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MONADIC EQUATIONAL LOGIC

by

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INTRODUCTION

Category theory is now playing a fundamental role in theoretical computer science (cf. [Gog89]). One of the branches of category theory which is most used in some recent developements of theoretical computer science is monad theory (cf. [BR85, Gog89]).

The fundamental aim of our paper is to give a generalization of equational logic in the categorical framework of monads. The logical system thus obtained is named "monadic equational logic" and we show that this logical system is an institution (cf. [GB85]), therefore it fits in the general framework of "abstract model theory for computer science". There are also proved some completeness results for this type of equational logic.

As signatures for monadic equational logic we take monads without respect to the underlying category in which they are defined. This ideea leads to the extension of the classical notion of monad morphism (see [MacL71, Man76] for the classical notion of monad morphism). The models of monadic equational logic are defined to be Eilenberg-Moore algebras.

A briefly exposition of the basic facts in both monad theory and the theory of institutions are the subject of the preliminary chapter.

In the second chapter, after the formal definition of monadic institutions which is based on the generalization of the classical notion of monad morphism, we discuss some examples. The example of many-sorted algebras is the leading one because of its importance for theoretical computer science and it is exposed in great detail. This example clearly suggests that our more general notion of monad morphism corresponds to the categorical passing from the homogenuous to the heterogenuous case in universal algebra.

Under very mild conditions one may consider internal equations as sentences in monadic institutions. This can be done due to the fact that the "satisfaction condition" holds for internal equations. We belive that this theorem is the heart of the second chapter. The satisfaction condition for many-sorted equational logic (which was first proved in [GB85]) appears as a set-theoretical corollary of this theorem.

The third chapter is devoted to the study of completeness of monadic equational logic by using the categorical generalization of some fundamental notions in universal algebra like relation, equivalence, congruence and fully invariant congruence. We give here a general categorical construction of the congruence and the fully invariant congruence generated by a relation. The completeness of monadic equational logic extends the classical equational completeness theorem which is due to Birkhoff (cf. [Grä79]) and also the completeness of many-sorted equational logic of [GM85].

The appendix is dedicated to those readers which are familiar with the abstract algebraic institutions of Tarlecki (introduced in [Tar85, 86]) There we show (without proofs) that under some mild and natural conditions monadic institutions become abstract algebraic.

1. PRELIMINARIES

Throughout the paper $|\underline{\underline{C}}|$ denotes the class of objects of the category $\underline{\underline{C}}$, while for any objects $A,B \in |\underline{\underline{C}}|$ the set of arrows (morphisms) $A \longrightarrow B$ is denoted by $\underline{\underline{C}}(A,B)$. Composition of arrows is written in the diagrammatic order, i.e. if $f:A \longrightarrow B$, $g:B \longrightarrow C$ then $fg:A \longrightarrow C$. The composition between a functor and a natural transformation is the only exception of this rule (because chasing diagrams needs the use of components of natural transformations), for example if $F \in \underline{\underline{Cat}}(A,B)$, $G,G' \in \underline{\underline{Cat}}(B,C)$ and $\theta: G \xrightarrow{\cdot} G'$ is a natural transformation then $\theta F: FG \xrightarrow{\cdot} FG'$. (Here the hypercategory of all categories is denoted by $\underline{\underline{Cat}}$).

For the basic notions of category theory (functor, natural transformation, limit, colimit, adjunction, Kan extension etc.) we use the same terminology as [MacL71].

MONADS

The role played by monads in universal algebra is well known (there are several monographies devoted to the subject, for example [Man76]). The increasing role played by monads in theoretical computer science was recently the subject of a survey [BR85]. Here we shall briefly recall some basic facts concerning monads, an introduction to the subject being the chapter devoted to monads in [MacL71].

A monad (T,μ,η) in the category X is a monoid in the strict monoidal category of endofunctors of X (see [MacL71]), otherwise said, it consists of a functor $T: X \longrightarrow X$ and two natural transformations $\mu: T^2 \xrightarrow{\cdot} T$ and $\eta: 1_X \xrightarrow{\cdot} T$ which satisfies

the associative law: $\mu T \cdot \mu = T\mu \cdot \mu$

and the unit law: $\eta T \cdot \mu = T\eta \cdot \mu = 1_T$.

Each adjunction $(F,G,\eta,\varepsilon): X \longrightarrow A$ gives rise to the monad $(FG, G\varepsilon F, \eta)$ in X and, conversely, these are all the monads. More precisely, given a monad (T,μ,η) in a category X one may built Eilenberg-Moore T-algebras category, which is denoted X^T, having T-algebras (x,ξ) as objects ($x \in |X|$ being the underlying object of algebra, $\xi: Tx \longrightarrow x$, subject to $T\xi \cdot \xi = \mu_x \cdot \xi$ and $\eta_x \cdot \xi = 1_x$, being the structure map of algebra) and T-morphisms $h: (x,\xi) \longrightarrow (x',\xi')$ as arrows ($h \in X(x,x')$ commutes with the algebraic structure, i.e. $Th \cdot \xi' = \xi \cdot h$). The forgetful functor $G^T: X^T \longrightarrow X$ has a left-adjoint $F^T: X \longrightarrow X^T$, $F^T x = (Tx, \mu_x)$ on objects and $F^T h = Th$ on arrows. The monad defined by this adjunction $X \longrightarrow X^T$ is just (T,μ,η).

If a right-adjoint functor $G: A \longrightarrow X$ behaves like a G^T (modulo an isomorphism, of course) G is said to be monadic in [MacL71] or algebraic in [Man76]. Monadic functors have a lot of important properties which are discussed in the books devoted to the subject.

A lot of examples of monadic functors could be found in [Man76], an important one for theoretical computer science being the forgetful from any variety of many-sorted algebras to many-sorted sets. This notion is so general that it includes some surprising examples, one of them being the forgetful from compact spaces to sets.

INSTITUTIONS

The theory of institutions was started in [GB85] as an abstract model theory for computer science. Some recent developements of the theory of institutions are oriented through pure model theory (see [Tar86,87]).

The notion of institution is an elegant categorical formalization of the intuitive notion of "logical system". It consists of

a category Sign of signatures,

a functor $Mod: Sign^o \longrightarrow \underline{\underline{Cat}}$ giving models,

a functor $Sen: Sign \longrightarrow \underline{\underline{Cat}}$ giving sentences and proofs (usualy $Sen: Sign \longrightarrow \underline{\underline{Set}}$ for the semantic-oriented approaches)

a satisfaction relation $\models_\Sigma \subseteq |Mod\,\Sigma| \times |Sen\,\Sigma|$ for each $\Sigma \in |Sign|$ such that for each $A' \in |Mod\,\Sigma'|$, $\varphi \in |Sen\,\Sigma|$ and $\phi \in Sign(\Sigma, \Sigma')$

$$A' \models_{\Sigma'} (Sen\,\phi)\varphi \text{ iff } (Mod\,\phi)A' \models_\Sigma \varphi$$

This is called the <u>satisfaction condition</u> and it express our insight that the truth does not depend on the actual notation.

"Any logical system is an institution" is the thesis formulated in the pioneering article [GB85], where there are also explored some examples, the leading ones being those of many-sorted equational logic (which is relevant for theoretical computer science) and many-sorted first-order logic (relevant for classical model theory). A further serie of articles on this subject ([GB86, Tar85, 86, 87] etc.) develops this notion with the aim of applying the resulting theory to abstract data types, artificial inteligence or model theory (for the model-theoretic oriented develo-pements, abstract algebraic institutions [Tar85, 86] are especially relevant).

