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ABSORBENT, PARABOLIC, ELLIPTIC AND QUASIELLIPTIC BALAYAGES IN H-CONES.II; THE RELATION WITH THE GREEN FUNCTION.

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In this paper we continue the study of absorbent, parabolic, elliptic and quasielliptic balayages started in [3] . We develop now the theory in the frame of standard H-cones of functions, especially for those H-cones which are represented on a Green set. In this case we characterize the parabolicity, ellipticity and quasiellipticity in terms of the Green function associated with the given H-cone.

In the first section, preliminaries results on absorbent sets are given. Particularly we remark that if $\underline{\underline{S}}$ is a standard H-cone of functions on a set X, then the absorbent balayages on $\underline{\underline{S}}$ (defined in [3]) are corresponding to the absorbent subsets of X (i.e. the zero sets of the elements of $\underline{\underline{S}}$). In the second section we consider the parabolic and elliptic subsets of X introduced and studied in [2] . Recall that a subbasic subset E of X is called elliptic if for any absorbent set $A \subseteq X$ we have either $A \cap E = \phi$ or $E \subseteq A$. A subbasic subset P of X is parabolic if for any absorbent sets A_1, A_2 with $A_1 \cap P \subsetneq A_2 \cap P$ there exists an absorbent set A with

$$A_1 \cap P \subsetneq A \cap P \subsetneq A_2 \cap P,$$

where $\subsetneq$ denotes the strict inclusion. We show that the above notions are strongly related with the similar ones of elliptic and parabolic balayages on $\underline{\underline{S}}$ considered and studied in [3] . We prove that if M is a subbasic subset of X then M will be parabolic (resp. elliptic) iff the corresponding balayage B^M is of the same type. A characterization of parabolicity in terms of harmonic carrier of the elements of $\underline{\underline{S}}$ is also given. If X is nearly saturated then a basic subset F of X is called quasielliptic if the are no non empty parabolic subsets of F. We show that if X is nearly saturated and P is the greatest parabolic subset of X then the fine open set $G:= X \setminus P$ is quasielliptic with respect to the localized $\underline{\underline{S}}(G)$ of $\underline{\underline{S}}$ on G (i.e. the H-cone of functions on G generated by the functions of the form $s-B^{X \setminus G}s$, $s \in \underline{\underline{S}}$).

In the third section we suppose that the standard H-cone $\underline{\underline{S}}$ and its dual $\underline{\underline{S}}^*$ are represented as standard H-cones of functions on the same set X which is a Green set associated with $(\underline{\underline{S}}, \underline{\underline{S}}^*)$.(We denote by $g(\cdot,\cdot)$ the associated Green function.) We show that if A is an absorbent set with respect to $\underline{\underline{S}}$ then the fine closure with respect to $\underline{\underline{S}}^*$ of the complement of A is an absorbent set with respect to $\underline{\underline{S}}^*$. We also prove that X is quasiellastic (resp.elliptic, parabolic) iff $g(x,x) > 0$ (resp. $g(x,y) > 0$, $g(x,x) = 0$) for any $x \in X$ without a semi-polar set (resp. for any $x,y \in X$). Generally, the essential base of the set $\{x \in X/\ g(x,x) = 0\}$ is the greatest parabolic subset of X with respect to $\underline{\underline{S}}$. Particularly if the fine topologies on X with respect to $\underline{\underline{S}}$ and $\underline{\underline{S}}^*$ coincide, then X is quasielliptic. Consequently, if $\underline{\underline{S}}$ is autodual then X is always quasielliptic.

In the last section we analyse the special case of totally parabolic H-cones. If $\underline{\underline{S}}$ and $\underline{\underline{S}}^*$ are as above, we say that $\underline{\underline{S}}$ is totally parabolic if X is parabolic and the set of all absorbent subsets of X is totally ordered. We show that $\underline{\underline{S}}$ is totally parabolic iff for any $x \in X$, without a semi-polar set, the set $\{y \in X/ g(x,y)=0\}$ is the smallest absorbent set containing x. The totally parabolic H-cones are illustrated by the standard H-cone associated with the heat equation on $\mathbb{R}^n \times \mathbb{R}$.

Finally, we remark that the contents of many results are clarified by suitable examples.

§ 1. Absorbent sets with respect to a standard H-cone of functions.

In this section $\underline{\underline{S}}$ will be a standard H-cone of functions on a set X.

We recall now some results concerning the balayages on a standard H-cone of functions $\underline{\underline{S}}$ on a set X (cf. [5]), the absorbent subsets of X with respect to $\underline{\underline{S}}$ (cf. [1] and [2]) and their relations with the absorbent balayages on $\underline{\underline{S}}$ (cf. [3]).

A subset M of X is called subbasic (with respect to $\underline{\underline{S}}$) if

$$B^M s(x) = s(x) \quad , \text{ for any } x \in M \text{ and } s \in \underline{\underline{S}} \ ,$$

where

$$B^M s := \bigwedge \{ s' \in \underline{\underline{S}} \ / \ s \leqslant s' \text{ on } M \} .$$

A subbasic set $M \subseteq X$ which is fine closed is termed basic. Obviously, the fine closure of any subbasic set is basic. If M is a subbasic set then B^M is a balayage, called the balayage on M with respect to $\underline{\underline{S}}$. A subset M of X will be subbasic iff M is not thin at any point $x \in M$. Consequently, if M is a subbasic set and U is a fine open subset of X then $M \cap U$ is also a subbasic set.

For any balayage B on $\underline{\underline{S}}$, the set

$$b(B) := \{ x \in X \ / \ Bs(x) = s(x), \text{ for any } s \in \underline{\underline{S}} \}$$

is called the base of B. If M is a basic set then

$$M = b(B^M)$$

and the map $M \longrightarrow B^M$ from the set of all basic set to the set of all balayages on $\underline{\underline{S}}$ is such that

$$M_1 \subseteq M_2 \Longleftrightarrow B^{M_1} \leqslant B^{M_2} .$$

It will be important the case when X is such that for any balayage B on $\underline{\underline{S}}$ there exists a basic subset M of X with $B = B^M$. It is known that this property holds iff X is nearly saturated (i.e. any universally continuous element of the dual $\underline{\underline{S}}^{\times}$ of $\underline{\underline{S}}$ is represented as a measure on X). In this case, the correspondence $B \longrightarrow b(B)$ between the set of all balayages on $\underline{\underline{S}}$ and the set of all basic subsets of X is a bijection such that

$$B_1 \leqslant B_2 \Longleftrightarrow b(B_1) \subseteq b(B_2)$$

For any positive numerical function f on X such that there exist $t_1, t_2 \in \underline{\underline{S}}$, $t_2 < \infty$ with $f = t_1 - t_2$, we have denoted by B_f the balayage on $\underline{\underline{S}}$ given by

$$B_f s = \bigvee_{n \in \mathbb{N}} R(s \wedge nf).$$

Since for any $s, t \in \underline{\underline{S}}$ we have

$$t \geqslant s \wedge nf \ , \text{ for any } n \in \mathbb{N},$$

iff $t \geqslant s$ on the fine open set $[f > 0]$, it follows

$$B_f s = B^{[f>0]} s \ , \text{ for any } s \in \underline{\underline{S}}.$$

Proposition 1.1. For any balayage B on $\underline{\underline{S}}$ and any $u \in \underline{\underline{S}}$ which is a finite generator of $\underline{\underline{S}}$ we have

$$B' = B^{[Bu < u]}.$$

Particularly, for any basic set M we have

$$(B^M)' = B^{X \setminus M}.$$

Proof. From [3, Proposition 1.5] we have

$$B' = \bigvee \{B_g \ / \ g = t - Bt, \ t \in \underline{\underline{S}}, \ t < \infty\} .$$

Since u is a finite generator of $\underline{\underline{S}}$ we get

$$[g > 0] \subseteq [Bu < u] \ ,$$

for any $g = t - Bt$, $t \in \underline{\underline{S}}$, $t < \infty$ and therefore

$$B' = B_{u - Bu} = B^{[Bu < u]} .$$

If M is a basic set then

$$M = \{x \in M \ / \ B^M u(x) = u(x)\}$$

and consequently $X \setminus M = [Bu < u]$, $(B^M)' = B^{X \setminus M}$.

Corollary 1.2. If B is a balayage on $\underline{\underline{S}}$ then

$$B = B''$$

iff there exists a basic set M on X such that $B = B^M$ and such that M is the fine clusure of the fine interior of M.

We remember now the notion of absorbent set. A subset A of X is called absorbent (with respect to $\underline{\underline{S}}$) if there exists $s \in \underline{\underline{S}}$ (or only a bounded element $s \in S$) such that

$$A = [s = 0].$$

If A is absorbent then it is closed (in the natural topology on X) and fine open and therefore a basic set.

Proposition 1.3. If $s \in \underline{\underline{S}}$ and $A := [s = 0]$ then

$$(B_s)' = B^A.$$

Proof. We have $B_s = B^{[s > 0]}$ and since the set $[s > 0]$ is a basic set, from Proposition 1.1 we conclude

$$(B_s)' = B^A.$$

A balayage B on $\underline{\underline{S}}$ is called absorbent (cf. [3]) if $Bs \preccurlyeq s$ for any $s \in \underline{\underline{S}}$, where $\preccurlyeq$ is the specific order on $\underline{\underline{S}}$.

Proposition 1.4. a) For any balayage B on $\underline{\underline{S}}$ we have: B is an absorbent balayage iff $b(B)$ is an absorbent subset of X and $B^{b(B)} = B$.

b) For any basic subset A of X we have: A is an absorbent set iff B^A is an absorbent balayage.

Proof. The assertions follow from Proposition 1.3 and Corollarly 1.2, using also [3, Theorem 2.2].

Remark 1.5. The map

$$A \longrightarrow B^A$$

between the set of all absorbent subsets of X and the set of all absorbent balayages on $\underline{\underline{S}}$ is a bijection and

$$A_1 \subseteq A_2 \Longleftrightarrow B^{A_1} \preccurlyeq B^{A_2}.$$

Moreover, if $(A_i)_{i \in I}$ is a family of absorbent subsets of X then the fine closure $\overline{\bigcup_{i \in I} A_i}^f$ of $\bigcup_{i \in I} A_i$ and $\bigcap_{i \in I} A_i$ are also absorbent sets and

$$B^{\overline{\bigcup_{i \in I} A_i}^f} = \bigvee_{i \in I} B^{A_i}, \quad B^{\bigcap_{i \in I} A_i} = \bigwedge_{i \in I} B^{A_i}.$$

Proposition 1.6. Let A be a basic subset of X. Then A is an absorbent set iff for any $s \in \underline{\underline{S}}$ we have

$$(B^A)'s = \begin{cases} s, & \text{on } X \setminus A \\ 0, & \text{on } A \end{cases} .$$

Proof. From Proposition 1.1 we have

$$(B^A)' = B^{X \setminus A}$$

and by Proposition 1.4 it follows that A is an absorbent set iff B^A is an absorbent balayage on $\underline{\underline{S}}$. From [3, Theorem 2.1] we deduce now that A is absorbent iff

$$B^A(B^{X \setminus A}) = 0$$

or equivalently iff

$$B^{X \setminus A}s = 0 \quad \text{on } A, \text{ for any } s \in \underline{\underline{S}}.$$

Proposition 1.7. Let A be a basic subset of X. Then A is an absorbent set iff for any subbasic subset M of X we have

$$B^{A \cap M} = B^A \wedge B^M = B^A B^M.$$

Proof. From Proposition 1.4, Corollary 1.2 and [3, Proposition 2.8 and Theorem 2.9] it follows that A is absorbent iff

$$B^A B^M = B^A \wedge B^M,$$

for any subbasic subset M of X.

