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# GLOBAL STABILIZING SOLUTIONS TO RICCATI EQUATIONS VIA STABILIZING COMPENSATORS WITH DISTURBANCE ATTENUATION

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ABSTRACT

For linear systems with time varying coefficients it is proved that existence of a stabilizing compensator with disturbance attenuation implies existence of global,positive semidefinite stabilizing solutions to three matrix Riccati differential equations.

## 1. THE PROBLEM. MAIN RESULT.

Consider the system

$$
\begin{aligned}
x' &= A(t)x+B_1(t)u_1+B_2(t)u_2 \\
y_1 &= C_1(t)x+D_{11}(t)u_1+D_{12}(t)u_2 \\
y_2 &= C_2(t)x+D_{21}(t)u_1
\end{aligned}
\tag{1.1}
$$

where $A:\mathbb{R}\to\mathcal{M}_{n\times n}(\mathbb{R})$, $B_i:\mathbb{R}\to\mathcal{M}_{n\times m_i}$ ,$C_i:\mathbb{R}\to\mathcal{M}_{p_i\times n}$ $i=1,2$ ,$D_{ij}:\mathbb{R}\to\mathcal{M}_{p_i\times m_j}$ $i,j=1,2$ (matrices of corresponding dimensions with real entires) are continuous and bounded.

Assume also that $D_{12}^*(t)D_{12}(t)$ and $D_{21}(t)D_{21}^*(t)$ are invertible with bounded inverses.

Here $u_2$ is a control input, $y_2$ a measured output ,$u_1$ is a disturbance and $y_1$ a quality output.

A compensator for (1.1) is a system

$$
\begin{aligned}
x_c' &= A_c(t)x_c+B_c(t)u_c \\
y_c &= C_c(t)x_c+D_c(t)u_c
\end{aligned}
\tag{1.2}
$$

$A_c, B_c, C_c, D_c$ being continuous and bounded on $\mathbb{R}$.

The compensator is coupled to the system by taking $u_c=y_2$, $u_2=y_c$. A compensator (1.2) is stabilizing for (1.1) if the system

$$\begin{aligned} x' &= [A(t)+B_2(t)D_c(t)C_2(t)]x + B_2(t)C_c(t)x_c \\ x_c' &= B_c(t)C_2(t)x + A_c(t)x_c \end{aligned} \qquad (1.3)$$

has an exponentially stable evolution.

After coupling the compensator (1.2) to (1.1) we obtain

$$\begin{aligned} x' &= [A(t)+B_2(t)D_c(t)C_2(t)]x + B_2(t)C_c(t)x_c + [B_1(t)+ \\ &\quad +B_2(t)D_c(t)D_{21}(t)]u_1 \\ x_c' &= B_c(t)C_2(t)x + A_c(t)x_c + B_c(t)D_{21}(t)u_1 \qquad (1.3') \\ y_1 &= [C_1(t)+D_{12}(t)D_c(t)C_2(t)]x + D_{12}(t)C_c(t)x_c + [D_{11}(t)+ \\ &\quad +D_{12}(t)D_c(t)D_{21}(t)]u_1 \end{aligned}$$

Since the compensator is stabilizing the system (1.3′) defines an operator $T_{y_1u_1} : L^2(R,R^{m_1}) \to L^2(R,R^{p_1})$ defined by $T_{y_1u_1}u_1=y_1$.

$$y_1(t) = [C_1(t)+D_{12}(t)D_c(t) \quad D_{12}(t)C_c(t)] \cdot \int_{-\infty}^{\infty} X_{\mathcal{A}}(t,s) \begin{pmatrix} B_1(s)+B_2(s)D_c(s)D_{21}(s) \\ B_c(s)D_{21}(s) \end{pmatrix} u_1(s)ds +$$

$$+[D_{11}(t)+D_{12}(t)D_c(t)D_{21}(t)] \; u_1(t)$$

where we have denoted by $\mathcal{A}(t)$ the matrix of system (1.3) and by $X_{\mathcal{A}}(\cdot,\cdot)$ the corresponding evolution operator.

A stabilizing compensator is disturbance attenuating or $\gamma$-contracting if $\|T_{y_1u_1}\| < \gamma$.

We associate to (1.1) the following Riccati equations

$$R'+A^*(t)R+RA(t)-(RB(t)+C_1^*(t)D_1(t))K^{-1}(t)(B^*(t)R+D_1^*(t)C_1(t))+$$

$$+C_1^*(t)C_1(t)=0 \qquad (1.4)\ .$$

$$S'=A(t)S+SA^*(t)-(SC^*(t)+B(t)D_2^*(t))\hat{K}^{-1}(t)(C(t)S+D_2(t)B_1^*(t)) \qquad (1.5)$$

where $B=(B_1 \quad B_2)$, $C=\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$, $D_1=(D_{11} \quad D_{12})$, $D_2=\begin{pmatrix} D_{11} \\ D_{21} \end{pmatrix}$

$K=D_1^*D_1-\gamma^2\begin{pmatrix} I_{m_1} & 0 \\ 0 & 0 \end{pmatrix}$, $\hat{K}=D_2D_2^*-\gamma^2\begin{pmatrix} I_{p_1} & 0 \\ 0 & 0 \end{pmatrix}$.