At the general level, any institution yields, for any signature $\Sigma \in |\mathrm{Sign}|$, the well known Galois connection between models and sentences

$$\mathcal{P}|\mathrm{Mod}\,\Sigma| \underset{*}{\overset{*}{\rightleftarrows}} \mathcal{P}|\mathrm{Sen}\,\Sigma|$$

defined by $E^* = \{A \in |\mathrm{Mod}\Sigma| \,:\, A \models_\Sigma \varphi \text{ for any } \varphi \in E\}$ for any $E \subseteq |\mathrm{Sen}\Sigma|$, and

$$M^* = \{\varphi \in |\mathrm{Sen}\Sigma| \,:\, A \models_\Sigma \varphi \text{ for any } A \in M\} \text{ for any } M \subseteq |\mathrm{Mod}\Sigma|.$$

A <u>presentation</u> in the institution $\underline{\underline{I}}$ is a pair (Σ, E), $\Sigma \in |\mathrm{Sign}|$ and $E \subseteq |\mathrm{Sen}\Sigma|$. A <u>morphism of presentations</u> $(\Sigma, E) \xrightarrow{\lambda} (\Sigma', E')$ is just a signature morphism $\Sigma \xrightarrow{\lambda} \Sigma'$ which maps E to E', more precisely $(\mathrm{Sen}\lambda)E \subseteq E'$. This yields a category $\mathrm{Pres}_{\underline{\underline{I}}}$ of all presentations in $\underline{\underline{I}}$. The full subcategory of all presentations (Σ, E) which are closed under semantical consequence, $E^{**} = E$, is called the category of theories in $\underline{\underline{I}}$ and it is denoted $\mathrm{Th}_{\underline{\underline{I}}}$. Some results in this area could be found in [GB85].

2. MONADIC INSTITUTIONS

The basic ideea which lies behind the notion of monadic institution is to look after monads as signatures for "logical systems". It is thus necessarily to consider morphisms between monads defined in different categories. This is done by the generalization of the classical notion of monad morphism.

THE HYPERCATEGORY OF ALL THE MONADS

Classicaly, monad morphisms are just monoid morphisms in the strict monoidal category of endofunctors of some category (see [MacL71] or [Man76]).

2.1 Definition Let (T,μ,η) be a monad in the category X and (T',μ',η') be a monad in other category X'. A morphism $(T,\mu,\eta)\longrightarrow(T',\mu',\eta')$ is a pair $\langle\lambda,G\rangle$, with $G:X'\longrightarrow X$ functor and $\lambda:GT\overset{\cdot}{\longrightarrow}T'G$ natural transformation s.th. $\eta G\cdot\lambda = G\eta'$

and $\mu G\cdot\lambda = \lambda^2\cdot G\mu'$, where $\lambda^2 = T\lambda\cdot\lambda T':GT^2\longrightarrow T'^2G$. ∎

Observe that the classical notion of monad morphism is obtained from the preceeding one when X=X' and G is the identity functor.

2.2 Definition The composition of monad morphisms is defined by

$$(T,\mu,\eta)\xrightarrow{\langle\lambda,G\rangle}(T',\mu',\eta')\xrightarrow{\langle\lambda',G'\rangle}(T'',\mu'',\eta'') = \langle\lambda G'\cdot G\lambda',G'G\rangle. ∎$$

A short diagram chase shows us that this composition is associative and thus we have built the hypercategory MON of ALL the monads (without respect to the category they are defined in).

At the end of [BR85] one can find a brief discussion on the relevance of definition 2.1 for theoretical computer science.

The following proposition is crucial for the definition of monadic institutions:

2.3 Proposition Let (T,μ,η) and (T',μ',η') be monads in X and X' respectively. Every monad morphism $\langle\lambda,G\rangle:(T,\mu,\eta)\longrightarrow(T',\mu',\eta')$ gives a functor $ALG\langle\lambda,G\rangle:X'^{T'}\longrightarrow X^T$ defined by

$(a',\alpha')\longmapsto(Ga',\lambda_{a'}\cdot G\alpha')$ on algebras and $f\longmapsto Gf$ on morphisms

Proof: The fact that $(Ga',\lambda_{a'}\cdot G\alpha')$ is indeed T-algebra immediately follows by monad morphism properties of $\langle\lambda,G\rangle$ (see def.2.1) and by the fact that (a',α') is a T'-algebra. Adding the naturality of λ to T'-morphism property of f we obtain T-morphism property for Gf. ALG$\langle\lambda, G\rangle$ is a functor since G is a functor. ■

Observe that ALG is a functor $\text{MON}^0 \longrightarrow \text{Cat}$, otherwise said ALG is a MON-indexed category (see [TBG88] for the notion of indexed category). We are now ready to give the definition of monadic institutions.

2.4 Definition An institution (Sign, Mod, Sen, $\models$) is called monadic iff Sign is a subcategory of MON and Mod is the restriction of ALG to Sign. ■

The very semantic nature of this definition is obvious, since it says nothing about syntax. Nevertheless, the discussion concerning syntax in monadic institutions will be the subject of the last paragraph of this section.

Some examples of this general definition are discussed in the next paragraph. The most important of them is that of many-sorted algebraic theories.

MANY-SORTED ALGEBRAIC THEORIES

Every morphism of many-sorted algebraic theories is seen as a (generalized - in the sense of definition 2.1) monad morphism, while morphisms of homogenuous (one-sorted) algebraic theories are seen as classical monad morphisms. This is perhaps an argument to see the extension of the classical notion of monad morphism (def.2.1) as a categorical passing from the homogenuous to the heterogenuous case.

Let Set be the category of all sets. For any set S we have the functor category Set^S of S-sorted sets. Any function $G:S \longrightarrow S'$ induces a functor $\underline{G}=G\_:\text{Set}^{S'} \longrightarrow \text{Set}^S$ (the left-composition with G).

2.5 Proposition For any $G:S \longrightarrow S'$ the induced functor $\underline{G}:\text{Set}^{S'} \longrightarrow \text{Set}^S$ has a left and a right-adjoint. The left-adjoint of $\underline{G}$ maps any S-sorted set X to the S'-sorted set X' defined by

$$X'_{s'} = \coprod_{Gs=s'} X_s$$

<u>Proof:</u> Since S is small (discrete) category and <u>Set</u> is small complete and cocomplete any $G:S \longrightarrow S'$ has a right and a left Kan-extension along any S-sorted set X (see [MacL71]). The expression of the left adjoint of <u>G</u> is given by the construction of left Kan-extensions as pointwise colimits (cf. [MacL71]). ∎

A <u>S-sorted signature</u> is a ranked set $\Sigma \xrightarrow{r} S^{*}\times S$ of operation symbols (the free monoid generated by S is denoted by S^{*}). $r(\nabla)=(s1\ldots sn,s)$ is often denoted as $\nabla: s1\ldots sn \longrightarrow s$, the interpretation ∇_A of ∇ in any S-sorted Σ-algebra $(A,(\nabla_A)_{\nabla\in\Sigma})$ being a function $A_{s1}\times A_{s2}\times\ldots\times A_{sn} \longrightarrow A_s$.

<u>Algebraic signatures</u> are pairs of the form $(S, \Sigma \xrightarrow{r} S^{*}\times S)$, where S is the set of sorts and $\Sigma \xrightarrow{r} S^{*}\times S$ is a S-sorted signature. An algebraic signature morphism from $(S, \Sigma \xrightarrow{r} S^{*}\times S)$ to $(S', \Sigma' \xrightarrow{r'} S'^{*}\times S')$ consists of a renaming of sorts $G:S \longrightarrow S'$ and a renaming of operation symbols $\lambda:\Sigma \longrightarrow \Sigma'$ preserving their ranks, i.e. $\lambda\cdot r'=r\cdot(G^{*}\times G)$

$$\begin{array}{ccc} \Sigma & \xrightarrow{\quad r \quad} & S^{*}\times S \\ \lambda\downarrow & & \downarrow G^{*}\times G \\ \Sigma' & \xrightarrow{\quad r' \quad} & S'^{*}\times S' \end{array}$$

Let <u>AlgSig</u> be the category of algebraic signatures (see [GB85]). There is a functor $Alg:\underline{AlgSig}^{o} \longrightarrow \underline{Cat}$ which maps any algebraic signature (S,Σ) to Alg_{Σ} - the category of S-sorted Σ-algebras and which maps any signature morphism $(G,\lambda):(S,\Sigma) \longrightarrow (S',\Sigma')$ to the functor $Alg(G,\lambda):Alg_{\Sigma'} \longrightarrow Alg_{\Sigma}$ defined as

$(A',(\nabla'_{A'})_{\nabla'\in\Sigma'}) \longmapsto (GA',((\lambda\nabla)_{A'})_{\nabla\in\Sigma})$ on algebras

and $f \longmapsto fG$ on morphisms.

