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V.E. CAZANESCU and GH. STEFANESCU

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V.E. CAZANESCU*) and Gh. STEFANESCU**)

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*) *Faculty of Mathematics, University of Bucharest, Str. Academiei No. 14, 70109 Bucharest, Romania.*

**) *Department of Mathematics, INCREST, Bd. Pacii 220, 79622 Bucharest, Romania.*

TREE MODELS FOR REDUCED AND MINIMAL FLOWCHART SCHEMES

Virgil-Emil CĂZĂNESCU

Faculty of Mathematics, University of Bucharest,
70109 Bucharest, Str. Academiei 14
ROMANIA

Gheorghe ŞTEFĂNESCU

Department of Mathematics, The National Institute
for Scientific and Technical Creation,
Bd. Păcii 220, 79622 Bucharest,
ROMANIA.

Elgot has considered a natural equivalence on flowchart schemes called "strong equivalence" [9]. Roughly speaking two schemes are strongly equivalent iff they have the same sequences of computation. In [11] it was constructed a tree model for the classes of equivalent schemes; more precisely two schemes are strongly equivalent iff they unfold into the same rational tree. The rational trees were further studied in [13, 12, 8, 1]. In particular [12] gives an equational axiomatization for the rational trees.

Turning to flowchart schemes, in [10] and [14] it is shown that the strong equivalence relation may be generated using simulation via functions. The indirect proof of this affirmation given in [14] uses Ésik's Theorem [12]. An aim of the present paper is to give a direct proof of this result.

A flowchart scheme is a notation for a sequential computation process. The (step by step) behaviour of a vertex v is the unfoldment starting form v of the scheme. The input behaviour of a scheme is the tuple of the behaviours of its input vertices. From the computational viewpoint only the input behaviour does interest. All the transformations on schemes we made in this paper preserves the input behaviour. For example the accessible part of a scheme, i.e. the part obtained by deletion of the statements which cannot be reached from an input, has the same input behaviour as the whole scheme.

The internal behaviour of a scheme is the set of the behaviours of the internal vertices (i.e. labeled by statements). A flowchart scheme is said to be reduced if it has no different internal verticed having the same behaviour. If we identify in a flowchart scheme the internal vertices having the same behaviour we obtain another scheme which is reduced and have the same input behaviour and the same internal behaviour as the given scheme.

Regarding the simulation [3] we prove the following results: a) two schemes are equivalent via the least congruence relation including the simulation via surjections if and only if they have the same input behaviour and the same internal behaviour; b) two schemes are equivalent via the least congruence relation including the simulation via functions if and only if they have the same input behaviour.

Another aim of this paper is to give tree models for (the surflow [3,7] of) the reduced flowchart schemes and for (the funflow [3,7] of) the minimal (i.e. accessible and reduced) flowchart schemes, models which are not obtained by a factorization as in our previous papers.

1. Introduction

In this section we establish the notation and we recall some results.

For a nonnegative integer n let $[n] = \{1,2,\ldots,n\}$.

If S^* is the free monoid on the set S, for $a \in S^*$ we denote its length by $|a|$ and for $i \in [|a|]$ we denote by a_i the i^{th} letter of a. As we use the additive notation we have $a = a_1 + a_2 + \ldots + a_{|a|}$.

Suppose $r : O \to S \times S^*$ is a function. Every $o \in O$ with $r(o) = (s, s_1 + s_2 + \ldots + s_n)$ is regarded as an operation symbol having n arguments of sort s_i and a result of sort s. For $b \in S^*$, let $\mathbf{Tr}_O(b)$ be the set of all the infinite partial trees with operation symbols in O and variables in the S-sorted set $b : [|b|] \to S$. From an algebraic viewpoint $\mathbf{Tr}_O(b)$ is the ω-continuous O-algebra freely generated by the S-sorted set b, see [15].

As the schemes we work with are multi-input we rather work with n-tuples of trees. From an algebraic viewpoint they form an ω-continuous algebraic theory freely generated by O [15] which is denoted by $\mathbf{CT}_O$. We recall that for $a,b \in S^*$

$$\mathbf{CT}_O(a,b) = \{f : [|a|] \to \mathbf{Tr}_O(b) \mid f(i) \text{ has the sort } a_i \text{ for } i \in [|a|]\}.$$

Let $I_O : O \to \mathbf{CT}_O$ be the standard interpretation of O into $\mathbf{CT}_O$, i.e. for $o \in O$ with $r(o) = (s, s_1 + s_2 + \ldots + s_n)$ $I_O(o) \in \mathbf{CT}_O(s, s_1 + s_2 + \ldots + s_n)$ is defined by

$$I_O(o)(1) = \begin{tikzcd}[column sep=small] & o \arrow[dl] \arrow[d] \arrow[dr] & \\ 1 & 2 \ldots & n \end{tikzcd}.$$

The algebraic theory of S-sorted partial functions $\mathbf{Pfn}_S$ is embedded in $\mathbf{CT}_O$. Every $f \in \mathbf{Pfn}_S(a,b)$ may by seen as the $|a|$-tuple of trees which maps $i \in [|a|]$ in the tree defined as follows:

"if f(i) is not defined then the empty tree else the tree consisting from a root labeled by f(i)."

In an ordered algebraic theory we denote by $0_{a,b}$ the least morphism from a to b.

1.1. Proposition. Let T be an ω-continuous algebraic theory. If $u \in T(a,b)$ fulfills $u0_{b,c} = 0_{a,c}$ for every object c then u is functorial.

Proof. For $f \in T(a,a+c)$ and $g \in T(b,b+c)$ such that $f(u+I_c) = ug$ we have to prove that $f^\dagger = ug^\dagger$. Recall that $f^\dagger = \bigvee_{n\in\omega} f^{(n)}$ where $f^{(0)} = 0_{a,c}$ and $f^{(n+1)} = f\langle f^{(n)}, I_c\rangle$. We prove by induction that $f^{(n)} = ug^{(n)}$. As for $n = 0$ the equality follows from the hypothesis we do the inductive step

$$f^{(n+1)} = f\langle f^{(n)},I_c\rangle = f\langle ug^{(n)},I_c\rangle = f(u+I_c)\langle g^{(n)},I_c\rangle = ug\langle g^{(n)}, I_c\rangle = ug^{(n+1)}.$$

As the composition is ω-continuous we conclude that $ug^\dagger = u\bigvee_{n\in\omega} g^{(n)} = \bigvee_{n\in\omega} ug^{(n)} = \bigvee_{n\in\omega} f^{(n)} = f^\dagger$. □

1.2. Corollary. In $\mathbf{CT}_O$ every partial function is functorial. □

When we unfold a scheme we get rational O-trees, therefore we prefer to work with $\mathbf{Rat}_O$ the algebraic theory of rational partial O-trees. We recall an infinite partial tree is rational if and only if the set of its subtrees is finite. Remark $\mathbf{Pfn}_S$ is included in $\mathbf{Rat}_O$. Using Corollary 1.2 we deduce $\mathbf{Rat}_O$ is a funflow [3,7].

In this paper we work with flowchart schemes having statements in a set Σ and connections from $\mathbf{Pfn}_S$. As usual $i : \Sigma^* \longrightarrow S^*$ and $o : \Sigma^* \longrightarrow S^*$ are two monoid morphisms. We denote by **Fl** the flow of the flowchart scheme representations [2]. Recall that for $a,b \in S^*$

$$\mathbf{Fl}(a,b) = \{(x,f) \mid x \in \Sigma^*,\ f \in \mathbf{Pfn}_S(a+o(x),\ b+i(x))\}.$$

$\mathbf{Pfn}_S$ is embedded in **Fl** identifying $f \in \mathbf{Pfn}_S(a,b)$ with the representation $(\varepsilon,f) \in \mathbf{Fl}(a,b)$ where ε is the empty word of Σ^*. Σ^* is embedded in **Fl** identifying $x \in \Sigma^*$ with the representation $(x, {}^{i(x)}X^{o(x)}) \in \mathbf{Fl}(i(x),o(x))$.

From the main result in [2] we get the following proposition. If B is an S^*-biflow which includes $\mathbf{Pfn}_S$ and if $I : \Sigma^* \longrightarrow B$ is a monoid morphism such that $I(x) \in B(i(x), o(x))$ for every $x \in \Sigma^*$ then there exists a unique flow morphism $I^{\#} : \mathbf{Fl} \rightarrow B$ such that $I^{\#}(x) = I(x)$ for every $x \in \Sigma^*$ and $I^{\#}(f) = f$ for every f in $\mathbf{Pfn}_S$.