Consider also the orthogonal projections

$$\Pi_1(t)=I_{p_1}-D_{12}(t)(D_{12}^*(t)D_{12}(t))^{-1}D_{12}^*(t)$$
$$\Pi_2(t)=I_{m_1}-D_{21}^*(t)(D_{21}(t)D_{21}^*(t))^{-1}D_{21}(t) \qquad (1.6)$$

We are now in position to start the main result

<u>Theorem 1.1</u> Assume

a) $(\Pi_1 C_1\ ,\ A-B_2(D_{12}^*D_{12})^{-1}D_{12}^*C_1)$ is detectable

b) $(A-B_1D_{21}^*(D_{21}D_{21}^*)^{-1}C_2,\ B_1\Pi_2)$ is stabilizable.

Let $\gamma>0$. If there exists a stabilizing and $\gamma$-contracting compensator the following conditions are satisfied :

1) $D_{11}^*(t)\Pi_1(t)D_{11}(t) \leq kI_{m_1} < \gamma^2 I_{m_1}$

$D_{11}(t)\Pi_2(t)D_{11}^*(t) \leq kI_{p_1} < \gamma^2 I_{p_1}$

for all $t\in\mathbb{R}$.

2) Equation (1.4) has a bounded on $\mathbb{R}$ solution $R$ with the properties $\alpha$) $R(t)\geq 0$ for all $t\in\mathbb{R}$

$\beta$) $A+BF$ defines an exponentially stable evolution for $F=-K^{-1}(B^*R+D_1^*C_1)$

3) Equation (1.5) has a bounded on $\mathbb{R}$ solution $S$ with the properties $\alpha$) $S(t)\geq 0$ for all $t\in R$

$\beta$) $A+HC$ defines an exponentially stable evolution for $H=-(SC^*+B_1D_2^*)\hat{K}^{-1}$.

4) With $F$ in 2) written as $F=\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ tha Riccati equation

$$\hat{S}'=(A+B_1F_1)\hat{S}+\hat{S}(A+B_1F_1)^*+B_1B_1^*-\left\{\begin{pmatrix} -D_{12}F_2 \\ C_2+D_{21}F_1 \end{pmatrix}\hat{S}+\begin{pmatrix} D_{11} \\ D_{21} \end{pmatrix}B_1^*\right\}\hat{K}^{-1}\left\{\begin{pmatrix} -D_{12}F_2 \\ C_2+D_{21}F_1 \end{pmatrix}\hat{S}+\begin{pmatrix} D_{11} \\ D_{21} \end{pmatrix}B_1^*\right\}$$

has a global, positive semidefinite, stabilizing solution. For constant coefficients the result is in [3], [6], [7]. The ideas of the proof follow the ones in [7]. Under stronger assumptions, the existence of global, positive, stabilizing solutions for the first two Riccati equations has been obtained in [4], where ideas in [3] were used. The time-varying coefficients case was considered also in [1], [7] but only for finite horizon problems.

## 2. NECESSITY OF CONDITION 1).

Proposition 2.1 If the compensator (1.2) is stabilizing and $\gamma$-contracting for (1.1)

$$[D_{11}(t)+D_{12}(t)D_c(t)D_{21}(t)]^*[D_{11}(t)+D_{12}(t)D_c(t)D_{21}(t)]\leq$$
$$\leq\|T_{y_1u_1}\|^2 I_{m_1}<\gamma^2 I_{m_1}$$

$$[D_{11}(t)+D_{12}(t)D_c(t)D_{21}(t)][D_{11}(t)+D_{12}(t)D_c(t)D_{21}(t)]^*\leq$$
$$\leq\|T_{y_1u_1}\|^2 I_{p_1}<\gamma^2 I_{p_1} \qquad \text{for all } t\in\mathbb{R}.$$

Proof. Let $\hat{u}_1\in R^{m_1}$ and $u_1^{\varepsilon\tau}(t)=\begin{cases} 0 & t<\tau \\ \frac{1}{\sqrt{\varepsilon}}\exp(-(t-\tau)/2\varepsilon)\hat{u}_1 & t\geq\tau \end{cases}$

Let $y_1^{\varepsilon\tau}=T_{y_1u_1}(u_1^{\varepsilon\tau})$ ; we have $y_1^{\varepsilon\tau}(t)=0$ for $t<\tau$ and for $t\geq\tau$

$$y_1^{\varepsilon\tau}(t)=\mathcal{C}(t)\int_\tau^t X_{\mathcal{A}}(t,s)\mathcal{B}(s)u_1^{\varepsilon\tau}(s)ds+\mathcal{D}(t)u_1^{\varepsilon\tau}(t)$$

$$\mathcal{C}(t)=(C_1(t)+D_{12}(t)D_c(t) \qquad D_{12}(t)C_c(t)),$$

$$\mathcal{B}(s)=\begin{pmatrix} B_1(s)+B_2(s)D_c(s)D_{21}(s) \\ B_c(s)D_{21}(s)\end{pmatrix} \qquad \mathcal{D}(t)=D_{11}(t)+D_{12}(t)D_c(t)D_{21}(t)$$