Suppose now that A is absorbent. Then $A \cap M$ is a subbasic set and we have

$$B^{A \cap M} \leqslant B^A \wedge B^M .$$

To prove the converse inequality it will be sufficient to show that if $s,t \in \underline{\underline{S}}$ then

$$s \leqslant t \text{ on } A \cap M \Longrightarrow B^M s \leqslant t \text{ on } A.$$

Indeed, from Proposition 1.6 we have

$$B^{X \setminus A}s = \begin{cases} s, & \text{on } X \setminus A \\ 0, & \text{on } A \end{cases} .$$

It follows

$$s \leqslant t + B^{X \setminus A}s \qquad \text{on } M,$$

$$B^M s \leqslant t + B^{X \setminus A}s \qquad \text{on } X,$$

$$B^M s \leqslant t \qquad \text{on } A$$

and the proof is complete.

Let M be a subbasic subset of X. Then the H-cone

$$B^M(\underline{\underline{S}}) := \{ B^M s \ / \ s \in \underline{\underline{S}} \}$$

is a standard H-cone (cf. [5, Corollary 5.2.6]). Since for any $s, t \in \underline{\underline{S}}$ we have

$$B^M s = s \text{ on } M$$

and

$$B^M s \leq B^M t \Longleftrightarrow s \leq t \text{ on } M$$

and since the infimum in $B^M(\underline{\underline{S}})$ of $B^M s$ and $B^M t$ is equal to $B^M(s \wedge t)$, it follows that the set

$$\underline{\underline{S}}_{|M} := \{ s_{|M} \ / \ s \in \underline{\underline{S}} \}.$$

is a standard H-cone of functions on the set M which is isomorphic with $B^M(\underline{\underline{S}})$.

Note that if M is a subbasic subset of X and A is a subset of M then A is semi-polar with respect to $\underline{\underline{S}}$ iff A is semi-polar with respect to $B^M(\underline{\underline{S}})$. We also remark that if A is a subset of M then A is subbasic with respect to $B^M(\underline{\underline{S}})$ iff A is a subbasic subset of X with respect to $\underline{\underline{S}}$. Moreover the balayage on A with respect to $B^M(\underline{\underline{S}})$ coincides with the restriction to $B^M(\underline{\underline{S}})$ of the balayage on A with respect to $\underline{\underline{S}}$.

Proposition 1.8. Let M be a subbasic subset of X and $A \subseteq M$. Then A is an absorbent set with respect to $B^M(\underline{\underline{S}})$ iff there exists an absorbent set (with respect to $\underline{\underline{S}}$) $\tilde{A}$ such that

$$A = \tilde{A} \cap M.$$

Proof. If $A_1 \subseteq X$ is absorbent with respect to $\underline{\underline{S}}$ and $s \in \underline{\underline{S}}$ is such that $A_1 = [s = 0]$ then we have

$$A_1 \cap M = [s_{|M} = 0]$$

and therefore $A_1 \cap M$ is absorbent with respect to $B^M(\underline{\underline{S}})$, Conversely, let $A \subseteq M$ be an absorbent set with respect to $B^M(\underline{\underline{S}})$ and let $s \in B^M(\underline{\underline{S}})$ such that

$$A = [s_{|M} = 0].$$

If we put $\tilde{A} := [s = 0]$ it follows that $\tilde{A}$ is absorbent with respect to $\underline{\underline{S}}$ and $A = \tilde{A} \cap M$.

Remark. The relation

$$A = \tilde{A} \cap M$$

from Proposition 1.8 is equivalent with the following one (cf. Proposition 1.7)

$$B^A = B^{\tilde{A}} \wedge B^M$$

and therefore the above proposition may be regarded as a consequence of a general assertion which holds on an H-cone and for an arbitrary balayage B instead of B^M (see [3, Theorem 2.15]).

§ 2. Parabolic, elliptic and quasielliptic subsets with respect to a standard H-cone of functions.

Definition. Let $\underline{\underline{S}}$ be a standard H-cone of functions on a set X. The set X is called parabolic with respect to $\underline{\underline{S}}$ (cf. [1]) if there exists a strictly increasing family $(A_t)_{t\in[0,1]}$ of absorbent sets such that $A_0 = \phi$, $A_1 = X$ and

$$\bigcap_{u > t} A_u = A_t, \quad \text{for any } t \in [0,1),$$

$$\overline{\bigcup_{u < t} A_u}^{f} = A_t, \quad \text{for any } t \in (0,1].$$

Remark. It is proved in [2] that X will be parabolic with respect to $\underline{\underline{S}}$ iff for any two absorbent sets $A_1, A_2 \subseteq X$, $A_1 \subsetneq A_2$ (i.e. $A_1 \subseteq A_2$ and $A_1 \neq A_2$) there exists an absorbent set A with $A_1 \subsetneq A \subsetneq A_2$. Keeping in mind this characterization of parabolicity, we recall the following definition:

Definition. An H-cone $\underline{\underline{S}}$ is called parabolic (cf. [3, §4]) if for any two absorbent balayages B_1, B_2 on $\underline{\underline{S}}$, $B_1 < B_2$, there exists an absorbent balayage B on $\underline{\underline{S}}$ with $B_1 < B < B_2$.

Proposition 2.1. Let $\underline{\underline{S}}$ be a standard H-cone. Then the following assertions are equivalent:

a) $\underline{\underline{S}}$ is parabolic.

b) There exists a set X such that $\underline{\underline{S}}$ is a standard H-cone of functions on X and X is parabolic with respect to $\underline{\underline{S}}$.

c) Whenever $\underline{\underline{S}}$ is a standard H-cone of functions on a set X then X is parabolic with respect to $\underline{\underline{S}}$.

Proof. Let X be a set such that $\underline{\underline{S}}$ is a standard H-cone of functions on X. From the preceding remark and Remark 1.5. it follows that $\underline{\underline{S}}$ is parabolic iff X is parabolic with respect to $\underline{\underline{S}}$.

Definition. Let $\underline{\underline{S}}$ be a standard H-cone of functions on a set X. The set X is called elliptic with respect to $\underline{\underline{S}}$ (cf. [2]) if there are no non empty absorbent sets $M \subseteq X$, $M \neq X$.

Definition. An H-cone $\underline{\underline{S}}$ is called elliptic (cf. [3, §4]) if there are no non zero absorbent balayages B on $\underline{\underline{S}}$, $B \neq I$.

Proposition 2.2. Let $\underline{\underline{S}}$ be a standard H-cone. Then the following assertions are equivalent:

a) $\underline{\underline{S}}$ is elliptic.

b) There exists a set X such that $\underline{\underline{S}}$ is a standard H-cone of functions on X and X is elliptic with respect to $\underline{\underline{S}}$.

c) Whenever $\underline{\underline{S}}$ is a standard H-cone of functions on a set X then X is elliptic with respect to $\underline{\underline{S}}$.

Proof. If X is a set such that $\underline{\underline{S}}$ is a standard H-cone of functions on X then from Remark 1.5 it follows that $\underline{\underline{S}}$ is elliptic iff X is elliptic with respect to $\underline{\underline{S}}$.

Definition. Let $\underline{\underline{S}}$ be a standard H-cone of functions on a set X. A subbasic subset $M \subset X$ is called parabolic (resp.elliptic) with respect to $\underline{\underline{S}}$ (cf. [2, § 2]) if M is parabolic (resp.elliptic) with respect to the standard H-cone of functions on M given by $\underline{\underline{S}}|_M$.

Definition. A balayage B on an H-cone $\underline{\underline{S}}$ is called parabolic (resp.elliptic)(cf. [3, § 4]) if the H-cone $B(\underline{\underline{S}})$ is parabolic (resp.elliptic).

In the sequel, if any confusion is avoid, we omit to specifie the standard H-cone $\underline{\underline{S}}$ with respect to which the parabolicity, ellipticity, the absorbent sets or other potential theoretic notions are considered.

Proposition 2.3. Let $\underline{\underline{S}}$ be a standard H-cone of functions on a set X and let M be a subbasic subset of X. Then M is parabolic (resp.elliptic) iff the balayage B^M on $\underline{\underline{S}}$ is parabolic (resp.elliptic).

Proof. The assertion follows from Proposition 2.1, Proposition 2.2 and from the fact that the H-cones $\underline{\underline{S}}|_M$ and $B^M(\underline{\underline{S}})$ are isomorphic.

Remark 2.4. Let $\underline{\underline{S}}$, X and M be as in Proposition 2.3. Then M is parabolic (resp. elliptic) iff the fine closure of M is parabolic (resp.elliptic).

Proposition 2.5. Suppose that $\underline{\underline{S}}$ is a standard H-cone of functions on a set X. Then the following assertions are equivalent:

a) X is parabolic.

b) There are no non empty elliptic subsets of X.

c) There are no non empty elliptic fine open subsets of X.

Proof. a) $\Longrightarrow$ b) follows from [3, Proposition 4.5] , b) $\Longrightarrow$ c) is obvious and c) $\Longrightarrow$ a) follows from [2, Theorem 2.3] , using also Proposition 2.3.

From the preceding considerations and from [3, § 4] the following assertions on the parabolicity and ellipticity holds:

Proposition 2.6. Suppose that $\underline{\underline{S}}$ is a standard H-cone of functions on a set X. We have:

a) For any family $(M_i)_{i \in I}$ of parabolic subset of X the set $\bigcup_{i \in I} M_i$ is also parabolic

b) For any family $(E_i)_{i \in I}$ of elliptic subsets of X such that $E_i \cap E_j \neq \emptyset$, for any $i,j \in I$, the set $\bigcup_{i \in I} E_i$ is also elliptic.

c) If M_1, M_2 are subbasic subsets of X, $M_1 \subseteq M_2$ and M_2 is parabolic (resp.elliptic) then M_1 is parabolic (resp.elliptic).

d) There exists the greatest parabolic subset P of X which is fine closed and

$$P = \bigcap \{X \setminus E \;/\; E \subseteq X,\ E \text{ is elliptic}\}.$$

c) Any elliptic subset of X is contained in a maximal elliptic subset of X; any two different maximal elliptic subsets of X are disjoint; the set of all maximal elliptic subsets of X is at most countable; an elliptic subset E of X, $E \neq \emptyset$ will be maximal iff there exists two absorbent subsets A_1, A_2 of X such that $A_1 \subsetneq A_2$ and $E = A_2 \setminus A_1$.

We give now an example of a standard H-cone of functions $\underline{\underline{S}}$ on a set X for which the greatest parabolic subset P of X is without fine interior points. Hence in this case $(B^P)^* = I$.

Example 2.7. Let us denote by $\underline{\underline{S}}$ the convex cone of all positive lower semi-continuous real functions s on the interval $(-1,1) =: X$, which are increasing and such that the restriction of s to the complement of Cantor set K is locally concave. We remark that $\underline{\underline{S}}$ is a standard H-cone of functions on the set X. More precisely there exists an harmonic space on X such that $\underline{\underline{S}}$ coincides with the set of all positive superharmonic functions on this space. A general construction may by found in [7] (see also [6, Exercice 3.1.47]).