By Theorem 6.4 in [6] the monoid morphisms i and o may be extended in a unique way to a δ-ssmc morphisms $i : \mathbf{Fn}_\Sigma \longrightarrow \mathbf{Pfn}_S$ and $o : \mathbf{Fn}_\Sigma \longrightarrow \mathbf{Pfn}_S$, respectively.

Suppose (x,f) and (y,g) are in $\mathbf{Fl}(a,b)$. If $u \in \mathbf{Fn}_\Sigma(x,y)$ and $f(I_b + i(u)) = (I_a + o(u))g$ we write $(x,f) \longrightarrow_u (y,g)$ and we say (x,f) <u>simulates</u> in (y,g) via the function u.

If there exists u in $\mathbf{Fn}$ such that $(x,f) \longrightarrow_u (y,g)$ we write $(x,f) \xrightarrow{a\delta} (y,g)$. If there exists an injection u such that $(x,f) \longrightarrow_u (y,g)$ we write $(x,f) \xrightarrow{a\beta} (y,g)$. If there exists a surjection u such that $(x,f) \longrightarrow_u (y,g)$ we write $(x,f) \xrightarrow{a\gamma} (y,g)$. If there exists a bijection u such that $(x,f) \longrightarrow_u (y,g)$ we write $(x,f) \xrightarrow{a\alpha} (y,g)$, or even $(x,f) \sim_{a\alpha} (y,g)$ as $\xrightarrow{a\alpha}$ is a congruence relation. We denote by $\sim_{a\beta}$, $\sim_{a\gamma}$, and $\sim_{a\delta}$ the least congruence relation including $\xrightarrow{a\beta}$, $\xrightarrow{a\gamma}$ and $\xrightarrow{a\delta}$, respectively.

The quotient of $\mathbf{Fl}$ by $\sim_{a\alpha}$ is denoted by **FS**. **FS** is a biflow and it is called the biflow of the flowchart schemes. The quotient of $\mathbf{Fl}$ by $\sim_{a\beta}$ is denoted **AFS**. It is an inflow and it is called the inflow of the accessible flowchart schemes. The quotient of $\mathbf{Fl}$ by $\sim_{a\gamma}$ is denoted by **RFS**. It is a surflow and it is called the surflow of the reduced flowchart schemes. The quotient of $\mathbf{Fl}$ by $\sim_{a\delta}$ is denoted by **MFS**. It is a funflow and it is called the funflow of minimal (accessible and reduced) flowcharts schemes.

Regarding the above $I^{\#}$ we remark that $(x,f) \sim_{a\alpha} (y,g)$ implies $I^{\#}(x,f) = I^{\#}(y,g)$, therefore $I^{\#}$ may be thought as a biflow morphism $I^{\#} : \mathbf{FS} \longrightarrow B$. If B is an inflow then $(x,f) \sim_{a\beta} (y,g)$ implies $I^{\#}(x,f) = I^{\#}(y,g)$, therefore $I^{\#}$ may be thought as an inflow morphism $I^{\#}: \mathbf{AFS} \longrightarrow B$. If B is a surflow $(x,f) \sim_{a\gamma} (y,g)$ implies $I^{\#}(x,f) = I^{\#}(y,g)$ therefore $I^{\#}$ may be thought as a surflow morphism $I^{\#}: \mathbf{RFS} \longrightarrow B$. If B is a funflow then $(x,f) \sim_{a\delta} (y,g)$ implies $I^{\#}(x,f) = I^{\#}(y,g)$ then $I^{\#}$ may be thought as a funflow morphism $I^{\#}: \mathbf{MFS} \longrightarrow B$.

2. Unfoldments

To unfold a flowchart scheme we replace every statement with n entries by n statements with one entry, therefore we define

$$O = \{(\sigma,k) \mid \sigma \in \Sigma, k \in [|i(\sigma)|]\}$$

and the function $r : O \longrightarrow S \times S^*$ by

$$r(\sigma,k) = (i(\sigma)_k, o(\sigma)) \quad \text{for} \quad (\sigma,k) \in O.$$

To define the unfoldment we use the interpretation I of Σ in $\mathbf{Rat}_O$ defined for $\sigma \in \Sigma$ by

$$I(\sigma) = \langle I_O(\sigma,1), \ldots, I_O(\sigma,|i(\sigma)|)\rangle.$$

and for $x \in \Sigma^*$ by $I(x) = \Sigma_{j \in [|x|]} I(x_j)$. Therefore there exists a unique flow morphism

$$\mathbf{U} : \mathbf{FI} \longrightarrow \mathbf{Rat}_O$$

such that $U(x) = I(x)$ for $x \in \Sigma^*$ and $U(f) = f$ for f in $\mathbf{Pfn}_S$. We recall that $U(x,f) = ((I_a + I(x))f)\uparrow^{i(x)}$ for $(x,f) \in \mathbf{FI}(a,b)$.

As $\mathbf{Rat}_O$ is a funflow $(x,f) \xrightarrow{a\delta} (y,g)$ implies $U(x,f) = U(y,g)$ therefore $\mathbf{U}$ may be thought as defined on $\mathbf{FS}$, $\mathbf{AFS}$, $\mathbf{RFS}$ or $\mathbf{MFS}$.

For a scheme represented by (x,f) in $\mathbf{FI}$, $U(x,f)$ is by definition its <u>input behaviour</u>.

In the sequel we are also interested in the unfoldments from the internal vertices of the scheme. As the roots of these unfoldments are given by the statements of the schemes we realy work with the unfoldments from the outputs of the statements which label the internal vertices. By definition

$$T(x,f) = U(x,(I_a + V_{o(x)})f)$$

is called the total unfoldment of (x,f). Remark that

$$(I_a + T_{o(x)})T(x,f) = U(x,f).$$

By definition

$$U_o(x,f) = (T_a + I_{o(x)})T(x,f)$$

is the unfoldment from the outputs of the statements.

2.1. Proposition. If $(x,f) \rightarrow_u (y,g)$ then

a) $T(x,f) = (I_a + o(u))T(y,g)$

b) $U_o(x,f) = o(u)U_o(y,g)$.

Proof. As

$$(I_a + V_{o(x)})f(I_a + i(u)) = (I_a + V_{o(x)}o(u))g =$$

$$= (I_{a+o(x)} + o(u))(I_a + o(u) + I_{o(y)})(I_a + V_{o(y)})g$$

we deduce that $(x,(I_a + V_{o(x)})f) \rightarrow_u (I_a + o(u))(y,(I_a + V_{o(y)})g)$ therefore $T(x,f) = (I_a + o(u))T(y,g)$. The other conclusion is an easy consequence of the first one. $\square$

For $a \in S^*$ and $i \in [|a|]$ we use the notation

$$x_i^a = T_{a_1+\ldots+a_{i-1}} + I_{a_i} + T_{a_{i+1}+\ldots+a_{|a|}} .$$

Suppose (y,g) in **Fl**. For $k \in [|y|]$, we define the unfoldment from internal vertex k by

$$U_k(y,g) = (y_k, o(x_k^y)U_o(y,g)).$$

We prefer the above definition instead of $U_k(y,g) = I(y_k)o(x_k^y)U_o(y,g)$ to include the case when there exists statements without inputs (i.e. $i(\sigma)$ is the empty word).

2.2. Lemma. If $(z,f) \rightarrow_u (y,g)$ then $U_k(z,f) = U_{u(k)}(y,g)$ for every $k \in [|z|]$.

Proof. From Proposition 2.1 we deduce

$$o(x_k^z)U_o(z,f) = o(x_k^z u)U_o(y,g) = o(x_{u(k)}^y)U_o(y,g)$$

therefore as $z_k = y_{u(k)}$ we get the conclusion. □

The internal behaviour of (x,f) is by definition

$$B(x,f) = \{U_k(x,f) \mid k \in [|x|]\}.$$

2.3. Corollary. If $(x,f) \sim_{a\gamma} (y,g)$ then $B(x,f) = B(y,g)$.

Proof. As $\sim_{a\gamma}$ is the least eqivalence relation including $\xrightarrow{a\gamma}$ it suffice to show $(x,f) \xrightarrow{a\gamma} (y,g)$ implies $B(x,f) = B(y,g)$. The last implication is an easy consequence of Lemma 2.2. □

We recall from [7] that (x,f) is reduced if $(x,f) \rightarrow_u (y,g)$ where u is a surjection implies u is a bijection.

2.4. Corollary. If the application which maps every internal vertex of a scheme in its unfoldment is injective then the scheme is reduced. □

2.5. Lemma. In a biflow over an algebraic theory for every $h : c \to b + c$ the following identities hold.

a) $(vh)\uparrow^c = ((I_c + h)({}^cX^b + I_c)(I_b + v))\uparrow^c$ for $v : c + c \to c$

b) $\langle g,h\rangle\uparrow^c = g\langle I_b, (V_c h)\uparrow^c\rangle$ for $g : a \to b + c$

c) $(V_c h)\uparrow^c = h\langle I_b,(V_c h)\uparrow^c\rangle$.