Alg is often denoted as $\models$ (see [GB85, Tar85, 86] and others).

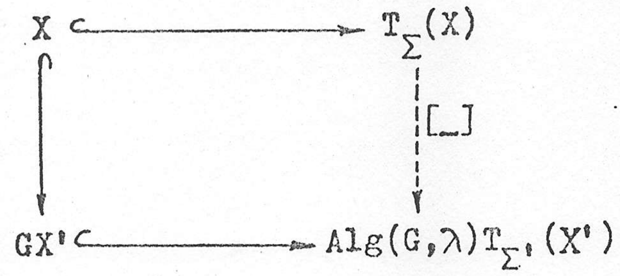
Given an S-sorted signature Σ and an S-sorted set X let $T_\Sigma(X)$ be the free Σ-algebra generated by X (the set of Σ-terms over the variables X). Following [GB85] or [GM83] (or others) a S-sorted equation in variables X is a pair $(l,r)\in T_\Sigma(X)^2$.

A Σ-algebra satisfies this equation, in symbols

$$(A,(\sigma_A)_{\sigma\in\Sigma})\models_\Sigma l=r, \quad \text{iff}$$

for any "valuation" of variables, $v:X\longrightarrow A$, (l,r) belongs to the kernel of $v^\#$, where $v^\#:T_\Sigma(X)\longrightarrow(A,(\sigma_A)_{\sigma\in\Sigma})$ is the unique extension of v to a Σ-morphism.

Let (G,λ) be an algebraic signature morphism $(S,\Sigma)\longrightarrow(S',\Sigma')$. Using the notations of proposition 2.5 define $[\_]:T_\Sigma(X)\longrightarrow T_{\Sigma'}(X')$ as the unique extension of $X\hookrightarrow GX'\hookrightarrow Alg(G,\lambda)T_{\Sigma'}(X')$ to a Σ-morphism $T_\Sigma(X)$—— $Alg(G,\lambda)T_{\Sigma'}(X')$



We thus obtain a translation of any Σ-equation in variables X to a Σ'-equation in variables X' by $(l,r)\longmapsto([l],[r])$.

All the facts presented from the begining of that paragraph are the ingredients of many-sorted equational logic [GB85]. The above definitions (algebraic signatures, the functor Alg, equations etc.) could be organized as an institution as follows:

- the category of signatures is that of algebraic signatures <u>AlgSig</u>

- the model functor is Alg

-the sentence functor $\underline{\underline{AlgSig}} \longrightarrow \underline{\underline{Set}}$ maps any signature (S,Σ) to the set of S-sorted Σ-equations, for any signature morphism the induced translations of equations being above defined.

-the satisfaction relation $\models$ is the usual one (being above presented).

A beautiful but expected result [GB85] says that many-sorted equational logic is an institution (i.e. the satisfaction condition holds).

A <u>many-sorted algebraic theory</u> (algebraic theory for short) is just a theory in the institution of many-sorted equational logic.

The theories of any institution can be viewed as signatures in other institutions in the following way:

<u>2.6 Remark</u> Given an institution $\underline{\underline{I}} = (\text{Sign}, \text{Mod}, \text{Sen}, \models)$ let $\text{Modt}: \text{Th}_{\underline{\underline{I}}} \longrightarrow \underline{\underline{Cat}}$ be defined by

$\text{Modt}(\Sigma,E)$ is the full subcategory of $\text{Mod}\Sigma$ having E^* as objects and for any theory morphism $(\Sigma,E) \longrightarrow (\Sigma',E')$, $\text{Modt}\lambda$ is the restriction of $\text{Mod}\lambda$.

Then Modt is well defined (in the virtue of satisfaction condition), thus Modt could play a model functor role. ☐

One may thus consider institutions having algebraic theories as signatures and varieties of algebras as models. This ideea seems to be close related to Lawvere algebraic theories [Law63].

<u>2.7 Proposition</u> Any institution having algebraic theories as signatures and varieties of algebras as models is a monadic institution.

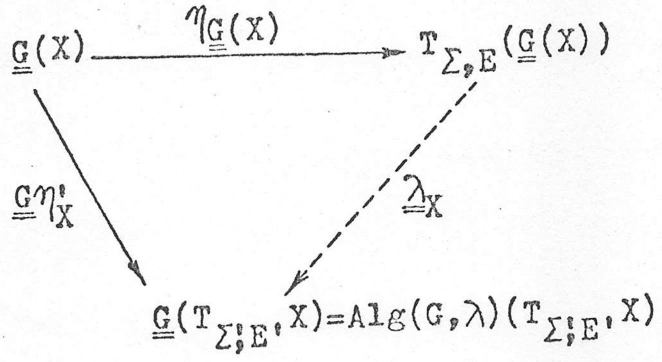
<u>Proof</u>: The first step is to show how algebraic theories morphisms could be naturally viewed as monad morphisms (in the sense of definition 2.1)

Let $(S,\Sigma,E) \xrightarrow{(G,\lambda)} (S',\Sigma',E')$ be an algebraic theory morphism. As known (see [Man76] or [BR85] for example), the forgetful functor $Alg_{\Sigma,E} \longrightarrow \underline{\underline{Set}}^S$ is monadic (algebraic), the monad $(T_{\Sigma,E},\mu,\eta)$ thus defined in $\underline{\underline{Set}}^S$ building (Σ,E)-terms (i.e. classes of Σ-terms which are equivalent under the congruence generated by E), and $Alg_{\Sigma,E}$ being just the category of $T_{\Sigma,E}$-algebras in $\underline{\underline{Set}}^S$. Let $(T_{\Sigma',E'},\mu',\eta')$ be the monad defined in $\underline{\underline{Set}}^{S'}$ by the forgetful $Alg_{\Sigma',E'} \longrightarrow \underline{\underline{Set}}^{S'}$.

(G,λ) induces a morphism $\langle \underline{\underline{\lambda}}, \underline{\underline{G}} \rangle : (T_{\Sigma,E},\mu,\eta) \longrightarrow (T_{\Sigma',E'},\mu',\eta')$ in the following manner:

$\underline{\underline{G}} : \underline{\underline{Set}}^{S'} \longrightarrow \underline{\underline{Set}}^S$ is $G\_$ (see prop.2.5)

$\underline{\underline{\lambda}} : \underline{\underline{G}}T_{\Sigma,E} \longrightarrow T_{\Sigma',E'}\underline{\underline{G}}$ is the natural transformation having $\underline{\underline{\lambda}}_X$, $X \in |\underline{\underline{Set}}^{S'}|$, as components. $\underline{\underline{\lambda}}_X$ is defined to be the unique extension of $\underline{\underline{G}}\eta'_X$ to a Σ-morphism.



The fundamental fact which has to be observed here is that $\underline{\underline{G}}(T_{\Sigma',E'}X) = Alg(G,\lambda)(T_{\Sigma',E'}X)$ is indeed a (Σ,E)-algebra (since $Sen(G,\lambda)E \subseteq E'$), and thus the universal property of $T_{\Sigma,E}(\underline{\underline{G}}X)$ can be applied.

$\underline{\underline{\lambda}}$ is natural in the virtue of naturality of η and η'.