If G is an open subset of X, we denote by $\mathcal{H}(G)$ the set of all real continuous functions h on G such that $x \in G \cap K \Rightarrow$ there exists $x' < x$ with $h|_{(x',x)}$ is constant, $h|_{G-K}$ is a locally affine function.
Obviously $\mathcal{H}(G)$ is a linear subspace of $\mathcal{C}(G)$ and for any increasing sequence $(h_n)_{n\in\mathbb{N}}$ from $\mathcal{H}(G)$ such that $\sup_{n\in\mathbb{N}} h_n$ is finite on a dense subset of G it follows that $\sup_{n\in\mathbb{N}} h_n$ belongs to $\mathcal{H}(G)$. Also the map $G \longrightarrow \mathcal{H}(G)$ is a sheaf $\mathcal{H}$ of linear spaces of real continuous functions. On the other hand let (a,b) be an open interval with $[a,b] \subseteq X$. If $b \notin K$ or $(a,b) \cap K = \emptyset$ then the open set (a,b) is regular with respect to $\mathcal{H}$ since $\mathcal{H}((a,b))$ coincides with the set of all continuous functions h on (a,b) such that h is linear on (c,b) and constant on (a,c) where

$$c = \begin{cases} a & , \text{ if } (a,b) \cap K = \emptyset \\ \sup((a,b)\cap K) & , \text{ if } (a,b)\cap K \neq \emptyset. \end{cases}$$

If $b \in K$ and $(a,b)\cap K \neq \emptyset$, then the interval (a,b) is semiregular since in this case $\mathcal{H}((a,b))$ coincides with the set of all constant functions on (a,b).

From the above considerations it is easy to see that a lower semi-continuous function s on X, $s > -\infty$ will be superharmonic with respect to the sheaf $\mathcal{H}$ iff s is finite, increasing and concave on any interval (a,b) such that $K \cap (a,b) = \emptyset$.
Hence $\underline{\underline{S}}$ is a standard H-cone of functions on X and a subset A of X will be absorbent (with respect to $\underline{\underline{S}}$) iff $A = (-1,c]$, where $c \in K$. From this fact it follows that a subset E of (-1,1) will be a maximal elliptic set with respect to $\underline{\underline{S}}$ iff $E = (a,b]$, where (a,b) is a component of the open set $X \setminus K$.
Moreover the greatest parabolic subset P of X with respect to $\underline{\underline{S}}$ is the set

$$K \cap (-1,1) \setminus M,$$

where

$$M = \{ b \in K \ / \text{ there exists } b' < b \text{ with } (b',b)\cap K = \emptyset \}.$$

Obviously the fine interior of P is empty and therefore the complement of the balayage B^P is the identity.

We extend now a result concerning the characterization of parabolicity in terms of harmonic carrier and in terms of balayages on compact subsets of X (see [1]).

If $\underline{\underline{S}}$ is a standard H-cone of functions on a set X and $s \in \underline{\underline{S}}$, the harmonic carrier of s on X is the set

$$\underline{carr}\ s = \{x \in X \ / \ B^{X \setminus V} s \neq s, \text{ for any } V \in \mathcal{V}_x\},$$

where $\mathcal{V}_x$ denotes the set of all natural neighbourhoods of x.

Theorem 2.8. Let $\underline{\underline{S}}$ be a standard H-cone of functions on a nearly saturated set X. Then the following assertions are equivalent:

a) X is parabolic.

b) For any universally continuous element p of $\underline{\underline{S}}$ we have

$$\inf\{p(x) \ / \ x \in \underline{carr}\ p\} = 0.$$

If moreover the topological space X (endowed with the natural topology) is universally measurable then each of the above two assertions is equivalent with each of the following ones:

c) For any universally continuous element p of $\underline{\underline{S}}$ and any compact subset K of X such that $\underline{carr}\ p \subseteq K$, there exists $x \in K$ with $p(x) = 0$.

d) For any compact subset K of X there exists $x_0 \in K$ such that $p(x_0) = 0$ for any universally continuous element p of $\underline{\underline{S}}$ with $\underline{carr}\ p \subseteq K$.

e) For any compact subset K of X there exists $x \in K$ with $B^K 1(x) = 0$.

f) For any compact subset K of X there exists $x_0 \in K$ such that $B^K s(x_0) = 0$, for any $s \in \underline{\underline{S}}$.

g) For any compact subset K of X and any universally continuous element p of $\underline{\underline{S}}$ there exists $x \in K$ with $B^K p(x) = 0$.

Proof. We suppose firstly that the topological space X is universally measurable. We denote by $\underline{\underline{S}}_0$ the set of all universally continuous elements of $\underline{\underline{S}}$.

The implications f) $\Rightarrow$ e) $\Rightarrow$ g), b) $\Rightarrow$ c) and d) $\Rightarrow$ c) are obvious. "g) $\Rightarrow$ f)". Since for any two universally continuous elements p,q of $\underline{\underline{S}}$ we have $B^K(p+q) = B^K p + B^K q$ it follows that the family $(\{x \in K \ / \ B^K p(x) = 0\})_{p \in \underline{\underline{S}}_0}$ is lower directed and therefore has non empty intersection. Hence there exists $x_0 \in K$ with $B^K p(x_0) = 0$ for any $p \in \underline{\underline{S}}_0$ and therefore $B^K s(x_0) = 0$ for any $s \in \underline{\underline{S}}$.

The proof of c) $\Rightarrow$ d) is similar to the above proof of g) $\Rightarrow$ f)."c) $\Rightarrow$ b)".

Let $p \in \underline{\underline{S}}_0$. Since X is neary saturated and universally measurable then there exists an increasing sequence $(K_n)_{n\in \mathbb{N}}$ of compact subsets of X such that the sequence $(p_{K_n})_{n \in \mathbb{N}}$ of specifically restrictions of p to K_n increases to p (see [5, §3.4]). Since for any $n \in \mathbb{N}$ we have

$$p = p_{K_n} + p_{X \setminus K_n}$$

and since $p \in S_0$, we deduce that the sequence $(p_{X\setminus K_n})_{n\in\mathbb{N}}$ decreases uniformly to 0. For any $n \in \mathbb{N}$ we get

$$\inf\{p(x) \,/\, x \in \underline{\text{carr}}\ p\} \leq \inf\{p_{K_n}(x) \,/\, x \in \underline{\text{carr}}\ p\} + \inf\{p_{X\setminus K_n} \,/ x \in \underline{\text{carr}}\ p\},$$

$$\inf\{p_{K_n}(x) \,/\, x \in \underline{\text{carr}}\ p\} \leq \inf\{p_{K_n}(x) \,/\, x \in \underline{\text{carr}}\ p_{K_n}\} = 0.$$

We conclude that $\inf\{p(x) \,/\, x \in \underline{\text{carr}}\ p\} = 0$.

"b) $\Rightarrow$ a)". Let G be a fine open subset of X, $G \neq \emptyset$. Since X is nearly saturated and in the same time universally measurable there exists a compact subset K of G which is not semi-polar. Thus there exists an universally continuous element p of $\underline{\underline{S}}$ with $\underline{\text{carr}}\ p \subseteq K$, $p \neq 0$. From b) we get $\inf\{p(x) \,/\, x \in K\} = 0$ and therefore there exists $x_0 \in K$ with $p(x_0) = 0$. On the other hand we have $\{x \in G \,/\, p(x) = 0\} \neq G$. Consequently G is not an elliptic subset of X. By Proposition 2.5 it follows that X is parabolic.

"a) $\Rightarrow$ e)". Let $(A_t)_{t\in[0,1]}$ be a strictly increasing family of absorbent subsets of X such that

$$t \in [0,1) \Rightarrow \bigcap_{s>t} A_s = A_t$$

$$t \in (0,1] \Rightarrow A_t = \overline{\bigcup_{s<t} A_s}^{f}$$

and let K be a compact subset of X such that $B^K 1 \neq 0$ on K. There exists $t \in (0,1)$ with $K \cap A_t \neq \emptyset$ since in the contrary case we have

$$B^K 1 \leq 1_{X\setminus A_t}, \text{ for any } t \in (0,1)$$

and therefore

$$B^K 1 = 0 \text{ on } A_t, \text{ for any } t \in (0,1),$$

$$B^K 1 = 0 \quad \text{on } X = \overline{\bigcup_{t<1} A_t}^{\,f}$$

Since K is compact and $\bigcap_{s>t} A_s = A_t$ it follows that there exists the smallest $t_0 \in (0,1)$ with

$$A_{t_0} \cap K \neq \emptyset.$$

Hence if $t < t_0$ then $A_t \cap K = \emptyset$ and thus $B^K 1 \leq 1_{X \setminus A_t}$. It follows

$$B^K 1 = 0 \quad \text{on } A_t, \text{ for any } t < t_0$$

$$B^K 1 = 0 \quad \text{on } A_{t_0} = \overline{\bigcup_{t<t_0} A_t}^{\,f}$$

and therefore there exists $x_0 \in K$ with $B^K 1(x_0) = 0$.

Suppose now that X is only nearly saturated and let X_1 be the saturated set with $X \subseteq X_1$. Since from Proposition 2.1 X and X_1 are simultaneously parabolic sets with respect to $\underline{\underline{S}}$, it follows that (using the above considerations applied to the universally measurable set X_1) b) ⟹ a). We also have a) ⟹ b) since for any $p \in \underline{\underline{S}}_0$

$$\underline{\text{carr}}\, p = (\underline{\text{carr}}_{X_1} p) \cap X,$$

$$\underline{\text{carr}}_{X_1} p = \overline{\underline{\text{carr}}\, p}\,.$$

(where $\underline{\text{carr}}_{X_1} p$ denotes the harmonic carrier of p on X_1) and therefore

$$\inf\{p(x) \,/\, x \in \underline{\text{carr}}\, p\} = \inf\{p(x) \,/\, x \in \underline{\text{carr}}_{X_1} p\}$$

Remark. The equivalence a) ⟺ e) was proved in [1, Theorem 4.3]. The arguments in the proof of a) ⟹ e) used above are the same as in [1].

Corollary 2.9. Let $\underline{\underline{S}}$ be a standard H-cone of functions on a set X and suppose that X is parabolic. Then for any universally continuous element p of $\underline{\underline{S}}$ there exists $x \in X$ with $p(x) = 0$

Proof. Let X_1 be the saturated set, $X \subseteq X_1$ and let $(p_n)_{n \in \mathbb{N}}$ be an increasing sequence, $p_n \in \underline{\underline{S}}_0$ $(n \in \mathbb{N})$, $\underline{\text{carr}}_{X_1} p_n$ is a compact subset of X_1 and $\sup_{n \in \mathbb{N}} p_n = p$. If $p > 0$ on X then $p > 0$ on X_1 and therefore p is a weak unit in $\underline{\underline{S}}$. Hence there exists $n_0 \in \mathbb{N}$ with

$$p \leqslant p_{n_o} + \frac{1}{2} p,$$

$$p \leqslant 2p_{n_0}$$

which contradicts the fact that $[p_{n_0} = 0]$ is non empty.

