 for $(j + I_c,\ I_a + k)\ \mathbf{Wpb}\ (I_b + k,\ j + I_d)$. $\square$

Proposition 7.4. The implication

$$I + i(Y)\,\mathbb{W}c\,i(Y) \text{ in } B \Longrightarrow I + iF_B(Y)\,\mathbb{W}c\,iF_B(Y) \text{ in } \mathbf{Fl}/\sim$$

is valid provided that $\sim_Y = {}_Y\!\longleftarrow \circ \longrightarrow_Y$.

Proof. For every $a \in Ob(B)$, $u \in Y(x',x)$ and $v \in Y(y',y)$ we have to prove $I_a + i(u)\,\mathbb{W}c\,i(v)$ in $\mathbf{Fl}/\sim$. To do it we apply Lemma 7.3 for $I_a + i(u)\,\mathbb{W}c\,i(v)$ in B. In our case conditions (ii) and (iii) in Lemma 7.3 becomes

$$\{I_{a+i(x)} + i(v),\ I_a + i(u + I_y)\}\,\mathbb{W}c\,i(Y) \quad \text{and}$$

$$I_a + i(u)\,\mathbb{W}c\,i(v + I_z) \quad \text{for every } z \in Ob(Y),$$

therefore they may be easily deduce from $I + i(Y)\,\mathbb{W}c\,i(Y)$ in B. □

Theorem 7.5. For every biflow B if

a) B fulfills the wpb-condition with respect to Y and i,

b) $I + i(Y)\,\mathbb{W}c\,i(Y)$

c) B satisfies the functoriality axiom (func : i(Y)).

then

a') $\mathbf{Fl}/\sim$ fulfills the wpb-condition with respect to Y and iF_B.

b') $I + iF_B(Y)\,\mathbb{W}c\,iF_B(Y)$

c') $\mathbf{Fl}/\sim$ satisfies the functoriality axiom (func : $iF_B(Y)$).

Proof. Using Proposition 6.1 we deduce $\sim_Y = {}_Y\!\longleftarrow \circ \longrightarrow_Y$. Conclusions a' and b' follows from Propositions 7.2 and 7.4 respectively. Using Proposition 6.6 we may apply Theorem 5.3 to get the last conclusion. □

This theorem answers the question asked after Proposition 4.4. Note that all the hypotheses in this theorem refer to the ssmc morphism $i : Y \longrightarrow B$. In the sequel we will give a slightly different version of Theorem 7.5 where hypotheses b and c are replaced by stronger hypotheses on B itself.

Definition 7.6. A biflow over a strong xy-ssmc is said to be an xy-flow if every xy-morphism is functorial. (Note that a α-flow means biflow.) □

Let B and B' be xy-flows. The biflow morphism $H : B \longrightarrow B'$ is said to be an xy-flow morphism if H is also an xy-ssmc morphism.

Definition 7.7. An xy-ssmc B is said to be weakly cartesian if f $\mathbf{Wc}$ g whenever f and g are xy-morphisms in B.

Theorem 7.8. Suppose Y is an xy-ssmc such that $Y_{xy} = Y$. Suppose $i : Y \longrightarrow B$ and $o : Y \longrightarrow B$ are xy-ssmc morphisms. If

B is an xy-flow,

B is weakly cartesian and

B fulfills the wpb-condition with respect to Y and i

then

$Fl/\sim$ is an xy-flow, F_B is an xy-flow morphism,

$Fl/\sim$ is weakly cartesian and

$Fl/\sim$ fulfills the wpb-condition with respect to Y and iF_B.

Proof. Propositions 4.5, 4.6 and 5.1 show $Fl/\sim$ is a biflow over a strong xy-ssmc and F_B has the required properties.

To get the other conclusions we use the following remarks

a) every morphism in i(Y) is an xy-morphism,

b) $I + f\ \mathbf{Wc}\ i(Y)$ whenever f is an xy-morphism is B

(by using remark a and the hypothesis that B is weakly cartesian),

c) $\sim_Y = {}_Y\!\longleftarrow \circ \longrightarrow_Y$ (by Proposition 6.1).

We show $Fl/\sim$ is weakly cartesian. Let j' and k' be xy-morphisms in $Fl/\sim$. Using Proposition 3.2 we deduce $j' = F_B(j)$ and $k' = F_B(k)$ where j and k are xy-morphisms in B. As B is weakly cartesian it follows $j \, \mathbf{Wc} \, k$. To finish we apply Lemma 7.3.

We show every xy-morphism in $\mathbf{Fl}/\sim$ is functorial. Let j' an xy-morphism in $\mathbf{Fl}/\sim$. Using Proposition 3.2 we get $j' = F_B(j)$ where j is an xy-morphism in B. Using remarks c and b and Proposition 6.6 we apply theorem 5.3 to prove j' is functorial.

Therefore $\mathbf{Fl}/\sim$ is an xy-flow and F_B is an xy-flow-morphism. To show $\mathbf{Fl}/\sim$ fulfills the wpo-condition with respect to Y and iF_B we apply Proposition 7.2 using remarks a and b. □

8. Duality

In the sequel we shall use a duality principle based on the following idea reverse all the arrows of a flowchart scheme. In this way we obtain from a flowchart scheme another flowchart scheme where the inputs and the outputs of the two schemes and even the inputs and the outputs of every statement are interchanged.

If Y is an smc then the dual of Y from the categorial point of view, denoted Y°, is an smc, too. Note that the monoid $Ob(Y^\circ)$ and $Ob(Y)$ are equal.

The dual Y° of an ssmc Y is an ssmc, too. Here we have to change ${}^aX^b$ with ${}^bX^a$.

The dual B° of a bilow B is a bilow, too. It is easy to see that a morphism is functorial in B if and only if it is functorial in B°.

[Note that this duality does not work well for flows, as the concept of flow is a nonpermutable one [CS89a]. Therefore to apply the duality principle to the flowchart scheme representations we must take some care.

Remark first that when we dualize i and o must be interchanged. Even if the following equality holds

$$\mathbf{Fl}_{X,B,i,o}(a,b) = \mathbf{Fl}_{X,B^\circ,o,i}(b,a)$$

$\mathbf{Fl}_{X,B^\circ,o,i}$ as a category is not dual to $\mathbf{Fl}_{X,B,i,o}$. To see this is enough to look at the composition. Nevertheless we may dualize $E_B(j)(x,f)$ by $(x,f)\,E_B(j)$; this is the case when one of the scheme representation have no statements.]

Passing to simulation relation remark that we may dualize $(x,f) \to_u (y,g)$ in $\mathbf{Fl}(a,b)$ by $(x,f)\ {}_u\!\!\leftarrow (y,g)$ in $\mathbf{Fl}_{X,B^\circ}(b,a)$ as the equality $f(I_b + i(u)) = (I_a + o(u))g$ becomes $g \circ (I_a + o(u)) = (I_b + i(u)) \circ f$ in B°.

Using the duality principle for the main results in sections 5-7 we obtain the following facts.

A pair (j',k') of morphisms $j' \in B(a_1,a')$ and $k' \in B(a_2,a')$ of a category B is said to be a <u>weak pushout</u> of the pair (j,k) of morphisms $j \in B(a,a_1)$ and $k \in B(a,a_2)$ and we write

$$(j,k)\ \mathbf{Wpo}\ (j',k')$$

if $jj' = kk'$ and for every $b \in Ob(B)$, $f \in B(a_1,b)$ and $g \in B(a_2,b)$ such that $jf = kg$ there exists $h \in B(a',b)$ such that $j'h = f$ and $k'h = g$. When the above h is unique we write $(j,k)\ \mathbf{Po}\ (j',k')$ and we say (j',k') is a <u>pushout</u> of (j,k).

Suppose $o : Y \to B$ is an ssmc-morphism. We say B fulfills the <u>wpo-condition</u> (weak pushout condition) with respect to Y and o if for every morphism $u \in Y(x,x_1)$ and $v \in Y(x,x_2)$ there exists $x' \in Ob(Y)$, $u' \in Y(x_1,x')$ and $v' \in Y(x_2,x')$ such that

(1) $uu' = vv'$

(2) $(I_a + o(u), I_a + o(v))\ \mathbf{Wpo}\ (I_a + o(u'), I_a + o(v'))$ for every $a \in Ob(B)$,

Propositin 8.1. If B fulfills the wpo-condition with respect to Y and o then $\sim_Y = \to_Y \circ {}_Y\!\!\leftarrow$. □

We say two morphism $j : a \to b$ and $k : c \to d$ in an ssmc B are <u>Wcc-connected</u> (weak cocartesianly connected) and we write

$$j\ \mathbf{Wcc}\ k \quad \text{if} \quad (j + I_c, I_a + k)\ \mathbf{Wpo}\ (I_b + k, j + I_d).$$

Lemma 8.2. If $jG \to_u F$ and $j\ \mathbf{Wcc}\ o(u)$ then there exists H such that $F = jH$ and $G \to_u H$.

Theorem 8.3. If the biflow B fulfills the wpo-condition with respect to Y and o, if j is functorial in B and if $I + j \,\mathbf{Wcc}\, o(Y)$ then j is functorial in $\mathbf{Fl}/\sim$.

Proposition 8.4. If B fulfills the wpo-condition with respect to Y and o, and if $I + o(Y) \,\mathbf{Wcc}\, o(Y)$ then $\mathbf{Fl}/\sim$ fulfills the wpo-condition with respect to Y and oF_B.

Proposition 8.5. If $\sim_Y = \longrightarrow_Y \circ {}_Y\!\longleftarrow$ then

$$I + o(Y) \,\mathbf{Wcc}\, o(Y) \text{ in } B \Longrightarrow I + oF_B(Y) \,\mathbf{Wcc}\, oF_B(Y) \text{ in } \mathbf{Fl}/\sim .$$

Theorem 8.6. For every biflow B if

a) B fulfills the wpo-condition with respect to Y and o,

b) $I + o(Y) \,\mathbf{Wcc}\, o(Y)$,

c) B satisfies the functoriality axiom (func : o(Y))

then

a') $\mathbf{Fl}/\sim$ fulfills the wpo-condition with respect to Y ano oF_B,

b') $I + oF_B(Y) \,\mathbf{Wcc}\, oF_B(Y)$,

c') $\mathbf{Fl}/\sim$ satisfies the functoriality axiom (func : $oF_B(Y)$).

Definition 8.7. An xy-ssmc B is said to be <u>weakly cocartesian</u> if f **Wcc** g whenever f and g are xy-morphism in B.

Theorem 8.8. Suppose Y is an xy-ssmc such that $Y_{xy} = Y$. Assume $i : Y \rightarrow B$ and $o : Y \rightarrow B$ are xy-ssmc morphisms. If

B is an xy-flow,

B is weakly cocartesian and

B fulfills the wpo-condition with respect to Y and o

then

$\mathbf{Fl}/\sim$ is an xy-flow, F_B is an xy-flow morphism,

$\mathbf{Fl}/\sim$ is weakly cocartesian and

$\mathbf{Fl}/\sim$ fulfills the wpo-condition with respect to Y and oF_B.

9. On xy-simulation

In the sequel we are interested in the study of some types of flowchart schemes. As we have mentioned in [CS87b] the study of each type in related to a certain type of simulation. Instead of using different categories to do simulations and different morphisms between these categories to compare different kind of simulation it is preferably to use for simulation different subcategories of a unique category Y.

This viewpoint agree with the case when our abstract flowchart schemes (abstract means the connections are taken from a biflow B) are build using statements in a set Σ. In this case Y is a subcategory of $\mathbf{Rel}_\Sigma$. For example (see the last section) when we study minimal schemes Y may be $\mathbf{Fn}_\Sigma$ and we may use four kinds of simulation, namely simulation using morphisms in $\mathbf{Bi}_\Sigma$, $\mathbf{In}_\Sigma$, $\mathbf{Sur}_\Sigma$ or $\mathbf{Fn}_\Sigma$.

The same viewpoint agree with another point of view, an algebraic one. To understand this algebraic point of view we need some preliminaries.

Let B be an xy-ssmc and let $h : X \longrightarrow Ob(B)$ be a monoid morphism. We define an X-xy-ssmc $h^\circ(B)$ as follows:

$h^\circ(B)(u,v) = B(h(u),h(u))$ for $u,v \in X$,

$fg = fg$ for $f \in h^\circ(B)(u,v)$ and $g \in h^\circ(B)(v,w)$,

$I_u = I_{h(u)}$ for $u \in X$,

$f + g = f + g$ for $f \in h^\circ(B)(u,v)$ and $g \in h^\circ(B)(u',v')$,

${}^u\mathsf{X}^v = {}^{h(u)}\mathsf{X}^{h(v)}$ for $u,v \in X$.

and for the additional distinguished morphisms for $u \in X$ we choose according to xy from

$$\mathsf{T}_u = \mathsf{T}_{h(u)},\ \perp^u = \perp^{h(u)},\ \vee_u = \vee_{h(u)} \text{ and } \wedge^u = \wedge^{h(u)}.$$

Let $\xi_h : h^\circ(B) \longrightarrow B$ be the xy-ssmc morphism defined by

$\varepsilon_h(u) = h(u)$ for $u \in X$ and

$\varepsilon_h(f) = f$ for each morphism f in $h^{\circ}(B)$.

For every monoid X we denote by xy_X the initial X-xy-ssmc. In xy_X every morphism in an xy-morphism. We have shown in [CS89a] that xy-$\mathbf{Rel}_{\Sigma}$ is a model for xy_{Σ^*}.

Proposition 9.1. Let B be an xy-ssmc. Every monoid morphism $h : X \longrightarrow Ob(B)$ can be uniquely extended to an xy-ssmc morphism $H : xy_X \longrightarrow B$.

Proof. As xy_X in the initial X-xy-ssmc there exists a unique X-xy-ssmc morphism $H' : xy_X \longrightarrow h^{\circ}(B)$. By definition $H = H'\varepsilon_h$. As $H(u) = \varepsilon_h(H'(u)) = \varepsilon_h(u) = h(u)$ for every $u \in X$ the xy-ssmc morphism H is the required extension of h. □

Proposition 9.2. Let Y be an X-xy-ssmc. Assume $i : Y \longrightarrow B$ and $o : Y \longrightarrow B$ are xy-ssmc morphisms. If $i^{xy} : xy_X \longrightarrow B$ and $o^{xy} : xy_X \longrightarrow B$ are the unique xy-ssmc morphisms which extend the monoid morphisms i and o, respectively then for every F and G in $Fl_{X,B}(a,b)$

$F \longrightarrow_{xy_X} G$ iff there exists an xy-morphism u in Y such that $F \longrightarrow_u G$.

Proof. Let $H : xy_X \longrightarrow Y$ be the unique X-xy-ssmc morphism. We deduce $Hi = i^{xy}$ and $Ho = o^{xy}$.

We prove only the more difficult implication. Suppose there exists an xy-morphism $u \in Y(x',x'')$ such that

$$F = (x',f) \longrightarrow_u G = (x'',g).$$

Form Proposition 3.2 there exists $j \in xy_X(x',x'')$ such that $u = H(j)$, therefore

$$f(I_b + i^{xy}(j)) = f(I_b + i(u)) = (I_a + o(u))g = (I_a + o^{xy}(j)).$$ □

We are now ready to explain the algebraic viewpoint. To build $\mathbf{Fl}_{\Sigma,B}$ we use two function $i : \Sigma \longrightarrow Ob(B)$ and $o : \Sigma \longrightarrow Ob(B)$ which give the input and the output of every statement in Σ. These functions may by extended in one way to monoid morphisms $i : \Sigma^* \longrightarrow Ob(B)$ and $o : \Sigma^* \longrightarrow Ob(B)$, and then to xy-ssmc morphisms $i : \text{xy-}\mathbf{Rel}_{\Sigma} \longrightarrow B$ and $o : \text{xy-}\mathbf{Rel}_{\Sigma} \longrightarrow B$. The simulation is made using morphisms in $\text{xy-}\mathbf{Rel}_{\Sigma}$.

To generalize we replace Σ^* by an arbitrary monoid X and we use two monoid morphisms $i : X \longrightarrow Ob(B)$ and $o : X \longrightarrow Ob(B)$ to build $\mathbf{Fl}_{X,B}$ as we already made in [CS87a]. In this case to simulate we use the category xy_X and the xy-ssmc morphisms $i^{xy} : xy_X \longrightarrow B$ and $o^{xy} : xy_X \longrightarrow B$.

To generalize we replace xy_X by an X-xy-ssmc Y and we use two xy-ssmc morphisms $i : Y \longrightarrow B$ and $o : Y \longrightarrow B$. Proposition 9.2 shows the simulation via xy_X-morphisms is equivalent to the simulation via xy-morphisms in Y. Hence the algebraic viewpoint agree with the point of view from the beginning of this section.