Denote $\hat{y}_1^{\varepsilon\tau}(t)=0$ for $t<\tau$ and for $t\geq\tau$

$$\hat{y}_1^{\varepsilon\tau}(t)=\mathcal{C}(t)\int_\tau^t X_{\mathcal{A}}(t,s)\mathcal{B}(s)u_1^{\varepsilon\tau}(s)ds$$

$$|\hat{y}_1^{\varepsilon\tau}(t)|\leq c\int_\tau^t e^{-\alpha(t-s)}\frac{1}{\sqrt{\varepsilon}}\, e^{-\frac{1}{2\varepsilon}(s-\tau)}|\hat{u}_1|\,ds \in$$

$$\leq \hat{c}\sqrt{\varepsilon}\, e^{-\alpha(t-\tau)}|\hat{u}_1|$$

We deduce that

$$\|y_1^{\varepsilon\tau}\|^2=\int_\tau^\infty (y_1^{\varepsilon\tau})^*(t)y_1^{\varepsilon\tau}(t)dt=\int_\tau^\infty\{[\hat{y}_1^{\varepsilon\tau}(t)]^*+[u_1^{\varepsilon\tau}(t)]^*\mathcal{D}^*(t)\}\cdot$$

$$\cdot\{\hat{y}_1^{\varepsilon\tau}(t)+\mathcal{D}(t)u_1^{\varepsilon\tau}(t)\}\,dt=\int_\tau^\infty[\hat{y}_1^{\varepsilon\tau}(t)]^*\hat{y}_1^{\varepsilon\tau}(t)dt+$$

$$+\int_\tau^\infty[u_1^{\varepsilon\tau}(t)]^*\mathcal{D}^*(t)\,\hat{y}_1^{\varepsilon\tau}(t)dt+$$

$$+\int_\tau^\infty[\hat{y}_1^{\varepsilon\tau}(t)]^*\mathcal{D}(t)u_1^{\varepsilon\tau}(t)dt+\int_\tau^\infty[u_1^{\varepsilon\tau}(t)]^*\mathcal{D}^*(t)\mathcal{D}(t)$$

$$\cdot\, u_1^{\varepsilon\tau}(t)dt$$

$$\int_\tau^\infty[\hat{y}_1^{\varepsilon\tau}(t)]^*\hat{y}_1^{\varepsilon\tau}(t)dt\leq\frac{1}{2\alpha}\hat{c}^2\varepsilon|u_1|^2$$

$$\left|\int_\tau^\infty[u_1^{\varepsilon\tau}(t)]^*\mathcal{D}^*(t)\hat{y}_1^{\varepsilon\tau}(t)dt\right|\leq\tilde{c}\int_\tau^\infty|\hat{u}_1|\frac{1}{\sqrt{\varepsilon}}\,e^{-\frac{t-\tau}{2\varepsilon}}\sqrt{\varepsilon}\,e^{-\alpha(t-\tau)}|\hat{u}_1|\,dt\leq$$

$$\leq\varepsilon\check{c}\,|\hat{u}_1|^2$$

$$\int_\tau^\infty[u_1^{\varepsilon\tau}(t)]^*\mathcal{D}^*(t)\mathcal{D}(t)u_1^{\varepsilon\tau}(t)dt=\frac{1}{\varepsilon}\hat{u}_1^*\int_\tau^\infty e^{-\frac{t-\tau}{\varepsilon}}\mathcal{D}^*(t)\mathcal{D}(t)dt\,\hat{u}_1$$

and by standard in singular perturbations procedures we have

$$\lim_{\varepsilon\to0}\int_\tau^\infty[u_1^{\varepsilon\tau}(t)]^*\mathcal{D}^*(t)\mathcal{D}(t)\,u_1^{\varepsilon\tau}(t)dt=\hat{u}_1^*\mathcal{D}^*(\tau)\mathcal{D}(\tau)\hat{u}_1^*$$

We deduce that $\|y_1^{\varepsilon\tau}\|^2=\hat{u}_1^*\mathcal{D}^*(\tau)\mathcal{D}(\tau)\hat{u}_1+\omega(\varepsilon)|\hat{u}_1|^2$

with $\lim_{\varepsilon\to0}\omega(\varepsilon)=0.$

On the other hand $\|u_1^{\varepsilon\tau}\|^2=\frac{1}{\varepsilon}\int_\tau^\infty e^{-\frac{t-\tau}{\varepsilon}}dt|\hat{u}_1|^2=|\hat{u}_1|^2$ and we deduce that $\hat{u}_1^*\mathcal{D}^*(\tau)\mathcal{D}(\tau)\hat{u}_1+\omega(\varepsilon)|\hat{u}_1|^2\leq\theta^2|\hat{u}_1|^2$,

$\theta=\|T_{y_1u_1}\|$, hence for $\varepsilon\to0$ $\mathcal{D}^*(\tau)\mathcal{D}(\tau)\leq\theta^2 I_{m_1}$.

Since $\tau$ was arbitrary the first assertion in the statement is proved. The second one is obtained in the same way by considering the adjoint operator $T^*_{y_1u_1}$.

Corollary 2.1 If there exists for (1.1) a stabilizing and $\gamma$-contracting compensator, then property 1) in Theorem 1.1 is satisfied.