Remark. Let $\underline{\underline{S}}$ be the set of all positive lower semi-continuous functions on $X := (-1,1)$ which are increasing on $(-1,0]$ and concave on $(0,1)$. It is known ([2, Example 1]) that $\underline{\underline{S}}$ is a standard H-cone of functions on X and X is not parabolic. On the other hand one can see that there are no universally continous weak units in $\underline{\underline{S}}$. Hence the condition

$$[p = 0] \neq \emptyset \quad , \text{ for any } p \in \underline{\underline{S}}_0$$

is not sufficient for the parabolicity of X (compare with Corollary 2.9).

Definition. Let $\underline{\underline{S}}$ be a standard H-cone functions on a set X. A subbasic subset $M \subseteq X$ is called nearly saturated (with respect to $\underline{\underline{S}}$) if M is nearly saturated with respect to the standard H-cone $\underline{\underline{S}}_{|M}$.

Remark. Let $\underline{\underline{S}}$ be a standard H-cone of functions on a set X and let M be a subbasic subset of X. Then the following assertions hold:

a) M is nearly saturated iff for any balayage B on $\underline{\underline{S}}$, $B \leqslant B^M$ there exists a subbasic subset L of X, $L \subseteq M$ with $B = B^L$.

b) If X is nearly saturated and M is a basic set or M differes from its fine closure with a semi-polar set then M is nearly saturated.

Definition. Let $\underline{\underline{S}}$ be a standard H-cone of functions on a nearly saturated set X. The set X is called quasielliptic with respect to $\underline{\underline{S}}$ if there are no nonempty parabolic subsets of X.

We recall the following definition (cf. [3, §5]):

Definition. An H-cone $\underline{\underline{S}}$ is called quasielliptic if there are no non zero parabolic balayages on $\underline{\underline{S}}$.

Proposition 2.10. Suppose that $\underline{\underline{S}}$ is a standard H-cone. Then the following assertions are equivalent:

a) $\underline{\underline{S}}$ is quasielliptic.

b) There exists a set X such that $\underline{\underline{S}}$ is a standard H-cone of functions on X, X is nearly saturated and quasielliptic.

c) Whenever $\underline{\underline{S}}$ is a standard H-cone of functions on a nearly saturated set X then X is quasielliptic.

The proof follows immediately, using Proposition 2.3.

Remark. Suppose that $\underline{\underline{S}}$ is a standard H-cone of functions on a set X. If $\underline{\underline{S}}$ is quasielliptic then there are no parabolic subsets of X with respect to $\underline{\underline{S}}$. If X is not nearly saturated the converse is not true. Indeed the standard H-cone $\underline{\underline{S}}$ from Example 2.7 is not quasielliptic however if we consider $\underline{\underline{S}}$ as a standard H-cone of functions on the set $X = (-1,1) \setminus K$ then there are no parabolic subsets of X.

Definition. Let $\underline{\underline{S}}$ be a standard H-cone of functions on a set X and let M be a subbasic nearly saturated subset of X. We say that M is a quasielliptic set with respect to $\underline{\underline{S}}$ if M is quasielliptic with respect to the standard H-cone of functions $\underline{\underline{S}}_{|M}$ on the nearly saturated set M.

Definition. A balayage B on an H-cone $\underline{\underline{S}}$ is called quasielliptic (cf. [3, §5]) if the H-cone $B(\underline{\underline{S}})$ is quasielliptic.

Proposition 2.11. Let $\underline{\underline{S}}$ be a standard H-cone of functions on a set X and let M be a nearly saturated subset of X. Then M is quasielliptic iff the balayage B^M on $\underline{\underline{S}}$ is quasielliptic.

The proof follows from Proposition 2.10.

Proposition 2.12. Let $\underline{\underline{S}}$ be a standard H-cone of functions on a nearly saturated set X. Then X is quasielliptic iff

$$X = \bigcup \{E \;/\; E \subseteq X,\ E \text{ is elliptic}\} .$$

Proof. If P is the greatest parabolic subset of X, from Proposition 2.6 we get

$$P = \bigcap \{X \setminus E \;/\; E \subseteq X,\ E \text{ is elliptic}\} .$$

Hence X is quasielliptic iff $P \neq \emptyset$ and therefore iff

$$X = \bigcup \{E \;/\; E \subseteq X,\ E \text{ is elliptic}\} .$$

Remark. If X is not nearly saturated Example 2.7 shows that the relation

$$X = \bigcup \{E \,/\, E \subseteq X,\ E \text{ is elliptic}\}$$

is not sufficient to characterize the quasiellipticity of $\underline{\underline{S}}$.

Proposition 2.13. Suppose that $\underline{\underline{S}}$ is a standard H-cone of functions on a set X and let F_1, F_2 be two nearly saturated subsets of X. The following assertions hold:

a) If F_1 is elliptic then F_1 is quasielliptic.

b) If F_1 is parabolic and quasielliptic then $F_1 = \emptyset$.

c) If F_1, F_2 are quasielliptic then $F_1 \cup F_2$ is quasielliptic.

d) If $F_1 \subseteq F_2$ and F_2 is quasielliptic then F_1 is quasielliptic.

Proof. The assertions follows immediately from the above considerations and since $F_1 \cup F_2$ is also a nearly saturated subset of X.

Let $\underline{\underline{S}}$ be a standard H-cone of functions on a nearly saturated set X and let G be a fine open subset of X. We recall that the localized of $\underline{\underline{S}}$ on G is (cf. [4]) the standard H-cone of functions on G denoted by $\underline{\underline{S}}(G)$, which is the cone of all positive functions f on G which are finite on a fine dense subset of G such that

$$f = \sup \{s - B^{X \setminus G}s \,/\, s \in \underline{\underline{S}},\ s < \infty,\ s - B_s^{X \setminus G} \leq f\}$$

It is known (see [4, Theorem 2.1]) that G is nearly saturated with respect to $\underline{\underline{S}}(G)$.

Theorem 2.14. Suppose that $\underline{\underline{S}}$ is a standard H-cone of functions on a nearly saturated set X and let P be the greatest parabolic subset of X. Then the fine open set $G := X \setminus P$ is quasielliptic with respect to the localized $\underline{\underline{S}}(G)$ of $\underline{\underline{S}}$ on G.

Proof. We have remarked that G is nearly saturated with respect to $\underline{\underline{S}}(G)$. From [3, Theorem 5.16] it follows that the H-cone

$$\underline{\underline{S}}_{B^P} := \{s - B^P s \,/\, s \in \underline{\underline{S}}\}$$

is quasielliptic. Since $\underline{\underline{S}}_{B^P}$ is solid and increasingly dense in $\underline{\underline{S}}(G)$ we deduce that $\underline{\underline{S}}(G)$ is also a quasielliptic H-cone and therefore, by Proposition 2.10, it follows that G is quasielliptic with respect to $\underline{\underline{S}}(G)$.

Remark. Generally the fine open set $G := X \setminus P$ in Theorem 2.14 is not quasielliptic with respect to $\underline{\underline{S}}$ (see Example 2.7) More precisely, from Proposition 2.6 and Proposition 2.12 we deduce that G is quasielliptic with respect to $\underline{\underline{S}}$ iff G is nearly saturated with respect to $\underline{\underline{S}}$.

§ 3. Absorbent, parabolic, elliptic and quasielliptic subsets on a Green set and their relations with the Green function.

Let $\underline{\underline{S}}$ be a standard H-cone and let X be a set such that $\underline{\underline{S}}$ and its dual $\underline{\underline{S}}^{*}$ are represented as standard H-cones of functions on X. Since $\underline{\underline{S}}$ is a solid and increasingly dense convex subcone in $\underline{\underline{S}}^{**} := (\underline{\underline{S}}^{*})^{*}$ (the bidual of $\underline{\underline{S}}$) then without loss of generality we may suppose that $\underline{\underline{S}} = \underline{\underline{S}}^{**}$. In this way if $x \in X$ then the map

$$s \longrightarrow s(x) \quad , s \in \underline{\underline{S}},$$

is an H-integral on $\underline{\underline{S}}$ and therefore an element of $\underline{\underline{S}}^{*}$ for which the associated function on X is denoted by ${}^{*}g_x$. Analogously, for any $x \in X$ we denote by g_x the function on X, $g_x \in \underline{\underline{S}} = \underline{\underline{S}}^{**}$, which is the associated function on X of the H-integral on $\underline{\underline{S}}^{**}$ given by

$$t \longrightarrow t(x) \quad , t \in \underline{\underline{S}}^{*}.$$

If we denote by $[\cdot,\cdot]$ the canonical duality between $\underline{\underline{S}}$ and $\underline{\underline{S}}^{*}$ then for any $x \in X$ we have

$$[s, {}^{*}g_x] = s(x) \quad , s \in \underline{\underline{S}} ,$$

$$[g_x, t] = t(x) \quad , t \in \underline{\underline{S}}^{*} .$$

Therefore for any $x, y \in X$ we get

$$[g_x, {}^{*}g_y] = g_x(y) = {}^{*}g_y(x).$$

The function on $X \times X$ with values in $\overline{\mathbb{R}}_+$ given by

$$g(x,y) := g_x(y) = {}^{*}g_y(x), \quad x, y \in X$$

is called the Green function on X associated with $(\underline{\underline{S}}, \underline{\underline{S}}^{*})$.

In the sequel we mark with the prefix "co" the potential theoretic notions related with $\underline{\underline{S}}^{*}$ as a standard H-cone of functions on X, in order to distinguish them from the similar notions related with the standard H-cone of functions $\underline{\underline{S}}$ on X.

Particularly we have on X the natural and conatural topologies, the fine and cofine topologies etc.

For any subset M of X, $\overline{M}^{f}$ and $\overset{\circ}{M}{}^{f}$ (resp. $\overline{M}^{cf}$ and $\overset{\circ}{M}{}^{cf}$) denote the fine closure and the fine interior (resp. the cofine closure and the cofine interior) of M.

If M is a subset of X, and $t \in \underline{\underline{S}}^{*}$ we put

$$ {}^{*}B^{M}t := \bigwedge \{ t' \in \underline{\underline{S}}^{*} \ / \ t \leq t' \text{ on } M \} $$

and we denote by carrt the harmonic carrier of t on X, i.e.

$$ \text{carr } t = \{ x \in X \ / \ {}^{*}B^{X \setminus V}t \neq t, \text{ for any } V \in \mathcal{V}^{*}_{x} \}, $$

when $\mathcal{V}^{*}_{x}$ denotes the set of all conatural neighbourhoods of x. Particularly, for any $x \in X$, since g_x (resp. ${}^{*}g_x$) is an extrem element of the convex set $\underline{\underline{S}}$ (resp. $\underline{\underline{S}}^{*}$) the set carr g_x (resp. carr ${}^{*}g_x$) is either empty or a singleton.

We remember that ([5, §5.5]) a set X is called a Green set associated with ($\underline{\underline{S}}$, $\underline{\underline{S}}^{*}$) if X is nearly saturated with respect to both $\underline{\underline{S}}$ and $\underline{\underline{S}}^{*}$ and if for any point $x \in X$ we have

$$ \text{carr } g_x = \{x\} \quad , \quad \text{carr } {}^{*}g_x = \{x\} \ . $$

It is known (cf. [5, §5.5]) that if $\underline{\underline{S}}$ is a standard H-cone there exists a set Y such that $\underline{\underline{S}}$ and $\underline{\underline{S}}^{*}$ are represented as standard H-cones of functions on Y and Y is a Lusin space with respect to the natural and conatural topologies and such that Y is a Green set associated with ($\underline{\underline{S}}$, $\underline{\underline{S}}^{*}$). Moreover we can choose Y such that the natural and conatural topologies on Y coincide (cf. [8]).