Suppose $i : Y \longrightarrow B$ and $o : Y \longrightarrow B$ are xy-ssmc morphisms and B is a biflow (over an xy-ssmc). The <u>xy-simulation</u>, i.e. the simulation via xy-morphisms in Y, is introduced in accordance with Definition 4.1 using the restrictin of i and o to Y_{xy} and is denoted by $\xrightarrow{xy}$. We denote by $\xleftarrow{xy}$ its converce and by $\sim_{xy}$ the least flow congruence relation which includes $\xrightarrow{xy}$. As $\sim_{a\alpha} \subset \sim_{xy}$ the quotient $\mathbf{Fl}/\sim_{xy}$ of $\mathbf{Fl}_{X,B}$ by $\sim_{xy}$ is a biflow over an xy-ssmc. By Proposition 4.6 if B is a strong xy-ssmc then $\mathbf{Fl}/\sim_{xy}$ is a strong xy-ssmc. The morphisms in $\mathbf{Fl}/\sim_{xy}$ are called <u>xy-schemes</u> and $\mathbf{Fl}/\sim_{xy}$ is called the <u>biflow of the xy-schemes.</u>

Definition 9.3. A monoid morphism $I : X \longrightarrow Mor(B)$ is said to be an <u>interpretation</u> of X in B with respect to i and o if $I(x) \in B(i(x), o(x))$ for every x in X. $\square$

Let $E_X^{xy} : X \longrightarrow \mathbf{Fl}/\sim_{xy}$ and $E_B^{xy} : B \longrightarrow \mathbf{Fl}/\sim_{xy}$ be the composite of E_X and E_B with the factorization morphism from $\mathbf{Fl}$ to $\mathbf{Fl}/\sim_{xy}$. Remark E_X^{xy} is an interpretation of X in $\mathbf{Fl}/\sim_{xy}$ with respect to iE_B^{xy} and oE_B^{xy}. Remark E_B^{xy} is an biflow morphism and an xy-ssmc morphism.

Proposition 9.4. If B is a strong xy-ssmc then the congruence relation $\sim_{xy}$ fulfills

(XX) $(z+t)^{o(z)}X^{o(t)} \equiv {}^{i(z)}X^{i(t)}(t+z)$ for t,z $\in$ X,

(TX) if $y \in \{\beta,\delta\}$ then $T_{i(z)}z \equiv T_{o(z)}$ for z $\in$ X,

(VX) if $y \in \{\gamma,\delta\}$ then $V_{i(z)}z \equiv (z+z)V_{o(z)}$ for z $\in$ X,

($\perp$X) if $x \in \{b,d\}$ then $x\perp^{o(z)} \equiv \perp^{i(z)}$ for z $\in$ X and

($\wedge$X) if $x \in \{c,d\}$ then $z\wedge^{o(z)} \equiv \wedge^{i(z)}(z+z)$ for z $\in$ X.

Proof. The same proof as for Proposition 4.6.

Lemma 9.5. If I is an interpretation of X in B with respect to the xy-ssmc morphisms i and o such that for every z $\in$ X

T) if $y \in \{\beta,\delta\}$ then $T_{i(z)}I(z) = T_{o(z)}$,

V) if $y \in \{\gamma,\delta\}$ then $V_{i(z)}I(z) = (I(z)+I(z))V_{o(z)}$,

$\perp$) if $x \in \{b,d\}$ then $I(z)\perp^{o(z)} = \perp^{i(z)}$ and

$\wedge$) if $x \in \{c,d\}$ then $I(z)\wedge^{o(z)} = \wedge^{i(z)}(I(z)+I(z))$

then $I(z)\,o(f) = i(f)\,I(t)$ for every $f \in Y_{xy}(z,t)$.

Proof. It is easy to show all the morphisms from Y fulfilling the above equality form a sub-xy-ssmc of Y. $\square$

Definition 9.6. A congruence relation in B is said to be <u>xy-functorial</u> if it fulfills (func : B_{xy}).

Remark 9.7. A congruence relation $\equiv$ in **Fl** is xy-functorial if and only if every xy-morphism from **Fl**/$\equiv$ is functorial.

Proof. Easy using Proposition 3.2. $\square$

Proposition 9.8. If the xy-functorial congruence relation $\equiv$ in **Fl** fulfills (XX), (TX),

$(\vee X)$, $(\perp X)$ and $(\wedge X)$ then $\equiv$ includes $\sim_{xy}$.

Proof. As $\equiv$ fulfills (XX) we deduce $\mathbf{Fl}/\equiv$ is a biflow [Lemma 7.5 in CS88b] over an xy-ssmc.

Let $G : \mathbf{Fl} \to \mathbf{Fl}/\equiv$ be the factorization morphism. As $E_X G$ is an interpretation of X in $\mathbf{Fl}/\equiv$ with respect to $iE_B G$ and $oE_B G$ we deduce from Lemma 9.5 that $z\,o(u) \equiv i(u)\,t$ for every $u \in Y_{xy}(z,t)$.

To get the conclusion it suffices to show $\xrightarrow{xy} \subset \equiv$. Suppose $(z,f) \to_u (t,g)$ in $\mathbf{Fl}(a,b)$ where $u \in Y_{xy}(z,t)$. As $i(u)$ is $\equiv$-functorial and as

$$(I_a + z)f(I_b + i(u)) = (I_a + zo(u))g \equiv (I_a + i(u))(I_a + t)g$$

we deduce $[(I_a + z)f]\uparrow^{i(z)} \equiv [(I_a + t)g]\uparrow^{i(t)}$ hence $(z,f) \equiv (t,g)$. $\square$

10. A universal theorem

Assume $i : Y \to B$ and $o : Y \to B$ are ssmc morphisms and $\sim$ is the least congruence relation in $\mathbf{Fl}$ including $\to_Y$.

Let $F_X : X \to \mathbf{Fl}/\sim$ be the composite of the embedding $E_X : X \to \mathbf{Fl}$ with the factorization morphism from $\mathbf{Fl}$ to $\mathbf{Fl}/\sim$. Note that F_X is an interpretation of X in $\mathbf{Fl}/\sim$ with respect to iF_B and oF_B.

Lemma 10.1. For every biflow morphism $H : B \to B'$ and for every interpretation I of X in B' with repect to iH and oH if for every $j \in Y(x,y)$

1) $I(x)H(o(j)) = H(i(j))I(y)$ and

2) $H(i(j))$ is functorial

then there exists a unique biflow morphism $(I,H) : \mathbf{Fl}/\sim \to B'$ such that $F_X(I,H) = I$ and $F_B(I,H) = H$.

Proof. We have proved [Theorem 5.2 in CS87a] that there exists a unique flow morphism $(I,H)^f : \mathbf{Fl} \to B'$ such that $E_X(I,H)^f = I$ and $E_B(I,H)^f = H$. Recall that for every $(x,g) \in \mathbf{Fl}(a,b)$

$$(I,H)^f(x,g) = [(I_{H(a)} + I(x))H(g)]\uparrow^{H(i(x))}$$

If $(x,g) \xrightarrow[j]{} (y,h)$ in $\mathbf{FI}(a,b)$ then using hypothesis 1 we deduce

$$[(I_{H(a)} + I(x))H(g)](I_{H(b)} + H(i(j))) =$$

$$= (I_{H(a)} + I(x))H(g(I_b + i(j))) = (I_{H(a)} + I(x))H((I_a + o(j))h) =$$

$$= (I_{H(a)} + H(i(j))I(y))H(h) = (I_{H(a)} + H(i(j)))[(I_{H(a)} + I(y))H(h)]$$

therefore as $H(i(j))$ is functorial we conclude $(I,H)^f(x,g) = (I,H)^f(y,h)$.

As $(I,H)^f(x,g) = (I,H)^f(y,g)$ whenever $(x,g) \sim (y,h)$ there is a unique flow morphism $(I,H) : \mathbf{FI}/\sim \longrightarrow B'$ such that the composite of the factorization morphism from $\mathbf{FI}$ to $\mathbf{FI}/\sim$ with (I,H) is equal to $(I,H)^f$. The other conclusions easily follows. □

Proposition 10.2. Assume $i : Y \longrightarrow B$ and $o : Y \longrightarrow B$ are xy-ssmc morphisms and I an interpretation of X in B with respect to i and o. If B is a strong xy-ssmc then

$$I(z)o(u) = i(u)I(t) \text{ for every } u \in Y_{xy}(z,t).$$

Proof. Apply Lemma 9.5. □

Theorem 10.3. Assume $i : Y \longrightarrow B$ and $o : Y \longrightarrow B$ are xy-ssmc morphisms. For every xy-flow morphism $H : B \longrightarrow B'$ and for every interpretation I of X in B' with respect to iH and oH there exists a unique xy-ssmc and biflow morphism $(I,H) : \mathbf{FI}/\sim_{xy} \longrightarrow B'$ such that $F_X(I,H) = I$ and $F_B(I,H) = H$.

Proof. To apply Lemma 10.1 for the restrictions of i and o to Y_{xy} we must show hypotheses 1 and 2 hold for every morphism j in Y_{xy}.

As B' is a strong xy-ssmc we apply Proposition 10.2 to show hypothesis 1 holds.

As $H(i(j))$ is an xy-morphism we deduce it is functorial, hence hypothesis 2 holds, too.

Apply Lemma 10.1 and remark the equality $F_B(I,H) = H$ implies (I,H) is an xy-ssmc morphism.

11. Some examples

The abstract theory written in the previous section is used in Sections 13, 15 and 16 to study three classes of flowchart schemes. In Sections 11, 12 and 14 we give examples for the concepts introduced in the previous sections.

Proposition 11.1. In a biflow over a complete matrix theory every morphism in functorial.

Proof. In [CS88a] we have proved in a matrix theory [El76b, CS89a] a morphism $j : a \to b$ is functorial if and only if $fj = jg$ implies $f^*j = jg^*$ for every $f : a \to a$ and $g : b \to b$.

In a complete matrix theory [CS88b] the repetition is defined by $f^* = \bigcup_{n \geq 0} f^n$ for every $f : a \to a$.

Assume $fj = jg$. By induction we deduce $f^n j = jg^n$ for every $n \geq 0$. Therefore

$$f^* j = \bigcup_{n \geq 0} f^n j = \bigcup_{n \geq 0} jg^n = jg^*. \quad \square$$

This proposition shows every morphism is functorial in **Rel**(S) and in $\mathbf{Rel}_S$ as well as in every subbiflow of them, for example **Pfn**(S), $\mathbf{Pfn}_S$, $\mathbf{In}_S$, etc.

Proposition 11.2. Assume T is an algebraic theory (i.e. a strong a δ-ssmc). If B is an sub-ssmc of T such that "st in B implies t in B whenever s and t are composable morphisms of T" then $j \,\mathbf{W}cc\, k$ for every $j \in B(a,b)$ and $k \in B(c,d)$.

Proof. Suppose $f \in B(b + c,u)$, $g \in B(a + d,u)$ and $(j + I_c)f = (I_a + k)g$. As $f = \langle f',f''\rangle$ where $f' \in T(b,u)$ and $f'' \in T(c,u)$, and $g = \langle g',g''\rangle$ where $g' \in T(a,u)$ and $g'' \in T(d,u)$ we deduce $jf' = g'$ and $f'' = kg''$. For $h = \langle f',g''\rangle \in T(b + d,u)$ we get $(I_b + k)h = f$ and $(j + I_d)h = g$. The morphism h is in B as $(I_b + k)h$ is in B. $\square$

11.3. The sixteen sub-ssmc-ies of $\mathbf{Rel}_S$. In [CS89a] we have studied sixteen sub-S*-ssmc-ies of $\mathbf{Rel}_S$ formed by all the morphisms in $\mathbf{Rel}_S$ having the properties given in Table 6.

The properties used for $r \in \mathbf{Rel}_S(a,b)$ are:

T (total) : $(\forall i \in [|a|])(\exists j \in [|b|])\ (i,j) \in r$,

S (surjective) : $(\forall j \in [b])(\exists i \in [a])\ (i,j) \in r$,

P (partial function) : $(\forall i \in [|a|])(\forall j,k \in [|b|])((i,j) \in r$ and $(i,k) \in r$ imply $j = k)$,

I (injective) : $(\forall j,k \in [|a|])(\forall i \in [|b|])((j,i) \in r$ and $(k,i) \in r$ imply $j = k)$.

Name	Properties	A	B	Name	Properties	A	B
$a\alpha\text{-}\mathbf{Rel}_S = \mathbf{Bi}_S$	T,P,S,I	Y	Y	$c\alpha\text{-}\mathbf{Rel}_S = \mathbf{Sur}_S^{-1}$	T,S,I	Y	Y
$a\beta\text{-}\mathbf{Rel}_S = \mathbf{In}_S$	T,P,I	Y	N	$c\beta\text{-}\mathbf{Rel}_S = \mathbf{PSur}_S^{-1}$	T,I	Y	N
$a\gamma\text{-}\mathbf{Rel}_S = \mathbf{Sur}_S$	T,P,S	Y	Y	$c\gamma\text{-}\mathbf{Rel}_S = \mathbf{STRel}_S$	T,S	Y	Y
$a\delta\text{-}\mathbf{Rel}_S = \mathbf{Fn}_S$	T,P	Y	Y	$c\delta\text{-}\mathbf{Rel}_S = \mathbf{TRel}_S$	T	Y	Y
$b\alpha\text{-}\mathbf{Rel}_S = \mathbf{In}_S^{-1}$	P,S,I	N	Y	$d\alpha\text{-}\mathbf{Rel}_S = \mathbf{Fn}_S^{-1}$	S,I	Y	Y
$b\beta\text{-}\mathbf{Rel}_S = \mathbf{PIn}_S$	P,I	N	N	$d\beta\text{-}\mathbf{Rel}_S = \mathbf{Pfn}_S^{-1}$	I	Y	N
$b\gamma\text{-}\mathbf{Rel}_S = \mathbf{PSur}_S$	P,S	N	Y	$d\gamma\text{-}\mathbf{Rel}_S = \mathbf{SRel}_S$	S	Y	Y
$b\delta\text{-}\mathbf{Rel}_S = \mathbf{Pfn}_S$	P	N	Y	$d\delta\text{-}\mathbf{Rel}_S = \mathbf{Rel}_S$		Y	Y

Table 6. Column A : Is xy-$\mathbf{Rel}_S$ weakly cartesian? (Y = yes, N = no)
Column B : Is xy-$\mathbf{Rel}_S$ weakly cocartesian?

Note that xy-$\mathbf{Rel}_S$ is a strong xy-ssmc in which every morphism is an xy-morphism. In Column A of Table 6 there is the answer (Y = yes, N = no) to the question "Is xy-$\mathbf{Rel}_S$ a weakly cartesian xy-ssmc?" and in column B of Table 6 is the answer to the question "Is xy-$\mathbf{Rel}_S$ a weakly cocartesian xy-ssmc?". We give the proofs only for the answers in column B. The proofs for the answers in column A are dual.

Let $x \in \{a,b,c,d\}$. In the cases $x\delta$ we apply Proposition 11.2 for $B = T$. In the cases $x\gamma$ we apply Proposition 11.2 for $T = x\delta\text{-}\mathbf{Rel}_S$. In the cases $x\alpha$ the proof is an easy consequence of the following remark.

If $j \in d\alpha\text{-}\mathbf{Rel}_S(a,b)$, $k \in d\alpha\text{-}\mathbf{Rel}_S(c,d)$, $f \in d\alpha\text{-}\mathbf{Rel}_S(b+c,u)$, $g \in d\alpha\text{-}\mathbf{Rel}_S(a+d,u)$ and $(j + I_c)f = (I_a + k)g$ then

$$h = \langle (I_b + T_c)f, (T_a + I_d)g \rangle \in d\alpha\text{-}\mathbf{Rel}_S(b+d,u)$$

The difficult part of the proof is to show h has property I. Assume $(n,i) \in h$ and $(m,i) \in h$. If $n \leq |b|$ and $m \leq |b|$ then $(n,i) \in f$ and $(m,i) \in f$ hence $n = m$. If $n > |b|$ and $m > |b|$ then $(n - |b| + |a|, i) \in g$ and $(m - |b| + |a|, i) \in g$ hence $n = m$. The other cases lead to a contradiction. Suppose for example $n \leq |b|$ and $m > |b|$. As above we deduce $(n,i) \in f$ and $(m - |b| + |a|, i) \in g$. As j and k have property S there exists $p \in [|a|]$ and $q \in [|c|]$ such that $(p,n) \in j$ and $(q, m - |b|) \in k$. Therefore $(p,i) \in (j + I_c)f$ and $(q + |a|, i) \in (I_a + k)g$. As $(j + I_c)f = (I_a + k)g$ has property I, we deduce $p = q + |a|$, a contradiction.

For the four answers "no" we give the following contraexample: $j = k = T_S$, $f = g = I_S$ and $h = V_S$. $\square$

In [CS89c] we have proved in $\mathbf{Bi}_S$, $\mathbf{In}_S$, $\mathbf{In}_S^{-1}$, $\mathbf{PIn}_S$, $\mathbf{PSur}_S$, $\mathbf{PSur}_S^{-1}$, $\mathbf{Pfn}_S$, $\mathbf{Pfn}_S^{-1}$ and $\mathbf{Rel}_S$ there is only one feedback to make them biflows. As all of them are subbiflow of $\mathbf{Rel}_S$ which is a complete matrix theory we deduce from Proposition 11.1 in all these biflows all the morphisms are functorial.

In conclusion

a) $xy\text{-}\mathbf{Rel}_S$ is an xy-flow for $xy \in \{a\alpha, a\beta, b\alpha, b\beta, b\gamma, b\delta, c\beta, d\beta, d\delta\}$,

b) $xy\text{-}\mathbf{Rel}_S$ is a weakly cartesian xy-flow for $xy \in \{a\alpha, a\beta, c\beta, d\beta, d\delta\}$,

c) $xy\text{-}\mathbf{Rel}_S$ is a weakly cocartesian xy-flow for $xy \in \{a\alpha, b\alpha, b\gamma, b\delta, d\delta\}$.

These examples motivate the neccesity of two variants: Theorem 7.8 and its dual 8.8. In the case $a\beta$ we may use only Theorem 7.8 but in the case $b\gamma$ we may use only Theorem 8.8. In the case $b\beta$ Theorem 7.8 and 8.8 cannot be used. This case will be studied in a forthcoming paper.

11.4. The semantic models are used to interpret statements, therefore they must can be substitute for the xy-flow B' in Theorem 10.3.