Proof. The right-hand side of a) is equal to

$$((I_c + vh)({}^cX^b + I_c))\uparrow^{c+c} = ((I_c + (vh)\uparrow^c)\,{}^cX^b)\uparrow^c =$$

$$({}^cX^c((vh)\uparrow^c + I_c))\uparrow^c = (vh)\uparrow^c.$$

Using a) the right-hand side of b) is equal to

$$g(I_b + ((I_c + h)({}^cX^b + I_c)(I_b + V_c))\uparrow^c)V_b = g((I_{b+c} + h)V_{b+c})\uparrow^c = \langle g,h\rangle\uparrow^c.$$

The last identity follows from b) and $V_c h = \langle h,h\rangle$. ☐

For $(x,f) \in Fl(a,b)$ we use the following notation

$$f_a = (I_a + T_{o(x)})f \quad \text{and} \quad f_x = (T_a + I_{o(x)})f.$$

2.6. Proposition. If $(x,f) \in Fl(a,b)$ then

a) $I(x)U_o(x,f) = (V_{i(x)}I(x)f_x)\uparrow^{i(x)}$

b) $T(x,f) = f\langle I_b, I(x)U_o(x,f)\rangle$.

Proof. From $(I_a + V_{o(x)})f = \langle f,f_x\rangle$ using Lemma 2.5.b we deduce

$$T(x,f) = \langle f,I(x)f_x\rangle\uparrow^{i(x)} = f\langle I_b,(V_{i(x)}I(x)f_x)\uparrow^{i(x)}\rangle.$$

Using Lemma 2.5.c we prove **a)**

$$I(x)U_o(x,f) = I(x)f_x\langle I_b,(V_{i(x)}I(x)f_x)\uparrow^{i(x)}\rangle = (V_{i(x)}I(x)f_x)\uparrow^{i(x)}$$

therefore **b)** follows from the above two equalities. $\square$

2.7. Lemma. If $f \in Fn_{\Sigma}(x,y)$, $u \in [|x|]$ and $u' \in [|i(x_u)|]$ then

$$i(f)(\Sigma_{r \in [u-1]}|i(x_r)| + u') = \Sigma_{r \in [f(u)-1]}|i(y_r)| + u'. \quad \square$$

2.8. Lemma. Suppose (z,f) and (y,g) in $\mathbf{FI}(a,b)$, $p \in \mathbf{Pfn}_S(c,b + i(z))$, and $q \in \mathbf{Pfn}_S(c,b + i(y))$ fulfill

$$p\langle I_b,I(z)U_o(z,f)\rangle = q\langle I_b, I(y)U_o(y,g)\rangle.$$

If $t \in Fn_{\Sigma}(z,x)$ and $s \in Fn_{\Sigma}(y,x)$ fulfill

$$(\forall u \in [|z|])(\forall v \in [|y|])\ [U_u(z,f) = U_v(y,g) \text{ implies } t(u) = s(v)]$$

then $p(I_b + i(t)) = q(I_b + i(s))$.

Proof. Suppose $k \in [c]$. Remark that

$$x_k^c p\langle I_b,I(z)U_o(z,f)\rangle = \begin{cases} 0_{c_k,b} & \text{if } p \text{ is not defined for } k \\ x^b_{p(k)} & \text{if } p(k) \le |b| \\ x^{i(z)}_{p(k)-|b|}I(z)U_o(z,f) & \text{if } p(k) > |b|. \end{cases}$$

and an analogous equality holds for q.

From the first hypothesis we obtain only three cases.

1) p and q are not defined for k. In this case $p(I_b + i(t))$ and $q(I_b + i(s))$ are not defined for k.

2) $p(k) \le |b|$, $q(k) \le |b|$ and $p(k) = q(k)$. In this case $(p(I_b + i(t)))(k) = p(k) = q(k) = q(I_b + i(s)))(k)$.

3) $p(k) > |b|$, $q(k) > |b|$ and

$$x^{i(z)}_{p(k)-|b|}I(z)U_o(z,f) = x^{i(y)}_{q(k)-|b|}I(y)U_o(y,g).$$

Using the notation

$$p(k) = |b| + \Sigma_{r\in[u-1]}|i(z_r)| + u' \text{ where } u\in[|z|] \text{ and } u'\in[|i(z_u)|],$$

$$q(k) = |b| + \Sigma_{r\in[v-1]}|i(y_r)| + v' \text{ where } v\in[|y|] \text{ and } v'\in[|i(y_v)|]$$

we deduce

$$x^{i(z_u)}_{u'}\, i(x^z_u)I(z)U_o(z,f) = x^{i(y_v)}_{v'}\, i(x^y_v)I(y)U_o(y,g)$$

therefore using $i(x^z_u)I(z) = I(z_u)o(x^z_u)$ we obtain

$$I_O(z_u,u')o(x^z_u)U_o(z,f) = I_O(y_v,v')o(x^y_v)U_o(y,g).$$

This equality implies

$$o(x^z_u)U_o(z,f) = o(x^y_v)U_o(y,g)$$

and

$$I_O(z_u,u') = I_O(y_v,v'), \text{ therefore } z_u = y_v \text{ and } u' = v'.$$

As $U_u(z,f) = U_v(y,g)$ the second hypothesis implies $t(u) = s(v)$ therefore using Lemma 2.7 we get

$$(p(I_b + i(t))(k) = |b| + i(t)(\Sigma_{r\in[u-1]}|i(z_r)| + u') = |b| + \Sigma_{r\in[t(u)-1]}|i(x_r)| + u' =$$

$$= |b| + \Sigma_{r\in[s(v)-1]}|i(x_r)| + v' = |b| + i(s)(\Sigma_{r\in[v-1]}|i(y_r)| + v') = (q(I_b + i(s)))(k). \quad \square$$

2.9. Proposition. Suppose $(z,f)\in FI(a,b)$. There exists a surjection $r\in Fn_\Sigma(z,y)$ such that

$$(\forall k,j\in[|z|])\,[r(k) = r(j) \iff U_k(z,f) = U_j(z,f)]$$

and (z,f) simulates via r in a reduced schemes.

Proof. As the existence of r fulfilling the first condition is obvious we prove the second one. We choose an injection $q \in \mathbf{Fn}_\Sigma(y,z)$ such that $qr = I_y$, we define $(y,g) \in \mathbf{FI}(a,b)$ by

$$g = (I_a + o(q))f(I_b + i(r))$$

and we prove $(z,f) \to_r (y,g)$, i.e.

$$f(I_b + i(r)) = (I_a + o(rq))f(I_b + i(r)).$$

As by composition to the left with $I_a + T_{o(z)}$ we get an equality it suffices to show

$$f_z(I_b + i(r)) = o(rq)f_z(I_b + i(r)).$$

To do it we apply Lemma 2.8 for $t = s = r$. As its second hypothesis follows from the first condition imposed on r we only have to prove its first hypothesis, i.e.

$$f_z\langle I_b, I(z)U_o(z,f)\rangle = o(rq)f_z\langle I_b, I(z)U_o(z,f)\rangle$$

equality which is equivalent via Proposition 2.6.b to $U_o(z,f) = o(rq)U_o(z,f)$.

For every $n \in [|z|]$ as $r(n) = r((rq)(n))$ we deduce $U_n(z,f) = U_{(rq)(n)}(z,f)$ therefore $o(x_n^z)U_o(z,f) = o(x_n^z)o(rq)U_o(z,f)$. Hence $U_o(z,f) = o(rq)U_o(z,f)$.

To finish the proof we show using Corollary 2.4 that (y,g) is reduced. For $j,k \in [|y|]$ suppose $U_j(y,g) = U_k(y,g)$.

Using Lemma 2.2. we deduce

$$U_{q(j)}(z,f) = U_{r(q(j))}(y,g) = U_j(y,g) = U_k(y,g) = U_{q(k)}(z,f)$$

From $U_{q(j)}(z,f) = U_{q(k)}(z,f)$ and the property of r we deduce $r(q(j)) = r(q(k))$ hence $j = k$.

2.10. Corollary. If a scheme is reduced then the application which maps every interval vertex in its unfoldment is injective. □

Corollaries 2.4 and 2.10 give an equivalent condition for a scheme to be reduced. Corollary 2.3 and the next proposition give an equivalent condition for two schemes to be $\sim_{a\gamma}$ equivalent.

2.11. Proposition. Assume (x,f) and (y,g) in $\mathbf{Fl}(a,b)$. If $\mathbf{U}(x,f) = \mathbf{U}(y,g)$ and $\mathbf{B}(x,f) = \mathbf{B}(y,g)$ then $(x,f) \sim_{a\gamma} (y,g)$.