From this fact it follows that whenever $\underline{\underline{S}}$ is a standard H-cone of functions on a nearly saturated set X there exists a subset Y of X such that $\underline{\underline{S}}^{*}$ may be represented as a standard H-cone of functions on Y and Y becomes a Green set associated with ($\underline{\underline{S}}$, $\underline{\underline{S}}^{*}$).

Consequently if $\underline{\underline{S}}$ and $\underline{\underline{S}}^{*}$ are represented as standard H-cones of functions on a set X then X is nearly saturated with respect to $\underline{\underline{S}}$ iff it is nearly saturated with respect to $\underline{\underline{S}}^{*}$.

In the sequel instead of "X is a Green set associated with ($\underline{\underline{S}}$, $\underline{\underline{S}}^{*}$)" we say simply "X is a Green set" if there is no any ambiguity concerning the pair ($\underline{\underline{S}}$, $\underline{\underline{S}}^{*}$).

We recall now some results concerning the theory of balayages on a Green set X associated ($\underline{\underline{S}}$, $\underline{\underline{S}}^{*}$) (cf. [5, §5.5]).

If A is a subset of X then:

1) For any $s \in \underline{\underline{S}}$ and $t \in \underline{\underline{S}}^{*}$ we have:

$$\left[B^{A}s, t\right] = \left[s, {}^{*}B^{A}t\right] .$$

2) A is semi-polar (resp. polar) iff A is cosemi-polar (resp. copolar).

3) A is thin (resp. cothin) at $x \in X$ iff ${}^{*}B^{A}({}^{*}g_{x}) \neq {}^{*}g_{x}$ (resp. $B^{A}g_{x} \neq g_{x}$).

As a consequence we have that:

a) Any natural (resp. conatural) open set is cofine (resp. fine) open.

b) Any fine (resp. cofine) open set is a cofine (resp. fine) neighbourhood for all its points without a semi-polar set.

Proposition 3.1. Suppose that $\underline{\underline{S}}$ and $\underline{\underline{S}}^{*}$ are represented as standard H-cones of functions on a nearly naturated set X. Then X is a Green set iff any natural (resp.conatural) open subset of X is cofine (resp. fine) open.

Proof. From the preceding considerations the "only if" part of the proof is clear. Further we want to show that for any subset A of X and any $s \in \underline{\underline{S}}$, $t \in \underline{\underline{S}}^{*}$ we have:

$$\left[B^{A}s, t\right] = \left[s, {}^{*}B^{A}t\right] .$$

Obviously it will be sufficient to suppose that $s \in \underline{\underline{S}}_{0}$ and $t \in \underline{\underline{S}}_{0}^{*}$. Since X is nearly saturated with respect to $\underline{\underline{S}}$ then there exists a subset Y of X which is a Green set associated with $(\underline{\underline{S}}, \underline{\underline{S}}^{*})$.

Obviously since $s \in \underline{\underline{S}}_{0}$ we have

$$B^{A}s = \bigwedge\left\{B^{U}s \;/\; A \subseteq U,\ U \text{ natural open in } X\right\} =$$

$$= \bigwedge\left\{B^{U\cap Y}s \;/\; A \subseteq U,\ U \text{ natural open in } X\right\}$$

and therefore, t being universally continuous,

$$\left[B^{A}s, t\right] = \inf\left\{\left[B^{U\cap Y}s, t\right] \;/\; A \subseteq U,\ U \text{ natural open in } X\right\} =$$

$$= \inf\left\{\left[s, {}^{*}B^{U\cap Y}t\right] / A \subseteq U,\ U \text{ natural open in } X\right\} .$$

Since any natural open set U in X is cofine open we get

$${}^{*}B^{U\cap Y}t = {}^{*}B^{U}t \geq {}^{*}B^{A}t$$

and therefore, using the above considerations,

$$\left[B^{A}s, t\right] \geq \left[s, {}^{*}B^{A}t\right] .$$

Analogously we get

$$\left[B^{A}s, t\right] \leq \left[s, {}^{*}B^{A}t\right] .$$

We show now that if A is a subset of X and $x \in X$ then we have

$$A \text{ is thin at } x \iff {}^{*}B^{A}({}^{*}g_x) \neq {}^{*}g_x,$$

$$A \text{ is cothin at } x \iff B^{A}g_x \neq g_x.$$

For any $x, y \in X$ we have

$$ {}^{*}B^{A}({}^{*}g_x)(y) = \left[g_y, {}^{*}B^{A}({}^{*}g_x)\right] = \left[B^{A}g_y, {}^{*}g_x\right] = B^{A}g_y(x).$$

Since X is nearly saturated with respect to $\underline{\underline{S}}$ then for any $s \in \underline{\underline{S}}_0$ there exists a measure μ on X such that

$$[s,t] = \int t(y)\,d\mu(y) \quad , \text{ for any } t \in \underline{\underline{S}}^{*}.$$

Hence for any $x \in X$,

$$s(x) = \left[s, {}^{*}g_x\right] = \int {}^{*}g_x(y)\,d\mu(y) = \int g_y(x)\,d\mu(y)$$

and therefore A is not thin at x iff

$$B^{A}g_y(x) = g_y(x), \text{ for any } y \in X$$

or equivalently

$$ {}^{*}B^{A}({}^{*}g_x)(y) = {}^{*}g_x(y) \quad , \text{ for any } y \in X.$$

Analogously, using the fact that X is also nearly saturated with respect to $\underline{\underline{S}}^{*}$, we get

$$A \text{ is cothin at } x \iff B^{A}g_x \neq g_x.$$

If $x \in X$ and U is a natural neighbourhood of x then U is a cofine neighborhood of x and therefore $X \setminus U$ is cothin at x. Hence

$$B^{X \setminus U}g_x \neq g_x$$

and therefore

$$\underline{\text{carr}}\; g_x = \{x\}.$$

Analogously for any $x \in X$ we have

$$\underline{\text{carr}}\; {}^{*}g_x = \{x\}.$$

Hence X is a Green set.

<u>Theorem 3.2.</u> Suppose thet X is a Green set. If $A \subseteq X$ is absorbent (with respect to $\underline{\underline{S}}$) then $\overline{X \setminus A}^{cf}$ is coabsorbent (i.e. absorbent with respect to $\underline{\underline{S}}^{*}$). Moreover we have

$$A = \overline{\overset{c}{A}^{cf}}^{f}.$$

Proof. Let A be an absorbent subset of X. Then A is a basic set and therefore from Proposition 1.4 it follows that the balayage B^A is absorbent. From [3, Theorem 3.2] we deduce that the balayage on $\underline{\underline{S}}^*$ given by $(B^A)^{*'}$ is coabsorbent. We have

$$(B^A)^{*'} = (B^A)'^{*} = (B^{\overline{X\setminus A}^f})^* = (B^{X\setminus A})^* = {}^*B^{X\setminus A} = {}^*B^{\overline{X\setminus A}^{cf}}.$$

Since $X\setminus A$ is natural open it is cofine open and therefore $\overline{X\setminus A}^{cf}$ is a cobasic set. Again from Proposition 1.4. we deduce that $\overline{X\setminus A}^{cf}$ is coabsorbent.

From the preveous considerations we have

$$(B^A)^{*'} = {}^*B^{\overline{X\setminus A}^{cf}} = (B^{\overline{X\setminus A}^{cf}})^*$$

$$B^A = ((B^A)^{*'})^{*'} = ({}^*B^{\overline{X\setminus A}^{cf}})^{*'} = B^{\overline{X\setminus(\overline{X\setminus A}^{cf})}^f}.$$

Since $\overline{X\setminus(\overline{X\setminus A}^{cf})}^f$ is a basic set, we deduce that

$$A = \overline{X\setminus(\overline{X\setminus A}^{cf})}^f = \overline{\overset{\circ}{A}^{cf}}^f .$$

Proposition 3.3. Let X be a Green set. Then for any $x \in X$ the following assertions are equivalent:

a) $g(x,x) = 0$.

b) The complement of the smallest absorbent set which contains x is not cothin at x.

c) The set $[g_x = 0]$ is the greatest absorbent set which contains x such that its complement is not cothin at x.

Proof. 1) ⟹ 3). We already remarked that $X\setminus[g_x = 0]$ is not cothin at x iff

$$B^{[g_x > 0]} g_x = g_x .$$

Or from $g_x = 0$ on the set $[g_x = 0]$ it follows that

$$B^{[g_x > 0]} g_x = g_x \quad \text{on } [g_x = 0]$$

and therefore

$$B^{[g_x > 0]} g_x = g_x .$$

Let A be an absorbent set which contains x and such that its complement is not cothin at x. Therefore

$$B^{X\setminus A} g_x = g_x .$$

Since A is absorbent we have

$$B^{X\setminus A} g_x = 0 \text{ on } A$$

and thus

$$[g_x = 0] \subseteq A .$$

Hence $[g_x = 0]$ is the greatest absorbent set which contains x and such that its complement is not cothin at x.

3) $\Rightarrow$ 2) is obvious.

2) $\Rightarrow$ 1). Let A_x be the smallest absorbent set which contains x. Since $X\setminus A_x$ is not cothin at x we get

$$B^{X\setminus A_x} g_x = g_x .$$

Since A_x is absorbent we have

$$B^{X\setminus A_x} g_x = 0 \text{ on } A_x$$

and therefore

$$g(x,x) = g_x(x) = 0 .$$

Corollary 3.4. If X is a Green set then the following assertions are equivalent:

a) $g(x,x) > 0$, for any $x \in X$.

b) Any absorbent subset of X is cofine open.

c) Any coabsorbent subset of X is fine open.

Proposition 3.5. Suppose that X is a Green set and let $x \in X$ be that $g(x,x) = 0$. Then the set $\overline{[{}^x g_x > 0]}^f$ is the smallest absorbent set which contains x.

Proof. By Proposition 3.3 the set $[{}^x g_x > 0]$ is not thin at x and therefore, using Theorem 3.2, the set $\overline{[{}^x g_x > 0]}^f$ is an absorbent set containing x and we have

$$[{}^x g_x = 0] = \overline{(X\setminus \overline{[{}^x g_x > 0]}^f)}^{cf} .$$

It follows that the set $X\setminus \overline{[{}^x g_x > 0]}^f$ is not cothin at x.

Hence

$$\overline{[{}^x g_x > 0]}^f \subseteq [g_x = 0] .$$

Let now A be an absorbent set containing x and such that $A \subseteq [g_x = 0]$. Since $[g_x > 0]$ is not cothin at x we get that $X\setminus A$ has the same property. Since

$$A = X \setminus \overline{(\overline{X \setminus A}^{cf})}^{f}$$

it follows that the coabsorbent set $\overline{X \setminus A}^{cf}$ is such that its complement is not thin at x and therefore

$$\overline{X \setminus A}^{cf} \subseteq [{}^{x}g_x = 0],$$

$$[{}^{x}g_x > 0] \subseteq {}'A, \quad \overline{[{}^{x}g_x > 0]}^{f} \subseteq A.$$

<u>Corollary 3.6.</u> Suppose that X and $x \in X$ are as in Proposition 3.5. Then for any $y \in X$ we have

$$g(x,y) = 0 \text{ or } g(y,x) = 0.$$

<u>Proposition 3.7.</u> Let X be a Green set and let $x \in X$ be such that $g(x,x) > 0$. Then there exists an elliptic subset of X which contains x.