Let S be the set of value-vectors denoting the states of the memory in a computing device. Recall that the basic semantic model in the nondeterministic case **Rel**(S) is defined by

$$\mathbf{Rel}(S)(m,n) = \{ r \mid r \subset ([m] \times S) \times ([n] \times S) \} \text{ for } m,n \in N.$$

For more details see [CS87b]. It is a complete matrix theory over the complete semiring $(\mathcal{P}(S \times S), \cup, \emptyset, \cdot, I_S)$.

From Propositions 11.1, 11.2 and their duals we deduce **Rel**(S) is a weakly cartesian and a weakly cocartesian dδ-flow.

Recall that the basic semantic model in the deterministic case **Pfn**(S) is defined by

$$\mathbf{Pfn}(S)(m,n) = \{ f \mid f : [m] \times S \longrightarrow [n] \times S \text{ partial function} \} \text{ for } n,m \in N.$$

As **Pfn**(S) is a subbiflow of **Rel**(S) we deduce **Pfn**(S) is a weakly cocartesian bδ-flow.

12. On wpb-condition (case aβ)

The study of the wpb-condition is difficult. We shall do it in this section only in the aβ-case. Namely we shall prove the following theorem.

Theorem 12.1. Assume that the aβ-ssmc Y fulfills

1) every morphism in Y is an aβ-morphism

2) the monoid of objects of Y is equidivisible.

If B is a weakly cartesian aβ-ssmc such that every aβ-morphism of B is a monomorphism then B fulfills the wpb-condition with respect to Y and i for every aβ-ssmc morphism $i : Y \longrightarrow B$.

We recall that a monoid $(M, +)$ is equidivisible [KS69] if for every $a,b,c,d \in M$

from $a + b = c + d$ one deduce

$$(\exists e)(a = c + e \text{ and } e + b = d) \text{ or } (\exists e)(c = a + e \text{ and } b = e + d).$$

We mention that the free monoids and the groups are equidivisible.

Lemma 12.2. In an equidivisible monoid M if $a_1 + a_2 + \ldots + a_n = b_1 + b_2 + \ldots + b_m$ then there exist $c_1, c_2, \ldots, c_r \in M$ and the integers $0 = i_0 < i_1 < i_2 < \ldots < i_{n-1} < i_n = r$ and $0 = j_0 < j_1 < j_2 < \ldots < j_{m-1} < j_m = r$ such that

$$a_k = c_{i_{k-1}+1} + c_{i_{k-1}+2} + \ldots + c_{i_k} \text{ for every } k \in [n] \text{ and}$$

$$b_k = c_{j_{k-1}+1} + c_{j_{k-1}+2} + \ldots + c_{j_k} \text{ for every } k \in [m]. \quad \square$$

Proposition. 12.3. If the monoid M of the objects of an $a\beta$-ssmc is equidivisible then every $a\beta$-morphism may be written as

$$j(f_1 + f_2 + \ldots + f_n)$$

where j is an $a\alpha$-morphism and f_i is of type I_a or T_a for every $i \in [n]$.

Proof. Remark first that every morphism of type $I_a + {}^bX^c + I_d$ or $I_a + T_b + I_c$ may be easily written as above.

Then we suppose $f = j(f_1 + f_2 + \ldots + f_n)$ where j is an $a\alpha$-morphism and $f_i \in \{I_{a_i}, T_{a_i}\}$ for $i \in [n]$ and we show $f(I_a + {}^bX^c + I_d)$ and $f(I_a + T_b + I_c)$ are of the same type.

a) For $f(I_a + {}^bX^c + I_d)$ as $a_1 + a_2 + \ldots + a_n = a + b + c + d$ there exists $b_1, b_2, \ldots, b_r \in M$, $0 = i_0 < i_1 < i_2 < \ldots < i_{n-1} < i_n = r$ and $1 \leq s < t < u \leq r$ such that

$$a_k = b_{i_{k-1}+1} + b_{i_{k-1}+2} + \ldots + b_{i_k} \text{ for } k \in [n],$$

$$a = b_1 + \ldots + b_s,\ b = b_{s+1} + \ldots + b_t,\ c = b_{t+1} + \ldots + b_u \text{ and } d = b_{u+1} + \ldots + b_r.$$

In the expression of f using B5 in Table 1 and SV2 in Table 3 we may write

$f = j(g_1 + g_2 + \ldots + g_r)$ where $g_k \in \{I_{b_k}, T_{b_k}\}$ for $k \in [r]$. Therefore, denoting by x,y,z and w the source of the morphisms $g_1 + \ldots + g_s$, $g_{s+1} + \ldots + g_t$, $g_{t+1} + \ldots + g_u$ and $g_{u+1} + \ldots + g_r$, respectively we deduce that

$$f(I_a + {}^b X^c + I_d) =$$

$$= j(I_x + {}^y X^z + I_w)(g_1 + \ldots + g_s + g_{t+1} + \ldots + g_u + g_{s+1} + \ldots + g_t + g_{u+1} + \ldots + g_r).$$

b) For $f(I_a + T_b + I_c)$ as $a_1 + a_2 + \ldots + a_n = a + c$ there exists $i \in [n]$ and the objects u and v such that

$$a = a_1 + \ldots + a_{i-1} + u,\ a_i = u + v \text{ and } c = v + a_{i+1} + \ldots + a_n.$$

If $f_i = I_{a_i}$ let $f' = I_u$ and $f'' = I_v$ else $f' = T_u$ and $f'' = T_v$. Therefore $f(I_a + T_b + I_c) = j(f_1 + \ldots + f_{i-1} + f' + T_b + f'' + f_{i+1} + \ldots + f_n)$. □

Proposition 12.4. Suppose B is an $a\beta$-ssmc and Ob(B) is equidivisible. If $u \in B(a,c)$ and $v \in B(b,c)$ are $a\beta$-morphisms then there exists the $a\alpha$-morphisms $p \in B(d + a', a)$, $q \in B(b' + d, b)$ and $j \in B(c, b' + d + a' + r)$ such that

$$puj = T_{b'} + I_{d+a'} + T_{r'},\quad qvj = I_{b'+d} + T_{a'+r} \text{ and } (I_d + T_{a'})pu = (T_{b'} + I_d)qv$$

Proof. First we use the previous proposition to write u and v as a composite of an $a\alpha$-morphism with a sum of morphisms of type I_a or T_a. Using Lemma 12.2 and the identities $I_{a+b} = I_a + I_b$ and $T_{a+b} = T_a + T_b$ we may write $u = f(f_1 + f_2 + \ldots + f_n)$ and $v = g(g_1 + g_2 + \ldots + g_n)$ where f and g are $a\alpha$-morphisms, $\{f_i, g_i\} \subset \{I_{c_i}, T_{c_i}\}$ and $c_1 + c_2 + \ldots + c_n = c$.

Starting from $f^{-1}u = f_1 + f_2 + \ldots + f_n$ and $g^{-1}v = g_1 + g_2 + \ldots + g_n$ we use $a\alpha$-morphisms to permute simultaneously the terms of the two sums to order them in the following way

- at the beginning those that satisfy $f_i = T_{c_i}$ and $g_i = I_{c_i}$,
- then those that satisfy $f_i = g_i = I_{c_i}$,
- then those that satisfy $f_i = I_{c_i}$ and $g_i = T_{c_i}$
- at the end those that satisfy $f_i = g_i = T_{c_i}$.

Then we group, using $I_a + I_b = I_{a+b}$ and $T_a + T_b = T_{a+b}$, the terms of the same type.

Using an induction we may suppose that there exists the aα-morphisms p,q and j such that $puj = T_{b'} + I_{d+a'} + T_r + f_n$ and $qvj = I_{b'+d} + T_{a'+r} + g_n$.

If $f_n = T_{c_n}$ and $g = I_{c_n}$ then $puj(I_{b'} + {}^{d+a'+r}X^{c_n}) = T_{b'+c_n} + I_{d+a'} + T_r$

and $$(I_{b'} + {}^{c_n}X^{d})qvj(I_{b'} + {}^{d+a'+r}X^{c_n}) = I_{(b'+c_n)+d} + T_{a'+r}.$$

If $f_n = g_n = I_{c_n}$ then $(I_d + {}^{c_n}X^{a'})puj(I_{b'+d} + {}^{a'+r}X^{c_n}) = T_{b'} + I_{(d+c_n)+a'} + T_r$

and $$qvj(I_{b'+d} + {}^{a'+r}X^{c_n}) = I_{b'+(d+c_n)} + T_{a'+r}.$$

If $f_n = I_{c_n}$ and $g = T_{c_n}$ then $puj(I_{b'+d+a'} + {}^{r}X^{c_n}) = T_{b'} + I_{d+(a'+c_n)} + T_r$

and $$qvj(I_{b'+d+a'} + {}^{r}X^{c_n}) = I_{b'+d} + T_{(a'+c_n)+r}.$$

If $f_n = g_n = T_{c_n}$ then $puj = T_{b'} + I_{d+a'} + T_{(r+c_n)}$ and $qvj = I_{b'+d} + T_{a'+(r+c_n)}$.

Using the first and the second conclusion we prove the third one.

$(I_d + T_{a'})pu = (T_{b'} + I_d + T_{a'+r})j^{-1} = (T_{b'} + I_d)qv.$ □

Proof of Theorem 12.1. Let $u \in Y(x_1,x')$ and $v \in Y(x_2,x')$. Applying Proposition 12.4 we may write

$$puj = T_z + I_{x+y} + T_{x''},\ qvj = I_{z+x} + T_{y+x''} \text{ and } (I_x + T_y)pu = (T_z + I_x)qv$$

where $p \in Y(x+y, x_1)$, $q \in Y(z + x,x_2)$ and $j \in Y(x',z + x + y + x'')$ are aα-morphisms.

For $u' = (I_x + T_y)p$ and $v' = (T_z + I_x)q$ we deduce $u'u = v'v$.

Assume $f \in B(b,a + i(x_1))$, $g \in B(b,a + i(x_2))$ and $f(I_a + i(u)) = g(I_a + i(v))$. From

$$f(I_a + i(p^{-1}))(I_a + T_{i(z)} + I_{i(x+y)} + T_{i(x'')}) =$$

$$= f(I_a + i(p^{-1}(T_z + I_{x+y} + T_{x''}))) = f(I_a + i(uj)) = g(I_a + i(vj)) =$$

$$= g(I_a + i(q^{-1}))(I_{a+i(z+x)} + T_{i(y+x'')})$$

as $I_{a+i(z+x+y)} + T_{i(x'')}$ is a monomorphism we deduce

$$f(I_a + i(p^{-1}))(I_a + T_{i(z)} + I_{i(x+y)}) = g(I_a + i(q^{-1}))(I_{a+i(z+x)} + T_{i(y)}).$$

As $I_a + T_{i(z)} \mathbf{W} \subset I_{i(x)} + T_{i(y)}$ there exists $h \in B(b, a + i(x))$ such that

$$h(I_{a+i(x)} + T_{i(y)}) = f(I_a + i(p^{-1})) \text{ and } h(I_a + T_{i(z)} + I_{i(x)}) = g(I_a + i(q^{-1}))$$

therefore $f = h(I_a + i(u'))$ and $g = h(I_a + i(v'))$. □

13. Accessible flowchart schemes

In this section we apply our abstract theorems from the first part of the paper to study accessible flowchart schemes.

The (internal) vertices that can be reached by paths going from inputs together with the inputs and the exists from the accessible part of a flowchart scheme. A scheme is said to be accessible if it is equal to its accessible part.

In this section we consider as equal two schemes that have the same accessible part.

For the motivation we work with a flowchart scheme having statements from a set Σ and connections from **Pfn**, i.e. the theory of the finite partial functions. For every $\sigma \in \Sigma$, $i(\sigma)$ and $o(\sigma)$ show the number of the inputs and of the outputs of σ, respectively. The functions $i,o : \Sigma \longrightarrow N$ are extended to monoid morphism $i,o : \Sigma^* \longrightarrow (N,+,0)$. For every nonnegative integers n and m, $f \in \mathbf{Pfn}(n,m)$ if and only if f is a partial function from $[n]$ to $[m]$.

Suppose our scheme is not accessible. We choose a statement σ which is on no path begining with an input of the scheme. Let $y \in \Sigma^*$ be a string containing all the statements τ in the scheme for which there exists at least a path from τ to σ. We mention y contains σ. Let $x \in \Sigma^*$ be a string containing all the statements of the scheme which are not in y. Let $(x + y, h) \in \mathbf{Fl}_{\Sigma,\mathbf{Pfn}}(a,b)$ be a representation of the scheme. From the above choice of y we deduce

- there is no arrow from an input of the scheme to a statement in y,
- there is no arrow from an exit of a statement in x to a statement in y.

The two facts are equivalent to the next property of $h \in \mathbf{Pfn}(a + o(x + y), b + i(x + y))$: $(j,k) \in h$ and $j \in [a + o(x)]$ imply $k \in [b + i(x)]$.

Therefore there exists $f \in \mathbf{Pfn}(a + o(x),\ b + i(x))$ such that $(I_{a+o(x)} + T_{o(y)})h = f + T_{i(y)}$. Remark that $(x,f) \in \mathbf{Fl}_{\Sigma,\mathbf{Pfn}}$ represents the scheme obtained from the initial one eliminating the statements in y and all the arrows which go from a statements in y.

Remark that a scheme is accessible if and only if the eliminations of a group of nonaccessible statements as above cannot be make. We prefer this definition for the concept of accessible scheme as at an abstract level it may be easier formalize (see Definition 13B3) than the definition using paths.

Coming back to the above example we remark in the equality $(I_{a+o(x)} + T_{o(y)})h = f + T_{i(y)}$ the presence of the functions having an empty source which from a technical viewpoint leads to the concept of aβ-ssmc. Extending the morphisms $i,o : \Sigma^* \longrightarrow (N,+,0)$ to the aβ-ssmc morphisms $i,o : \mathbf{In}_\Sigma \longrightarrow \mathbf{Pfn}$ [Theorem 6.4 in CS89a] we remark that the above equality becomes $(I_a + o(I_x + T_y))h = f(I_b + i(I_x + T_y))$, i.e. $(x,f) \longrightarrow_{I_x+T_y} (x + y,h)$. The particular form of this simulation is due to our choice of the representation $(x + y,h)$ where the vertices to eliminate y are isolated. Generally using a bijection $u \in \mathbf{Bi}_\Sigma(x + y, z)$ we may replace the particular representation $(x + y, h)$ of our scheme by an arbitrary one (z,g), i.e. $h(I_b + o(u)) = (I_a + i(u))g$ [CS88b]. For $v = (I_x + T_y)u \in \mathbf{In}_\Sigma(x,z)$ we remark that $f(I_b + o(v)) = (I_a + i(v))g$, i.e. $(x,f) \longrightarrow_v (z,g)$. This comment shows the study of the accessibility may be made using a particualr case of simulation (Definition 4.1) and proves some affirmations from Example 4.2.

A simulation $\longrightarrow_v$ where v in $\mathbf{In}_\Sigma$ is said to be a simulation via injections. Passing to the flow congruence relation generated by the simulation via injections we remark that two flowchart schemes are congruente if and only if they have the same accessible part. For the more difficult implication one use Theorem 12.1 and Proposition 6.1.

13A. Introduction to the algebra of accessibility

The useful algebraic concept to study accessibility is that of weakly cartesian aβ-flow which we name in the sequel inflow. We have no intention to do a deep algebraic study of the inflows as we only are interested in that aspects which are connected to the accessibility.

First aspect we are interested in is the simplication of the definition of the inflow. It is given in Propositions 13A3 and 13A4 below and it is based on the following property.

Lemma 13A1. In an aβ-ssmc B if $f \in B(a,b)$ is an aβ-morphism then there exists an aα-morphism $j \in B(a + c, b)$ such that $f = (I_a + T_c)j$.

Proof. As the aβ-morphisms of B from the least sub-aβ-ssmc of B, it suffices to prove that all the morphisms of type $(I_a + T_b)k$ where k is an aα-morphism from a sub-aβ-ssmc. □

Lemma 13A2. Let $\equiv$ be a congruence relation in a biflow over an aβ-ssmc. If T_u is $\equiv$-functorial for every object u of B then $\equiv$ is aβ-functorial.

Proof. Assume $f \in B(c + a,\ d + a)$, $g \in B(c + b,\ d + b)$, $j \in B_{a\beta}(a,b)$ and $f(I_d + j) \equiv$ $\equiv (I_c + j)g$. Using Lemma 13A1 we may write $j = (I_a + T_u)k$ where k is an aα-morphism, therefore

$$f(I_{d+a} + T_u) \equiv (I_{c+a} + T_u)(I_c + k)g(I_d + k^{-1}).$$

As T_u is $\equiv$-functorial we deduce $f \equiv [(I_c + k)g(I_c + k^{-1})]\uparrow^u$ hence $f\uparrow^a \equiv [(I_c + k)g(I_c + k^{-1})]\uparrow^{a+u} = g\uparrow^b$. □

Recall that in an ssmc the neutral element of the monoid of objects is denoted by e. Remark that an ssmc B is a strong aβ-ssmc if and only if B(e,a) is a singleton for every object a of B.

Proposition 13A3. A biflow B is aβ-flow if and only if for every object a of B there exists a distinguished morphism $T_a \in B(e,a)$ such that

1) $T_e f = T_a$ for every $f \in B(e,a)$
2) T_a is functorial for every object a of B.

Proof. As by hypothesis 1 we have $T_e(T_a + I_e) = (T_a + I_e)I_a$ we deduce using hypothesis 2 that $T_e = \uparrow^a I_a = I_e$. For $f \in B(e,a)$ we deduce $f = I_e f = T_e f = T_a$, hence B is a strong aβ-ssmc. From hypothesis 2 and Lemma 13A2 we deduce every aβ-morphism is functorial. $\square$

Proposition 13A4. An aβ-ssmc B is weakly cartesian if and only if for every $f \in B(a, d + b)$ and $g \in B(a, b + c)$ if $f + T_c = T_d + g$ then there exists $h \in B(a,b)$ such that $f = T_d + h$ and $g = h + T_c$.