We have now to check that $\mu\underline{\underline{G}} \cdot \underline{\underline{\lambda}} = \underline{\underline{\lambda}}^2 \cdot \underline{\underline{G}}\mu'$. Indeed, for any S'-sorted set X we have:

$$\eta_{T_{\Sigma,E}(\underline{\underline{G}}(X))} \cdot T_{\Sigma,E}\underline{\underline{\lambda}}_X \cdot \underline{\underline{\lambda}}_{T_{\Sigma',E'}X} \cdot \underline{\underline{G}}\mu'_X = \text{(naturality of } \eta \text{)}$$

$$= \underline{\underline{\lambda}}_X \cdot \eta_{\underline{\underline{G}}(T_{\Sigma',E'}X)} \cdot \underline{\underline{\lambda}}_{T_{\Sigma',E'}X} \cdot \underline{\underline{G}}\mu'_X = \text{(definition of } \underline{\underline{\lambda}} \text{)}$$

$$= \underline{\underline{\lambda}}_X \cdot \underline{\underline{G}}\eta'_{T_{\Sigma',E'}X} \cdot \underline{\underline{G}}\mu'_X = \text{(monad properties)} \ \underline{\underline{\lambda}}_X = \text{(monad properties)}$$

$= \eta_{T_{\Sigma,E}(\underline{\underline{G}}(X))} \cdot \mu_{\underline{\underline{G}}(X)} \cdot \underline{\underline{\lambda}}_X.$

Since $T_{\Sigma,E}\underline{\underline{\lambda}}_X \cdot \underline{\underline{\lambda}}_{T_{\Sigma',E'},X} \cdot \underline{\underline{G}}\mu'_X$ and $\mu_{\underline{\underline{G}}(X)} \cdot \underline{\underline{\lambda}}_X$ are Σ-morphisms (as composites of Σ-morphisms) it follows that they are equal.

The remark that Alg:algebraic theories ⟶ Cat is just the restriction of ALG (see prop.2.3) will finish the proof of this proposition. ∎

When one-sorted (homogenuous) algebraic theories morphisms are involved remark that the induced monad morphism are classical monad morphisms (as in [MacL71, Man76]).

Instead of many-sorted algebras one may consider many-sorted continuous algebras (see [BR85] or [ADJ77]). Here monads over categories of sorted complete partial orders have to be considered. It is also possible to consider mixtures between ordinary algebras and continuous algebras.

An interesting example of monadic institution is provided by linear algebra:

An institution (CRng, Module, Sen, $\models$) in linear algebra has the category of commutative rings, CRng, as signatures category.
The model functor Module: CRng^{o} ⟶ Cat gives for any commutative ring R its corresponding module category R-Mod and for any ring morphism R ⟶ R' it gives the natural translation R'-Mod ⟶ R-Mod. CRng can be embedded in MON by R ⟼ (T^R, μ^R, η^R), where (T^R, μ^R, η^R) is the monad defined by the (monadic) forgetful R-Mod ⟶ Set.
Module is just the restriction of ALG to CRng.

This example is only a particular case of a more general one. That one is constituted by actions over objects in monoidal categories and it is strongly connected to categorical automata theory.

MONADIC EQUATIONAL LOGIC

The categorical approach of syntax of equational logic has a long story (see [Hat70, HR72, Man76] or more recently [BR85]); The central notion of this approach is given by the following definition:

2.8 Definition Let (T,μ,η) be a monad in a category X.

A T-equation having $x\in|X|$ as the "object of variables" is a parallel pair $k \overset{l}{\underset{r}{\rightrightarrows}} Tx$ in X (also denoted as (l,r) or l=r).

This T-equation is satisfied by a T-algebra (a,α) w.r.t. the variables valuation $v:x\longrightarrow a$ (in symbols $(a,\alpha)\models l=r[v]$) iff $lv^{\#}=rv^{\#}$, where $v^{\#}:(Tx,\mu_x)\longrightarrow(a,\alpha)$ is the unique T-morphism extension of v.

The equation l=r is satisfied by (a,α) iff it is satisfied by (a,α) w.r.t. any valuation of variables. ∎

Clearly, the kind of logic involved by this definition is an equational one. However, observe that the sentences shaped in definition 2.8 seem to be families of equations over the same set of variables rather than classical equations, k playing the role of "object of indices". The aim of this paragraph is to investigate the conditions which make this type of logic an institution. It is obvious that the institutions involved here will be monadic ones (in the sense of definition 2.4).

2.9 Definition Let $\langle\lambda,G\rangle:(T,\mu,\eta)\longrightarrow(T',\mu',\eta')$ be a monad morphism such that $G:X'\longrightarrow X$ has a left-adjoint $F:X\longrightarrow X'$. If ϕ^G is the unit of adjunction $X\longrightarrow X'$ and $\_^{\square}:X(x,Ga)\xrightarrow{\simeq}X'(Fx,a)$ is the natural bijection defined by this adjunction, then we define

$$[\_]^{k,x}_{\lambda}:X(k,Tx)\longrightarrow X'(Fk,T'(Fx)) \quad \text{by} \quad h\longmapsto(h\cdot T\phi^G_x\cdot\lambda_{Fx})^{\square}. \blacksquare$$

If $\underline{MON}_0$ is the subcategory of $\underline{MON}$ of all monad morphisms $\langle\lambda,G\rangle$ having right-adjoint functors as functor components G, then one may define a functor $\underline{EQN}:\underline{MON}_0\longrightarrow\underline{Set}$ which will play the sentence-functor role.

2.10 Definition $\underline{EQN}$: $\underline{MON} \longrightarrow \underline{Set}$ is defined as follows:

for any monad (T,μ,η) having X as its underlying category:

$$\underline{EQN}(T,\mu,\eta) = \{k \overset{r}{\underset{l}{\rightrightarrows}} Tx \; : \; k,x \in |X|\}, \text{ and}$$

for any monad morphism $(T,\mu,\eta) \xrightarrow{\langle\lambda,G\rangle} (T',\mu',\eta')$ in $\underline{MON}_0$:

$$(\underline{EQN}\langle\lambda,G\rangle)(k \overset{l}{\underset{r}{\rightrightarrows}} Tx) = ([l]_\lambda^{k,x}, [r]_\lambda^{k,x}) \text{ for any } (l,r) \in \underline{EQN}(T,\mu,\eta). \blacksquare$$

The following lemma proves the correctness of the previous definition:

2.11 Lemma Let $(T,\mu,\eta) \xrightarrow{\langle\lambda,G\rangle} (T',\mu',\eta') \xrightarrow{\langle\lambda',G'\rangle} (T'',\mu'',\eta'')$ be monad morphisms in $\underline{MON}_0$. Using notations of definition 2.9 we have

$$[[h]_\lambda^{k,x}]_{\lambda'}^{Fk,Fx} = [h]_{\lambda G'\cdot G\lambda'}^{k,x} \text{ for any } h \in X(k,Tx) .$$

Proof: The unit of the composed adjunction $X \longrightarrow X' \longrightarrow X''$ is $\phi^{G'G} = \phi^G \cdot G\phi^{G'}_F$ (see composition of adjoints [MacL71]).

$$\phi^G_k \cdot G\phi^{G'}_{Fk} \cdot G(G'[[h]_\lambda]_{\lambda'}) = (\text{definition of } [\_]_{\lambda'}^{Fk,Fx})$$

$$= \phi^G_k \cdot G[h]_\lambda \cdot G(T'\phi^{G'}_{Fx}) \cdot G\lambda'_{F'(Fx)} = (\text{definition of } [\_]_\lambda^{k,x})$$

$$= h \cdot T\phi^G_x \cdot \lambda_{Fx} \cdot G(T'\phi^{G'}_{Fx}) \cdot G\lambda'_{F'(Fx)} = (\text{naturality of } \lambda)$$

$$= h \cdot T\phi^G_x \cdot T(G\phi^{G'}_{Fx}) \cdot \lambda_{G'(F'(Fx))} \cdot G\lambda'_{F'(Fx)} = (\text{definition of } \phi^{G'G})$$

$= h \cdot T\phi^{G'G}_x \cdot (\lambda G' \cdot G\lambda')_{F'(Fx)}$. Since $\phi^{G'G}_k$ is an universal arrow we have that

$$[[h]_\lambda]_{\lambda'} = (h \cdot T\phi^{G'G}_x \cdot (\lambda G' \cdot G\lambda')_{F'(Fx)})^\square = [h]_{\lambda G'\cdot G\lambda'} . \blacksquare$$

Definition 2.10 could be also regarded as the internalization of the notion of equation in monadic institutions. Let $\underline{ALG}_0$ be the restriction of $\underline{ALG}$ to $\underline{MON}$. The main result of this paragraph lies in the following theorem:

2.12 THEOREM $(\underline{MON}_0, \underline{ALG}_0, \underline{EQN}, \models)$ is a monadic institution.

Proof: We have to check that satisfaction condition holds, that is for any $\langle\lambda,G\rangle : (T,\mu,\eta) \longrightarrow (T',\mu',\eta')$ in $\underline{MON}_0$, for any T'-algebra (a,α) and for any T-equation $k \overset{l}{\underset{r}{\rightrightarrows}} Tx$

$$(a,\alpha)\models [l]_\lambda=[r]_\lambda \text{ iff } (Ga,\lambda_a\cdot G\alpha)\models l=r \,.$$

Using the notations of definition 2.9, observe that $\_^\Box : X(x,Ga) \longrightarrow X'(Fx,a)$ can be regarded as an one-one correspondence between the valuations $Fx \longrightarrow a$ and the valuations $x \longrightarrow Ga$. It thus suffices to show that for any $x \xrightarrow{v} Ga$

$$(a,\alpha)\models [l]_\lambda=[r]_\lambda[v^\Box] \text{ iff } (Ga,\lambda_a\cdot G\alpha)\models l=r[v].$$

The first step is to prove that $T\phi^G_x\cdot\lambda_{Fx}$ is T-morphism $(Tx,\mu_x)\longrightarrow(\underline{\underline{ALG}}\langle\lambda,G\rangle)(T'(Fx),\mu'_{Fx})$.

$$T^2\phi^G_x\cdot T\lambda_{Fx}\cdot\lambda_{T'(Fx)}\cdot G\mu'_{Fx} = (\text{definition of } \lambda^2)$$

$$= T^2\phi^G_x\cdot\lambda^2_{Fx}\cdot G\mu'_{Fx} = (\langle\lambda,G\rangle \text{ is monad morphism})$$

$$= T^2\phi^G_x\cdot\mu_{G(Fx)}\cdot\lambda_{Fx} = (\mu \text{ natural})\ \mu_x\cdot T\phi^G_x\cdot\lambda_{Fx}\,.$$

Let $v^\#:(Tx,\mu_x)\longrightarrow(Ga,\lambda_a\cdot G\alpha)$ and $v^{\Box\#}:(T'(Fx),\mu'_{Fx})\longrightarrow(a,\alpha)$ be the unique extensions of v and $v^\Box$ to morphisms.

$$\eta_x\cdot T\phi^G_x\cdot\lambda_{Fx}\cdot Gv^{\Box\#} = (\eta \text{ is natural})$$

$$= \phi^G_x\cdot\eta_{G(Fx)}\cdot\lambda_{Fx}\cdot Gv^{\Box\#} = (\lambda \text{ is monad morphism})$$

$$= \phi^G_x\cdot G\eta'_{Fx}\cdot Gv^{\Box\#} = \phi^G_x\cdot Gv^\Box = v = \eta_x\cdot v^\#.$$

Since $T\phi^G_x\cdot\lambda_{Fx}\cdot Gv^{\Box\#}$ is a T-morphism (as composition of T-morphisms) it is just $v^\#$ (η_x being universal arrow).

It follows that $(Ga,\lambda_a\cdot G\alpha)\models l=r[v]$ iff $lv^\#=rv^\#$ iff

$$l\cdot T\phi^G_x\cdot\lambda_{Fx}\cdot Gv^{\Box\#} = r\cdot T\phi^G_x\cdot\lambda_{Fx}\cdot Gv^{\Box\#} \text{ iff}$$

$$\phi^G_k\cdot G[l]_\lambda\cdot Gv^{\Box\#} = \phi^G_k\cdot G[r]_\lambda\cdot Gv^{\Box\#} \text{ iff}$$

$$[l]_\lambda\cdot v^{\Box\#} = [r]_\lambda\cdot v^{\Box\#} \text{ (the universal property of } \phi^G_k) \text{ iff}$$

$$(a,\alpha)\models [l]_\lambda=[r]_\lambda[v^\Box].\ \blacksquare$$

This theorem could be considered as a categorical generalization of the main theorem concerning many-sorted equational logic in [GB85] (asserting that many-sorted equational logic is an institution). This remark is due to proposition 2.5. Moreover, the restriction of EQN to AlgSig is just the translation of equations as defined in [GB85].

3. COMPLETENESS

The study of completeness of logical systems plays a central role in any kind of model theory. Equational logic has its own completeness theory. The first result was obtained in the field of homogenuous universal algebra by Birkhoff(1935) (see [Grä79]). This result was recently extended to many-sorted universal algebra in [GM85]. The purpose of this chapter is to extend these results to monadic equational logic.

THE FULLY INVARIANT CONGRUENCE GENERATED BY A RELATION

The following three definitions agree to the set-theoretic notion of relation, equivalence and congruence in use in classical universal algebra. They are part of the folklore of categorical universal algebra.

3.1 Definition Let X be a category and let a be an object in X. A relation on a is a parallel pair $k \overset{l}{\underset{r}{\rightrightarrows}} a$. □

Recall (cf. [MacL71] or [Man76]) that given any arrow $a \xrightarrow{f} b$ its kernell pair (kernel, for short) is the pullback of the pair (f,f).

3.2 Definition A relation $k \overset{l}{\underset{r}{\rightrightarrows}} a$ on the object a in the category X is an equivalence iff there is an arrow $a \xrightarrow{f} b$ s.th. (l,r) is the kernel of f, i.e. (l,r)=kerf. □

3.3 Definition Let (T,μ,η) be a monad in X. A relation $k \overset{l}{\underset{r}{\rightrightarrows}} (a,\alpha)$ on the underlying object a of a T-algebra (a,α) is a congruence iff there is a T-morphism $(a,\alpha) \xrightarrow{f} (b,\beta)$ s.th. (l,r)=kerf. □

Observe that any congruence is an equivalence (since the forgetful functor from algebras to the underlying category X preserves limits as a right-adjoint functor).

The following definition is only a categorical formalization of the set-theoretic inclusion between relations.

<u>3.4 Definition</u> Let $k \overset{l}{\underset{r}{\rightrightarrows}} a$ and $k' \overset{l'}{\underset{r'}{\rightrightarrows}} a$ be relations on the same object a. Then (l',r') <u>includes</u> (l,r), in symbols $(l,r)\subseteq(l',r')$, iff there is $h: k \longrightarrow k'$ s.th. $l=hl'$ and $r=hr'$. ■

Observe that $\subseteq$ is a preorder and that we may identify relations which are equivalent under the equivalence $\subseteq\cap\supseteq$.

<u>3.5 Remark</u> Let (T,μ,η) be a monad in a category X. Any T-equation over the object of variables x is just a relation on Tx.

Given a T-algebra (a,α), for any valuation of variables $x \xrightarrow{v} a$, if $(k \overset{l}{\underset{r}{\rightrightarrows}} Tx) \subseteq (k' \overset{l'}{\underset{r'}{\rightrightarrows}} Tx)$ then $(a,\alpha)\models l'=r'[v]$ implies $(a,\alpha)\models l=r[v]$, and thus $(a,\alpha)\models l'=r'$ implies $(a,\alpha)\models l=r$. ■

This corresponds to the intuition that any model which satisfies a set of sentences should also satisfy any subset of this sentences. Observe also that equivalent (under $\subseteq\cap\supseteq$) T-equations are satisfied by the same class of T-algebras.

Birkhoff Completeness Theorem is sometimes formulated using the notion of fully invariant congruence (see [Grä79] for example). This is a more algebraic treatment of syntax and it seems to be also adequate for our abstract categorical framework too. The following definition is a categorical generalization of the classical set-theoretic notion of fully invariant congruence (see [Grä79] for the set-theoretic definition).

<u>3.6 Definition</u> Let (T,μ,η) be a monad in X. A congruence (l,r) on T-algebra (a,α) is <u>fully invariant</u> iff $(lf,rf)\subseteq(l,r)$ for any endomorphism $(a,\alpha) \xrightarrow{f} (a,\alpha)$. ■

3.7 Definition Let $k \overset{l}{\underset{r}{\rightrightarrows}} (a,\alpha)$ be a relation on the underlying object of T-algebra (a,α).

The congruence generated by (l,r), which is denoted as $C(l,r)$, is a congruence on (a,α) containing (l,r) (i.e. $(l,r)\subseteq C(l,r)$) s.th. any other congruence (l',r') containing (l,r) (i.e. $(l,r)\subseteq(l',r')$) also contains $C(l,r)$ (i.e. $C(l,r)\subseteq(l',r')$).