<u>Proof.</u> Let us denote by A_x (resp A_x^*) the smallest absorbent (resp. coabsorbent) subset of X containing x. We put

$$E := A_x \setminus \overline{(X \setminus A_x^*)}^{f}.$$

Obviously E is fine open and fine closed. Further $x \in E$. Indeed, since $g(x,x) > 0$, using Proposition 3.3, we get that $X \setminus A_x^*$ is thin at x and therefore $x \notin \overline{X \setminus A_x^*}^{f}$, $x \in E$.

We show now that E is an elliptic subset of X. Let A be an absorbent subset of X. If $x \in A$ then $A_x \subseteq A$ and therefore $E \subseteq A$. If $x \notin A$ then $x \in X \setminus A$ and therefore

$$A_x^* \subseteq \overline{X \setminus A}^{cf}.$$

Since $A = X \setminus \overline{(\overline{X \setminus A}^{cf})}^{f}$ we deduce now that

$$A \subseteq X \setminus \overline{A_x^*}^{f}$$

and we conclude that

$$A \cap E = A \cap A_x \cap (X \setminus \overline{(X \setminus A_x^*)}^{f}) \subseteq A \cap (X \setminus A) = \emptyset.$$

<u>Proposition 3.8.</u> Suppose that X is a Green set. Let E be an elliptic subset of X and $x \in E$ be such that E is a cofine neighborhood of x. Then $g(x,x) > 0$.

<u>Proof.</u> Let A_x be the smallest absorbent set containing x. Since E is elliptic we have $E \subseteq A_x$. If $g(x,x) = 0$ then, by Proposition 3.3 it follows that $X \setminus A_x$ is not cothin at x and therefore $X \setminus E$ is also not cothin at x, which contradicts the

fact that E is a cofine neighbourhood of x.

Theorem 3.9. Let $\underline{\underline{S}}$ be a standard H-cone. Then the following assertions are equivalent:

a) $\underline{\underline{S}}$ is quasielliptic.

b) There exists a Green set X such that $g(x,x) > 0$ for any $x \in X$.

c) There exists a Green set X such that $g(x,x) > 0$ for any $x \in X$ without a semi-polar set.

d) For any Green set X we have $g(x,x) > 0$ for any $x \in X$ without a semi-polar set.

Proof. a) $\Longrightarrow$ d). Let X be a Green set. If we denote by $(E_i)_{i \in I}$ the family of all maximal elliptic subsets of X, since $\underline{\underline{S}}$ is quasielliptic it follows that this family is at most countable and by Proposition 2.10 and Proposition 2.12 we get

$$X = \bigcup_{i \in I} E_i$$

On the other hand, for any $i \in I$ the fine open set E_i is a cofine neighbourhood for any $x \in E_i$ without a semi-polar subset of E_i and therefore, by Proposition 3.8, the set

$$\{x \in E_i \;/\; g(x,x) = 0\}$$

is semi-polar. We conclude that the set

$$\{x \in X \;/\; g(x,x) = 0\}$$

is semi-polar.

Obviously d) $\Longrightarrow$c) $\Longrightarrow$b). The implication b) $\Longrightarrow$a) follows from Proposition 3.7, Proposition 2.12 and Proposition 2.10.

Corollary 3.10. Suppose that X is a Green set such that the fine and cofine topologies on X coincide. Then $\underline{\underline{S}}$ is quasielliptic. Particularly if $\underline{\underline{S}}$ is an autodual standard H-cone then $\underline{\underline{S}}$ is quasielliptic.

Proof. From Corollary 3.4 it follows that $g(x,x) > 0$ for any $x \in X$ and therefore by Theorem 3.9, X is quasielliptic.

Proposition 3.11. Let $\underline{\underline{S}}$ be a standard H-cone. Then the following assertions are equivalent:

a) $\underline{\underline{S}}$ is elliptic.

b) There exists a Green set X such that $g(x,y) > 0$ for any $x,y \in X$.

c) For any Green set X we have $g(x,y) > 0$ for any $x,y \in X$.

Proof. a) $\Longrightarrow$ c) follows from the fact that for any $x \in X$ we have $g_x \neq 0$. c) $\Longrightarrow$ b) is trivial.

b) $\Longrightarrow$ a). For any universally continuous element s of $\underline{\underline{S}}$ there exists a measure μ on X such that

$$s(x) = \int g(y,x)\,d\mu(y) \quad , \text{ for any } x \in X$$

and therefore $s > 0$ if $s \neq 0$. Hence there are no absorbent subsets A of X, $A \neq \emptyset$, $A \neq X$. We conclude that X is elliptic and by Proposition 2.2 it follows that $\underline{\underline{S}}$ is elliptic.

Theorem 3.12. Let $\underline{\underline{S}}$ be a standard H-cone. Then the following assertions are equivalent:

a) $\underline{\underline{S}}$ is parabolic.

b) There exists a Green set X such that $g(x,x) = 0$ for any $x \in X$.

c) For any Green set X we have $g(x,x) = 0$ for any $x \in X$.

d) There exists a Green set X such that for any $x,y \in X$ we have

$$g(x,y) = 0 \quad \text{or} \quad g(y,x) = 0.$$

e) For any Green set X we have for any $x,y \in X$

$$g(x,y) = 0 \quad \text{or} \quad g(y,x) = 0 .$$

Proof. a) $\Longrightarrow$ c). Let X be a Green set and let $x \in X$. Then $g(x,x) = 0$ since in the contrary case from Proposition 3.7 there exists an elliptic subset E of X, $x \in E$ which contradicts the fact that X is parabolic.

c) $\Longrightarrow$ e) follows from Corollary 3.6. The implications e) $\Longrightarrow$ d) $\Longrightarrow$ b) are trivial.

b) $\Longrightarrow$ a). Let X be Green set such that $g(x,x) = 0$ for any $x \in X$ and suppose that $\underline{\underline{S}}$ is not parabolic. Then from Proposition 2.1, Proposition 2.5 and Proposition 2.6 it follows that there exists a maximal elliptic subset E of X, $E \neq \emptyset$. Since E is fine open and since E is a cofine neighbourhood for any $x \in E$ without a semi-polar set, from Proposition 3.8 we deduce that there exists $x \in E$ with $g(x,x) > 0$, which contradicts the hypothesis.

Definition. Let X be a Green set associated with $(\underline{\underline{S}},\underline{\underline{S}}^*)$. A subset M of X is called a Green subset of X if M is a subbasic and nearly saturated subset of X

We remember now some remarks concerning the duality between H-cones (cf. [3, §6]).

If B is a balayage on a standard H-cone $\underline{\underline{S}}$ then the dual $(B(\underline{\underline{S}}))^*$ of $B(\underline{\underline{S}})$ is isomorphic with the H-cone $B^*(\underline{\underline{S}}^*)$, where B^* is the adjoint of B. The restriction to $B(\underline{\underline{S}})\times B^*(\underline{\underline{S}}^*)$ of the canonical map defining the duality between $\underline{\underline{S}}$ and $\underline{\underline{S}}^*$ is the canonical map defining the duality between $B(\underline{\underline{S}})$ and $(B(\underline{\underline{S}}))^*$.

Suppose now that X is Green set and let M be a subset of X which is subbasic with respect to both $\underline{\underline{S}}$ and $\underline{\underline{S}}^*$. In this case we always consider that the standard H-cones $B^M(\underline{\underline{S}})$ and $(B^M)^*(\underline{\underline{S}}^*) = {}^*B^M(\underline{\underline{S}}^*)$ are represented as standard H-cones of functions on the set M. In this way $B^M(\underline{\underline{S}})$ (resp. ${}^*B^M(\underline{\underline{S}}^*)$) is identified with the set $\underline{\underline{S}}_{|M}$ (resp. $\underline{\underline{S}}^*_{|M}$) of the restrictions of $s \in \underline{\underline{S}}$ (resp. $s \in \underline{\underline{S}}^*$) to M. The above notion of Green subset is strongly related with the case when M becames a Green set associated with $(B^M(\underline{\underline{S}}), (B^M(\underline{\underline{S}}))^*)$.

<u>Proposition 3.13</u>. Suppose that X is a Green set associated with $(\underline{\underline{S}}, \underline{\underline{S}}^*)$ and let M be a subset of X which is subbasic with respect to both $\underline{\underline{S}}$ and $\underline{\underline{S}}^*$. Then M is a Green subset of X iff M is a Green set associated with $(B^M(\underline{\underline{S}}), (B^M(\underline{\underline{S}}))^*)$. Moreover if M is a Green subset of X then the Green function on M associated with $(B^M(\underline{\underline{S}}), (B^M(\underline{\underline{S}}))^*)$ is the restriction to $M \times M$ of the Green function on X associated with $(\underline{\underline{S}}, \underline{\underline{S}}^*)$.

<u>Proof</u>. If M is a Green set with respect to $(B^M(\underline{\underline{S}}), (B^M(\underline{\underline{S}}))^*)$ then M is nearly saturated with respect to $\underline{\underline{S}}$ and since $(B^M(\underline{\underline{S}}))^* = {}^*B^M(\underline{\underline{S}}^*)$, M is also nearly saturated with respect to $\underline{\underline{S}}^*$. Hence M is a Green subset of X.

Suppose now that M is a Green subset of X. Then M is nearly saturated with respect to the standard H-cones of functions on M given by $B^M(\underline{\underline{S}})$ and ${}^*B^M(\underline{\underline{S}}^*) = (B^M(\underline{\underline{S}}))^*$. To obtain that M is a Green set associated to $(B^M(\underline{\underline{S}}), (B^M(\underline{\underline{S}}))^*)$ we apply Proposition 3.1. Hence it will be sufficient to show that: any natural open subset of M with respect to $B^M(\underline{\underline{S}})$ is fine open with respect to ${}^*B^M(\underline{\underline{S}}^*)$. This assertion follows from the fact that the natural and the fine topologies on M associated with $B^M(\underline{\underline{S}})$ (resp. ${}^*B^M(\underline{\underline{S}}^*)$) are the traces on M of the corresponding ones associated on X with $\underline{\underline{S}}$ (resp. $\underline{\underline{S}}^*$).

Proposition 3.14. Suppose that X is Green set. Then the following assertions hold:

a) For any balayage B on $\underline{\underline{S}}$ the set

$$b(B) \cap b(B^{*})$$

is a Green subset of X and a Green set associated with $(B(\underline{\underline{S}}), (B(\underline{\underline{S}}))^{*})$.

b) If M is a Green subset of X then

$$\overline{M}^{f} \cap \overline{M}^{cf}$$

is the greatest Green subset of X which contains M.

c) If M is a nearly saturated subbasic subset of X with respect to $\underline{\underline{S}}$ then the set

$$M \cap b^{*}(M)$$

is the greatest Green subset of X containing M and a Green set associated with $(B^{M}(\underline{\underline{S}}), (B^{M}(\underline{\underline{S}}))^{*})$. ($b^{*}(M)$ denotes the base of M with respect to $\underline{\underline{S}}^{*}$ i.e. $b^{*}(M) := \{x \in X \,/\, {}^{*}B^{M}s(x) = s(x), \text{ for any } s \in \underline{\underline{S}}^{*}\}$.)