Proof. The necessity follows from $T_d \,\mathrm{Wc}\, I_b + T_c$.

To prove the converse we show $j \,\mathrm{Wc}\, k$ for every $j \in B_{a\beta}(a,b)$ and $k \in B_{a\beta}(c,d)$. Using Lemma 13A1 we may write $j = (T_{a'} + I_a)p$ and $k = (I_c + T_{c'})q$ where p and q are aα-morphisms.

Assume $u \in B(m,b + c)$, $v \in B(m, a + d)$ and $u(I_b + k) = v(j + I_d)$. By composition to the right with $p^{-1} + q^{-1}$ we get

$$u(p^{-1} + I_c) + T_{c'} = T_{a'} + v(I_a + q^{-1})$$

therefore using the hypothesis there exists $h \in B(m, a + c)$ such that $u(p^{-1} + I_a) = T_{a'} + h$ and $v(I_a + q^{-1}) = h + T_{c'}$. Therefore $u = h(j + I_c)$ and $v = h(I_a + k)$. $\square$

To apply Theorem 12.1 the next proposition is useful.

Proposition 13A5. In an $a\beta$-flow every $a\beta$-morphism is a monomorphism.

Proof. Let $j \in B_{a\beta}(b,c)$. Suppose $fj = gj$ where $f : a \to b$ and $g : a \to b$. As $j = (T_d + I_b)k$ where $k \in B_{a\alpha}(d + b, c)$ we deduce

$$f(T_d + I_b) = g(T_d + I_b) = (T_d + I_a)(I_d + g).$$

As T_d is functorial we get $f = \uparrow^d(I_d + g) = g$. $\square$

Another aspect we are interested in is the connection to other concepts and the examples.

Proposition 13A6. Every $b\beta$-ssmc is a weakly cartesian $a\beta$-ssmc.

Proof. We use Proposition 13A4. Assume $f : a \to d + b$, $g : a \to b + c$ and $f + T_c = T_d + g$. By composition to the right with $\perp^d + I_b + \perp^c$ we get

$$f(\perp^d + I_b) = g(I_b + \perp^c) : a \to b$$

The next equalities finishes the proof

$$T_d + g(I_b + \perp^c) = (T_d + g)(I_{d+b} + \perp^c) = (f + T_c)(I_{d+b} + \perp^c) = f$$

$$f(\perp^d + I_b) + T_c = (f + T_c)(\perp^d + I_{b+c}) = (T_d + g)(\perp^d + I_{b+c}) = g. \quad \square$$

Proposition 13A7. In a biflow over an $a\gamma$-ssmc if we define

$$\perp^a = \uparrow^a V_a$$

then we get a $b\alpha$-ssmc.

Proof. $\perp^e = \uparrow^e V_e = V_e = I_e$.

$$\perp^a + \perp^b = \perp^a + \uparrow^b V_b = \uparrow^b[({}^bX^a + I_b)(\uparrow^a V_a + V_b)] =$$

$$= \uparrow^b \uparrow^a[(I_a + {}^bX^a + I_b)(V_a + V_b)] = \uparrow^{a+b} V_{a+b} = \perp^{a+b}. \quad \square$$

Theorem 13A8. [Es80, CU82, St87a] Every biflow B over an algebraic theory is an inflow.

Proof. To show B is an $a\beta$-flow we use Proposition 13A3, therefore we only have to show T_a is functorial. If $f \in B(b,c)$, $g \in B(a+b, a+c)$ and $f(T_a + I_c) = (T_a + I_b)g$ then

$$\uparrow^a g = \uparrow^a < (I_a + T_b)g, (T_a + I_b)g > = \uparrow^a < (I_a + T_b)g, f(T_a + I_c) > =$$

$$= f \cdot \uparrow^a < (I_a + T_b)g, T_a + I_c > = f \cdot \uparrow^a [((I_a + T_b)g + T_a + I_c)(I_a + {}^c X^a + I_c)(V_a + V_c)] =$$

$$= f \cdot \uparrow^a [((I_a + T_b)g + I_c)(I_a + V_c)] = f(T_b \cdot \uparrow^a g + I_c)V_c = f.$$

Using Proposition 13A7 we deduce B is an $b\beta$-ssmc, therefore from Proposition 13A6 we get the conclusion. □

From this theorem we deduce $\mathbf{Pfn}_S$, $\mathbf{Rel}_S$, $\mathbf{Pfn}(S)$ and $\mathbf{Rel}(S)$ are inflows. Other examples of inflows are $\mathbf{In}_S$, $\mathbf{PIn}_S$, $\mathbf{PSur}_S^{-1}$ and $\mathbf{Pfn}_S^{-1}$. Among them $\mathbf{In}_S$ has a special place as we can see from the next theorem.

A <u>inflow morphism</u> is by definition a biflow morphism between two inflows. Every inflow morphism is an $a\beta$-ssmc morphism.

Proposition 13A9. Suppose B is an biflow over an $a\beta$-ssmc. If $H : \mathbf{In}_S \longrightarrow B$ is an $a\beta$-ssmc morphism then H is a biflow morphism.

Proof. Let $f \in \mathbf{In}_S(s + a, s + b)$ where $s \in S$. We study three cases.

i) There exists $g \in \mathbf{In}_S(s + a, b)$ such that $f = T_s + g$. We deduce $\uparrow^s f = (T_s + I_a)g$ and

$$\uparrow^{H(s)} H(f) = \uparrow^{H(s)}[H(g)(T_{H(s)} + I_{H(b)})] = \uparrow^e[(T_{H(s)} + I_{H(a)})H(g)] = H(\uparrow^s f).$$

ii) there exists $g \in \mathbf{In}_S(a,b)$ such that $f = I_s + g$. We get $\uparrow^s f = g$ and

$$\uparrow^{H(s)} H(f) = \uparrow^{H(s)}(I_{H(s)} + H(g)) = H(g) = H(\uparrow^s f).$$

iii) There exist $j \in [|a|]$ such that $f(1 + j) = 1$. In this case there exists $i \in [|b|]$ such that $f(1) = 1 + i$, therefore denoting $a = a' + a_j + a''$ with $|a'| = j - 1$ and $b = b' + b_i + b''$ with $|b'| = i - 1$ there exists $g \in \mathbf{In}_S(a' + a'', b' + b'')$ such that

$$f = (I_s + {}^{a'}X^s + I_{a''})({}^sX^s + g)(I_s + {}^sX^{b'} + I_{b''})$$

hence $\uparrow^s f = ({}^{a'}X^s + I_{a''})(I_s + g)({}^sX^{b'} + I_{b''})$ and

$$\uparrow^{H(s)} H(f) =$$

$$= \uparrow^{H(s)}[(I_{H(s)} + {}^{H(a')}X^{H(s)} + I_{H(a'')})({}^{H(s)}X^{H(s)} + H(g))(I_{H(s)} + {}^{H(s)}X^{H(b')} + I_{H(b'')})] =$$

$$= ({}^{H(a')}X^{H(s)} + I_{H(a'')})(I_{H(s)} + H(g))({}^{H(s)}X^{H(b')} + I_{H(b'')}) = H(\uparrow^s f). \quad \square$$

The computation in the above proof may be used to see the feedback in $\mathbf{In}_S$ is unique.

Theorem 13A10. $\mathbf{In}_S$ is an initial object in the category of the S^*-biflows over an $a\beta$-ssmc as well as in the category of the S^*-inflows.

Proof. We use Corollary 6.5 case $a\beta$ in [CS89a] and Proposition 13A9.

13B. Abstract accessible flowchart schemes

We suppose until the end of this subsection X is an equidivisible monoid, Y is an X-$a\beta$-ssmc, B is an inflow, and $i : Y \longrightarrow B$ and $o : Y \longrightarrow B$ are $a\beta$-ssmc morphisms. We require the equidivisibility of X to apply Theorem 12.1.

We have seen in the introduction of Section 13 for concrete flowchart schemes (i.e. X is the free monoid on the set of statements and $B \subset \mathbf{Rel}$) that $(x,f) \sim_{a\beta} (y,g)$ if and only if the schemes represented by (x,f) and (y,g) have the same accessible part.

As by Proposition 13A5 every $a\beta$-morphism from B is a monomorphism we may apply Theorem 12.1 to deduce B fulfills the wpb-condition with respect to $Y_{a\beta}$ and the restriction of i to $Y_{a\beta}$. Therefore from Proposition 6.1 we deduce $\sim_{a\beta} = \xleftarrow{a\beta}\ \xrightarrow{a\beta}$.

We denote by $\mathbf{AFS}_{X,B}$ the quotient of $\mathbf{Fl}_{X,B}$ by $\sim_{a\beta}$. From Theorem 7.8 applied to the restriction of i to $Y_{a\beta}$ we deduce $\mathbf{AFS}_{X,B}$ is an inflow. The morphisms in $\mathbf{AFS}_{X,B}$ are called <u>$a\beta$-schemes</u>.

Let $A_X : X \longrightarrow \mathbf{AFS}_{X,B}$ and $A_B : B \longrightarrow \mathbf{AFS}_{X,B}$ be the composites of $E_X : X \longrightarrow \mathbf{Fl}_{X,B}$ and of $E_B : B \longrightarrow \mathbf{Fl}_{X,B}$ with the factorization morphism from $\mathbf{Fl}_{X,B}$ to $\mathbf{AFS}_{X,B}$, respectively. Remark A_B is an inflow morphism and A_X is an interpretation of X with respect to iA_B and oA_B. The next theorem is an instance of Theorem 10.3.

Theorem 13B1. If $H : B \longrightarrow B'$ is an inflow morphism and if I is an interpretation of X in B' with respect to iH and oH then there exists a unique inflow morphism $(I,H) : \mathbf{AFS}_{X,B} \longrightarrow B'$ such that $A_X(I,H) = I$ and $A_B(I,H) = H$. $\square$

Corollary 13B2. For every S^*-inflow B and for every interpretation I of X in B with respect to i and o there exists a unique S^*-inflow morphism $I^{\#} : \mathbf{AFS}_{X,\mathbf{In}_S} \longrightarrow B$ such that $A_X I^{\#} = I$.

Proof. Apply Theorems 13A10 and 13B1.

Definition 13B3. A representation F from $\mathbf{Fl}$ is said to be <u>accessible</u> if and only if $G \xrightarrow{a\beta} F$ imply $G \sim_{a\alpha} F$. $\square$

As F is accessible and $F \sim_{a\alpha} F'$ imply F' is accessible, we deduce the accessibility is a property of the schemes.

Lemma 13B4. If F is accessible and $F \sim_{a\beta} G$ then $F \xrightarrow{a\beta} G$.

Proof. As $\sim_{a\beta} = \xleftarrow{a\beta} \xrightarrow{a\beta}$ there exists F' such that $F' \xrightarrow{a\beta} F$ and $F' \xrightarrow{a\beta} G$. As F is accessible we deduce $F' \sim_{a\alpha} F$ therefore $F \xrightarrow{a\beta} G$. □

Proposition 13B5. If F and G are accessible and $F \sim_{a\beta} G$ then $F \sim_{a\alpha} G$. □

Proposition 13B6. If $F \in \mathbf{Fl}(a,b)$ is accessible and if $j \in B(b,c)$ fulfills $j\,\mathbf{W}c\,i(Y_{a\beta})$ then Fj is accessible.

Proof. Suppose $G \in \mathbf{Fl}(a,c)$ and $G \xrightarrow{a\beta} Fj$. By Lemma 6.2 there exists $G' \in \mathbf{Fl}(a,b)$ such that $G = G'j$ and $G' \xrightarrow{a\beta} F$. As F is accessible we deduce $G' \sim_{a\alpha} F$ therefore $G \sim_{a\alpha} Fj$. □

Corollary 13B7. If $F \in \mathbf{Fl}(a,b)$ is accessible and $j \in B_{a\beta}(b,c)$ then Fj is accessible. □

Proposition 13B8. If the monoid X is free on a set Σ then the scheme represented by (x,f) is accessible if and only if $(x,f) \sim_{a\beta} (x',f')$ implies $|x| \leq |x'|$.

Proof. Suppose (x,f) is accessible. If $(x,f) \sim_{a\beta} (x',f')$ from Lemma 13B4 we get $(x,f) \xrightarrow{a\beta} (x',f')$, hence $|x| \leq |x'|$.

Conversely, suppose $(x',f') \xrightarrow{a\beta} (x,f)$, i.e. there exists $u \in \mathbf{In}_\Sigma(x',x)$ such that $(x',f') \longrightarrow_u (x,f)$. As $(x',f') \sim_{a\beta} (x,f)$ implies $|x| \leq |x'|$, we deduce $u \in \mathbf{Bi}_\Sigma(x',x)$ hence $(x',f') \sim_{a\alpha} (x,f)$. □

The biflow of flowchart schemes $FS_{X,B}$ is the quotient of **Fl** by $\sim_{a\alpha}$. As $\sim_{a\alpha} \subset \sim_{a\beta}$ there exists a unique biflow morphism

$$\mathbf{AP}_{X,B} : \mathbf{FS}_{X,B} \longrightarrow \mathbf{AFS}_{X,B}$$

such that $E_X^{a\alpha}\mathbf{AP}_{X,B} = A_X$ and $E_B^{a\alpha}\mathbf{AP}_{X,B} = A_B$. The morphism $\mathbf{AP}_{X,B}$ is called "Accessible Part" as it maps every concrete flwchart scheme in its accessible part. Proposition 13B5 tell us that if two accessible schemes have the same accessible part (i.e. the same image by $\mathbf{AP}_{X,B}$) then they are equal. In other words in every coset of the kernel of $\mathbf{AP}_{X,B}$ there is at most one accessible flowchart scheme. The next proposition shows for concrete schemes that in every coset of the kernel of $\mathbf{AP}_{X,B}$ there exists one accessibe flowchart sheme, therefore in the concrete cases the $a\beta$-schemes and the accessible schemes coincide.

Proposition 13B9. If the monoid X is free and if $B \subset \mathbf{Rel}_S$ then for every $F \in \mathbf{Fl}(a,b)$ there exists $G \in \mathbf{Fl}(a,b)$ such that G is accessible and $F \sim_{a\beta} G$.

Proof. To determine the accessible part of $F = (x,f)$ we may procced in the following way. We forget the sorts and the exits of the scheme, we identify all the inputs of the scheme in one input, we identify the inputs of every statements in one input and we identify the outputs of every statements in one output to obtain the relation

$$h = (\Lambda_{|a|} + \Sigma_{k\in[|x|]}\Lambda_{|o(x_k)|})f(\perp^{|b|} + \Sigma_{k\in[|x|]}V^{|i(x_k)|}) \in \mathbf{Rel}(1 + |x|,|x|),$$

where Λ_n and V^n are define by induction $\Lambda_0 = \perp^1$, $\Lambda_{n+1} = \Lambda(I_1 + \Lambda_n)$; $V^0 = T_1$, $V^{n+1} = (I_1 + V^n)V$. The image of the relation $(h\Lambda^{|x|})\uparrow^{|x|}$ gives the accessible part of F. $\square$

Proposition 13B10. Suppose X is a free monoid on a set Σ and $B \subset \mathbf{Rel}_S$. If $F \in \mathbf{Fl}(a,b)$ is accessible and if $F \xrightarrow{a\gamma} G$ then G is accessible.

Proof. Using the same notation as above as $F = (x,f)$ is accessible we deduce $(h\Lambda^{|x|})\uparrow^{|x|} = \Lambda_{|x|}$. As $F \xrightarrow{a\gamma} G = (y,g)$ there exists $u \in \mathbf{Sur}_{\Sigma}(x,y)$ such that $f(I_b + i(u)) = (I_a + o(u))g$. As we forget the sorts all the computations we make are in **Rel.** First remark that

$$u(\Sigma_{k\in[|y|]}\Lambda_{|o(y_k)|}) = (\Sigma_{k\in[|x|]}\Lambda_{|o(x_k)|})o(u) \quad \text{and}$$

$$(\Sigma_{k\in[|x|]}\vee^{|i(x_k)|})u = i(u)(\Sigma_{k\in[|y|]}\vee^{|i(y_k)|}).$$

For $h' = (\Lambda_{|a|} + \Sigma_{k\in[|y|]}\Lambda_{|o(y_k)|})g(\perp^{|b|} + \Sigma_{k\in[|y|]}\vee^{|i(y_k)|})$ we deduce

$$(I_1 + u)h' = (\Lambda_{|a|} + \Sigma_{k\in[|x|]}\Lambda_{|o(x_k)|})(I_a + o(u))g(\perp^{|b|} + \Sigma_{k\in[|y|]}\vee^{|i(y_k)|}) =$$

$$= (\Lambda_{|a|} + \Sigma_{k\in[|x|]}\Lambda_{|o(x_k)|})f(I_b + i(u))(\perp^{|b|} + \Sigma_{k\in[|y|]}\vee^{|i(y_k)|}) = hu$$

therefore

$$h\Lambda^{|x|}(u + I_{|x|})(I_{|y|} + u) = hu\Lambda^{|y|} = (I_1 + u)h'\Lambda^{|y|}.$$

As u is functorial we deduce

$$(h'\Lambda^{|y|})\uparrow^{|y|} = (h\Lambda^{|x|}(u + 1_{|x|}))\uparrow^{|x|} = (h\Lambda^{|x|})\uparrow^{|x|}\cdot u = \Lambda_{|x|}u = \Lambda_{|y|}$$

hence G is accessible.