The fully invariant congruence generated by (l,r) which is denoted as $FIC(l,r)$ is similarly defined (we have only to replace the word "congruence" by "fully invariant congruence" in the definition of the congruence generated by (l,r)).■

Observe that the [fully invariant] congruence generated by a relation, if exists, is unique up to the equivalence defined by the preorder $\subseteq$.

The following two propositions (having an algebraic rather than a logic nature) show that under some mild and natural conditions the congruence and the fully invariant congruence generated by a relation exist and, moreover, they give us a method to construct them in particular cases.

The construction of the congruence generated by a relation is a two-step process.

3.8 Proposition Let (T,μ,η) be a monad in X. Suppose that X has small limits and that T-algebras category X^T has coequalizers. Let $k \overset{l}{\underset{r}{\rightrightarrows}} (a,\alpha)$ be a relation on the underlying object of the T-algebra (a,α).

I. let $l^{\#}$ and $r^{\#}$ be the unique extensions of l and r to T-morphisms $(Tk,\mu_k) \longrightarrow (a,\alpha)$

II. let e be the coequalizer of $(l^{\#},r^{\#})$ in X^T, and let $(\bar{l},\bar{r})$ be the kernel of e, that is $(\bar{l},\bar{r})=\ker(\mathrm{coeq}(l^{\#},r^{\#}))$.

Then $(\bar{l},\bar{r}) = C(l,r)$.

Proof: $(\bar{l},\bar{r})$ is a congruence by construction (as a kernel pair). We only have to show that for any T-morphism $h:(a,\alpha)\longrightarrow(b,\beta)$, if $(l,r)\subseteq$ $\subseteq$ kerh then $(\bar{l},\bar{r})\subseteq$ kerh.

$(l,r)\subseteq$ kerh implies $lh=rh$, which implies $\eta_k l^{\#}h=\eta_k r^{\#}h$. Since η_k is universal it follows that $l^{\#}h=r^{\#}h$.

Since e is the coequalizer of $(l^{\#},r^{\#})$, there is h' s.th. eh'=h. But $\bar{l}e=\bar{r}e$ (by construction). It follows $\bar{l}h=\bar{r}h$ (since h=eh'). Universal property of kerh (as pullback) implies $(\bar{l},\bar{r})\subseteq$ kerh.∎

At set-theoretic level $Tk\overset{l^{\#}}{\underset{r^{\#}}{\rightrightarrows}}(a,\alpha)$ is the closure of $k\overset{l}{\underset{r}{\rightrightarrows}}(a,\alpha)$ under algebra operations and $(\bar{l},\bar{r})$ is the reflexive-symmetric-tranzitive closure of $(l^{\#},r^{\#})$.

3.9 Proposition Let (T,μ,η) be a monad in X. Suppose that X has small limits and small coproducts and that T-algebras category X^T has coequalizers. Let $k\overset{l}{\underset{r}{\rightrightarrows}}(a,\alpha)$ be a relation on the underlying object of T-algebra (a,α)

I. let $End(a,\alpha)$ be the set of endomorphisms of (a,α), that is

$End(a,\alpha)=\{f : f:(a,\alpha)\longrightarrow(a,\alpha)\}$.

Construct the coproduct $(k\xrightarrow{s_f}\coprod_{f\in End(a,\alpha)}k = k^*)$ and let

$k^*\overset{l^*}{\underset{r^*}{\rightrightarrows}}a$ be the unique arrows which satisfies $s_f\cdot l^*= l\cdot f$ and $s_f\cdot r^*= rf$ for any $f\in End(a,\alpha)$

II. let $(\overline{l^*},\overline{r^*}) = C(l^*,r^*)$ be the congruence generated by (l^*,r^*). Then $(\overline{l^*},\overline{r^*}) = FIC(l,r)$.

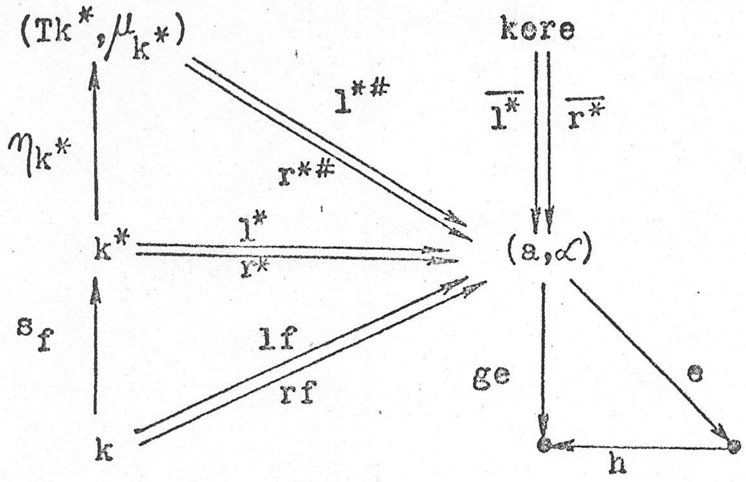
Proof: $(\overline{l^*},\overline{r^*})$ is by definition a congruence.

Let us show that it is fully invariant:

Let $g:(a,\alpha)\longrightarrow(a,\alpha)$ be an endomorphism of (a,α). Using the same notations as in proposition 3.8 we succesively deduce for any $f:(a,\alpha)\longrightarrow(a,\alpha)$:

$s_f\cdot l^*\cdot g\cdot e = l\cdot f\cdot g\cdot e = s_{fg}\cdot l^*\cdot e$ (since e is the coequalizer of $(l^{*\#},r^{*\#})) = s_{fg}\cdot r^*\cdot e = r\cdot f\cdot g\cdot e = s_f\cdot r^*\cdot g\cdot e$.

By coproduct universal property it follows that $l^* \cdot g \cdot e = r^* \cdot g \cdot e$.



Then $l^{*\#} \cdot g \cdot e = r^{*\#} \cdot g \cdot e$ in the virtue of the universal property of free T-algebra (Tk^*, μ_{k^*}).

There exists h s. th. eh=ge since e is the coequalizer of $(l^{*\#}, r^{*\#})$.

Then $\overline{l^*} \cdot g \cdot e = \overline{l^*} \cdot e \cdot h$ ($(\overline{l^*}, \overline{r^*})$ is the kernel of e) $= \overline{r^*} \cdot e \cdot h = \overline{r^*} \cdot g \cdot e$.

In the virtue of the universal property of kernels we may conclude that $(\overline{l^*}g, \overline{r^*}g) \subseteq (\overline{l^*}, \overline{r^*})$. This proof is summarized in the previous diagram.

It remains to prove that given any fully invariant congruence $k' \overset{l'}{\underset{r'}{\rightrightarrows}} (a, \alpha)$ containing (l,r) it also contains $(\overline{l^*}, \overline{r^*})$.

If $(l,r) \subseteq (l',r')$ then for any $f:(a,\alpha) \longrightarrow (a,\alpha)$ we have $(lf,rf) \subseteq (l'f,r'f)$ (since (l',r') is fully invariant) $\subseteq (l',r')$. Let $h_f : k \longrightarrow k'$ s.th. $lf = h_f l'$ and $rf = h_f r'$ and let $h: k^* \longrightarrow k'$ s.th. $s_f h = h_f$ for any $f:(a,\alpha) \longrightarrow$ $\longrightarrow (a,\alpha)$.

By coproduct universal property $hl' = l^*$ and $hr' = r^*$, therefore $(l^*, r^*) \subseteq$ $\subseteq (l',r')$. Using proposition 3.8 we have $(\overline{l^*}, \overline{r^*}) = C(l^*, r^*) \subseteq (l',r')$. ◻

Again we mention here that the set-theoretic level construction of the fully invariant congruence generated by a relation is just a special case of the previous proposition.

COMPLETENESS OF MONADIC EQUATIONAL LOGIC

The results of the previous paragraph are used here for proving some Birkhoff-like theorems concerning the completeness of monadic equational logic.

When reading [Tar85] or [Tar86] one immediately observes the crescent role played in theoretical computer science (and especially in abstract data types and artificial inteligence) by the ground equational logic.