Proof. Assertion a) follows from the fact that

$$b(B) \setminus b(B^{*}) \quad , \quad b(B^{*}) \setminus b(B)$$

are semi-polar subset of X. Assertion b) follows from a) using the obvious relations $\overline{M}^{f} = b(B^{M})$, $\overline{M}^{cf} = b({}^{*}B^{M}) = b((B^{M})^{*})$.

c) We have

$$ {}^{*}B^{\overline{M}^{f}} = (B^{\overline{M}^{f}})^{*} = (B^{M})^{*} = {}^{*}B^{M}$$

and therefore

$$b({}^{*}B^{\overline{M}^{f}}) = b({}^{*}B^{M}) = \bigcap_{s \in \underline{\underline{S}}} [{}^{*}B^{M}s = s] = b^{*}(M) = b^{*}(\overline{M}^{f}).$$

Hence we get

$$b(B^{M}) \cap b((B^{M})^{*}) = \overline{M}^{f} \cap b^{*}(M) = (M \cap b^{*}(M)) \cup ((\overline{M}^{f} \setminus M) \cap b^{*}(M)).$$

Since $\overline{M}^{f} \setminus M$ is neglijable (i.e. any compact subset of $\overline{M}^{f} \setminus M$ is semi-polar), $b(B^{M}) \cap b((B^{M})^{*}))$ is nearly saturated and $\overline{M} \setminus b^{*}(M)$ is semi-polar, we deduce that $M \cap b^{*}(M)$ is nearly saturated. The assertion c) follows now from a).

Proposition 3.15. Let X be a Green set and let M be a Green subset of X. Then we have

$$M \text{ is elliptic} \iff M \text{ is coelliptic} \iff g(x,y) > 0, \text{ for any } x,y \in M.$$

$$M \text{ is parabolic} \iff M \text{ is coparabolic} \iff g(x,x) = 0, \text{ for any } x \in M.$$

M is quasielliptic $\Longleftrightarrow$ M is coquasielliptic $\Longleftrightarrow g(x,x) > 0$, for any $x \in M$ without a semi-polar set.

Proof. The assertion follows from Proposition 3.13 and from Proposition 3.9, Proposition 3.11 and Theorem 3.12.

Corollary 3.16. Let X be a Green set and let M be a nearly saturated subbasic subset of X with respect to $\underline{\underline{S}}$. Then M is elliptic (resp. parabolic, quasielliptic) iff there exists a semi-polar set $A \subseteq M$ such that

$$g(x,y) > 0 \text{ (resp. } g(x,x) = 0,\ g(x,x) > 0)$$

for any $x,y \in M \setminus A$.

Theorem 3.17. Suppose that X is a Green set associated with $(\underline{\underline{S}}, \underline{\underline{S}}^{*})$.

We put

$$X_p := \{x \in X \,/\, g(x,x) = 0\},$$
$$X_e := \{x \in X \,/\, g(x,x) > 0\},$$

and let P be the greatest parabolic subset of X with respect to $\underline{\underline{S}}$ and $(E_i)_{i \in I}$ be the family of all maximal elliptic subsets of X with respect to $\underline{\underline{S}}$. Then X_p is a fine and cofine closed subset of X and P coincides with the essential base of X_p. Particularly

$$X_e \subseteq \bigcup_{i \in I} E_i$$

$$P \subseteq X_p$$

and the sets

$$\bigcup_{i \in I} E_i \setminus X_e,\quad X_p \setminus P$$

are semi-polar.

Proof. Since the functions

$$x \longrightarrow (x,x)$$
$$(x,y) \longrightarrow g(x,y)$$

are fine lower semi-continuous it follows that the function

$$x \longrightarrow g(x,x)$$

is also fine lower semi-continuous and therefore the set X_p is fine and cofine closed. Let us denote by P_0 the essential base of X_p (i.e. the greatest basic subset of X_p) and let

$$Y := P_0 \cap b^*(P_0)$$

From Proposition 3.14 it follows that Y is a Green subset of X and a Green set associated with $(B^{P_0}(\underline{\underline{S}}), (B^{P_0}(\underline{\underline{S}}))^*)$. If g is the Green function on X associated with $(\underline{\underline{S}}, \underline{\underline{S}}^*)$ then by Proposition 3.13 it follows that its restriction to Y x Y is the Green function on Y associated with $(B^{P_0}(\underline{\underline{S}}), (B^{P_0}(\underline{\underline{S}}))^*)$. Since

$$g(x,x) = 0 \quad , \text{ for any } x \in P_0$$

it follows by Theorem 3.12 that $B^{P_0}(\underline{\underline{S}})$ is parabolic and therefore

$$P_0 \subseteq P.$$

By the definition of P_0 we get that $X_p \setminus P_0$ is semi-polar.

On the other hand, from Proposition 3.7 we deduce

$$X_e \subseteq \bigcup_{i \in I} E_i.$$

By Proposition 2.6 we get

$$P = X \setminus \bigcup_{i \in I} E_i \subseteq X \setminus X_e = X_p$$

and therefore

$$P \subseteq P_0 \quad , \quad P = P_0$$

From

$$\bigcup_{i \in I} E_i \setminus X_e = (\bigcup_{i \in I} E_i) \cap X_p \subseteq X_p \setminus P$$

it follows that the set $\bigcup_{i \in I} E_i \setminus X_e$ is also semi-polar.

§ 4. Totally parabolic standard H-cones.

Definition. An H-cone $\underline{\underline{S}}$ is called totally parabolic if it is parabolic and the set of all absorbent balayages on $\underline{\underline{S}}$ is totally ordered.

Definition. A balayage B on a given H-cone $\underline{\underline{S}}$ is called totally parabolic if the H-cone $B(\underline{\underline{S}})$ is totally parabolic.

Remark. 1. Generally, the notion of totally parabolic H-cone is more restrictive than the parabolic H-cone one. For example if X is a Stonian space which has no isolated points and $\underline{\underline{S}}$ is the convex cone of all positive real continuous functions on X, then $\underline{\underline{S}}$ is an H-cone such that any non zero balayage on $\underline{\underline{S}}$ is parabolic without beeing totally parabolic.

2. If $\underline{\underline{S}}$ and $\underline{\underline{T}}$ are two H-cones in duality (cf. [3, §6]) then $\underline{\underline{S}}$ and $\underline{\underline{T}}$ are simulteneously totally parabolic. Particularly a standard H-cone $\underline{\underline{S}}$ will be totally parabolic iff $\underline{\underline{S}}^{*}$ is totally parabolic.

3. Suppose that $\underline{\underline{S}}$ is a parabolic H-cone such that there exists an absorbent control function on $\underline{\underline{S}}$ (see [3, §4]). Then $\underline{\underline{S}}$ will be totally parabolic iff there exists an increasing bijection $t \to A_t$ from $[0,1]$ on the set of all absorbent balayages on $\underline{\underline{S}}$.

From now on $\underline{\underline{S}}$ will be a given parabolic standard H-cone.

If X is a Green set associated with $(\underline{\underline{S}}, \underline{\underline{S}}^{*})$ and $g : X \times X \to \overline{\mathbb{R}}_+$ is the Green function on X associated with $(\underline{\underline{S}}, \underline{\underline{S}}^{*})$, for any $x \in X$ we denote by G_x (resp. G_x^{*}) the absorbent (resp. coabsorbent) set given by

$$G_x := [g_x = 0] \quad (\text{resp. } G_x^{*} := [{}^{*}g_x = 0]).$$

Also we denote by A_x (resp. A_x^{*}) the smallest absorbent (resp. coabsorbent) subset of X containing the point x.

Remark. Since $\underline{\underline{S}}$ is parabolic from Theorem 3.12 and Proposition 3.3 it follows that for any $x \in X$ we have

$$A_x \subseteq G_x, \quad A_x^{*} \subseteq G_x^{*}.$$

and G_x (resp. G_x^*) is the greatest absorbent (resp. coabsorbent) subset of X such that $X \setminus G_x$ (resp. $X \setminus G_x^*$) is not cothin (resp. not thin) at x. We have also

$$A_x = \overline{X \setminus G_x^*}^{f} \quad , \quad A_x^* = \overline{X \setminus G_x}^{cf} .$$

Proposition 4.1. The following assertions are equivalent:

a) $\underline{\underline{S}}$ is totally parabolic.

b) There exists a Green set X such that the set

$$\{G_x \ / \ x \in X\}$$

is totally ordered.

c) There exists a Green set X such that the set

$$\{A_x \ / \ x \in X\}$$

is totally ordered.

Proof. a) $\Rightarrow$b) and a) $\Rightarrow$c) are obvious.

c) $\Rightarrow$a). Let A_1, A_2 be two absorbent subset of X and suppose that there exists $x_0 \in A_2$ with $x_0 \notin A_1$. We show that $A_1 \subseteq A_2$. If $x \in A_1$ we have $A_x \subseteq A_{x_0}$ and therefore

$$A_1 = \bigcup_{x \in A_1} A_x \subseteq A_{x_0} \subseteq A_2 .$$

b) $\Rightarrow$ a). Since for any $x \in X$ we have $A_x^* = \overline{X \setminus G_x}^{cf}$ it follows that the set

$$\{A_x^* \ / \ x \in X\}$$

is totally ordered and therefore from c)$\Rightarrow$a) applied to $\underline{\underline{S}}^*$ we get that $\underline{\underline{S}}$ is totally parabolic.

Proposition 4.2. Let X be a Green set. Then for any $x \in X$ the following assertions are equivalent:

a) $A_x = G_x$.

b) $G_x \cap G_x^*$ is semi-polar.

c) There exists an unique pair (A, A^*) where A (resp. A^*) is absorbent (resp. coabsorbent) with

$$x \in A \cap A^* \quad , \quad X = A \cup A^*$$

and $A \cap A^*$ is semi-polar.

Proof. a)$\Longleftrightarrow$b). Since A_x is fine open it follows that $A \setminus \mathring{A}^{cf}$ is semi-polar and therefore from

$$G_x \cap G_x^* = (G_x \setminus A_x) \cup (A_x \setminus \mathring{A}_x^{cf})$$

we get that $G_x \cap G_x^*$ is semi-polar iff $G_x \setminus A_x$ is semi-polar or equivalently $A_x = G_x$.

a) $\Longrightarrow$c). If (A, A^*) is a pair as in the assertion c) then we have $X \setminus A$ (resp. $X \setminus A^*$) is not cothin (resp. not thin) at x and therefore from Proposition 3.3 it follows

$$A \subseteq G_x \quad , \quad A^* \subseteq G_x^* .$$

Since

$$A_x \subseteq A \text{ and } A_x^* \subseteq A^* \text{ we get}$$

$$A = A_x = G_x \ , \ A^* = A_x^* = G_x^*.$$

Obviously the pair (A_x, G_x^*) verifies the conditions from c).

c) $\Longrightarrow$a). Since (A_x, G_x^*) and (G_x, A_x^*) are two pairs which verify the conditions from c) we deduce $A_x = G_x$.