13C. Three characterizations of $\sim_{a\beta}$-equivalence

In [CS88b, Proposition 7.6] we have proved $\sim_{a\alpha}$ is the least congruence relation satisfying (XX) in Proposition 9.4. In this subsection we give analogous characterizations for $\sim_{a\beta}$.

Assume $i : Y \longrightarrow B$ and $o : Y \longrightarrow B$ are $a\beta$-ssmc morphisms and B is an inflow. As usual $X = Ob(Y)$.

Proposition 13C1. If X is equidivisible then $\sim_{a\beta}$ is the least $a\beta$-functorial congruence relation in **Fl** satisfying (XX) and (TX).

Proof. As $\mathbf{AFS}_{X,B}$ is an inflow we deduce from Remark 9.7 that $\sim_{a\beta}$ is $a\beta$-functorial. For the remainder we use Propositions 9.4 and 9.8.

Lemma 13C2. If a congruence relation $\equiv$ in $\mathbf{Fl}$ has the property (XX) and

$$(x,f) \longrightarrow_{I_x+T_y} (x + y, g) \text{ implies } (x,f) = (x + y, g)$$

then $\equiv$ includes $\sim_{a\beta}$.

Proof. It suffices to show $\equiv$ includes $\xrightarrow{a\beta}$. Assume $(x,f) \longrightarrow_u (y,g)$ in $\mathbf{Fl}(a,b)$ where u is in $Y_{a\beta}$. Using Lemma 13A1 we may write $u = (I_x + T_z)k$ where $k \in Y_{a\alpha}(x + z, y)$. For $h = (I_a + o(k))g(I_b + i(k^{-1}))$ we deduce $(x,f) \longrightarrow_{I_x+T_z} (x + z, h)$ and $(x + z, h) \longrightarrow_k (y,g)$. By hypothesis $(x,f) = (x + z, h)$. As $=$ fulfills (XX) we get $\equiv$ includes $\sim_{a\alpha}$, therefore $(x + z, h) \equiv (y,g)$, hence $(x,f) = (y,g)$. $\square$

Proposition 13C3. $\sim_{a\beta}$ is the least congruence relation $\equiv$ in $\mathbf{Fl}$ fulfilling (XX) and

$$(x,f) \longrightarrow_{I_x+T_y} (x + y, g) \text{ implies } (x,f) \equiv (x + y, g). \quad \square$$

Proposition 13C4. If B is an inflow oven an algebraic theory then $\sim_{a\beta}$ is the least congruence relation $\equiv$ in $\mathbf{Fl}$ with the properties (XX) and $T_a \equiv (x,f)$ for every (x,f) in $\mathbf{Fl}(e,a)$.

Proof. It is easy to see that $\sim_{a\beta}$ has the above properties.

Let $\equiv$ be a congruence relation in $\mathbf{Fl}$ having the above properties. To show $\equiv$ includes $\sim_{a\beta}$ we use Lemma 13C2 therefore we have to prove its hypothesis.

Suppose $(x,f) \rightarrow_{I_x+T_y} (x + y,g)$ in $\mathbf{Fl}(a,b)$. By hypothesis

$$T_{b+i(x)} \equiv [(T_{a+o(x)} + y)g]\uparrow^{i(y)}.$$

Adding to the left $(I_a + x)f$ we deduce

$$(I_a + x)(f + T_{b+i(x)}) \equiv [(I_a + x + y)(f + (T_{a+o(x)} + I_{o(y)})g)]\uparrow^{i(y)}$$

Composing to the right with $V_{b+i(x)}$ we deduce

$$(I_a + x)f \equiv [(I_a + x + y)(f + (T_{a+o(x)} + I_{o(y)})g)(V_{b+i(x)} + I_{i(y)})]\uparrow^{i(y)} =$$

$$= [(I_a + x + y)(f + T_{i(y)} + (T_{a+o(x)} + I_{o(y)})g)V_{b+i(x+y)}]\uparrow^{i(y)}.$$

As $f + T_{i(y)} = (I_{a+o(x)} + T_{o(y)})g$ we deduce

$$(I_a + x)f \equiv [(I_a + x + y) < (I_{a+o(x)} + T_{o(y)})g, (T_{a+o(x)} + I_{o(y)})g >]\uparrow^{i(y)} =$$

$$= [(I_a + x + y)g]\uparrow^{i(y)}.$$

Applying $\uparrow^{i(x)}$ we get $(x,f) \equiv (x + y, g)$. $\square$

14. On wpo-condition (cases aγ and aδ)

The study of the wpo-condition is more difficult than the study of the wpb-condition owing to the pushouts which have a more complicated construction than the pullbacks. To overcome this difficulty we suppose the monoid of statements is free and even more hypotheses in case aγ which is more difficult than case aδ.

The study of case aδ is made using the duals of conditions wpb_{1-3} in Section 6.

We begin to study the dual of wpb_1 for cases aγ and aδ.

The concept of (weak) pushout is the dual of the concept of (weak) pullback. We use the same notation as in Section 8.

It is known that in the category of sets, denoted **Set**, there exist pushouts. For $p : B \longrightarrow D$ and $q : C \longrightarrow D$, we mention that (f,g) **Po** (p,q) in **Set** implies

A) $(\forall d \in D)[(\exists b \in B)p(b) = d$ or $(\exists c \in C)\, q(c) = d]$.

To prove that the pushouts exist in $\mathbf{Sur}_S$ and in $\mathbf{Fn}_S$ we recall an old proposition from the theory of the categories.

Assume C is a category and $S \in Ob(C)$. The definition of the comma category $C \downarrow S$ is:

- $(A,a) \in Ob(C \downarrow S) \Longleftrightarrow A \in Ob(C)$ and $a \in C(A,S)$,
- $C \downarrow S((A,a),(B,b)) = \{ f \in C(A,B) \mid fb = a \}$,
- composition in $C \downarrow S$ is induced by the composition in C.

Proposition 14.1. Assume C is a category having pushouts and E is the subcategory of its epimorphisms. For every $S \in Ob(C)$ the comma categories $C \downarrow S$ and $E \downarrow S$ have pushouts and the forgetful functors from $E \downarrow S$ to $C \downarrow S$ and from $C \downarrow S$ to C preserve the pushouts.

Proof. Suppose $f \in C \downarrow S((A,a),(B,b))$ and $g \in C \downarrow S((A,a),(R,r))$.

As C has pushouts there exist $p \in C(B,D)$ and $q \in C(R,D)$ such that $(f,g)\,\mathbf{Po}\,(p,q)$ in C. As $fb = a = gr$ there exists $d \in C(D,S)$ such that $pd = b$ and $qd = r$, therefore $p \in C \downarrow S((B,b),(D,d))$ and $q \in C \downarrow S((R,r),(D,d))$.

We prove $(f,g)\,\mathbf{Po}\,(p,q)$ in $C \downarrow S$. Assume $u \in C \downarrow S((B,b),(E,e))$, $v \in C \downarrow S((R,r),(E,e))$ and $fu = gv$. As $(f,g)\,\mathbf{Po}\,(p,q)$ in C there exists a unique $h \in C(D,E)$ such that $ph = u$ and $qh = v$. As $phe = ue = b = pd$ and $qhe = ve = r = qd$ we deduce $he = d$ therefore $h \in C \downarrow S((D,d),(E,e))$. Hence $C \downarrow S$ has pushouts and the forgetful functor from $C \downarrow S$ to C preserves them.

To get the other conclusion, keeping the above notation we remark that if f and g are epimorphisms then p and q are epimorphisms and if u and v are epimorphisms then h is an epimorphism. $\square$

Corollary 14.2. The categories $\mathbf{Sur}_\Sigma$ and $\mathbf{Fn}_\Sigma$ have pushouts. The forgetful functors from $\mathbf{Sur}_\Sigma$ to $\mathbf{Fn}_\Sigma$ and from $\mathbf{Fn}_\Sigma$ to $\mathbf{Set}$ preserve the pushouts. $\square$

The next proposition covers the dual of wpb_2 in case $a\delta$.

Proposition 14.3. If T is an algebraic theory then every a δ-ssmc morphism $o : \mathbf{Fn}_{\Sigma} \longrightarrow T$ preserves the pushouts.

Proof. We assume $(p,q)\,\mathbf{Po}\,(p',q')$ where $p \in \mathbf{Fn}_{\Sigma}(x,y)$, $q \in \mathbf{Fn}_{\Sigma}(x,z)$, $p' \in \mathbf{Fn}_{\Sigma}(y,x')$ and $q' \in \mathbf{Fn}_{\Sigma}(z,x')$ and we prove

$$(o(p), o(q))\,\mathbf{Po}\,(o(p'), o(q')).$$

For $y \in \Sigma^*$ and $i \in [|y|]$ we use the notation

$$x_i^y = T_{y_1+\ldots+y_{i-1}} + I_{y_i} + T_{y_{i+1}+\ldots+y_{|y|}}.$$

Suppose $f \in T(o(y),a)$, $g \in T(o(z),a)$ and $o(p)f = o(q)g$. We define the functions u and v by

$$u(i) = o(x_i^y)f \quad \text{for } i \in [|y|],$$

$$v(j) = o(x_j^z)g \quad \text{for } j \in [|z|]$$

and we remark that $pu = qv$. Indeed for $i \in [|x|]$

$$u(p(i)) = o(x^y_{p(i)})f = o(x_i^x)o(p)f = o(x_i^x)o(q)g = o(x^z_{q(i)})g = v(q(i)).$$

As by Corollary 14.2 we get $(p,q)\,\mathbf{Po}\,(p',q')$ in **Set** there exists a unique function w defined on $[|x'|]$ such that $p'w = u$ and $q'w = v$.

Denote $h = \langle w(1), w(2), \ldots, w(|x'|)\rangle$. We show $h \in T(o(x'), a)$. For every $k \in [|x'|]$ as $(p,q)\,\mathbf{Po}\,(p',q')$ in **Set** we deduce from A there exists $i \in [|y|]$ such that $p'(i) = k$ or there exists $j \in [|z|]$ such that $q'(j) = k$. We deduce in the first case $w(k) = w(p'(i)) = u(i) \in T(o(y_i), a) = T(o(x'_k), a)$ and in the second case $w(k) = w(q'(j)) = v(j) \in T(o(z_j), a) = T(o(x'_k), a)$, hence $h \in T(o(x'), a)$.

For $i \in [|y|]$ we deduce $o(x_i^y)o(p')h = o(x^{x'}_{p'(i)})h = w(p'(i)) = u(i) = o(x_i^y)f$ therefore $o(p')h = f$ and for $j \in [|z|]$ we deduce $o(x_j^z)o(q')h = o(x^{x'}_{q'(j)})h = w(q'(j)) = v(j) = o(x_j^z)g$ therefore $o(q')h = g$. Hence $(o(p), o(q))\,\mathbf{Wpo}\,(o(p'), o(q'))$.

To prove the uniqueness of h suppose $h' \in T(o(x'), a)$, $o(p')h' = f$ and $o(q')h' = g$. It suffices to show $o(x^{x'}_k)h' = w(k)$ for every $k \in [|x'|]$. If $k = p'(i)$ where $i \in [|y|]$ then

$$o(x^{x'}_k)h' = o(x^{x'}_{p'(i)})h' = o(x_i^y)o(p')h' = o(x_i^y)f = u(i) = w(p'(i)) = w(k)$$

and if $k = q'(j)$ where $j \in [iz\ell]$ then

$$o(x_k^{x'})h' = o(x_{q'(j)}^{x'})h' = o(x_j^z)o(q')h' = o(x_j^z)g = v(j) = w(q'(j)) = w(k). \quad \square$$

The next proposition is useful in the two cases. In case $a\delta$ it covers the dual of wpb_3.

Proposition 14.4. Assume T is an algebraic theory and P is a sub-ssmc of T such that fg in P implies g in P for every pair of composable morphisms f and g from T. If the inclusion functor from P to T preserves the (weak) pushouts then the functor $I_a + \_ : P \longrightarrow P$ preserves the (weak) pushouts for every object a of T.

Proof. Assume $(p,q)\,\mathbf{Wpo}\,(p',q')$ in P where $p \in P(b,c)$, $q \in P(b,c')$, $p' \in P(c,d)$ and $q' \in P(c',d)$ and we prove $(I_a + p, I_a + q)\,\mathbf{Wpo}\,(I_a + p', I_a + q')$ in P.

Suppose $f \in P(a + c,\ d')$, $g \in P(a + c',\ d')$ and $(I_a + p)f = (I_a + q)g$, therefore $(I_a + T_c)f = (I_a + T_{c'})g$ and $p(T_a + I_c)f = q(T_a + I_{c'})g$. As $(p,q)\,\mathbf{Wpo}\,(p',q')$ in T there exists $w \in T(d,d')$ such that $p'w = (T_a + I_c)f$ and $q'w = (T_a + I_{c'})g$. For $h = \langle (I_a + T_c)f, w\rangle \in T(a + d,\ d')$ we deduce

$$(I_a + p')h = \langle (I_a + T_c)f,\ (T_a + I_c)f\rangle = f \text{ and}$$

$$(I_a + q')h = \langle (I_a + T_{c'})g,\ (T_a + I_{c'})g\rangle = g.$$

As $(I_a + p')h$ is in P we deduce $h \in P(a + d,\ d')$. $\square$

Corollary 14.5. The functor $I_a + \_ : \mathbf{Sur}_S \longrightarrow \mathbf{Sur}_S$ preserves the (weak) pushouts for every $a \in S^*$.

Proof. Apply Proposition 14.4 for $P = \mathbf{Sur}_S$ and $T = \mathbf{Fn}_S$.

Theorem 14.6. (case $a\delta$). If T is an algebraic theory and if the Σ^*-$a\delta$-ssmc Y fulfills $Y = Y_{a\delta}$ then T fulfills the wpo-condition with respect to Y and o for every a δ-ssmc morphism $o : Y \longrightarrow T$.

Proof. As $\mathbf{Fn}_\Sigma$ is the initial Σ^*-aδ-ssmc there exists a unique Σ^*-aδ-ssmc morphism $O : \mathbf{Fn}_\Sigma \longrightarrow Y$.

Suppose $u \in Y(x,x_1)$ and $v \in Y(x,x_2)$. As every morphism in Y is an aδ-morphism by Proposition 3.2 there exists $u_1 \in \mathbf{Fn}_\Sigma(x,x_1)$ and $v_1 \in \mathbf{Fn}_\Sigma(x,x_2)$ such that $u = O(u_1)$ and $v = O(v_1)$. By Corollary 14.2 there exist $p \in \mathbf{Fn}_\Sigma(x_1,x')$ and $q \in \mathbf{Fn}_\Sigma(x_2,x')$ such that $(u_1,v_1)\,\mathbf{Po}\,(p,q)$.

For $u' = O(p)$ and $v' = O(q)$ we deduce $uu' = vv'$.

From Proposition 14.3 applied for $Oo : \mathbf{Fn}_\Sigma \longrightarrow T$ we deduce $(o(u), o(v))\,\mathbf{Po}\,(o(u'), o(v'))$. From Proposition 14.4 for $P = T$ we deduce $(I_a + o(u), I_a + o(v))\,\mathbf{Po}\,(I_a + o(u'), I_a + o(v'))$. $\square$

Theorem 14.7. (case aγ). Assume B is an aγ-ssmc and Y is a Σ^*-aγ-ssmc such that $Y_{a\gamma} = Y$. If for every aγ-ssmc morphism $G : \mathbf{Sur}_\Sigma \longrightarrow B$ and for every $a \in Ob(B)$

$$(u,v)\,\mathbf{Po}\,(u',v') \text{ in } \mathbf{Sur}_\Sigma \text{ implies } (I_a + G(u), I_a + G(v))\,\mathbf{Wpo}\,(I_a + G(u'), I_a + G(v'))$$

then B fulfills the wpo-condition with respect to Y and o for every aγ-ssmc morphism $o : Y \longrightarrow B$.

Proof. Assume $H : \mathbf{Sur}_\Sigma \longrightarrow Y$ is the unique Σ^*-aγ-ssmc morphism.

Suppose $u \in Y(x,x_1)$ and $v \in Y(x,x_2)$. As $Y_{a\gamma} = Y$ by Proposition 3.2 there exist $u_1 \in \mathbf{Sur}_\Sigma(x,x_1)$ and $v_1 \in \mathbf{Sur}_\Sigma(x,x_2)$ such that $H(u_1) = u$ and $H(v_1) = v$. By Corollary 14.2 there exist $p \in \mathbf{Sur}_\Sigma(x_1,x')$ and $q \in \mathbf{Sur}_\Sigma(x,x')$ such that $(u_1,v_1)\,\mathbf{Po}\,(p,q)$.

For $u' = H(p)$ and $v' = H(q)$ we get $uu' = vv'$.

Applying the hypothesis for $G := Ho$ we get

$(I_a + o(u), I_a + o(v))\, \mathbf{Wpo}\, (I_a + o(u'), I_a + o(v'))$. $\square$

15. Reduced Flowchart Schemes

In this section we apply our abstract theorems from the first part of the paper to study reduced flowchart schemes. As this concept is not as well known as the concept of accessible flowchart scheme we give some explanations.

To <u>reduce</u> a scheme we identify interval vertices which are labeled by the same statements which have coherent continuations, i.e. the arrows going from the same output of two statements which are identified must go to the same output of the scheme or to the same input of two statements which are identified. The simplest example of <u>reduction</u> is $(x + x)V_{o(x)} \longrightarrow V_{i(x)}x$. Remark that by reduction the behaviour of the scheme does not change.

For concrete schemes (statements in a set Σ and connection in **Rel**) we show that $(x,f) \in \mathbf{Fl}_{\Sigma,\mathbf{Rel}}(a,b)$ can be reduced to $(y,g) \in \mathbf{Fl}_{\Sigma,\mathbf{Rel}}(a,b)$ if and only if there exists $u \in \mathbf{Sur}_{\Sigma}(x,y)$ such that $(x,f) \longrightarrow_u (y,g)$.