3.10 Definition Let (T,μ,η) be a monad in X. Suppose that X has an initial object O. A ground T-equation is a T-equation having O as the object of variables. ◻

This definition agree to the set-theoretic notion of ground equation, that is equations having no symbols of variables. In this special case the complete sets of ground equations are the congruences on the initial algebra. The following theorem generalizes this fact.

3.11 Definition Let (T,μ,η) be a monad in X. We say that the T-equation (l',r') is a semantical consequence of the T-equation (l,r) ($(l,r)\models(l',r')$, in symbols) iff for any T-algebra (a,α) if $(a,\alpha)\models(l,r)$ then $(a,\alpha)\models$ $\models(l',r')$. ◻

3.12 THEOREM[completeness of monadic ground equational logic] Let (T,μ,η) be a monad in X. Assume that X has small limits and an initial object O and that X^T has coequalizers. Let $k\overset{l}{\underset{r}{\rightrightarrows}}TO$ and $k'\overset{l'}{\underset{r'}{\rightrightarrows}}TO$ be ground T-equations. Then $(l,r)\models(l',r')$ iff $(l',r')\subseteq C(l,r)$.

Proof: We shall use the notations made in propositions 3.8 and 3.9. Let (a_e,α_e) be the codomain of e, where e is the coequalizer of $(l^{\#},r^{\#})$. Recall that (TO,μ_O) is the initial T-algebra (left-adjoint functors preserve initial objects). For any T-algebra (b,β) the unique T-morphism

$(TO,\mu_0) \Longrightarrow (b,\beta)$ is denoted as $!_{(b,\beta)}$.

Now, suppose that $(l,r) \models (l',r')$. Since $(a_e,\alpha_e) \models l=r$ ($\eta_0 e$ is the unique valuation of variables) it follows that $(a_e,\alpha_e) \models l'=r'$, therefore $l'e=r'e$. Applying the universal property of kernels for e we conclude that $(l',r') \subseteq \ker e = C(l,r)$.

Now, suppose $(l',r') \subseteq C(l,r)$. Then $(l',r') \subseteq \ker e$, therefore $l'e=r'e$. Let $(b,\beta) \models l=r$, that is $l \cdot !_{(b,\beta)} = r \cdot !_{(b,\beta)}$. $l^{\#} \cdot !_{(b,\beta)} = r^{\#} \cdot !_{(b,\beta)}$ since η_k is universal. There is $h:(a_e,\alpha_e) \longrightarrow (b,\beta)$ s.th. $eh = !_{(b,\beta)}$ because e is the coequalizer of $(l^{\#},r^{\#})$. Then $l' \cdot !_{(b,\beta)} = l'eh = r'eh = r' \cdot !_{(b,\beta)}$, that is $(b,\beta) \models l'=r'$. The soundness part of the theorem is thus also proved. ∎

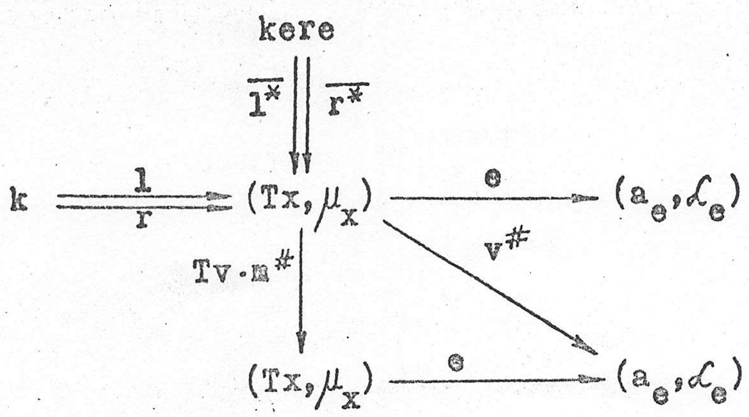
The steps involved in the construction of the congruence generated by a ground T-equation (see proposition 3.8) can be used in certain particular situations to produce explicit deduction rules.

A method often used in logic is to fix the set of variables. We shall prove a Birkhoff-like completeness theorem for monadic equational logic using a fixed object of variables. The syntactic part of the theorem is formulated in terms of fully invariant congruences on the free T-algebra over the object of variables. However, a Choice Axiom-like supplementary condition is needed.

<u>3.13 THEOREM [completeness of monadic equational logic]</u> Let (T,μ,η) be a monad in X. Suppose that X has small limits and coproducts and that X^T has coequalizers. Assume also that every coequalizer e in X^T is a G^T-split epi (i.e. e has a left-inverse in X; see also [Man76]).

Let $k \underset{r}{\overset{l}{\rightrightarrows}} Tx$ and $k' \underset{r'}{\overset{l'}{\rightrightarrows}} Tx$ be T-equations over a fixed object of variables $x \in |X|$. Then $(l,r) \models (l',r')$ iff $(l',r') \subseteq FIC(l,r)$.

<u>Proof:</u> As usual we shall use the previous notations made in propositions 3.8 and 3.9. Let (a_e,α_e) be the codomain of e, where e is the coequalizer of $(l^{*\#},r^{*\#})$



Suppose that $(l,r)\models(l',r')$. We shall prove that $(a_e,\alpha_e)\models l=r$. Let $v:x\longrightarrow a_e$ be any valuation of variables and let $v^{\#}:(Tx,\mu_x)\longrightarrow(a_e,\alpha_e)$ be its unique T-morphism extension.

In the virtue of Choice Axiom-like assumption (e is split epi in X) there is $m:a_e\longrightarrow Tx$ s.th. $me=1_{a_e}$. Let $m^{\#}:(Ta_e,\mu_{a_e})\longrightarrow(Tx,\mu_x)$ be the unique T-morphism extension of m.

$$v = v\cdot m\cdot e = v\cdot\eta_{a_e}\cdot m^{\#}\cdot e \ (\eta \text{ is natural}) = \eta_x\cdot Tv\cdot m^{\#}\cdot e\ .$$

But $v=\eta_x\cdot v^{\#}$. Since η_x is universal it follows that $v^{\#}=Tv\cdot m^{\#}\cdot e$.

Then $(\overline{l^*}v^{\#},\overline{r^*}v^{\#}) = (\overline{l^*}\cdot Tv\cdot m^{\#}\cdot e,\overline{r^*}\cdot Tv\cdot m^{\#}\cdot e)\subseteq(\overline{l^*}e,\overline{r^*}e)$ (since $(\overline{l^*},\overline{r^*}) = FIC(l,r)$ is fully invariant we have that $(\overline{l^*}\cdot Tv\cdot m^{\#},\overline{r^*}\cdot Tv\cdot m^{\#})\subseteq(\overline{l^*},\overline{r^*})$, see also the previous diagram).

Since $\overline{l^*}e=\overline{r^*}e$ it follows that $\overline{l^*}v^{\#}=\overline{r^*}v^{\#}$, therefore $lv^{\#}=rv^{\#}$.

Now $(l,r)\models(l',r')$ implies $(a_e,\alpha_e)\models l'=r'$ which also implies $l'e=r'e$.

Since $FIC(l,r)=\ker e$ we can conclude that $(l',r')\subseteq FIC(l,r)$.

Now suppose that $(l',r')\subseteq FIC(l,r)$ and some T-algebra (b,β) satisfies $l=r$. We have to prove that $(b,\beta)\models l'=r'$. Let $v:x\longrightarrow b$ be any valuation of variables.

For any $f:(Tx,\mu_x)\longrightarrow(Tx,\mu_x)$ we have $l\cdot f\cdot v^{\#} = r\cdot f\cdot v^{\#}$ (since $(b,\beta)\models l=r$ and $fv^{\#}:(Tx,\mu_x)\longrightarrow(Tx,\mu_x)$). Immediately follows (universal property of coproducts) that $l^*v^{\#}=r^*v^{\#}$, and then $l^{*\#}v^{\#}=r^{*\#}v^{\#}$, which implies the existence of $v':(a_e,\alpha_e)\longrightarrow(b,\beta)$ s.th. $ev'=v^{\#}$.

Since $l'e=r'e$ (by assumption $(l',r')\subseteq FIC(l,r)=\ker e$) we deduce $l'v^{\#}=r'v^{\#}$.