Proposition 4.3. Let X be a Green set. Then the following assertions are equivalent:

a) $A_x = G_x$ for any $x \in X$.

b) $G_x \cap G_x^*$ is semi-polar for any $x \in X$.

c) $G_x \cap G_x^*$ is totally thin for any $x \in X$.

d) For any absorbent set A we have $A = A_x$ if $x \in A \setminus \mathring{A}^{cf}$.

Particularly if one of the above conditions is verified then for any absorbent set A have

$$\mathring{A}^{cf} = \bigcup \{A_x \ / \ x \in \mathring{A}^{cf}\} .$$

and therefore A^{cf} is fine open and the set $A \setminus \mathring{A}^{cf}$ is polar in A with respect to $\underline{\underline{S}}|_A \equiv B^A(\underline{\underline{S}})$.

Proof. Suppose that a) is fulfiled and let A be an absorbent subset of X. If $x \in \mathring{A}^{cf}$ then $A_x \subseteq A$. If $y \in A_x$ then $y \in \mathring{A}^{cf}$ since in the contrary case we have by Proposition 3.3 that $A \subseteq G_y$ and from a) we get

$$A \subseteq G_y = A_y \subseteq A_x \subseteq A$$

which contradicts the fact that $X \setminus A_x$ is not cothin at x. From the above considerations we deduce that $A_x \subseteq \mathring{A}^{cf}$ if $x \in \mathring{A}^{cf}$ and therefore

$$\overset{\circ}{A}{}^{cf} = \bigcup \{ A_x \,/\, x \in \overset{\circ}{A}{}^{cf} \}.$$

For any $s \in \underset{\sim}{S}$ we have

$$B^{A \setminus \overset{\circ}{A}{}^{cf}} s \leq \bigwedge \{ B^{A \setminus A_x} s \,/\, x \in \overset{\circ}{A}{}^{cf} \} = 0 \text{ on } \bigcup_{x \in \overset{\circ}{A}{}^{cf}} A_x = \overset{\circ}{A}{}^{cf}$$

and therefore, since $A \setminus \overset{\circ}{A}{}^{cf}$ is semi-polar,

$$B^{A \setminus \overset{\circ}{A}{}^{cf}} = 0 \text{ on } A.$$

a) <=> b) follows from Proposition 4.2 and c) => b) is trivial.

a) => d). If A is an absorbent subset of X and $x \in A \setminus \overset{\circ}{A}{}^{cf}$ from Proposition 3.3. we have

$$A_x \subseteq A \subseteq G_x$$

and therefore

$$A_x = A = G_x.$$

d) => a). From Proposition 3.3 we get $x \in G_x \setminus \overset{\circ}{G}_x{}^{cf}$ and consequently

$$A_x = G_x.$$

a) => c). From the first part of the proof we have for any $x \in X$

$$G_x \cap G_x^* = A_x \setminus \overset{\circ}{A}_x{}^{cf}$$

and therefore

$$B^{G_x \cap G_x^*} 1 = 0 \text{ on } G_x .$$

We conclude that $G_x \cap G_x^*$ is totally thin.

Theorem 4.4. The following assertions are equivalent:

a) S is totally parabolic.

b) There exists a Green set X such that:

$$A_x = G_x \text{ for any } x \in X.$$

c) There exists a Green set X such that:

$$G_x \cap G_x^* \text{ is totally thin for any } x \in X.$$

d) For any Green set X we have:

$$A_x = G_x \text{ for any } x \in X \text{ without a semi-polar set.}$$

e) For any Green set X we have:

$$G_x \cap G_x^* \text{ is semi-polar}$$

for any $x \in X$ without a semi-polar set.

Proof. b) $\Longleftrightarrow$ c) follows from Proposition 4.3 and d) $\Longleftrightarrow$ e) follows from Proposition 4.2.

b) $\Longrightarrow$a) Let $x, y, \in X$. Since $\underline{\underline{S}}$ is parabolic, from Theorem 3.12 we get that either $g(x,y) = 0$ or $g(y,x) = 0$. Therefore

$$x \in G_y \quad \text{or} \quad y \in G_x \,.$$

If $x \in G_y$ we have $A_x \subseteq G_y$ and consequently

$$G_x = A_x \subseteq G_y \,.$$

By Proposition 4.1 we deduce now that a) is true.

d) $\Longrightarrow$ b) follows from the fact that if X is a Green set associated with $(\underline{\underline{S}}, \underline{\underline{S}}^{*})$ and M is a semi-polar subset of X then $X \setminus M$ is also a Green set associated with $(\underline{\underline{S}}, \underline{\underline{S}}^{*})$.

a) $\Longrightarrow$e). Let X be a Green set associated with $(\underline{\underline{S}}, \underline{\underline{S}}^{*})$ and let E be a Green set associated with $(\underline{\underline{S}}, \underline{\underline{S}}^{*})$ such that E is a Lusin space (cf. [5, § 5.5]).

Let D be a countable set of non zero H-measures such that does not charge the semi-polar subsets of X and such that the Green potentials (resp. the Green copotentials) on E, given by

$$G^{\mu}(.) = \int g_y(.)d\mu(y) \quad (\text{resp. } {}^{*}G^{\mu}(.) = \int {}^{*}g_y(.)d\mu(y)),$$

where $\mu \in D$, form an increasing dense subset of $\underline{\underline{S}}$ (resp. $\underline{\underline{S}}^{*}$).

We denote by M the set of all points $x \in E$ such that the set $G_x \cap G_x^{*}$ is not semi-polar.

We remark that for any $x \in E$ we have:

$G_x \cap G_x^{*}$ is semi-polar (or equivalently $x \notin M$) iff for any $\mu \in D$ we have

$$G^{\mu}(x) > 0, \quad \text{or} \quad {}^{*}G^{\mu}(x) > 0.$$

Therefore

$$M = \bigcup_{\mu \in D} ([G^{\mu} = 0] \cap [{}^{*}G^{\mu} = 0]).$$

It remains to show that for any $\mu \in D$ the set

$$L := [G^{\mu} = 0] \cap [{}^{*}G^{\mu} = 0]$$

is semi-polar. Since the sets

$$\left[G^{\mu} = 0\right] \quad , \quad \overline{\left[{}^{x}G^{\mu} > 0\right]}^{f}$$

are absorbent, from hypothesis a) we get

$$\left[G^{\mu} = 0\right] \subseteq \overline{\left[{}^{x}G^{\mu} > 0\right]}^{f} .$$

Indeed in the contrary case we have

$$\overline{\left[{}^{x}G^{\mu} > 0\right]}^{f} \subseteq \left[G^{\mu} = 0\right].$$

We remark that a measure ν on E such that $G^{\nu} = 0$ on suppν and ν does not charges any semi-polar set is equal to zero. From this fact we may suppose that μ is charged only by the set $\left[G^{\mu} > 0\right] \cap \left[{}^{x}G^{\mu} > 0\right]$ and therefore $\mu = 0$, which is a contradiction.

Hence

$$\left[G^{\mu} = 0\right] \cap \left[{}^{x}G^{\mu} = 0\right] \subseteq \left[{}^{x}G^{\mu} = 0\right] \cap \overline{\left[{}^{x}G^{\mu} > 0\right]}^{f} = \left[{}^{x}G^{\mu} = 0\right] \setminus \overset{of}{\left[{}^{x}G^{\mu} = 0\right]}$$

and we conclude that the set

$$\left[G^{\mu} = 0\right] \cap \left[{}^{x}G^{\mu} = 0\right]$$

is semi-polar.

Corollary 4.5. The following assertions are equivalent:

a) $\underline{\underline{S}}$ is totally parabolic.

b) There exists a Green set X such that for any $x \in X$ there exists a unique pair (A, A^{x}) where A (resp. A^{x}) is an absorbent (resp. coabsorbent) set with

$$x \in A \cap A^{x} \quad , \quad X = A \cup A^{x}$$

and $A \cap A^{x}$ is semi-polar.

c) For any Green set X and any $x \in X$ without a semi-polar set, there exists a unique pair (A, A^{x}) where A (resp. A^{x}) is an absorbent (resp. coabsorbent) set with

$$x \in A \cap A^{x} \quad , \quad X = A \cup A^{x}$$

and $A \cap A^{x}$ is semi-polar.

Proof. It follows from Theorem 4.4 and Proposition 4.2.

Remark. 1. If $\underline{\underline{S}}$ is the standard H-cone of functions associated with the heat equation on $X := \mathbb{R}^{n} \times \mathbb{R}$, $n \geq 1$, then $\underline{\underline{S}}$ is totally parabolic and the totally thin set (see Theorem 4.4)

$$G_{x} \cap G^{x}_{x} \quad , \quad x \in X$$

is exactly the horizontal line

$$\{(z,t) \;/\; z \in \mathbb{R}^n\},$$

where $x = (y,t) \in \mathbb{R}^n \times \mathbb{R}$.

2. We give now an example of totally parabolic standard H-cone $\underline{\underline{S}}$ and a Green set X associated with $(\underline{\underline{S}}, \underline{\underline{S}}^{\times})$ such that there exist points $x \in X$ for which

$$G_x \cap G_x^{\times} \text{ is not semi-polar.}$$

We consider $X := \mathbb{R} \times \mathbb{R}$ and we distinguish in X the following three regions:

$$D_1 = \{(x,y) \;/\; y \geqslant 0\},$$

$$D_2 = \{(x,y) \;/\; y \leqslant 0,\ x \geqslant 0\},$$

$$D_3 = \{(x,y) \;/\; y \leqslant 0,\ x \leqslant 0\} \setminus \{(0,0)\}.$$

We identify D_2 with

$$\{(x,y) \;/\; y \leqslant 0\}$$

by a homeomorphism φ such that

$$\varphi((x,0)) = (x,0) \quad \text{if } x \geqslant 0$$

$$\varphi((0,y)) = (y,0) \quad \text{if } y \leqslant 0$$

and we identify D_3 with the band

$$\{(x,y) \;/\; 0 \leqslant y \leqslant 1\}$$

by a homeomorphism ψ such that

$$\psi((y,0)) = \psi((0,y)) = y - \frac{1}{y} \quad \text{if } y < 0$$

and such that

$$\lim_{(x,y) \to (0,0)} |\psi((x,y))| = +\infty.$$

On X we consider now a sheaf $\mathcal{H}$ of linear vector spaces of real continuous functions defined by: if U is an open subset of X, a real continuous functions h on U belongs to $\mathcal{H}(U)$ iff

$$h|_{U \cap D_1},\ h|_{D_2} \circ \varphi,\ h|_{D_3} \circ \psi$$

are harmonic for the heat equation. It is easy to see that $(X, \mathcal{H})$ is a Bauer space. If we denote by $\underline{\underline{S}}$ the standard H-cone of functions on X of all positive superharmonic functions on X then one can see that $\underline{\underline{S}}$ is parabolic and X is a Green set associated with $(\underline{\underline{S}}, \underline{\underline{S}}^{\times})$. We also remark that if $a = (x,0)$, $x \geqslant 0$ then we have

$$A_a = (x,0) + D_2 \quad , \quad G_a = D_2 \cup D_3$$

and therefore

$$A_a \neq G_a .$$

On the other hand for any $a \neq (x,0)$ with $x \geqslant 0$ we have

$$A_a = G_a .$$

Obviously the set $\{(x,0) \;/\; x \geqslant 0\}$ is semi-polar and $\underline{\underline{S}}$ is totally parabolic.

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