Let $(x,f) \in \mathbf{Fl}_{\Sigma,\mathbf{Rel}}(a,b)$. To identify vertices having common labels we may use an equivalence relation $\equiv$ on $[|x|]$ such that $j \equiv k$ implies $x_j = x_k$, or equivalently but more useful in the sequel we may use a surjection $u \in \mathbf{Sur}_{\Sigma}(x,y)$ to identify the vertices j and k if and only if $u(j) = u(k)$. For $j \in [|x|]$ we use the notation

$$h_j = T_{o(x_1+\ldots+x_{j-1})} + I_{o(x_j)} + T_{o(x_{j+1}+\ldots+x_{|x|})}.$$

Remark that v and w in $[|b + i(x)|]$ become equal after identification if and only if $(I_b + i(u))(v) = (I_b + i(u))(w)$. To understand this the following lemma is useful.

Lemma 15.1. Assume $o : \mathbf{Sur}_{\Sigma} \longrightarrow \mathbf{Sur}_S$ is an aγ-ssmc morphism where Σ and S are sets. If $u \in \mathbf{Sur}_{\Sigma}(x,y)$, $i \in [|x|]$ and $j \in [|o(x_i)|]$ then

$$o(u)(\Sigma_{k\in[i-1]}|o(x_k)| + j) = \Sigma_{k\in[u(i)-1]}|o(y_k)| + j.$$

Proof. By Theorem 3.1 in [CS89a] we may write $u = g\Sigma_{r\in[|y|]} V_{y_r}^{n_r}$ where g is in $\mathbf{Bi}_\Sigma$ and $n_r \geq 1$ for $r \in [|y|]$. As $g(i) = \Sigma_{r\in[u(i)-1]} n_r + q$ where $q \in [n_{u(i)}]$ from Corollary 2.4 in [CS89a] we deduce

$$o(g)(\Sigma_{k\in[i-1]}|o(x_k)| + j) = \Sigma_{r\in[u(i)-1]} n_r|o(y_r)| + (q-1)|o(y_{u(i)})| + j$$

therefore

$$o(u)(\Sigma_{k\in[i-1]}|o(x_k)| + j) =$$

$$= (\Sigma_{r\in[|y|]} V_{o(y_r)}^{n_r})(\Sigma_{r\in[u(i)-1]} n_r|o(y_r)| + (q-1)|o(y_{u(i)})| + j) =$$

$$= \Sigma_{r\in[u(i)-1]}|o(y_r)| + V_{o(y_{u(i)})}^{n_{u(i)}}((q-1)|o(y_{u(i)})| + j) =$$

$$= \Sigma_{r\in[u(i)-1]}|o(y_r)| + j. \quad \Box$$

The identification may be made only if the identified statements have coherent continuations, that is we must have

I. $(T_a + h_j)f(I_b + i(u)) = (T_a + h_k)f(I_b + i(u))$ whenever $u(j) = u(k)$.

This condition is equivalent to

II. $(T_a + o(uu'))f(I_b + i(u)) = (T_a + I_{o(x)})f(I_b + i(u))$

for every $u' \in \mathbf{In}_\Sigma(y,x)$ such that $u'u = I_y$.

We prove I implies II. Assume $u' \in \mathbf{In}_\Sigma(y,x)$ and $u'u = I_y$. For $j \in [|x|]$ as $u((uu')(j)) = u(j)$ we deduce from I that

$$(T_a + h_{(uu')(j)})f(I_b + i(u)) = (T_a + h_j)f(I_b + i(u))$$

therefore as by Lemma 15.1 $h_j o(uu') = h_{uu'(j)}$ we get

$$h_j(T_a + o(uu'))f(I_b + i(u)) = h_j(T_a + I_{o(x)})f(I_b + i(u)).$$

Hence II is proved.

We prove II implies I. Assume $u(j) = u(k)$. We may choose $u' \in \mathbf{In}_{\Sigma}(y,x)$ such that $u'u = I_y$ and $u'(u(k)) = j$. Form II we deduce composing to the left with h_k

$$(T_a + h_k o(uu'))f(I_b + i(u)) = (T_a + h_k)f(I_b + i(u))$$

therefore $(T_a + h_{uu'(k)})f(I_b + i(u)) = (T_a + h_k)f(I_b + i(u))$ hence I.

As $(I_a + T_{o(x)})(I_a + o(uu'))f(I_b + i(u)) = (I_a + T_{o(x)})f(I_b + i(u))$ we deduce the identified statements have coerent continuations if and only if

III. $(I_a + o(uu'))f(I_b + i(u)) = f(I_b + i(u))$ for every $u' \in \mathbf{In}_{\Sigma}(y,x)$ such that $u'u = I_y$.

The reduction of (x,f) is (y,g) where $g = (I_a + o(u'))f(I_b + i(u))$, $u' \in \mathbf{In}_{\Sigma}(y,x)$ and $u'u = I_y$. It does not depend on u' as if $u'' \in \mathbf{In}_{\Sigma}(y,x)$ and $u''u = I_y$ we deduce

$$(I_a + o(u''))f(I_b + i(u)) = (I_a + o(u''))(I_a + o(uu'))f(I_b + i(u)) = g.$$

Moreover as $(I_a + o(u))g = (I_a + o(uu'))f(I_b + i(u)) = f(I_b + i(u))$ we deduce that $(x,f) \longrightarrow_u (y,g)$.

Suppose $u \in \mathbf{Sur}_{\Sigma}(x,y)$ and $(x,f) \longrightarrow_u (y,g)$. We deduce for every $u' \in \mathbf{In}_{\Sigma}(y,x)$ such that $u'u = I_y$ that

$$(I_a + o(u'))f(I_b + i(u)) = (I_a + o(u'u))g = g,$$

therefore as III holds we may use u to reduce (x,f) and its reduction is (y,g).

By definition a scheme represented by (x,f) is said to be reduced if $(x,f) \longrightarrow_u (y,g)$ and $u \in \mathbf{Sur}_{\Sigma}(x,y)$ imply $u \in \mathbf{Bi}_{\Sigma}(x,y)$.

15A. Introduction to the algebra of reduction

The algebraic structure we use to study reduction, called surflow, consists in a weakly cocartesian aγ-flow B such that for every set S, for every aγ-ssmc morphism $G : \mathbf{Sur}_S \to B$ and for every $a \in Ob(B)$

$$(u,v)\,\mathbf{Po}\,(u',v') \text{ in } \mathbf{Sur}_S \Longrightarrow (I_a + G(u), I_a + G(v))\,\mathbf{Wpo}\,(I_a + G(u'), I_a + G(v')).$$

Proposition 15A1. Suppose P is a sub-aγ-ssmc of an algebraic theory T such that fg in P implies g in P for every pair of composable morphisms f and g from T. For every set S, every aγ-ssmc morphism $G : \mathbf{Sur}_S \to P$ and every $a \in Ob(T)$

$$(u,v)\,\mathbf{Po}\,(u',v') \text{ in } \mathbf{Sur}_S \Longrightarrow (I_a + G(u), I_a + G(v))\,\mathbf{Po}\,(I_a + G(u'), I_a + G(v'))$$

Proof. Assume $F : \mathbf{Fn}_S \to T$ is the unique aδ-ssmc morphism such that $F(x) = G(x)$ for every $x \in S^*$. As the restriction of F to $\mathbf{Sur}_S$ is equal to the composite of G with the inclusion of P in T we deduce $F(f) = G(f)$ for every morphism f in $\mathbf{Sur}_S$.

Suppose $(f,g)\,\mathbf{Po}\,(p,q)$ where $f \in \mathbf{Sur}_S(x,x')$, $g \in \mathbf{Sur}_S(x,x'')$, $p \in \mathbf{Sur}_S(x',y)$ and $q \in \mathbf{Sur}_S(x'',y)$. Using Corollary 14.2 and Proposition 14.3 we deduce $(F(f),F(g))\,\mathbf{Po}\,(F(p),F(q))$. By Proposition 14.4 we get $(I_a + G(f), I_a + G(g))\,\mathbf{Po}\,(I_a + G(p), I_a + G(q))$ in T. If $u \in P(a + G(x'), b)$, $v \in P(a + G(x''),b)$ and $(I_a + G(f))u = (I_a + G(g))v$ then there exists a unique $h \in T(a + G(y),b)$ such that $(I_a + G(p))h = u$ and $(I_a + G(q))h = v$. Note that from hypothesis we get h is in P. ☐

Corollary 15A2. Let Σ and S be sets. If $G : \mathbf{Sur}_\Sigma \to \mathbf{Sur}_S$ is an aγ-ssmc morphism then G preserves the pushouts. ☐

Proposition 15A3. If B is a biflow over an algebraic theory and if every aγ-morphism is functorial then B is a surflow.

Proof. As every algebraic theory is a strong aγ-ssmc we deduce B is an aγ-flow. The conclusion follows from Propositions 11.2 and 15A1. ☐

From this proposition we deduce $\mathbf{Rel}(S)$, $\mathbf{Pfn}(S)$, $\mathbf{Rel}_S$ and $\mathbf{Pfn}_S$ are surflows. Another example of surflow is $\mathbf{PSur}_S$ as follows from Subsection 11.3 and Proposition 15A1.

A <u>surflow morphism</u> is by definition a biflow morphism between two surflows which is also an aγ-ssmc morphism.

Proposition 15A4. In an aγ-flow B if we define $\perp^a = \uparrow^a V_a$ then B becomes a bγ-ssmc such that

$$\uparrow^a V_a^{n+1} = \perp^{na} \quad \text{for} \quad n \geq 1.$$

Proof. By Proposition 13A7 B is a bα-ssmc.

We show $p \perp^b = \perp^a$ for every aγ-morphism $p : a \to b$. Using Theorem 3.7 we deduce $V_a p = (p + I_a)(I_b + p)V_b$. As p is functorial we obtain $\uparrow^a V_a = \uparrow^b((I_b + p)V_b)$ hence $\perp^a = p \perp^b$.

Therefore $V_b \perp^b = \perp^{b+b} = \perp^b + \perp^b$ hence B is a bγ-ssmc.

The last conclusion is proved by inductin on n:

$\uparrow^a V_a^1 = \uparrow^a I_a = I_e = \perp^e = \perp^{0a}$ and

$\uparrow^a V_a^{n+1} = \uparrow^a[(I_a + V_a^n)V_a] = V_a^n \perp^a = \perp^{na}$. $\square$

If $H : B \to B'$ is a surflow morphism then for $\perp^a$ defined as above H becomes a bγ-ssmc morphism.

Remark in $\mathbf{PSur}_S$ that $\perp^a = \uparrow^a V_a$ is an identify.

Theorem 15A5. Let B a surflow. If $H : \mathbf{PSur}_S \to B$ is a bγ-ssmc morphism then H is a surflow morphism.

Proof. It suffices to show H preserves the scalar feedback. Assume $f \in \mathbf{PSur}_S(s + a, s + b)$ where $s \in S$.

If $f = \perp^s + g$ where $g \in \mathbf{PSur}_S(a,s+b)$ then $\uparrow^s f = g(\perp^s + I_b)$ and

$$\uparrow^{H(s)}H(f) = \uparrow^{H(s)}((\perp^{H(s)} + I_{H(a)})H(g)) = \uparrow^e(H(g)(\perp^{H(s)} + I_{H(b)})) = H(\uparrow^s f).$$

If $f(1) = 1$ using the standard representation of f [CS89a] we may write

$$f = (I_s + w)(I_s + g)(V_s^n + h)$$

where w is in $\mathbf{In}_S^{-1}$, g is in $\mathbf{Bi}_S$, $n \geq 1$ and h is in $\mathbf{Sur}_S$, therefore $\uparrow^s f = wg(\perp^{(n-1)s} + h)$ hence

$$\uparrow^{H(s)}H(f) = H(wg)(\uparrow^{H(s)}V^n_{H(s)} + H(h)) = H(wg)(\perp^{(n-1)H(s)} + H(h)) = H(\uparrow^s f).$$

In the other cases we use again the standard representation of f to write

$$f = (I_s + w)g(V_s^n + \Sigma_{i \in [|b|]}p_i)$$

where w is in $\mathbf{In}_S^{-1}$, g is in $\mathbf{Bi}_S$, $g(1) > n \geq 1$ and $p_i = V_{b_i}^{n_i}$ with $n_i \geq 1$ for $i \in [|b|]$. As $g(1) = n + \Sigma_{j \in [i-1]}n_j + 1$ where $i \in [|b|]$ there exists h in $\mathbf{Bi}_S$ such that

$$g = (I_s + {}^cX^s + I_d)(I_{2s} + h)({}^sX^{ns+b'} + I_{b''})$$

where $b' = \Sigma_{j \in [i-1]}n_j b_j$ and $b'' = (n_i - 1)s + n_{i+1}b_{i+1} + \ldots + n_{|b|}b_{|b|}$. Using the notation $q = w({}^cX^s + I_d)(I_s + h)$, $p' = \Sigma_{j \in [i-1]}p_j$, $p'' = p_{i+1} + \ldots + p_{|b|}$ and $u = \Sigma_{j \in [i-1]}b_j$ we deduce

$$f = (I_s + q)[{}^sX^{ns+b'}(V_s^n + p' + I_s) + I_{b''}](I_{s+u} + V_s^{n_i} + p'') =$$

$$= (I_s + q)[(I_s + V_s^n + p')^sX^{s+u} + I_{b''}](I_{s+u} + V_s^{n_i} + p'')$$

therefore

$$\uparrow^s f = q[(V_s^n + p')^sX^u + I_{b''})](I_u + V_s^{n_i} + p'')$$

hence

$$\uparrow^{H(s)}H(f) = H(q)[H(V_s^n + p')(\uparrow^{H(s)}\,{}^{H(s)}X^{H(s+u)}) + H(I_{b''})]H(I_u + V_s^{n_i} + p'') =$$

$$= H(q)[(V^n_{H(s)} + H(p'))^{H(s)}X^{H(u)} + H(I_{b''})]H(I_u + V_s^{n_i} + p'') = H(\uparrow^s f). \quad \Box$$

The computations in the above proof may be used to see the feedback in $\mathbf{PSur}_S$ is unique.

Theorem 15A6. $\mathbf{PSur}_S$ is the initial S^*-surflow.

Proof. We use Corollary 6.5 case bγ in [CS89a] and Theorem 15A5.

15B. Abstract Reduced Flowchart Schemes

We suppose until the end of this section that X is a free monoid on a set Σ, Y is an X-aγ-ssmc, B is a surflow, and $i : Y \longrightarrow B$ and $o : Y \longrightarrow B$ are aγ-ssmc morphisms.

By Theorem 14.7 B fulfills the wpo-condition with respect to $Y_{a\gamma}$ and the restriction of o to $Y_{a\gamma}$. From Proposition 8.1 we deduce $\sim_{a\gamma} = \xrightarrow{a\gamma} \xleftarrow{a\gamma}$.

We denote by $\mathbf{RFS}_{X,B}$ the quotient of $\mathbf{Fl}_{X,B}$ by $\sim_{a\gamma}$. From Theorem 8.8 applied to the restriction of o to $Y_{a\gamma}$ we deduce $\mathbf{RFS}_{X,B}$ is a weakly cocartesian aγ-flow. The morphisms in $\mathbf{RFS}_{X,B}$ are called <u>aγ-schemes.</u>

Let $R_X : X \longrightarrow \mathbf{RFS}_{X,B}$ and $R_B : B \longrightarrow \mathbf{RFS}_{X,B}$ be the composites of $E_X : X \longrightarrow \mathbf{Fl}_{X,B}$ and of $E_B : B \longrightarrow \mathbf{RFS}_{X,B}$ with the factorization morphism from $\mathbf{Fl}_{X,B}$ to $\mathbf{RFS}_{X,B}$, respectively. Remark R_B is an aγ-flow morphism and R_X is an interpretation of X in $\mathbf{RFS}_{X,B}$ with respect to iR_B and oR_B.

Proposition 15B1. $\mathbf{RFS}_{X,B}$ is a surflow.

Proof. We only have to show the last condition in the definition of the surflows. Suppose $G : \mathbf{Sur}_S \to \mathbf{RFS}_{X,B}$ is an $a\gamma$-ssmc morphism, $a \in Ob(B)$ and $(u,v)\,\mathbf{Po}\,(u',v')$ where $u \in \mathbf{Sur}_S(b,c')$, $v \in \mathbf{Sur}_S(b,c'')$, $u' \in \mathbf{Sur}_S(c',d)$ and $v' \in \mathbf{Sur}_S(c'',d)$. We have to show

$$(I_a + G(u), I_a + G(v))\,\mathbf{Wpo}\,(I_a + G(u'), I_a + G(v')).$$

Let $F : \mathbf{Sur}_S \to B$ be the unique $a\gamma$-ssmc morphism such that $F(b) = G(b)$ for every $b \in S^*$. Remark that $FR_B = G$.