It means that $(b,\beta)\models l'=r'$. ■

The classical set-theoretical results on equational completeness (of equational logic [Grä79] and of many-sorted equational logic [GM85]) are instances of the previous theorem.

The completeness of monadic equational logic also provides us with the appropriate framework to treat equational completeness in the field of infinitary theories. It also gives us the possibility to develop a complete equational logic in other categories than those based on Set. An example in this sense is provided by topological algebras. We think that the previous theorem has many other concrete applications.

SOME CONCLUDING REMARKS

We think that monadic equational logic projects a new light on the general foundations of equational logic and on its connections with monad theory:

A. trying to fit monadic equational logic in the general framework of abstract model theory (in its institutional version) leads to the natural generalization of the classical notion of monad morphism which permits us to connect all the monads (without respect to the category they are defined in) in a hypercategory. Monads are thus playing the role of signatures for logical systems.

B. families of equations seems to be more appropriate than singular equations for the notion of sentence of (classical) equational logic

C. monadic equational logic seems to destill the essence of equational logic and to purify it from its set-theoretical aspects. For example, completeness of equational logic is treated using a pure category-theoretical line of thought (cf. [MacL71] , category theory is the art of "living without elements, using arrows instead"). We belive that this treatement of completeness is the most important mathematical message of this paper.

APPENDIX

This chapter is devoted to the study of the connections between monadic institutions and Tarlecki "abstract algebraic institutions". The axioms of abstract algebraic institutions are introduced in [Tar85] and [Tar86].

Here we shall present (without proof) some natural conditions which make a monadic institution to be abstract algebraic. Each axiom in the definition of abstract algebraic institutions is independently treated. We shall use the concept of regular category as defined in [Man76].

The following proposition is well known in categorical universal algebra:

4.1 Proposition [Man76] Let (T,μ,η) be a monad in a regular category (X,E,M) having 0 as initial object. If T preserves E (that is $TE \subseteq E$) then $(X^T, E^T=\{e \in X^T : G^T e \in E\}, M^T=\{m \in X^T : G^T m \in M\})$ is regular having $(T0,\mu_0)$ as initial object. □

From now on, any monad defined in a regular caregory is supposed to preserve regular epis.

4.2 Proposition Let (T,μ,η) and (T',μ',η') be monads in regular categories (X,E,M) and (X',E',M') respectively. Let $\langle\lambda,G\rangle:(T,\mu,\eta) \longrightarrow (T',\mu',\eta')$ be monad morphism.

I. for any small category J, if $G:X' \longrightarrow X$ preserves J-limits, then $\underline{ALG}\langle\lambda,G\rangle : X'^{T'} \longrightarrow X^T$ also preserves J-limits

II. if G preserves regular monos $(GM' \subseteq M)$ then $\underline{ALG}\langle\lambda,G\rangle$ preserves subalgebras (submodels). □

Thus, if the functor component of a monad morphism preserves products and regular monos then the model functor also preserves products and regular monos. Note that if the functor component of the monad morphism is a right-adjoint functor then it automatically preserves all small limits.

We use the same category-theoretical notions of variety, ground variety, reachable objects etc. (which are defined in regular categories) as defined in [Tar85].

The following proposition shows us that the ground varieties in monadic institutions are exactly those classes of Eilenberg-Moore algebras which are defined by ground equations.

<u>4.3</u> <u>Proposition</u> Let (T,μ,η) be a monad in a regular category (X,E,M) having initial object O.

I. if every $e \in E$ is a coequalizer then every ground variety is definable by a ground equation

II. if X^T has coequalizers then every ground equation defines a ground variety. ◻

The final proposition express the possibility to use the model-theoretical method of diagrams in monadic institutions.

<u>4.4</u> <u>Proposition</u> Let (T,μ,η) be a monad in a regular category (X,E,M) having finite coproducts. Then for any T-algebra (a,α) there are

a/. a monad $(T^{\alpha},\mu^{\alpha},\eta^{\alpha})$ in X

b/. a monad morphism $\langle\iota,1_X\rangle:(T,\mu,\eta)\longrightarrow(T^{\alpha},\mu^{\alpha},\eta^{\alpha})$

c/. a reachable T^{α}-algebra $(a,\alpha)_{a,\alpha}$

s.th.

I. T^{α} preserves E

II. $\underline{ALG}\langle\iota,1_X\rangle$ induces an isomorphism $((a,\alpha)_{a,\alpha}\downarrow X^{T^{\alpha}})\simeq((a,\alpha)\downarrow X^T)$

III. $(\underline{ALG}\langle\iota,1_X\rangle)(a,\alpha)_{a,\alpha}=(a,\alpha)$

IV. if $f\in E^{T^{\alpha}}$ then $(\underline{ALG}\langle\iota,1_X\rangle)f\in E^T$. ◻

The only axiom from the definition of abstract algebraic institutions which is not treated here is that one asserting that the category of signatures must be finitely cocomplete and that the model functor must preserve finite limits (see [Tar85, 86]). The reason is that there are many results in the theory of abstract algebraic institutions [Tar85, 86, 87] which does not use this axiom and that this is the only axiom which can not be treated at the general level of monadic institutions because it depends on more particular characteristics of institutions.

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REFERENCES

[ADJ77] J.A.GOGUEN, J.W.THATCHER, E.G.WAGNER, J.B.WRIGHT, Initial algebra semantics and continuous algebras, J.Assoc.Comput.Mach. 24(1977), 68-95

[BR85] R.M.BURSTALL, D.E.RYDEHEARD, Monads and Theories: A Survey for Computation, Algebraic Methods in Semantics, Semin. Fontainbleu/ /France82; 575-605

[CK73] C.C.CHANG, H.J.KEISLER, "Model Theory", North-Holland, Amsterdam (1973)

[GB85] J.A.GOGUEN, R.M.BURSTALL, Institutions: Abstract Model Theory for Computer Science. Technical Report CSLI-85-30, Center for the Study of Language and Information, Stanford University (1985)

[GB86] J.A.GOGUEN, R.M.BURSTALL, A Study in the Foundations of Programming Mehtodology: Specifications, Institutions, Charters and Parchments. Technical Report CSLI-86-54, Center for the Study of Language and Information, Stanford University (1986)

[GM82] J.A.GOGUEN, J.MESEGUER, An Initiality Primer. Application of Algebra to Language Definition and Compilation, ed. by M.Nivat & J.Reynolds, North-Holland(1982)

[GM85] J.A.GOGUEN, J.MESEGUER, Completeness of Many-Sorted Equational Logic, Houston J. of Math. 11(3) (1985), 307-334

[Gog89] J.A.GOGUEN, A Categorical Manifesto, Technical Monograph PRG-70, Oxford University Computing Laboratory (1989)

[Grä79] G.GRÄTZER, "Universal Algebra". 2nd ed., Springer-Verlag, New York (1979)

[Hat70] W.S.HATCHER, Quasiprimitive Categories, Math.Ann.190 (1970), 93-96

[HR72] H.HERRLICH, C.M.RINGEL, Identities in Categories, Can.Math.Bull. 15(1972), 297-299

[Law63] F.LAWVERE, Functorial Semantics of Algebraic Theories, Proc.Nat. Acad.Sci. USA 50 (1963), 869-872

[MacL71] S.MACLANE, "Categories for the Working Mathematician", Springer-Verlag, New York (1971)

[Man76] E.G.MANES, "Algebraic Theories", Springer-Verlag, New York (1976)

[Tar85] A.TARLECKI, On the Existence of Free Models in Abstract Algebraic Institutions, Theor.Comput.Sci.37 (1985), 269-304

[Tar86] A.TARLECKI, Quasi-varieties in Abstract Algebraic Institutions, Jour.Comput.Syst.Sci.33 (1986), 333-359

[Tar87] A.TARLECKI, Bits and Pieces of the Theory of Institutions ....?

[TBG88] A.TARLECKI, R.M.BURSTALL, J.A.GOGUEN, Some Fundamental Algebraic Tools for the Semantics of Computation PartIII: Indexed Categories , Technical Report ECS-LFCS-88-60, University of Edinburgh (1988)