Proof. Let $\tilde{v}_1(t)=\Pi_1(t)D_{11}(t)\hat{u}_1$ , $\hat{v}_1(t)=\mathcal{D}(t)\hat{u}_1-\tilde{v}_1(t)=$
$=(I-\Pi_1(t))D_{11}(t)\hat{u}_1+D_{12}(t)D_c(t)D_{21}(t)\hat{u}_1=[I-\Pi_1(t)][D_{11}(t)+$
$+D_{12}(t)D_c(t)D_{21}(t)]\hat{u}_1$ since $\Pi_1(t)D_{12}(t)=0$ .
We deduce that $\tilde{v}_1^*(t)\hat{v}_1(t)=0$ hence

$$\hat{u}_1^*\mathcal{D}^*(t)\mathcal{D}(t)\hat{u}_1=\hat{u}_1^*D_{11}^*(t)\Pi_1(t)D_{11}(t)\hat{u}_1+\hat{v}_1^*(t)\hat{v}_1(t)\leq\theta^2|\hat{u}_1|^2$$

and from here $D_{11}^*(t)\Pi_1(t)D_{11}(t)<\theta^2 I_{m_1}$.

## 3. A FIRST RICCATI EQUATION

Theorem 3.1 Assume that the detectability condition in Theorem 1.1 is satisfied and that a stabilizing compensator exists for (1.1). Then there exists a global, bounded, positive semidefinite solution $\hat{R}$ to the Riccati equation

$$R'+A^*R+RA-(RB_2+C_1^*D_{12})(D_{12}^*D_{12})^{-1}(B_2^*R+D_{12}^*C_1)+C_1^*C_1=0 \qquad (3.1)$$

such that if $\hat{F}_2=-(D_{12}^*D_{12})^{-1}(B_2^*\hat{R}+D_{12}^*C_1)$ the evolution defined by $A+B_2\hat{F}_2$ is exponentially stable.

Proof. Let $\tau\in\mathbb{R}$ and denote by $\hat{R}_\tau$ the solution of the above Riccati equation with $\hat{R}_\tau(\tau)=0$. The equation may be written also in Liapunov form

$$R'+(A+B_2\hat{F}_2)^*R+R(A+B_2\hat{F}_2)+(C_1+D_{12}\hat{F}_2)^*(C_1+D_{12}\hat{F}_2)=0$$

hence $\hat{R}_\tau$ has for $t<\tau$ the representation

$$\hat{R}_\tau(t)=\int_t^\tau X^*_{A+B_2F_2}(s,t)[C_1(s)+D_{12}(s)\hat{F}_2(s)][C_1(s)+D_{12}(s)\hat{F}_2(s)]X_{A+B_2F_2}(s,t)ds$$

and we deduce that for all $t < \tau$ for which the solution is defined it is positive semidefinite. Consider a control $u_2$ and the associated solution $x$ (for $u_1=0$) ; we have

$$\frac{d}{dt}(x^*\hat{R}_\tau x)=(x^*\hat{R}_\tau(Ax+B_2u_2)+(x^*A^*+u_2^*B_2^*)\hat{R}_\tau x - x^*C_1^*C_1x -$$

$$-x^*A^*\hat{R}_\tau x - x^*\hat{R}_\tau A x + x^*(\hat{R}_\tau B_2+C_1^*D_{12})(D_{12}^*D_{12})^{-1}(B_2^*\hat{R}_\tau+D_{12}^*C_1)x =$$

$$=[u_2^*+x^*(\hat{R}_\tau B_2+C_1^*D_{12})(D_{12}D_{12})^{-1}](D_{12}^*D_{12})[u_2+(D_{12}D_{12})^{-1}(B_2^*\hat{R}_\tau +$$

$$+D_{12}^*C_1)x] - (x^*C_1+u_2^*D_{12}^*)(C_1x+D_{12}u_2)$$

We deduce from here that

$$-x^*(t_0)\hat{R}_\tau(t_0)x(t_0)=-\int_{t_0}^{t}|C_1x+D_{12}u_2|^2ds+\int_{t_0}^{\tau}|D_{12}[u_2+(D_{12}D_{12})^{-1}$$

$$(B_2^*\hat{R}_\tau+D_{12}^*C_1)x]|^2ds \qquad \text{hence} \qquad 0\le x^*(t_0)\hat{R}_\tau(t_0)x(t_0)\le$$

$$\le\int_{t_0}^{\tau}|C_1x+D_{12}u_2|^2\,ds \qquad t_0<\tau.$$

Let $x_0$ be arbitrary, $\hat{x}$, $\hat{x}_c$ be the solution of the compensated system with $\hat{x}(t_0)=x_0$, $\hat{x}_c(t_0)=0$ ; let $\hat{u}_2=C_c\hat{x}_c+D_cC_2\hat{x}$ . For this solution and the corresponding control $\hat{u}_2$ we have

$$0\le x_0^*\hat{R}_\tau(t_0)x_0\le\int_{t_0}^{\tau}|C_1x+D_{12}\hat{u}_2|^2\,ds\le\int_{t_0}^{\infty}|C_1\hat{x}+D_{12}\hat{u}_2|^2\,ds\le$$

$$\le c\,|x_0|^2$$

because of the exponential stability of the compensated system. Such inequalities prove that $\hat{R}_\tau(t)$ is defined for all $t<\tau$. On the other hand $\tau \longrightarrow \hat{R}_\tau(t)$ is increasing, $\lim\limits_{\tau\to\infty}\hat{R}_\tau(t)$ exists and defines a global, bounded solution $\hat{R}$ to the Riccati equation. We have further , with the same computations as above