Assume $(I_a + F(u))(y,g) \sim_{a\gamma} (I_a + F(v))(z,h)$ where $(y,g) \in \mathbf{Fl}(a + G(c'),d')$ and $(z,h) \in \mathbf{Fl}(a + G(c''),d')$. As $\sim_{a\gamma} = \xrightarrow{a\gamma} \xleftarrow{a\gamma}$ there exists $(x,f) \in \mathbf{Fl}(a + G(b),d')$ such that

$$(I_a + F(u))(y,g) \xrightarrow{a\gamma} (x,f) \text{ and } (I_a + F(v))(z,h) \xrightarrow{a\gamma} (x,f).$$

Applying Lemma 8.2 twice, there exist $(x,f') \in \mathbf{Fl}(a + G(c'),d')$ and $(x,f'') \in \mathbf{Fl}(a + G(c''),d')$ such that

$$(y,g) \xrightarrow{a\gamma} (x,f'),\ (z,h) \xrightarrow{a\gamma} (x,f'') \text{ and } (I_a + F(u))(x,f') = (x,f) = (I_a + F(v))(x,f'').$$

As B is a surflow we deduce

$$(I_a + F(u) + I_{o(x)}, I_a + F(v) + I_{o(x)})\,\mathbf{Wpo}\,(I_a + F(u') + I_{o(x)}, I_a + F(v') + I_{o(x)})$$

therefore as $(I_a + F(u) + I_{o(x)})f' = (I_a + F(v) + I_{o(x)})f''$ there exists $r \in B(a + F(d) + o(x), d' + i(x))$ such that

$$(I_a + F(u') + I_{o(x)})r = f' \text{ and } (I_a + F(v') + I_{o(x)})r = f''.$$

We deduce $(x,r) \in \mathbf{Fl}(a + F(d),d')$, $(I_a + F(u'))(x,r) = (x,f')$ and $(I_a + F(v'))(x,r) = (x,f'')$ therefore

$$(y,g) \xrightarrow{a\gamma} (I_a + F(u'))(x,r) \text{ and } (z,h) \xrightarrow{a\gamma} (I_a + F(v'))(x,r)$$

hence

$$(I_a + F(u'))(x,r) \sim_{a\gamma} (y,g) \text{ and } (I_a + F(v'))(x,r) \sim_{a\gamma} (z,h). \quad \square$$

The next theorem is an instance of Theorem 10.3.

Theorem 15B2. If $H : B \longrightarrow B'$ is a surflow morphism and if I is an interpretation of X in B' with respect to iH and oH then there exists a unique surflow morphism $(I,H) : \mathbf{RFS}_{X,B} \longrightarrow B'$ such that $R_X(I,H) = I$ and $R_B(I,H) = H$. $\square$

Corollary 15B3. For every S^*-surflow B and for every interpretation I of X in B with respect to i and o there exists a unique S^*-surflow morphism $I^{\#} : \mathbf{RFS}_{X,\mathbf{PSur}_S} \longrightarrow B$ such that $R_X I^{\#} = I$.

Proof. Apply Theorems 15A6 and 15B2. $\square$

Definition 15B4. A representation F from $\mathbf{Fl}_{X,B}$ is said to be <u>reduce</u> if and only if $F \xrightarrow{a\gamma} G$ implies $F \sim_{a\gamma} G$.

As F is reduced and $F \sim_{a\alpha} F'$ imply F' is reduced, we deduce the reduction is a property of the schemes.

Lemma 15B5. If F is reduced and if $F \sim_{a\gamma} G$ then $G \xrightarrow{a\gamma} F$.

Proposition 15B6. If F and G are reduced and $F \sim_{a\gamma} G$ then $F \sim_{a\alpha} G$.

Proposition 15B7. If $j \in B_{a\gamma}(c,b)$ and if $F \in \mathbf{Fl}_{X,B}(b,a)$ is reduced then jF is reduced. $\square$

Proposition 15B8. If B is a surflow over an algebraic theory, $j \in B(a,b)$ and $F \in \mathbf{Fl}_{X,B}(b,c)$ is reduced then jF is reduced.

Proof. Suppose $jF \xrightarrow{a\gamma} G$. By Proposition 11.2 and Lemma 8.2 there exists H such that $G = jH$ and $F \xrightarrow{a\gamma} H$. As F is reduced we get $F \sim_{a\alpha} H$ hence $jF \sim_{a\alpha} G$. $\square$

Proposition 15B9. The scheme represented by (x,f) is reduce if and only if $(x,f) \sim_{a\gamma} (y,g)$ implies $|x| \leq |y|$.

Proof. If (x,f) is reduced we get $(y,g) \xrightarrow{a\gamma} (x,f)$ by Lemma 15B5, therefore $|x| \leq |y|$.

Conversely, suppose $(x,f) \xrightarrow{a\gamma} (y,g)$, i.e. there exists $u \in \mathbf{Sur}_{\Sigma}(x,y)$ such that $(x,f) \longrightarrow_u (y,g)$. As $(x,f) \sim_{a\gamma} (y,g)$ implies $|x| \leq |y|$ we deduce $u \in \mathbf{Bi}_{\Sigma}(x,y)$ hence $(x,f) \sim_{a\gamma} (y,g)$.

15C. A characterization of $\sim_{a\gamma}$-equivalence

15C1. Theorem. $\sim_{a\gamma}$ is the least $a\gamma$-functorial congruence relation in **Fl** satisfying (XX) and (VX).

Proof. As $RFS_{X,B}$ is a surflow we deduce from Remark 9.7 that $\sim_{a\gamma}$ is a γ-functorial. For the remainder we use Proposition 9.4 and 9.8. □

16. Minimal flowchart schemes (with respect to the input-behaviour)

For the motivation of this Section we sent to the introduction of Section 7 in [CS87b] and to [El77] where the simulation by functions was introduced.

16A. Introduction to the algebra of minimization

The algebraic structure used in this section is an $a\delta$-flow, which is called in the sequel a <u>funflow.</u> The concept of strong iteration theory was introduced in [St87a]. In [CS88a] it is proved that the concepts of funflow and strong iteration theory coincide. As examples of funflow we mention **Pfn**(S), **Rel**(S), $\mathbf{Pfn}_S$ and $\mathbf{Rel}_S$.

Lemma 16A1. In an $a\delta$-ssmc B if $f \in B_{a\delta}(a,b)$ then there exists $p \in B_{a\gamma}(c+a, b)$ such that $f = (T_c + I_a)p$.

Proof. As $B_{a\delta}$ is the least sub-aδ-ssmc of B, it suffices to prove that all the morphisms of type $(T_c + I_a)p$ where p is $B_{a\gamma}$ form a sub-aδ-ssmc of B. $\square$

Lemma 16A2. In an aδ-ssmc B if $p \in B_{a\gamma}(a,b)$ there exists $u \in B_{a\beta}(b,a)$ such that $up = I_b$.

Proof. If $p = I_a + {}^bX^c + I_d$ or $p = I_a + V_b + I_c$ we take $u = I_a + {}^cX^b + I_d$ or $u = I_a + T_b + I_{b+c}$, respectively. If $p = p_1 p_2 \dots p_n$ where every p_i is of one of the above types then we take u_i as above and $u = u_n \dots u_2 u_1$. $\square$

Proposition 16A3. B is a funflow if and only if B is a biflow over an algebraic theory such that every aγ-morphism is functorial.

Proof. On implication is obvious. Suppose B is a biflow over an algebraic theory such that every aγ-morphism is functorial. We have to show every aδ-morphism is functorial. Suppose $f \in B(a + c,\ a + d)$, $g \in B(b + c,\ b + d)$, $v \in B_{a\delta}(a,b)$ and $f(v + I_d) = (v + I_c)g$. By Lemma 16A1 we may write $v = (T_r + I_a)p$ where $p \in B_{a\gamma}(r + a, b)$. By Lemma 16A2 there exists $u \in B_{a\beta}(b, r + a)$ such that $up = I_b$. Let

$$h = \langle (I_r + T_{a+c})(p + I_c)g(u + I_d),\ f(T_r + I_{a+d})\rangle \in B(r + a + c, r + a + d).$$

As $(T_r + I_{a+c})h = f(T_r + I_{a+d})$ and as by Theorem 13A8 $T_r + I_a$ is functorial we deduce $\uparrow^a f = \uparrow^{r+a} h$. As

$$h(p + I_d) = \langle (I_r + T_{a+c})(p + I_c)g,\ f(v + I_d)\rangle =$$

$$= \langle (I_r + T_{a+c})(p + I_c)g,\ (T_r + I_{a+c})(p + I_c)g\rangle = (p + I_c)g$$

we deduce from hypothesis $\uparrow^{r+a} h = \uparrow^b g$. Hence $\uparrow^a f = \uparrow^b g$. $\square$

Proposition 16A4. Every funflow is an inflow and a surflow. If B is an inflow and a surflow such that $(T_a + I_a)V_a = I_a$ for every $a \in Ob(B)$ then B is a funflow.

Proof. The first statement follows from Theorem 13A8 and Proposition 15A3.

The last conclusion follows from Proposition 16A3 and the following remark: if C is a strong aβ-ssmc and a strong aγ-ssmc such that $(T_a + I_a)V_a = I_a$ for every $a \in Ob(C)$ then C is an algebraic theory. □

Proposition 16A5. In a funflow B if we define $\perp^a = \uparrow^a V_a$ then B becomes a bδ-ssmc.

Proof. Using Proposition 15A4.

Theorem 16A6. If B is a funflow and $H : \mathbf{Pfn}_S \longrightarrow B$ is a bδ-ssmc morphism there H is a funflow morphism.

Proof. It suffice to show H preserves the scalar feedback. Suppose $f \in \mathbf{Pfn}_S(s + a, s + b)$ where $s \in S$.

If $f = \perp^s + g$ where $g \in \mathbf{Pfn}_S(a, s + b)$ then $\uparrow^s f = g(\perp^s + I_b)$ hence

$$\uparrow^{H(s)} H(f) = \uparrow^{H(s)}((\perp^{H(s)} + I_{H(a)})H(g)) = \uparrow^e(H(g)(\perp^{H(s)} + I_{H(b)})) = H(\uparrow^s f).$$

If $f = T_s + g$ where $g \in \mathbf{Pfn}_S(s + a, b)$ then $\uparrow^s f = (T_s + I_a)g$ hence

$$\uparrow^{H(s)} H(f) = \uparrow^{H(s)}(H(g)(T_{H(s)} + I_{H(b)})) = \uparrow^e((T_{H(s)} + I_{H(a)})H(g)) = H(\uparrow^s f).$$

In the other cases using the standard representation [CS89a] of f we may write

$$f = (I_s + w)g(V_s^n + \Sigma_{i \in [|b|]} p_i)$$

where w is in $\mathbf{In}_S^{-1}$, g is in $\mathbf{Bi}_S$, $n \geq 1$ and $p_i = V_{b_i}^{n_i}$ where $n_i \geq 0$ for $i \in [|b|]$. We study two cases.

a) If $g(1) = 1$ then $g = I_s + h$ therefore $\uparrow^s f = wh(\perp^{(n-1)s} + \Sigma p_i)$ hence

$$\uparrow^{H(s)} H(f) = H(wh)(\uparrow^{H(s)} V^n_{H(s)} + H(\Sigma p_i)) =$$

$$= H(wh)(\perp^{(n-1)H(s)} + H(\Sigma p_i)) = H(\uparrow^s f).$$

b) If $g(1) > 1$ we deduce from the properties of the standard representations there exists $i \in [|b|]$ such that $n_i \geq 1$ and $g(1) = n + \sum_{j \in [i-1]} n_j + 1$, therefore there exists h in $\mathbf{Bi}_S$ such that

$$g = (I_s + {}^cX^s + I_d)(I_{2s} + h)({}^sX^{ns+b'} + I_{b''})$$

where $b' = \sum_{j \in [i-1]} n_j b_j$ and $b'' = (n_i - 1)s + n_{i+1} b_{i+1} + \ldots + n_{|b|} b_{|b|}$.
Using the notation $q = w({}^cX^s + I_d)(I_s + h)$, $p' = \sum_{j \in [i-1]} p_j$, $p'' = p_{i+1} + \ldots + p_{|b|}$ and $u = b_1 + \ldots + b_{i-1}$ we deduce

$$f = (I_s + q)({}^sX^{ns+b'}(V_s^n + p' + I_s) + I_{b''})(I_{s+u} + V_s^{n_i} + p'') =$$

$$= (I_s + q)((I_s + V_s^n + p')^s X^{s+u} + I_{b''})(I_{s+u} + V_s^{n_i} + p'')$$

therefore

$$\uparrow^s f = q((V_s^n + p')^s X^u + I_{b''})(I_u + V_s^{n_i} + p'')$$

hence

$$\uparrow^{H(s)} H(f) = H(q)(H(V_s^n + p')(\uparrow^{H(s)}\, {}^{H(s)}X^{H(s+u)} + H(I_{b''}))H(I_u + V_s^{n_i} + p'') =$$

$$= H(q)(H(V_s^n + p')^{H(s)}X^{H(u)} + H(I_{b''}))H(I_u + V_s^{n_i} + p'') = H(\uparrow^s f). \quad \square$$

The computations in the above proof may be used to see the feedback in $\mathbf{Pfn}_S$ is unique.

Corollary 16A7. $\mathbf{Pfn}_S$ is the initial S^*-funflow.

16B. Abstract minimal flowchart schemes

We suppose in the sequel X is a free monoid on a set Σ, Y is an X-aδ-ssmc, B is a funflow, and $i : Y \longrightarrow B$ and $o : Y \longrightarrow B$ are aδ-ssmc morphisms.

Theorem 14.6 shows B fulfills the wpo-condition with respect to $Y_{a\delta}$ and the restriction of o to $Y_{a\delta}$. From Proposition 8.1 we deduce $\sim_{a\delta} = \xrightarrow{a\delta} \xleftarrow{a\delta}$.
Remark that $\sim_{a\beta} \subset \sim_{a\delta}$ and $\sim_{a\gamma} \subset \sim_{a\delta}$.

We denote $\mathbf{MFS}_{X,B}$ the quotient of $\mathbf{Fl}_{X,B}$ by $\sim_{a\delta}$. As by Proposition 11.2 every funflow is weakly cocartesian we may apply Theorem 8.8 to deduce $\mathbf{MFS}_{X,B}$ is a funflow. The morphisms in $\mathbf{MFS}_{X,B}$ are called <u>aδ-schemes.</u>

Let $M_X : X \longrightarrow \mathbf{MFS}_{X,B}$ and $M_B : B \longrightarrow \mathbf{MFS}_{X,B}$ be the composites of $E_X : X \longrightarrow \mathbf{Fl}_{X,B}$ and $E_B : B \longrightarrow \mathbf{Fl}_{X,B}$ with the factorization morphism from $\mathbf{Fl}_{X,B}$ to $\mathbf{MFS}_{X,B}$, respectively. Remark that M_B is a funflow morphism and M_X is an interpretation of X with respect to iM_B and oM_B.

The next theorem is an instance of Theorem 10.3.

Theorem 16B1. If $H : B \longrightarrow B'$ is a funflow morphism and if I is an interpretation of X in B' with respect to iH and oH then there exists a unique funflow morphism $(I,H) : \mathbf{MFS}_{X,B} \longrightarrow B'$ such that $M_X(I,H) = I$ and $M_B(I,H) = H$.

Corollary 16B2. For every S^*-funflow B and for every interpretation I of X in B with respect to i and o there exists a unique S^*-funflow morphism

$$I^{\#} : \mathbf{MFS}_{X,\mathbf{Pfn}_S} \longrightarrow B$$

such that $M_X I^{\#} = I$.

Proof. As by Corollary 16A7 $\mathbf{Pfn}_S$ is the initial S^*-funflow there exists a unique S^*-funflow morphism $H : \mathbf{Pfn}_S \longrightarrow B$. It suffice to apply the above theorem. □

Proposition 16B3. $\xrightarrow{a\delta} \subset \xrightarrow{a\gamma} \xrightarrow{a\beta}$.

Proof. Suppose (x,f) and (y,g) in $\mathbf{Fl}(a,b)$, $u \in Y_{a\delta}(x,y)$ and $f(I_b + i(u)) = (I_a + o(u))g$.

We prove $u = vw$ where $v \in Y_{a\gamma}(x,z)$ and $w \in Y_{a\beta}(z,y)$. Assume $F : \mathbf{Fn}_\Sigma \longrightarrow Y$ is the unique Σ^*-aδ-ssmc morphism. By Proposition 3.2 there exists u' in $\mathbf{Fn}_\Sigma$ such that $u = F(u')$. As there exists $v' \in \mathbf{Sur}_\Sigma(x,z)$ and $w' \in \mathbf{In}_\Sigma(z,y)$ such that $u' = v'w'$ for $v = F(v') \in Y_{a\gamma}(x,z)$ and $w = F(w') \in Y_{a\beta}(z,y)$ we obtain $u = vw$.

We show $r \in Y_{a\delta}(x,x)$ and $rv = v$ imply

$$(I_a + o(r))f(I_b + i(v)) = f(I_b + i(v)).$$

From

$$(I_a + o(r))f(I_b + i(v))(I_b + i(w)) = (I_a + o(r))f(I_b + i(u)) = (I_a + o(r))(I_a + o(u))g =$$
$$= (I_a + o(rvw))g = (I_a + o(u))g = f(I_b + i(v))(I_b + i(w))$$

as $I_b + i(w)$ is a monomorphism (cf. Propositions 16A4 and 13A5) we deduce the above equality.

As $v \in Y_{a\gamma}(x,z)$ by lemma 16A2 there exists $j \in Y_a(z,x)$ such that $jv = 1_z$. As $(vj)v = v$ we deduce

$$(I_a + o(vj))f(I_b + i(v)) = f(I_b + i(v)).$$

For $h = (I_a + o(j))f(I_b + i(v)) \in B(a + o(z), b + i(z))$ we deduce $(z,h) \in \mathbf{Fl}(a,b)$ and $(I_a + o(v))h = f(I_b + i(v))$ therefore $(x,f) \xrightarrow{a\gamma} (z,h)$. As B is a surflow from $(v',v')\,\mathbf{Po}\,(I_z,I_z)$ we deduce applying Fo that $(I_a + o(v), I_a + o(v))\,\mathbf{Wpo}\,(I_{a+o(z)}, I_{a+o(z)})$ hence $I_a + o(v)$ is an epimorphism. From

$$(I_a + o(v))h(I_b + i(w)) = f(I_b + i(vw)) = (I_a + i(u))g = (I_a + o(v))(I_a + o(w))g$$

as $I_a + o(v)$ is an epimorphism we deduce $h(I_b + i(w)) = (I_a + o(w))g$ hence $(z,h) \xrightarrow{a\beta} (y,g)$.