Proof. Using Proposition 2.9 and Corollary 2.3 it suffices to do the proof when (x,f) and (y,g) are reduced.

As the equal set $\mathbf{B}(x,f)$ and $\mathbf{B}(y,g)$ has $|x|$ and $|y|$ elements, respectively there exists a bijection $j \in Fn_{\Sigma}(x,y)$ such that

$$(\forall u \in [|x|])(\forall v \in [|y|])\ (U_u(x,f) = U_v(y,g) \iff j(u) = v),$$

therefore $U_o(x,f) = o(j)U_o(y,g)$.

Using Proposition 2.6.b we deduce

$$f\langle I_b, I(x)U_o(x,f)\rangle = T(x,f) = \langle \mathbf{U}(x,f), U_o(x,f)\rangle = \langle \mathbf{U}(y,g),\ o(j)U_o(y,g)\rangle =$$

$$= (I_a + o(j))T(y,g) = (I_a + o(j))g\langle I_b, I(y)U_o(y,g)\rangle.$$

From Lemma 2.8 for $t = j$ and $s = I_y$ we deduce $f(I_b + i(j)) = (I_a + o(j))g$. □

2.12. Proposition. If $(y,g) \in \mathbf{Fl}(a,b)$ is reduced and if $(x,f) \xrightarrow{a\beta} (y,g)$ then (x,f) is reduced.

Proof. To prove (x,f) is reduced we use Corollary 2.4. Assume $U_j(x,f) = U_k(x,f)$ where $k,j \in [|x|]$. As $(x,f) \rightarrow_u (y,g)$ for an injection u we deduce from Proposition

2.1.b that $U_j(x,f) = U_{u(j)}(y,g)$ and $U_k(x,f) = U_{u(k)}(y,g)$, therefore $U_{u(j)}(y,g) = U_{u(k)}(y,g)$. As (y,g) is reduced from Corollary 2.10 we get $u(j) = u(k)$ hence $j = k$ as u is injective. □

3. A model for the surflow of the reduced flowchart schemes

For $b \in S^*$ let $W_b = \{(\sigma,u) \mid \sigma \in \Sigma,\ u \in \mathbf{Rat}_O(o(\sigma),b)\}$. An element (σ,u) in W_b is seen as a tree with the root labeled by σ and u as the remainder of the tree. For $A \subset W_b$ and $g \in \mathbf{Rat}_O(b,c)$ by definition

$$Ag = \{(\sigma,ug) \mid (\sigma,u) \in A\}.$$

Before defining the model we define an aγ-flow C. For $a,b \in S^*$ by definition

$$C(a,b) = \{(f,F) \mid f \in \mathbf{Rat}_O(a,b),\quad F \subset W_b\}.$$

The composite of $(f,F) \in C(a,b)$ and $(g,G) \in C(b,c)$ is defined by

$$(f,F)(g,G) = (fg,\ Fg \cup G)$$

It is easy to see C is a category where the identity morphism of $a \in S^*$ is $(I_a,\emptyset)$.

For $(f,F) \in C(a,b)$ and $(g,G) \in C(c,d)$ we define their sum by

$$(f,F) + (g,G) = (f + g,\ F(I_b + T_d) \cup G(T_b + I_d)).$$

To show C is a strict monoidal category we do only the most difficult verification. For $(f,F) \in C(a,b)$, $(g,G) \in C(b,c)$, $(f',F') \in C(a',b')$ and $(g',G') \in C(b',c')$ we have

$$((f,F) + (f',F'))((g,G) + (g',G')) =$$

$$= (f + f',\ F(I_b + T_{b'}) \cup F'(T_b + I_{b'}))\ (g + g',\ G(I_c + T_{c'}) \cup G'(T_c + I_{c'})) =$$

$$= ((f + f')(g + g'),\ F(g + T_{c'}) \cup F'(T_c + g') \cup G(I_c + T_{c'}) \cup G'(T_c + I_{c'})) =$$

$= (fg + f'g', (Fg \cup G)(I_c + T_{c'}) \cup (F'g' \cup G')(T_c + I_{c'})) =$

$= (fg, Fg \cup G) + (f'g', F'g' \cup G') = (f,F)(g,G) + (f',F')(g',G').$

Using the following remarks

a) if $f \in \mathbf{Rat}_O(a,b)$ then $(f,\emptyset) \in C(a,b)$

b) $(f,\emptyset)(g,\emptyset) = (fg,\emptyset)$ for $f \in \mathbf{Rat}_O(a,b)$ and $g \in \mathbf{Rat}_O(b,c)$

c) $(f,\emptyset) + (g,\emptyset) = (f + g,\emptyset)$ for $f \in \mathbf{Rat}_O(a,b)$ and $g \in \mathbf{Rat}_O(c,d)$

we may identify $f \in \mathbf{Rat}_O(a,b)$ to $(f,\emptyset) \in C(a,b)$, therefore $\mathbf{Rat}_O \subset C$.

Using the distinguished morphism ${}^aX^b$, T_a, V_a and $\perp^a$ from $\mathbf{Pfn}_S$ one may prove C is an $a\gamma$-strong $b\delta$-ssmc [7]. We prove only some identities.

For $(f,F) \in C(a,b)$ and $(g,G) \in C(c,d)$

$((f,F) + (g,G))^bX^d = ((f + g)^bX^d, (F(I_b + T_d) \cup G(T_b + I_d))^bX^d) =$

$= ({}^aX^c(g + f), F(T_d + I_b) \cup G(I_d + T_b)) =$

$= {}^aX^c(g + f, G(I_d + T_b) \cup F(T_d + I_b)) = {}^aX^c((g,G) + (f,F)).$

For $(f,F) \in C(a,b)$

$((f,F) + (f,F))V_b = ((f + f)V_b, (F(I_b + T_b) \cup F(T_b + I_b))V_b) = (V_a f,F) = V_a(f,F).$

As in the sequel we need the iteration we recall from [4] that in an algebraic theory D the connection between feedback and iteration $\dagger : D(a,a + b) \rightarrow D(a,b)$ is done by the equalities

$\uparrow^a\langle f,g\rangle = g\langle f^\dagger,I_c\rangle$ for $f \in D(a,a + c)$ and $g \in D(b,a + c)$

$f^\dagger = \uparrow^a\langle f,I_a + T_b\rangle$ for $f \in D(a,a + b)$.

For $(\langle f,g\rangle,A) \in C(a + b, a + c)$ where $f \in \mathbf{Rat}_O(a,a + c)$ and $g \in \mathbf{Rat}_O(b,a + c)$ we define

$$\uparrow^a(\langle f,g\rangle,A) = (\uparrow^a\langle f,g\rangle, A\langle f^\dagger, I_c\rangle).$$

We prove C is a biflow. First we remark for $f \in \mathbf{Rat}_O(a+b,\ a+c)$ that $\uparrow^a(f,\emptyset) = (\uparrow^a f,\emptyset)$ therefore $\uparrow^a I_a = I_\lambda$ where λ is the empty word and $\uparrow^a\, {}^aX^a = I_a$.

If $(g,G) \in C(b',b)$, $(h,H) \in C(c,c')$ and $(\langle f,f'\rangle,F) \in C(a+b,a+c)$ where $f \in \mathbf{Rat}_O(a,a+c)$ and $f' \in \mathbf{Rat}_O(b,a+c)$ then

$$\uparrow^a[(I_a + (g,G))(\langle f,f'\rangle,F)(I_a + (h,H))] =$$

$$= \uparrow^a[(I_a + g,\ G(T_a + I_b))(\langle f,f'\rangle,F)(I_a + h,\ H(T_a + I_{c'}))] =$$

$$= \uparrow^a(\langle f,gf'\rangle(I_a + h),\ Gf'(I_a + h) \cup F(I_a + h) \cup H(T_a + I_{c'})) =$$

$$= (\uparrow^a\langle f,gf'\rangle h,\ (Gf'(I_a + h) \cup F(I_a + h) \cup H(T_a + I_{c'}))\langle f^\dagger h, I_{c'}\rangle) =$$

$$= (g(\uparrow^a\langle f,f'\rangle)h,\ Gf'\langle f^\dagger,I_c\rangle h \cup F\langle f^\dagger,I_c\rangle h \cup H) =$$

$$= (gf'\langle f^\dagger,I_c\rangle,\ Gf'\langle f^\dagger,I_c\rangle \cup F\langle f^\dagger, I_c\rangle)(h,H) =$$

$$= (g,G)(\uparrow^a\langle f,f'\rangle,\ F\langle f^\dagger,I_c\rangle)(h,H) = (g,G)(\uparrow^a(\langle f,f'\rangle,F))(h,H).$$