$$\frac{d}{dt}(x^*\hat{R}x)=-|C_1x+D_{12}u_2|^2+|D_{12}[u_2+(D_{12}^*D_{12})^{-1}(B_2^*\hat{R}+D_{12}^*C_1)x]|^2=$$

$$=-|C_1x+D_{12}u_2|^2+|D_{12}(u_2-\hat{F}_2x)|^2 .$$

If we take $u_2=\hat{F}_2x$ we deduce that

$$\int_{t_0}^{\tau}|(C_1+D_{12}\hat{F}_2)x|^2dt=x_0^*\hat{R}(t_0)x_0-x^*(\tau)\hat{R}(\tau)x(\tau)\le x_0^*\hat{R}(t_0)x_0\le$$

$$\le c|x_0|^2 \qquad \text{hence} \qquad \int_{t_0}^{\infty}|(C_1+D_{12}\hat{F}_2)x|^2dt\le c\,|x_0|^2 \quad \text{for}$$

$x(t_0)=x_0$ , $x'=(A+B_2\hat{F}_2)x$ .

But we have $C_1+D_{12}\hat{F}_2=\Pi_1C_1-D_{12}(D_{12}^*D_{12})^{-1}B_2\hat{R}$ . Since $\Pi_1^2=\Pi_1$ , $\Pi_1D_{12}=0$ we deduce that $|(C_1+D_{12}\hat{F}_2)x|^2=$

$=x^*C_1^*\Pi_1C_1x+x^*\hat{R}B_2^*(D_{12}^*D_{12})^{-1}D_{12}^*D_{12}(D_{12}^*D_{12})^{-1}B_2\hat{R}x$ .

It follows that $\int_{t_0}^{\infty}|\Pi C_1x|^2dt \le c|x_0|^2$

$\int_{t_0}^{\infty}|(D_{12}^*D_{12})^{-1/2}B_2\hat{R}x|^2dt \le c|x_0|^2$

Since ( $\Pi C_1$, $A-B_2(D_{12}^*D_{12})^{-1}D_{12}^*C_1$ ) is detectable there exists a bounded H such that $A-B_2(D_{12}^*D_{12})^{-1}D_{12}^*C_1+H\Pi C_1$ defines an exponentially stable evolution.

We may write $x'=(A+B_2\hat{F}_2)x=[A-B_2(D_{12}^*D_{12})^{-1}D_{12}^*C_1+H\ C_1]x-$

$-H\Pi C_1x-B_2(D_{12}^*D_{12})^{-1}B_2^*\hat{R}x$ .

Since $A-B_2(D_{12}^*D_{12})^{-1}D_{12}^*C_1+H\Pi C_1$ defines an exponentially stable evolution , the estimates obtained above lead to

$\int_{t_0}^{\infty}|x(t)|^2dt \le \hat{c}|x(t_0)|^2$ for all $t_0$ and this implies that $A+B_2\hat{F}_2$ defines an exponentially stable evolution.

The result is essentially the same as theone for infinite dimensional situation in [2] .

## 4. A MINIMIZATION PROBLEM.

We shall consider now , for a given $u_1\in L^2(t_0,\infty;R^{m_1})$ the problem of minimizing with respect to $u_2\in L^2(t_0,\infty;R^{m_2})$

$\int_{t_0}^{\infty}|C_1x+D_{11}u_1+D_{12}u_2|^2dt$ , $x'=Ax+B_1u_1+B_2u_2$ $x(t_0)=x_0$ .

We shall use the solution $\hat{R}$ of (3.1) and the unique solution $\hat{r}$ in $L^2(t_0,\infty;R^n)$ to the equation

$$r'+(A+B_2\hat{F}_2)^*r+\{\hat{R}[B_1-B_2(D_{12}^*D_{12})^{-1}D_{12}^*D_{11}]+C_1^*\Pi_1 D_{11}\}u_1=0 \qquad (4.1)$$

Using these solutions of (3.1) and (4.1) we have

Lemma 4.1 For every $(t_0,x_0)\in R\times R^n$ and every $u_i\in L^2(t_0,\infty;R^{m_i})$ i=1,2 such that $x\in L^2(t_0,\infty;R^n)$ where

$x' = Ax + B_1u_1 + B_2u_2$ $(x(t_o) = x_o)$ , the following identity holds:

$$\int_{t_o}^{\infty} |C_1x + D_{11}u_1 + D_{12}u_2|^2 dt = x_o^*\hat{R}(t_o)x_o + x_o^*\hat{r}(t_o) + \hat{r}^*(t_o)x_o +$$
$$+ \int_{t_o}^{\infty} [u_1^*D_{11}^*D_{11}u_1 + u_1^*B_1^*\hat{r} + \hat{r}^*B_2u_1 - (\hat{r}^*B_2 + u_1^*D_{11}^*D_{12})(D_{12}^*D_{12})^{-1}(B_2^*\hat{r} +$$
$$+ D_{12}^*D_{11}u_1)]\, dt + \int_{t_o}^{\infty} |D_{12}[u_2 - \hat{F}_2x + (D_{12}^*D_{12})^{-1}(B_2^*\hat{r} + D_{12}^*D_{11}u_1)]|^2 dt$$

Proof. The equality is obtained with standard computations for the linear quadratic problem.