Theorem 16B4. $\sim_{a\delta} = \xrightarrow{a\gamma}\xleftarrow{a\beta}\ \xrightarrow{a\beta}\xleftarrow{a\gamma}$.

Proof. As an inclusion is obvious we prove the other one. As $\sim_{a\delta} = \xrightarrow{a\delta}\xleftarrow{a\delta}$ we get from Proposition 16B3 $\sim_{a\delta} \subset \xrightarrow{a\gamma}\ \xrightarrow{a\beta}\xleftarrow{a\beta}\xleftarrow{a\gamma}$. At the beginning of subsection 13B we proved $\sim_{a\beta} = \xleftarrow{a\beta}\ \xrightarrow{a\beta}$. As $\xrightarrow{a\beta}\xleftarrow{a\beta} \subset \sim_{a\beta} = \xleftarrow{a\beta}\ \xrightarrow{a\beta}$ we deduce $\sim_{a\delta} \subset \xrightarrow{a\gamma}\xleftarrow{a\beta}\ \xrightarrow{a\beta}\xleftarrow{a\gamma}$. □

Definition 16B5. A representation F from **Fl** is said to be <u>minimal</u> if F is accessible and reduced.

As F is minimal and $F \sim_{a\alpha} G$ imply G is minimal the minimality is a propoerty of the schemes.

Lemma 16B6. If F is minimal and if F $\sim_{a\delta}$ F' then F $\xrightarrow{a\beta}\xleftarrow{a\gamma}$ F'.

Proof. Using Theorem 16B4 there exist F_1 and F_2 such that F $\xrightarrow{a\gamma}$ $F_1 \xleftarrow{a\beta}$ $F_2 \xrightarrow{a\beta}\xleftarrow{a\gamma}$ F'. As F is reduced from F $\xrightarrow{a\gamma}$ F_1 we deduce F $\sim_{a\alpha}$ F_1 therefore F $\xleftarrow{a\beta}$ $F_2 \xrightarrow{a\beta}\xleftarrow{a\gamma}$ F'. As F is accessible from F $\xleftarrow{a\beta}$ F_2 we deduce F $\sim_{a\alpha}$ F_2 therefore F $\xrightarrow{a\beta}\xleftarrow{a\gamma}$ F'. $\square$

Proposition 16B7. If F and F' are minimal then F $\sim_{a\delta}$ F' implies F $\sim_{a\alpha}$ F'.

Proof. From Lemma 16B6 we deduce F $\xrightarrow{a\beta}$ F" $\xleftarrow{a\gamma}$ F'. As F' is reduce we get F" $\sim_{a\alpha}$ F' therefore F $\xrightarrow{a\beta}$ F'. As F' is accessible we get F $\sim_{a\alpha}$ F'.

Proposition 16B8. The scheme represented by (x,f) is minimal if and only if (x,f) $\sim_{a\delta}$ (x',f') implies $|x| \leq |x'|$.

Proof. If (x,f) is minimal and if (x,f) $\sim_{a\delta}$ (x',f') we get (x,f) $\xrightarrow{a\beta}\xleftarrow{a\gamma}$ (x',f') therefore $|x| \leq |x'|$.

Conversely, suppose (x,f) $\sim_{a\delta}$ (x',f') implies $|x| \leq |x'|$. To show (x,f) is accessible and reduces we use the same proofs as in Propositions 13B8 and 15B9.

16C. A characterization of $\sim_{a\delta}$-equivalence

We assume the same hypotheses as in Section 16B.

Proposition 16C1. $\sim_{a\delta}$ is the least $a\delta$-functorial congruence relation in **Fl** such that (XX), (TX) and (VX) hold.

Proof. As $MFS_{X,B}$ is a funflow we deduce from Remark 9.7 $\sim_{a\delta}$ is a δ-functorial. For the remainder we use Propositions 9.4 and 9.8.

16D. A comment on iteration theories

Theorem 16B1 shows that a correct interpretation of an aδ-scheme may be given in a funflow. If we restrict the class of schemes to those over an S^*-funflow T satisfying:

(p) for every $f \in T(a,b+c)$ there exists $f^\circ \in T(a,b+c+d)$ such that

(i) there exists $u \in \mathbf{Fn}_S(d,c)$ with $f = f^\circ(I_b + \langle I_c, u\rangle)$,

(ii) for every $g \in T(a,b+c+d)$ and $v \in \mathbf{Sur}_S(c+d,c')$ such that $f^\circ(I_b + v) = g(I_b + v)$ there exists $v' \in \mathbf{Fn}_S(c+d,c+d)$ with $v'v = v$ and $g = f^\circ(I_b + v')$

then Theorem 16B1 may be made a bit stronger: a correct interpretation may be given in all iteration theories.

As in the sequel we use a right iteration $\_^\dagger : B(a,b+a) \to B(a,b)$ we recall some computation rules. From [CS88a] we know a biflow over an algebraic theory is equivalent to an algebraic theory where an iteration is defined and satisfies some axioms. Moreover if $f : b \to c + a$ and $g : a \to c + a$ then

$$\langle f,g\rangle\uparrow^a = f\langle I_c, g^\dagger\rangle.$$

If $H : T \to B$ is an S^*-biflow morphism and I is an interpretation of X in B with respect to iH and oH then the behaviour of $(x,f) \in \mathbf{Fl}_{X,T}(a,b)$ is

$$((I_a + I(x))H(f))\uparrow^{i(x)} = H((I_a + T_{o(x)})f)\langle I_b, (I(x)H((T_a + I_{o(x)})f))^\dagger\rangle.$$

An <u>iteration S-sorted algebraic theory</u> may be defined as a biflow over an S-sorted algebraic theory satisfying Ésik's commutativity axiom

$$u(f(I_c + u))^\dagger = \langle x^b_{u(1)} f(I_c + u_1), x^b_{u(2)} f(I_c + u_2), \ldots, x^b_{u(|a|)} f(I_c + u_{|a|})\rangle^\dagger$$

where $f : b \to c + a$, $u \in \mathbf{Sur}_S(a,b)$ and $u_i \in \mathbf{Fn}_S(a,a)$ satisfies $u_i u = u$ for every $i \in [|a|]$. (We recall that $x^b_j = T_{b_1+\ldots+b_{j-1}} + I_{b_j} + T_{b_{j+1}+\ldots+b_{|b|}}$ for every $j \in [|b|]$.)

Proposition 16D1. Suppose Σ is a one-ranked alphabet, i.e. $i(\sigma) \in S$ for every $\sigma \in \Sigma$, and T is an S^*-biflow over a strong a δ-ssmc satisfying (p). Assume B is an iteration S-sorted algebraic theory, $H : T \to B$ is an S^*-biflow morphism, and I is an interpretation of Σ in B with respect to iH and oH. If F and G in $\mathbf{Fl}_{\Sigma,T}(a,b)$ are similar via a surjection v then F and G have the same behaviour in B.

Proof. Suppose $F = (x,\langle f',f\rangle)$ and $G = (y,\langle g',g\rangle)$ where $x,y \in \Sigma^*$, $f' \in T(a,b + i(x))$, $f \in T(o(x), b + i(x))$, $g' \in T(a,b + i(y))$ and $g \in T(o(y), b + i(y))$. Assume $f = \langle f_1,f_2,\ldots,f_{|x|}\rangle$ where $f_k \in T(o(x_k), b + i(x))$ for $k \in [|x|]$ and $g = \langle g_1,g_2,\ldots,g_{|y|}\rangle$ where $g_j \in T(o(y_j), b + i(y))$ for $j \in [|y|]$.

From $F \to_v G$ where $v \in \mathbf{Sur}_\Sigma(x,y)$ we deduce

$$f'(I_b + i(v)) = g' \quad \text{and} \quad f(I_b + i(v)) = o(v)g.$$

With the above notation the last equality is equivalent to

$$f_k(I_b + i(v)) = g_{v(k)} \quad \text{for all } k \in [|x|].$$

Our first aim is to prove the following statements

A) There exist $w \in \Sigma^*$, $u \in \mathbf{Fn}_\Sigma(w,x)$ and $h_j \in T(o(y_j), b + i(x + w))$ for $j \in [|y|]$ such that:

1) $\langle h_1,h_2,\ldots,h_{|y|}\rangle (I_b + i(\langle I_x,u\rangle v)) = g$ and

2) for every $k \in [|x|]$ there exists $t_k \in \mathbf{Fn}_S(i(x + w), i(x + w))$ such that

$$t_k i(\langle I_x,u\rangle v) = i(\langle I_x,u\rangle v) \quad \text{and} \quad f_k + T_{i(w)} = h_{v(k)}(I_b + t_k).$$

As $v \in \mathbf{Sur}_\Sigma(x,y)$ there exists $q \in \mathbf{In}_\Sigma(y,x)$ such that $qv = I_y$.

For every $j \in [|y|]$ applying (p) for $f_{q(j)} \in T(o(y_j), b + i(x))$ we get $f_j^\circ \in T(o(y_j), b + i(x) + c^j)$ and $u^j \in \mathbf{Fn}_S(c^j,i(x))$ such that

$$f_{q(j)} = f_j^\circ(I_b + \langle I_{i(x)}, u^j\rangle).$$

For $j \in [|y|]$ we denote $w_j = x_{u^j(1)}x_{u^j(2)}\cdots x_{u^j(|c^j|)}$ and we remark $i(w_j) = c^j$ and

$u^j \in Fn_{\Sigma}(w_j,x)$. Let $w = w_1 + w_2 + \ldots + w_{|y|}$ and $u = \langle u^1,u^2,\ldots,u^{|y|}\rangle \in Fn_{\Sigma}(w,x)$.

For $j \in [|y|]$ we denote

$$h_j = f_j^{\circ}(I_{b+i(x)} + T_{c^1+\ldots+c^{j-1}} + I_{c^j} + T_{c^{j+1}+\ldots+c^{|y|}})$$

and we deduce

$$\langle h_1,h_2,\ldots,h_{|y|}\rangle(I_b + i(\langle I_x,u\rangle v)) =$$

$$= \langle f_1^{\circ}(I_b + i(\langle I_x,u^1\rangle v)), f_2^{\circ}(I_b + i(\langle I_x,u^2\rangle v)),\ldots,f_{|y|}^{\circ}(I_b + i(\langle I_x,u^{|y|}\rangle v))\rangle =$$

$$= \langle f_{q(1)}(I_b + i(v)), f_{q(2)}(I_b + i(v)),\ldots,f_{q(|y|)}(I_b + i(v))\rangle =$$

$$= \langle g_{v(q(1))},g_{v(q(2))},\ldots,g_{v(q(|y|))}\rangle = \langle g_1,g_2,\ldots,g_{|y|}\rangle = g.$$

For every $k \in [|x|]$ as $v(q(v(h))) = v(k)$ we deduce that $f_{q(v(k))}(I_b + i(v)) = f_k(I_b + i(v))$ hence

$$f^{\circ}_{v(k)}(I_b + i(\langle I_x,u^{v(k)}\rangle v)) = (f_k + T_{c^{v(k)}})(I_b + i(\langle I_x,u^{v(k)}\rangle v)).$$

Applying (p)(ii) there exists $r_k \in Fn_S(i(x) + c^{v(k)}, i(x) + c^{v(k)})$ such that

$$f^{\circ}_{v(k)}(I_b + r_k) = f_k + T_{c^{v(k)}} \text{ and } i(\langle I_x,u^{v(k)}\rangle v) = r_k i(\langle I_x,u^{v(k)}\rangle v).$$

Using the notation $c' = c^1c^2\ldots c^{v(k)-1}$ and $c'' = c^{v(k)+1}\ldots c^{|y|}$

we define

$$t_k = (I_{i(x)} + {}^{c'}X^{c^{v(k)}} + I_{c''})(r_k + I_{c'+c''})(I_{i(x)} + {}^{c^{v(k)}}X^{c'} + I_{c''}).$$

Therefore

$$t_k i(\langle I_x,u\rangle v) =$$

$$= (I_{i(x)} + {}^{c'}X^{c^{v(k)}} + I_{c''})(r_k + I_{c'+c''})i(\langle I_x,u^{v(k)},u^1,u^2,\ldots,u^{v(k)-1},u^{v(k)+1}\ldots u^{|y|}\rangle v) =$$

$$= (I_{i(x)} + {}^{c'}X^{c^{v(k)}} + I_{c''})\langle r_k i(\langle I_x,u^{v(k)}\rangle v), i(\langle u^1,\ldots,u^{v(k)-1},u^{v(k)+1},\ldots,u^{|y|}\rangle v)\rangle =$$

$$= i(\langle I_x,u\rangle v) \text{ and}$$

$$h_{v(k)}(I_b + t_k) = (f^{\circ}_{v(k)}(I_b + r_k) + T_{c'+c''})(I_{b+i(x)} + {}^{c^{v(k)}}X^{c'} + I_{c''}) = f_k + T_{i(w)}.$$

The proof of A) is finished.

Now, we apply Ésik's commutativity axiom for

$I(y)H(\langle h_1,h_2,\dots,h_{|y|}\rangle) \in B(i(y),\ b + i(x + w))$, $H(i(\langle I_x,u\rangle v)) \in \mathbf{Sur}_S(i(x + w),\ i(y))$ and $H(t_k) \in \mathbf{Fn}_S(i(x + w),\ i(x + w))$ for $k \in [|x|]$ to obtain

$$H(i(\langle I_x,u\rangle v))[I(y)H(\langle h_1,h_2,\dots,h_{|y|}\rangle(I_b + i(\langle I_x,u\rangle v)))]^\dagger =$$

$$=\langle I(y_{v(1)})H(h_{v(1)}(I_b + t_1)),\dots,I(y_{v(|x|)})H(h_{v(|x|)}(I_b + t_{|x|})),\dots\rangle^\dagger =$$

$$= \langle I(x_1)H(f_1 + T_{i(w)}),\dots,I(x_{|x|})H(f_{|x|} + T_{i(w)}),\dots\rangle^\dagger =$$

$$= \langle I(x)H(f) + T_{i(w)},\dots\rangle^\dagger = \langle (I(x)H(f))^\dagger,\dots\rangle$$

where in the last equality we used the next property of the iteration

$$(I_a + T_b)\langle f + T_b,g\rangle^\dagger = f^\dagger \text{ for } f : a \to c + a \text{ and } g : b \to c + a + b.$$

Therefore $(I(x)H(f))^\dagger = H(i(v))[I(y)H(g)]^\dagger$ hence

$$H(f')\langle I_b,(I(x)H(f))^\dagger\rangle = H(f'(I_b + i(v)))\langle I_b,\ (I(y)H(g))^\dagger\rangle = H(g')\ \langle I_b,(I(y)H(g))^\dagger\rangle. \quad \square$$

Corollary 16D2. An iteration S-sorted algebraic theory T which satisfies (p) is a funflow.

Proof. By Proposition 16A3 it suffices to show every $a\gamma$-morphism is functorial.

Suppose $f \in T(c,b + c)$, $g \in T(a,b + a)$, $v \in \mathbf{Sur}_S(c,a)$ and $f(I_b + v) = vg$.

We work in $\mathbf{FI}_{S^*,T}$ build for $i = o = 1_{S^*}$. Note that $(c,\langle T_b + I_c,f\rangle) \in \mathbf{FI}_{S^*,T}(c,b)$ and $(a,\langle T_b + I_a,g\rangle) \in \mathbf{FI}_{S^*,T}(a,b)$ fulfill

$$(c,\langle T_b + I_c,\ f\rangle) \longrightarrow_v v(a,\langle T_b + I_a,g\rangle).$$

Applying Proposition 16D1 for $H = 1_T$ and $I(a) = I_a$ for every $a \in S^*$ we get

$$(T_b + I_c)\langle I_b,\ (I(c)f)^\dagger\rangle = (T_b + v)\langle I_b,\ (I(a)g)^\dagger\rangle$$

hence $f^\dagger = vg^\dagger$. $\square$

Collecting all the above facts we get the following theorem.

Theorem 16D3. If T is an iteration theory fulfilling condition (p), then $\mathbf{Fl}_{\Sigma,T}/\sim_{a\delta}$ is an iteration theory which is the coproduct of T and of the free iteration theory generated by Σ. ☐

Observation 16D4. The condition (p) holds in $\mathbf{Pfn}_S$.

Proof. First note that (ii) holds if $f^\circ(\perp^b + I_{c+d})$ is an injective partial function. Such an f° obeying (i) may be obtained from f using the following procedure:

Start with $f^\circ := f$. For $i := 1, \ldots, |c|$ do

if $\{j \in [|a|] \mid f(j) = |b| + i\} = \{n_1, n_2, \ldots, n_s\}$ with $s > 1$ then

replace $f^\circ : a \to b + c + d$ by $f' : a \to b + c + d + (s-1)c_i$ defined by

$$f'(j) = \begin{cases} |b + c + d| + t \text{ if } j = n_t \text{ and } t \in [s-1] \\ \\ f^\circ(j) \text{ otherwise.} \end{cases}$$

Clearly there is an $u \in \mathbf{Fn}_S(d,c)$ such that $f = f^\circ(I_b + \langle I_c, u\rangle)$. ☐

Corollary 16D5. $\mathbf{Fl}_{\Sigma,\mathbf{Pfn}_S}/\sim_{a\delta}$ is the free iteration theory generated by Σ. ☐

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