If $(\langle f,f'\rangle,F) \in C(a+b,\ a+c)$ where $f \in \mathbf{Rat}_O(a,a+c)$ and $f' \in \mathbf{Rat}_O(b,a+c)$ then

$$\uparrow^a[(\langle f,f'\rangle,F) + I_d] = \uparrow^a(\langle f,f'\rangle + I_d, F(I_{a+c} + T_d)) =$$

$$= (\uparrow^a\langle f,f'\rangle + I_d,\ F(I_{a+c} + T_d)\langle((I_a + T_{b+d})(\langle f,f'\rangle + I_d))^\dagger,\ I_{c+d}\rangle) =$$

$$= (\uparrow^a\langle f,f'\rangle + I_d,\ F\langle f^\dagger + T_d,\ I_c + T_d\rangle) = (\uparrow^a\langle f,f'\rangle,\ F\langle f^\dagger,I_c\rangle) + I_d = \uparrow^a(\langle f,f'\rangle,F) + I_d.$$

If $(\langle f,g,h\rangle,A) \in C(b+a+c,b+a+d)$ where $f \in \mathbf{Rat}_O(b,b+a+d)$,

$g \in \mathbf{Rat}_O(a, b + a + d)$ and $h \in \mathbf{Rat}_O(c, b + a + d)$ then

$$\uparrow^a \uparrow^b(\langle f,g,h\rangle, A) = \uparrow^a(\langle g,h\rangle\langle f^\dagger, I_{a+d}\rangle, A\langle f^\dagger, I_{a+d}\rangle) =$$

$$= (\uparrow^a \uparrow^b \langle f,g,h\rangle, A\langle f^\dagger, I_{a+d}\rangle\langle (g\langle f^\dagger, I_{a+d}\rangle)^\dagger, I_d\rangle) =$$

$$= (\uparrow^{b+a}\langle f,g,h\rangle, A\langle f^\dagger\langle (g\langle f^\dagger, I_{a+d}\rangle)^\dagger, I_d\rangle, (g\langle f^\dagger, I_{a+d}\rangle)^\dagger, I_d\rangle) =$$

$$= (\uparrow^{b+a}\langle f,g,h\rangle, A\langle\langle f,g\rangle^\dagger, I_d\rangle) = \uparrow^{b+a}(\langle f,g,h\rangle, A)$$

and

$$\uparrow^{a+b}[({}^aX^b + I_c)(\langle f,g,h\rangle, A)({}^bX^a + I_d)] =$$

$$= \uparrow^{a+b}(({}^aX^b + I_c)\langle f,g,h\rangle({}^bX^a + I_d), A({}^bX^a + I_d)) =$$

$$= (\uparrow^{b+a}\langle f,g,h\rangle, A({}^bX^a + I_d)\langle(\langle g,f\rangle({}^bX^a + I_d))^\dagger, I_d\rangle) =$$

$$= (\uparrow^{b+a}\langle f,g,h\rangle, A\langle\langle f,g\rangle^\dagger, I_d\rangle) = \uparrow^{b+a}(\langle f,g,h\rangle, A).$$

We show that every $p \in \mathbf{Pfn}_S(a,b)$ is functorial in C. Suppose $(\langle f,f'\rangle, F) \in C(a + c, a + d)$ where $f \in \mathbf{Rat}_O(a, a + d)$ and $g \in \mathbf{Rat}_O(c, a + d)$, $(\langle g,g'\rangle, G) \in C(b + c, b + d)$ where $g \in \mathbf{Rat}_O(b, b + d)$ and $g' \in \mathbf{Rat}_O(c, b + d)$ and $(\langle f,f'\rangle, F)(p + I_d) = (p + I_c)(\langle g,g'\rangle, G)$. We deduce

$$\langle f,f'\rangle(p + I_d) = (p + I_c)\langle g,g'\rangle \text{ and } F(p + I_d) = G.$$

As by Corollary 1.2 p is functorial in $\mathbf{Rat}_O$ we get $\uparrow^a\langle f,f'\rangle = \uparrow^b\langle g,g'\rangle$, and from $f(p + I_d) = pg$ we get $f^\dagger = pg^\dagger$, therefore

$$\uparrow^a(\langle f,f'\rangle, F) = (\uparrow^a\langle f,f'\rangle, F\langle f^\dagger, I_d\rangle) =$$

$$= (\uparrow^b\langle g,g'\rangle, F\langle pg^\dagger, I_d\rangle) = (\uparrow^b\langle g,g'\rangle, G\langle g^\dagger, I_d\rangle) = \uparrow^b(\langle g,g'\rangle, G).$$

3.1. Proposition. If for every $(x,f) \in \mathbf{Fl}(a,b)$ we define

$Q(x,f) = (U(x,f), B(x,f)) \in C(a,b)$

then $Q : Fl \longrightarrow C$ is a flow morphism and

$(x,f) \sim_{a\gamma} (y,g) \iff Q(x,f) = Q(y,g).$

Proof. Suppose $(x,f) \in Fl(a,b)$ and $(y,g) \in Fl(b,c)$. As

$T((x,f)(y,g)) =$

$= U(x + y,(I_a + V_{o(x+y)})(f + I_{o(y)})(I_b + {}^{i(x)}X^{o(y)})(g + I_{i(x)})(I_c + {}^{i(y)}X^{i(x)})) =$

$= [(I_{a+o(x+y)} + I(x + y))(I_a + V_{o(x+y)})(f + I_{o(y)})(I_b + {}^{i(x)}X^{o(y)})$

$\quad (g + I_{i(x)})(I_c + {}^{i(y)}X^{i(x)})]\uparrow^{i(x+y)} =$

$= [(I_{a+o(x+y)} + I(x + y))(I_{a+o(x)} + {}^{o(y)}X^{o(x)} + I_{o(y)})[(I_a + V_{o(x)})f + V_{o(y)}]$

$\quad (I_b + {}^{i(x)}X^{o(y)})(g + I_{i(x)})(I_c + {}^{i(y)}X^{i(x)})]\uparrow^{i(x + y)} =$

$= [(I_{a+o(x)} + {}^{o(y)}X^{i(x)} + I_{i(y)})[(I_{a+o(x)} + I(x))(I_a + V_{o(x)})f + (I_{o(y)} + I(y))V_{o(y)}]$

$\quad (I_b + {}^{i(x)}X^{o(y)})(g + I_{i(x)})(I_c + {}^{i(y)}X^{i(x)})]\uparrow^{i(y)}\uparrow^{i(x)} =$

$= [(I_{a+o(x)} + {}^{o(y)}X^{i(x)})[(I_{a+o(x)} + I(x))(I_a + V_{o(x)})f + I_{o(y)}]$

$[(I_b + {}^{i(x)}X^{o(y)+i(y)}((I_{o(y)} + I(y))V_{o(y)} + I_{i(x)}))(g + I_{i(x)})(I_c + {}^{i(y)}X^{i(x)})]\uparrow^{i(y)}]\uparrow^{i(x)} =$

$= [(I_{a+o(x)} + {}^{o(y)}X^{i(x)})[(I_{a+o(x)} + I(x))(I_a + V_{o(x)})f + I_{o(y)}](I_b + {}^{i(x)}X^{o(y)})$

$[(I_{b+o(y)} + {}^{i(x)}X^{i(y)})[(I_{b+o(y)} + I(y))(I_b + V_{o(y)})g + I_{i(x)}](I_c + {}^{i(y)}X^{i(x)})]\uparrow^{i(y)}]\uparrow^{i(x)} =$

$= [(I_{a+o(x)} + {}^{o(y)}X^{i(x)})[(I_{a+o(x)} + I(x))(I_a + V_{o(x)})f + I_{o(y)}](I_b + {}^{i(x)}X^{o(y)})$