We see that $u_2 = \hat{F}_2x - (D_{12}^*D_{12})^{-1}(B_2^*\hat{r} + D_{12}^*D_{11}u_1)$ gives the minimum of $\int_{t_o}^{\infty} |C_1x + D_{11}u_1 + D_{12}u_2|^2 dt$ provided it is admisible .
But with such $u_2$ we have $x' = Ax + B_1u_1 + B_2\hat{F}_2x - B_2(D_{12}^*D_{12})^{-1}(B_2^*\hat{r} + D_{12}^*D_{11}u_1)$ and since $A + B_2\hat{F}_2$ defines an exponentially stable evolution and $u_1$ and $\hat{r}$ are in $L^2(t_o, \infty)$ we deduce that $x \in L^2(t_o, \infty)$ hence $u_2$ is admissible

Theorem 4.1 For every $t_o$ and every $u_1 \in L^2(t_o, \infty; R^{m_1})$ there exists $u_2$ that minimizes $\int_{t_o}^{\infty} |C_1x + D_{11}u_1 + D_{12}u_2|^2 dt$ with $x' = Ax + B_1u_1 + B_2u_2$ , $x(t_o) = x_o$ , this optimal $u_2$ is given by $\hat{u}_2 = \hat{F}_2x - (D_{12}^*D_{12})^{-1}(B_2^*\hat{r} + D_{12}^*D_{11}u_1)$ and the minimal value is

$$x_o^*\hat{R}(t_o)x_o + x_o^*\hat{r}(t_o) + \hat{r}^*(t_o)x_o + \int_{t_o}^{\infty} [u_1^*D_{11}^*D_{11}u_1 + u_1^*B_1^*\hat{r} +$$
$$+ \hat{r}^*B_1u_1 - (\hat{r}^*B_2 + u_1^*D_{11}^*D_{12})(D_{12}^*D_{12})^{-1}(B_2^*\hat{r} + D_{12}^*D_{11}u_1)]\, dt$$

The proof is performed by the preceeding computations.

## 5. A SECOND MINIMIZATION PROBLEM AND THE FIRST TWO RICCATI EQUATIONS IN THE STATEMENT OF THEOREM 1.1

We consider now the problem

$$\min_{u_1} \int_{t_o}^{\infty} \{ \gamma^2|u_1|^2 - |C_1x + D_{11}u_1 + D_{12}\hat{u}_2|^2\} dt \qquad (5.1)$$
$$x' = Ax + B_1u_1 + B_2\hat{u}_2 \quad , \ x(t_o) = x_o$$

Remember that for given $u_1$ we defined a unique $\hat{r}$ and that our minimization problem reads in fact

$$\min_{u_1} \int_{t_0}^{\infty} \{ \gamma^2 |u_1|^2 - u_1^* D_{11}^* D_{11} u_1 - u_1^* B_1^* \hat{r} - \hat{r}^* B_1 u_1 + (\hat{r}^* B_2 + u_1^* D_{11}^* D_{12})(D_{12}^* D_{12})^{-1}(B_2^* \hat{r} + D_{12}^* D_{11} u_1) \} dt - x_0^* \hat{r}(t_0) - \hat{r}^*(t_0) x_0 - x_0^* \hat{R}(t_0) x_0$$

and since $\hat{r}$ is defined as a linear operator on $u_1$ this is a Hilbert space minimization problem.

To have existence of the minimum we need a positivity property for the quadratic part of the functional corresponding to $x_0=0$.

Proposition 5.1. If there exists for (1.1) a stabilizing and $\gamma$-contracting compensator, then there exists $\delta > 0$ such that for every $t_0$ and every $u_1 \in L^2(t_0,\infty;R^{m_1})$, for zero initial conditions and for the corresponding solution of the compensated system we have

$$\int_{t_0}^{\infty} (\gamma^2 |u_1|^2 - |y_1|^2)\, dt \geq \delta^2 \int_{t_0}^{\infty} |u_1|^2\, dt.$$

Proof. It follows directly from $\gamma > \| T_{y_1 u_1} \|$ if we take $u_1(t)=0$ for $t < t_0$ and remark that exponential stability of the compensated system implies that the $L^2(\mathbb{R};\mathbb{R}^n)$ solution must have $x(t_0)=0$. Remark now that from the optimality of $\hat{u}_2$ we have

$$\int_{t_0}^{\infty} | C_1 x + D_{11} u_1 + D_{12} \hat{u}_2 |^2 dt \leq \int_{t_0}^{\infty} | C_1 x + D_{11} u_1 + D_{12} u_2 |^2 dt$$

for every other $u_2$; in particular if $u_2$ is defined by using the compensator, then

$$-\int_{t_0}^{\infty} | y_1 |^2 dt \leq -\int_{t_0}^{\infty} | C_1 x + D_{11} u_1 + D_{12} \hat{u}_2 |^2 dt$$

hence

$$\int_{t_0}^{\infty} \{ \gamma^2 |u_1|^2 - |C_1 x + D_{11} u_1 + D_{12} \hat{u}_2|^2 \} dt \geq$$

$$\geq \int_{t_0}^{\infty} (\gamma^2 |u_1|^2 - |y_1|^2)\, dt \geq \delta^2 \int_{t_0}^{\infty} |u_1|^2\, dt$$

and we have obtained the required positivity condition implying existence of the minimum. If we denote the optimal $u_1$ as $\tilde{u}_1^{(t_0,x_0)}$ and the corresponding $\hat{u}_2$ as $\tilde{u}_2^{(t_0,x_0)}$ we see directly that $\tilde{u}_1^{(t_0,x_0)}$ and $\tilde{u}_2^{(t_0,x_0)}$ are linear bounded operators of $x_0$ and the corresponding norms are uniformly bounded with respect to $t_0$.

Taking into account the form of $\tilde{u}_2^{(t_0,x_0)}$, we deduce that for the corresponding solution $\tilde{x}^{(t_0,x_0)}$ we shall have

$$\int_{t_0}^{\infty} |\tilde{x}^{(t_0,x_0)}(t)|^2\, dt \leq c|x_0|^2$$

We have thus proved

<u>Theorem 5.1</u> For every $(t_0,x_0)$ there exists a unique $\tilde{u}_1^{(t_0,x_0)}$ such that

$$\int_{t_0}^{\infty} \{\gamma^2 |u_1^{(t_0,x_0)}|^2 - |C_1\tilde{x}^{(t_0,x_0)} + D_{11}\tilde{u}_1^{(t_0,x_0)} + D_{12}\tilde{u}_2^{(t_0,x_0)}|^2\}\, dt =$$

$$= \min_{u_1} \int_{t_0}^{\infty} (\gamma^2|u_1|^2 - |C_1x + D_{11}u_1 + D_{12}\hat{u}_2|^2)\, dt$$

where $\hat{u}_2 = \hat{F}_2x - (D_{12}^*D_{12})^{-1}(B_2^*\hat{r} + D_{12}^*D_{11}u_1)$

and $\hat{r}$ is associated to $u_1$; by $\tilde{u}_2^{(t_0,x_0)}$ we have denoted the control $\hat{u}_2$ associated to $\tilde{u}_1^{(t_0,x_0)}$.

Moreover $\tilde{u}_1^{(t_0,x_0)} = \mathcal{H}_{t_0}^1 x_0$, $\mathcal{H}_{t_0}^1 : \mathbb{R}^n \to L^2(t_0,\infty;\mathbb{R}^{m_1})$ $\|\mathcal{H}_{t_0}^1\| \leq c$, hence $\tilde{u}_2^{(t_0,x_0)} = \mathcal{H}_{t_0}^2 x_0$ $\mathcal{H}_{t_0}^2 : \mathbb{R}^n \to L^2(t_0,\infty;\mathbb{R}^{m_2})$ $\|\mathcal{H}_{t_0}^2\| \leq c$ and for the correspondig solution $\tilde{x}^{(t_0,x_0)}$ we have $\int_{t_0}^{\infty} |\tilde{x}^{(t_0,x_0)}(t)|^2 dt \leq c|x_0|^2$

The optimal value is a quadratic form in $x_0$.

Denote by $\tilde{r}^{(t_0,x_0)}$ the solution $\hat{r}$ associated to $\tilde{u}_1^{(t_0,x_0)}$ and define $\tilde{\lambda}^{(t_0,x_0)}(t) = \hat{R}(t)\tilde{x}^{(t_0,x_0)}(t) + \tilde{r}^{(t_0,x_0)}(t)$.

We had $\tilde{u}_2^{(t_0,x_0)}(t) = \hat{F}_2(t)\tilde{x}^{(t_0,x_0)}(t) - [D_{12}^*(t)D_{12}(t)]^{-1}[B_2^*(t)\tilde{r}^{(t_0,x_0)}(t) + D_{12}^*(t)D_{11}(t)\tilde{u}_1^{(t_0,x_0)}(t)]$.

Remember that $\hat{F}_2 = -(D_{12}^*D_{12})^{-1}B_2^*\hat{R} - (D_{12}^*D_{12})^{-1}D_{12}^*C_1$ hence

$$\tilde{u}_2^{(t_0,x_0)} = -(D_{12}^*D_{12})^{-1}[B_2^*(\hat{R}\tilde{x}^{(t_0,x_0)} + \tilde{r}^{(t_0,x_0)}) -$$
$$-(D_{12}^*D_{12})^{-1}D_{12}^*(C_1\tilde{x}^{(t_0,x_0)} + D_{11}\tilde{u}_1^{(t_0,x_0)})$$

We deduce that

$$\tilde{u}_2^{(t_0,x_0)} = -(D_{12}^*D_{12})^{-1}[B_2^*\tilde{\lambda}^{(t_0,x_0)} + D_{12}^*(C_1\tilde{x}^{(t_0,x_0)} + D_{11}\tilde{u}_1^{(t_0,x_0)})] \qquad (5.2)$$

We obtain from the equations for $\hat{R}$, $\tilde{r}^{(t_0,x_0)}$, $\tilde{x}^{(t_0,x_0)}$

$$\frac{d}{dt}\tilde{\lambda}^{(t_0,x_0)} = -A^*\tilde{\lambda}^{(t_0,x_0)} - C_1^*\left[C_1\tilde{x}^{(t_0,x_0)} + D_{11}\tilde{u}_1^{(t_0,x_0)} + D_{12}\tilde{u}_2^{(t_0,x_0)}\right] = -A^*\tilde{\lambda}^{(t_0,x_0)} - C_1^*\tilde{y}_1^{(t_0,x_0)}$$

<u>Proposition 5.2</u> If $\tilde{u}_1^{(t_0,x_0)}(t)$ is the solution of the minimization problem (5.1) and $\tilde{u}_2^{(t_0,x_0)}(t)$ is given by (5.2) then we have $\begin{pmatrix}\tilde{u}_1^{(t_0,x_0)}\\ \tilde{u}_2^{(t_0,x_0)}\end{pmatrix} = -K^{-1}(B^*\tilde{\lambda}^{(t_0,x_0)} + D_1^*C_1)$

<u>Proof.</u> We know that $\frac{d}{dt}\tilde{\lambda}^{(t_0,x_0)} = -A^*\tilde{\lambda}^{(t_0,x_0)} - C_1^*\tilde{y}_1^{(t_0,x_0)}$.