$\quad (T(y,g) + I_{i(x)})]\uparrow^{i(x)} =$

$$= [(I_{a+o(x)} + {}^{o(y)}X^{i(x)})[(I_{a+o(x)} + I(x))(I_a + V_{o(x)})f + I_{o(y)}](I_b + {}^{i(x)}X^{o(y)})]\uparrow^{i(x)}T(y,g) =$$

$$= (T(x,f) + I_{o(y)})T(y,g)$$

we deduce

$$U_o((x,f)(y,g)) = (U_o(x,f) + I_{o(y)})T(y,g) = \langle U_o(x,f)U(y,g), U_o(y,g)\rangle$$

therefore

$$B((x,f)(y,g)) = \left\{((x + y)_k, o(x_k^{x+y})\langle U_o(x,f)U(y,g), U_o(y,g)\rangle) \,\middle|\, k \in [|x + y|]\right\} =$$

$$= B(x,f)U(y,g) \cup B(y,g)$$

hence

$$Q((x,f)(y,g)) = (U(x,f)U(y,g), B(x,f)U(y,g) \cup B(y,g)) =$$

$$= (U(x,f),B(x,f))(U(y,g),B(y,g)) = Q(x,f)Q(y,g).$$

Assume $(x,f) \in \mathbf{FI}(a,b)$ and $(y,g) \in \mathbf{FI}(c,d)$. As

$$T((x,f) + (y,g)) =$$

$$= U(x + y, (I_{a+c} + V_{o(x+y)})(I_a + {}^{c}X^{o(x)} + I_{o(y)})(f + g)(I_b + {}^{i(x)}X^{d} + I_{i(y)})) =$$

$$= [(I_{a+c+o(x+y)} + I(x + y))(I_{a+c+o(x)} + {}^{o(y)}X^{o(x)} + I_{o(y)})(I_{a+c} + V_{o(x)} + V_{o(y)})$$

$$(I_a + {}^{c}X^{o(x)} + I_{o(y)})(f + g)(I_b + {}^{i(x)}X^{d} + I_{i(y)})]\uparrow^{i(x+y)} =$$

$$= [(I_{a+c+o(x)} + {}^{o(y)}X^{i(x)} + I_{i(y)})[I_a + (I_c + (I_{o(x)} + I(x))V_{o(x)})^{c}X^{o(x)} + (I_{o(y)} + I(y))V_{o(y)}]$$

$$(f + g)(I_b + {}^{i(x)}X^{d} + I_{i(y)})]\uparrow^{i(y)}\uparrow^{i(x)} =$$

$$= [(I_{a+c+o(x)} + {}^{o(y)}X^{i(x)})(I_a + {}^{c}X^{o(x)+i(x)} + I_{o(y)})[(I_{a+o(x)} + I(x))(I_a + V_{o(x)})f +$$

$$[(I_{c+o(y)} + I(y))(I_c + V_{o(y)})g]\uparrow^{i(y)}](I_b + {}^{i(x)}X^{d})]\uparrow^{i(x)} =$$

$$= (I_a + {}^{c}X^{o(x)} + I_{o(y)})[(I_{a+o(x)} + {}^{c+o(y)}X^{i(x)})[(I_{a+o(x)} + I(x))(I_a + V_{o(x)})f + T(y,g)]$$

$$(I_b + {}^{i(x)}X^{d})]\uparrow^{i(x)} =$$

$$= (I_a + {}^{c}X^{o(x)} + I_{o(y)})(T(x,f) + T(y,g))$$

we deduce

$$U_o((x,f) + (y,g)) = U_o(x,f) + U_o(y,g)$$

therefore

$$B((x,f) + (y,g)) = B(x,f)(I_b + T_d) \cup B(y,g)(T_b + I_d)$$

hence

$$Q((x,f) + (y,g)) = (U(x,f) + U(y,g), B(x,f)(I_b + T_d) \cup B(y,g)(T_b + I_d)) =$$

$$= Q(x,f) + Q(y,g).$$

Suppose $(x,f) \in FI(a + b, a + c)$. As

$$T(\uparrow^a(x,f)) = U(x,(I_b + V_{o(x)})(\uparrow^a f)) = U(\uparrow^a(x,(I_{a+b} + V_{o(x)})f)) = \uparrow^a T(x,f)$$

we deduce

$$U_o(\uparrow^a(x,f)) = (T_b + I_{o(x)})(\uparrow^a\langle U(x,f),U_o(x,f)\rangle) =$$

$$= \uparrow^a\langle (I_a + T_b)U(x,f),U_o(x,f)\rangle = U_o(x,f)\langle [(I_a + T_b)U(x,f)]^\dagger,I_c\rangle$$

therefore

$$B(\uparrow^a(x,f)) = B(x,f)\langle [(I_a + T_b)U(x,f)]^\dagger,I_c\rangle$$

hence

$$Q(\uparrow^a(x,f)) = (\uparrow^a U(x,f),B(x,f)\langle [(I_a + T_b)U(x,f)]^\dagger,I_c\rangle) =$$

$$= \uparrow^a(U(x,f),B(x,f)) = \uparrow^a Q(x,f).$$

The last conclusion follows from Corollary 2.3 and Proposition 2.11. $\square$

From this proposition we deduce the surflow of the reduced flowchart schemes is isomorphic to the image of **Q**. Therefore, to find a model for the surflow of the reduced flowchart schemes we need to find the image of **Q**.

For $f \in \mathrm{Rat}_O(a,b)$ and $F \subset W_b$ we say <u>F contains the trees of f</u> if for every $j \in [|a|]$, $(\sigma,g) \in W_b$ and $k \in [|i(\sigma)|]$

$x_j^a f = I_O((\sigma,k))g$ implies $(\sigma,g) \in F$.

A subset F of W_b is said to be <u>hereditary</u> if for every $(\sigma,f) \in F$, F contains the trees of f.

We begin to construct the model, i.e. the image of Q. For $a,b \in S^*$ by definition

$R(a,b) = \{(f,A) \in C(a,b) | A$ is finite, hereditary and containes the trees of $f\}$.

3.2. Lemma. R is a subbiflow of C which includes $\mathbf{Pfn}_S$.

Proof. For $(f,F) \in R(a,b)$ and $(g,G) \in R(b,c)$ we show $(fg, Fg \cup G) \in R(a,c)$. As $Fg \cup G$ is finite we have to prove $Fg \cup G$ is hereditary and contains the trees of fg.

To show $Fg \cup G$ is hereditary as G is hereditary it suffices to show $Fg \cup G$ contains the trees of ug for every $(\sigma,u) \in F$. Suppose

$x_j^{o(\sigma)} ug = I_O((\tau,k))v$

where $(\sigma,u) \in F$, $j \in [|o(\sigma)|]$, $(\tau,v) \in W_c$ and $k \in [|i(\tau)|]$. There are two cases.

If $x_j^{o(\sigma)} u = x_r^b$ with $r \in [|b|]$ then as G contains the trees of g from $x_r^b g = I_O((\tau,k))v$ we get $(\tau,v) \in G$.

If $x_j^{o(\sigma)} u = I_O((\tau,k))v'$ then $v'g = v$, therefore as F is hereditary we get $(\tau,v') \in F$ hence $(\tau,v) \in Fg$.

We show $Fg \cup G$ contains the trees of fg. Suppose

$x_j^a fg = I_O((\sigma,k))u$

where $j \in [|a|]$, $(\sigma,u) \in W_c$ and $k \in [|i(\sigma)|]$. There are two cases.

If $x_j^a f = x_r^b$ where $r \in [|b|]$ as G contains the trees of g from $x_r^b g = I_O((\sigma,k))u$ we get $(\sigma,u) \in G$.

If $x_j^a f = I_O((\sigma,k))v$ then $vg = u$. As F contains the trees of f we get $(\sigma,v) \in F$ hence $(\sigma,u) = (\sigma,vg) \in Fg$.

For $(f,F) \in R(a,b)$ and $(g,G) \in R(c,d)$ we show that

$(f + g, F(I_b + T_d) \cup G(T_b + I_d)) \in R(a + c, b + d)$.

To show $F(I_b + T_d) \cup G(T_b + I_d)$ is hereditary we study two cases:

a) $(\sigma,u) \in F$ and $x_j^{o(\sigma)} u(I_b + T_d) = I_O((\tau,k))v$ where $j \in [|o(\sigma)|]$, $(\tau,v) \in W_{b+d}$ and $k \in [|i(\tau)|]$. From the above equality we deduce $v = v'(I_b + T_d)$ where $v' \in \mathbf{Rat}_O(o(\tau),b)$ and $x_j^{o(\sigma)} u = I_O((\tau,k))v'$. As F is hereditary we get $(\tau,v') \in F$ hence $(\tau,v) \in F(I_b + T_d)$.

b) If $(\sigma,u) \in G$ and $x_j^{o(\sigma)} u(T_b + I_d) = I_O((\tau,k))v$ where $j \in [|o(\sigma)|]$, $(\tau,v) \in W_{b+d}$ and $k \in [|i(\tau)|]$ the proof is analogous to the above one.

To show $F(I_b + T_d) \cup G(T_b + I_d)$ contains the trees of f + g we suppose

$$x_j^{a+c}(f + g) = I_O((\sigma,k))u$$

where $j \in [|a + c|]$, $(\sigma,u) \in W_{b+d}$ and $k \in [|i(\sigma)|]$.

If $j \leq |a|$ then $x_j^a f(I_b + T_d) = I_O((\sigma,k))u$ therefore $x_j^a f = I_O((\sigma,k))u'$ and $u'(I_b + T_d) = u$. As F contains the trees of f we deduce $(\sigma,u') \in F$ therefore $(\sigma,u) \in F(I_b + T_d)$.

If $j > |a|$ then $x_{j-|a|}^c g(T_b + I_d) = I_O((\sigma,k))u$ therefore $x_{j-|a|}^c g = I_O((\sigma,k))u'$ and $u'(T_b + I_d) = u$. As G contains the trees of g we deduce $(\sigma,u) \in G(T_b + I_d)$.

For $(\langle f,g\rangle,F) \in R(a + b, a + c)$ where $f \in \mathbf{Rat}_O(a,a + c)$ and $g \in \mathbf{Rat}_O(b,a + c)$ we show $(\uparrow^a\langle f,g\rangle, F\langle f^\dagger,I_c\rangle) \in R(b,c)$.

To show $F\langle f^\dagger,I_c\rangle$ contains the trees of $\uparrow^a\langle f,g\rangle$ we suppose

$x_j^b(\uparrow^a\langle f,g\rangle) = I_O((\sigma,k))h$ where $j \in [|b|]$, $(\sigma,h) \in W_c$ and $k \in [|i(\sigma)|]$, therefore

$$x_j^b g\langle f^\dagger, I_c\rangle = I_O((\sigma,k))h.$$

We study the two possible cases:

a) $x_j^b g = I_O((\sigma,k))h'$. We deduce $h = h'\langle f^\dagger, I_c\rangle$. As F contains the trees of $\langle f,g\rangle$ and $x_{|a|+j}^{a+b}\langle f,g\rangle = I_O((\sigma,k))h'$ we get $(\sigma,h') \in F$ hence $(\sigma,h) \in F\langle f^\dagger, I_c\rangle$,

b) $x_j^b g = x_{j_0}^{a+c}$ and $j_0 \in [|a|]$. As $x_{j_0}^a f^\dagger = I_O((\sigma,k))h$ there exist $n \geq 0$, $j_1, j_2, \ldots, j_n \subset [|a|]$ and $h' \in \mathbf{Rat}_O(o(\sigma), a + c)$ such that

$$x_{j_{r-1}}^a f = x_{j_r}^{a+c} \text{ for } r \in [n] \text{ and } x_{j_n}^a f = I_O((\sigma,k))h'.$$

As $x_{j_0}^a f^\dagger = x_{j_0}^a f\langle f^\dagger, I_c\rangle = x_{j_1}^a f^\dagger = \ldots = x_{j_n}^a f^\dagger = x_{j_n}^a f\langle f^\dagger, I_c\rangle = I_O((\sigma,k))h'\langle f^\dagger, I_c\rangle$ we get $h = h'\langle f^\dagger, I_c\rangle$. As F contains the trees of $\langle f,g\rangle$ from $x_{j_n}^{a+b}\langle f,g\rangle = I_O((\sigma,k))h'$ we get $(\sigma,h') \in F$, hence $(\sigma,h) \in F\langle f^\dagger, I_c\rangle$.

To prove $F\langle f^\dagger, I_c\rangle$ is hereditary we suppose $(\sigma,h) \in F$ and we show $F\langle f^\dagger, I_c\rangle$ contains the trees of $h\langle f^\dagger, I_c\rangle$. Assume

$$x_j^{o(\sigma)} h\langle f^\dagger, I_c\rangle = I_O((\tau,k))u$$

where $j \in [|o(\sigma)|]$, $(\tau,u) \in W_c$ and $k \in [|i(\tau)|]$. We study the two possible cases:

a) $x_j^{o(\sigma)} h = I_O((\tau,k))u'$. We deduce $u'\langle f^\dagger, I_c\rangle = u$. As F is hereditary we get $(\tau,u') \in F$, hence $(\tau,u) \in F\langle f^\dagger, I_c\rangle$.

b) $x_j^{o(\sigma)} h = x_{j_0}^{a+c}$ and $j_0 \in [|a|]$. As $x_{j_0}^a f^\dagger = I_O((\tau,k))u$ there exist $n \geq 0$ and $j_1, j_2, \ldots, j_n \subset [|a|]$ such that

$$x_{j_{r-1}}^a f = x_{j_r}^{a+c} \text{ for } r \in [n] \text{ and } x_{j_n}^a f = I_O((\tau,k))u'.$$

As $x_{j_0}^a f^\dagger = I_O((\tau,k))u'\langle f^\dagger, I_c\rangle$ we get $u = u'\langle f^\dagger, I_c\rangle$. As F contains the trees of $\langle f,g\rangle$ we get $(\tau,u') \in F$, hence $(\tau,u) \in F\langle f^\dagger, I_c\rangle$. $\square$

3.3. Theorem. R is a model for the surflow of the reduced flowchart schemes.

Proof. We show the image of Q is included in R. As for every $(x,f) \in \mathbf{Fl}(a,b)$, $(x,f) = ((I_a + \Sigma_{j\in[|x|]} x_j)f)\uparrow^{i(x)}$ and as $\mathbf{Pfn}_S$ is included in R, using Lemma 3.2 it suffices to show $Q(x, {}^{i(x)}X^{o(x)}) \in R(i(x), o(x))$ for every $x \in \Sigma$. As

$$T(x, {}^{i(x)}X^{o(x)}) = U(x, (I_{i(x)} + V_{o(x)})^{i(x)}X^{o(x)}) =$$

$$= ((I_{i(x)} + (I_{o(x)} + I(x))V_{o(x)})^{i(x)}X^{o(x)})\uparrow^{i(x)} =$$

$$= ({}^{i(x)}X^{o(x)+i(x)}(<I_{o(x)}, I(x)> + I_{i(x)}))\uparrow^{i(x)} = {}^{i(x)}X^{o(x)}<I_{o(x)}, I(x)> = <I(x), I_{o(x)}>$$

we deduce $Q(x, {}^{i(x)}X^{o(x)}) = (I(x), \{(x, I_{o(x)})\}) \in R(i(x), o(x))$.

We prove R is included in the image of Q. Assume $(g,G) \in R(a,b)$. As G is finite we may write

$$G = \{(\sigma_1, u_1), (\sigma_2, u_2), \ldots, (\sigma_k, u_k)\}.$$

We denote $y = \sigma_1 + \sigma_2 + \ldots + \sigma_k \in \Sigma^*$ and $u = <u_1, u_2, \ldots, u_k> \in \mathbf{Rat}_O(o(y), b)$.

We define $(y,f) \in \mathbf{Fl}(a,b)$ as follows

a) for $j \in [|a|]$

$$f(j) = \begin{cases} v \in [|b|] & \text{if } \mathbf{x}_j^a g = x_v^b \\ |b| + \Sigma_{t\in[q-1]} |i(\sigma_t)| + r & \text{if } \mathbf{x}_j^a g = I_O(\sigma_q, r)u_q \\ \text{nondefined} & \text{if } \mathbf{x}_j^a g = 0_{a_j, b} \end{cases}$$

b) for $s \in [k]$ and $n \in [|o(\sigma_s)|]$

$$f(a + \Sigma_{t\in[s-1]} |o(\sigma_t)| + n) = \begin{cases} v \in [|b|] & \text{if } \mathbf{x}_n^{o(\sigma_s)} u_s = \mathbf{x}_v^b \\ |b| + \Sigma_{t\in[q-1]} |i(\sigma_t)| + r & \text{if } \mathbf{x}_n^{o(\sigma_s)} u_s = I_O(\sigma_q, r)u_q \\ \text{nondefined} & \text{if } \mathbf{x}_n^{o(\sigma_s)} u_s = 0_{o(\sigma_s)_n, b} \end{cases}$$

We prove $f < I_b, I(y)u> = <g,u>$.

For $j \in [|a|]$:

- if $x_j^a g = 0_{a_j,b}$ then $x_j^{a+o(y)} f<I_b,I(y)u> = 0_{a_j,b} = x_j^{a+o(y)}<g,u>$

- if $x_j^a g = x_v^b$ then $x_j^{a+o(y)} f<I_b,I(y)u> = x_v^{b+i(y)}<I_b,I(y)u> = x_v^b = x_j^{a+o(y)}<g,u>$

- if $x_j^a g = I_O(\sigma_q,r)u_q$ then $x_j^{a+o(y)} f<I_b,I(y)u> =$

$$= x^{b+i(y)}_{|b|+\Sigma_{t\in[q-1]}|i(\sigma_t)|+r}<I_b,I(y)u> = x_r^{o(\sigma_q)} I(\sigma_q)u_q = I_O(\sigma_q,r)u_q = x_j^{a+o(y)}<g,u>.$$

And for $s \in [k]$ and $n \in [|o(\sigma_s)|]$:

- if $x_n^{o(\sigma_s)} u_s = 0_{o(\sigma_s)_n,b}$ then

$$x^{a+o(y)}_{|a|+\Sigma_{t\in[s-1]}|o(\sigma_t)|+n} f<I_b,I(y)u> = 0_{o(\sigma_s)_n,b} = x^{a+o(y)}_{|a|+\Sigma_{t\in[s-1]}|o(\sigma_t)|+n}<g,u>$$

- if $x_n^{o(\sigma_s)} u_s = x_v^b$ then

$$x^{a+o(y)}_{|a|+\Sigma_{t\in[s-1]}|o(\sigma_t)|+n} f<I_b,I(y)u> =$$

$$= x_v^{b+i(y)}<I_b,I(y)u> = x_n^{o(\sigma_s)}u_s = x^{a+o(y)}_{|a|+\Sigma_{t\in[s-1]}|o(\sigma_t)|+n}<g,u>$$

- if $x_n^{o(\sigma_s)} u_s = I_O(\sigma_q,r)u_q$ then

$$x^{a+o(y)}_{|a|+\Sigma_{t\in[s-1]}|o(\sigma_t)|+n} f<I_b,I(y)u> =$$

$$= x^{b+i(y)}_{|b|+\Sigma_{t\in[q-1]}|i(\sigma_t)|+r}<I_b,I(y)u> = x_r^{i(\sigma_q)} I(\sigma_q)u_q =$$

$$= I_O(\sigma_q,r)u_q = x_n^{o(\sigma_s)}u_s = x^{a+o(y)}_{|a|+\Sigma_{t\in[s-1]}|o(\sigma_t)|+n}<g,u>.$$

We prove $f(I_b + Q(y)u)V_b = (<g,u>,G)$. As $Q(y) = \Sigma_{j\in[k]}(I(\sigma_j), \{(\sigma_j, I_{o(\sigma_j)})\}) =$

$$= (I(y), \{(\sigma_j, T_{o(\sigma_1+\ldots+\sigma_{j-1})} + I_{o(\sigma_j)} + T_{o(\sigma_{j+1}+\ldots+\sigma_k)}) \mid j\in[k]\})$$

we deduce $Q(y)u = (I(y)u,G)$ therefore

$$f(I_b + Q(y)u)V_b = f(I_b + I(y)u,\ G(T_b + I_b))V_b = f(<I_b,I(y)u>,G) = (<g,u>,G).$$

We prove $I(y)U_o(y,f) = I(y)u$. From Proposition 2.6.b we get $I(y)U_o(y,f) = I(y)(T_a + I_{o(y)})T(y,f) = I(y)(T_a + I_{o(y)})f<I_b,I(y)U_o(y,f)>$. From $f<I_b,I(y)u> = <g,u>$ we get $I(y)(T_a + I_{o(y)})f<I_b,I(y)u> = I(y)u$. As there exists a unique $h \in Rat_O(o(y),b)$ such that $I(y)(T_a + I_{o(y)})f<I_b,h> = h$ we obtain $I(y)U_o(y,f) = I(y)u$.

From Proposition 2.6.b we deduce

$T(y,f) = f<I_b,I(y)U_o(y,f)> = f<I_b,I(y)u> = <g,u>$ therefore $U(y,f) = g$ and $U_o(y,f) = u$.

Hence $Q(y,f) = (U(y,f),B(y,f)) = (g,\{(\sigma_j,o(x_j^y)U_o(y,f)) \mid j \in [k]\}) = (g,G)$. $\square$

4. A model for the funflow of the minimal flowchart schemes

4.1. Lemma. If $(y,f) \in Fl(a,b)$ is accessible then $B(y,f)$ is the least hereditary subset of W_b which contains the trees of $U(y,f)$.

Proof. As $Q(y,f) = (U(y,f),\ B(y,f)) \in R(a,b)$ it follows that $B(y,f)$ is hereditary and contains the trees of $U(y,f)$.

Assume $F \subset W_b$ is hereditary and contains the trees of $U(y,f)$. To prove $B(y,f) \subset F$ we suppose $k \in [|y|]$ and we show

$$U_k(y,f) = (y_k,o(x_k^y)U_o(y,f)) \in F.$$

As (y,f) is accessible there exist $j \in [|a|]$; $r \geq 1$; $j_1,\ldots,j_{r-1},j_r = k$ in $[|y|]$; $t_m \in [|i(y_{j_m})|]$ for $m \in [r]$ and $u_m \in [|o(y_{j_m})|]$ for $m \in [r-1]$ such that

$f(j) = |b| + \Sigma_{n\in[j_1-1]}|i(y_n)| + t_1$

$f(|a| + \Sigma_{n\in[j_m-1]}|o(y_n)| + u_m) = |b| + \Sigma_{n\in[j_{m+1}-1]}|i(y_n)| + t_{m+1}$ for $m\in[r-1]$.

As F contains the trees of $U(y,f)$ we deduce from

$$x_j^a U(y,f) = x_j^{a+o(y)} T(y,f) = x_j^{a+o(y)} f\langle I_b, I(y)U_o(y,f)\rangle = x_{f(j)}^{b+i(y)}\langle I_b, I(y)U_o(y,f)\rangle =$$

$$= x^{i(y)}_{\Sigma_{n\in[j_1-1]}|i(y_n)|+t_1} I(y)U_o(y,f) =$$

$$= x_{t_1}^{i(y_{j_1})} I(y_{j_1}) o(x_{j_1}^y)U_o(y,f) = I_O((y_{j_1},t_1)) o(x_{j_1}^y)U_o(y,f)$$ that $U_{j_1}(y,f)\in F$.

The proof go on by induction on $m\in[r-1]$. Assume $U_{j_m}(y,f)\in F$. As F is hereditary and as $(y_{j_m}, o(x_{j_m}^y)U_o(y,f))\in F$ we deduce from

$$x_{u_m}^{o(y_{j_m})} o(x_{j_m}^y)U_o(u,f) =$$

$$= x^{a+o(y)}_{|a|+\Sigma_{n\in[j_m-1]}|o(y_n)|+u_m} f\langle I_b, I(y)U_o(y,f)\rangle =$$

$$= x^{b+i(y)}_{|b|+\Sigma_{n\in[j_{m+1}-1]}|i(y_n)|+t_{m+1}} \langle I_b, I(y)U_o(y,f)\rangle =$$

$$= x^{i(y)}_{\Sigma_{n\in[j_{m+1}-1]}|i(y_n)|+t_{m+1}} I(y)U_o(y,f) =$$

$$= x_{t_{m+1}}^{i(y_{j_{m+1}})} I(y_{j_{m+1}}) o(x_{j_{m+1}}^y)U_o(y,f) = I_O((y_{j_{m+1}}, t_{m+1})) o(x_{j_{m+1}}^y)U_o(y,f)$$

that $U_{j_{m+1}}(y,f)\in F$.

In conclusion $U_k(y,f)\in F$. $\square$

4.2. Theorem. Suppose (x,f) and (y,g) are in $Fl(a,b)$.

a) If $U(x,f) = U(y,g)$ then $(x,f) \sim_{a\delta} (y,g)$

b) $\sim_{a\delta} = \xleftarrow{a\beta}\ \xrightarrow{a\gamma}\ \xleftarrow{a\gamma}\ \xrightarrow{a\beta}$.

Proof. Let (x',f') and (y',g') be the accessible parts of (x,f) and (y,g), respectively. As $\mathbf{U}(x',f') = \mathbf{U}(x,f)$ and $\mathbf{U}(y',g') = \mathbf{U}(y,g)$ we deduce $\mathbf{U}(x',f') = \mathbf{U}(y',g')$. From Lemma 4.1 we deduce $\mathbf{B}(x',f') = \mathbf{B}(y',g')$. By Proposition 2.11 we get $(x',f') \sim_{a\gamma} (y',g')$. As $(x,f) \xleftarrow{a\beta} (x',f')$, $\sim_{a\gamma} = \xrightarrow{a\gamma}\xleftarrow{a\gamma}$ and $(y',g') \xrightarrow{a\beta} (y,g)$ we deduce the conclusions.

4.3. Theorem. Rat_O is a model for the minimal Σ-flowchart schemes.

Proof. Using Theorem 4.2.a we deduce that the model $\mathbf{Fl}/\sim_{a\delta}$ for the minimal Σ-flowchart schemes is isomorphic to the image of $\mathbf{U}$. Therefore it suffices to show the image of $\mathbf{U}$ is $\mathbf{Rat}_O$.

Suppose $g \in \mathrm{Rat}_O(a,b)$. As the trees in g are rational the least hereditary subset G of W_b which contains the trees of g is finite, therefore $(g,G) \in R(a,b)$. As R is the image of Q there exists $(x,f) \in \mathbf{Fl}(a,b)$ such that $Q(x,f) = (g,G)$ hence $U(x,f) = g$. $\square$

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