We have also $B_2^*\tilde{\lambda}^{(t_0,x_0)} + D_{12}^*(C_1\tilde{x}^{(t_0,x_0)} + D_{11}\tilde{u}_1^{(t_0,x_0)}) + D_{12}^*D_{12}\tilde{u}_2^{(t_0,x_0)} = 0$

hence $B_2^*\tilde{\lambda}^{(t_0,x_0)} + D_{12}^*\tilde{y}_1^{(t_0,x_0)} = 0$

Compute $\frac{d}{dt}\{[\tilde{x}^{(t_0x_0)}]^*\tilde{\lambda}^{(t_0x_0)}\} = [\tilde{u}_1^{(t_0x_0)}]^*B_1^*\tilde{\lambda}^{(t_0x_0)} - \{[\tilde{u}_2^{(t_0x_0)}]^*D_{12}^* + [\tilde{x}^{(t_0x_0)}]^*C_1^*\}\tilde{y}_1^{(t_0,x_0)}$.

It follows that

$$\gamma^2[\tilde{u}_1^{(t_0x_0)}]^*\tilde{u}_1^{(t_0x_0)} - [\tilde{y}_1^{(t_0x_0)}]^*\tilde{y}_1^{(t_0x_0)} - \frac{d}{dt}\{[\tilde{x}^{(t_0x_0)}]^*\tilde{\lambda}^{(t_0x_0)}\} - \frac{d}{dt}[\tilde{\lambda}^{(t_0x_0)}]^*\tilde{x}^{(t_0x_0)} = |\tilde{y}_1^{(t_0x_0)}|^2 + \gamma^2\{[\tilde{u}_1^{(t_0x_0)}]^* - \frac{1}{\gamma^2}[B_1^*\tilde{\lambda}^{(t_0x_0)} + D_{11}^*\tilde{y}_1^{(t_0x_0)}]^*\}\cdot\{\tilde{u}_1^{(t_0x_0)} - \frac{1}{\gamma^2}[B_1^*\tilde{\lambda}^{(t_0x_0)} + D_{11}^*\tilde{y}_1^{(t_0x_0)}]\} - \gamma^2\cdot\frac{1}{\gamma^4}[B_1^*\tilde{\lambda}^{(t_0x_0)} + D_{11}^*\tilde{y}_1^{(t_0x_0)}]^*[B_1^*\tilde{\lambda}^{(t_0x_0)} + D_{11}^*\tilde{y}_1^{(t_0x_0)}]$$

Introduce the notation $\hat{u}_1^{(t_0x_0)} = \frac{1}{\gamma^2}[B_1^*\tilde{\lambda}^{(t_0x_0)} + D_{11}^*\tilde{y}_1^{(t_0x_0)}]$ and write

$$\gamma^2|\tilde{u}_1^{(t_0x_0)}|^2 - |\tilde{y}_1^{(t_0x_0)}|^2 - \frac{d}{dt}\{[\tilde{x}^{(t_0x_0)}]^*\tilde{\lambda}^{(t_0x_0)} + [\tilde{\lambda}^{(t_0x_0)}]^*\tilde{x}^{(t_0x_0)}\} = \gamma^2|\tilde{u}_1^{(t_0x_0)} - \hat{u}_1^{(t_0x_0)}|^2 + |\tilde{y}_1^{(t_0x_0)}|^2 - \gamma^2|\hat{u}_1^{(t_0x_0)}|^2 \qquad (5.3)$$

To $\hat{u}_1^{(t_0x_0)}$ associate $\hat{r}^{(t_0x_0)}$ and $\hat{u}_2^{(t_0x_0)}$ in the usual way, then define corresponding $\hat{x}^{(t_0x_0)}$ and $\hat{y}_1^{(t_0x_0)}$. We have thus

$$\frac{d}{dt}\hat{x}^{(t_0x_0)} = A\hat{x}^{(t_0x_0)} + B_1\hat{u}_1^{(t_0x_0)} + B_2\hat{u}_2^{(t_0x_0)}$$

$$\hat{u}_2^{(t_0x_0)} = \hat{F}_2\hat{x}^{(t_0x_0)} - (D_{12}^*D_{12})^{-1}[B_2^*\hat{r}^{(t_0,x_0)} + D_{12}^*D_{11}\hat{u}_1^{(t_0x_0)}]$$

$$\frac{d}{dt}\{[\hat{x}^{(t_0x_0)}]^*\tilde{\lambda}^{(t_0x_0)}\}=\{[\hat{x}^{(t_0x_0)}]^*A^*+[\hat{u}_1^{(t_0x_0)}]^*B_1^*+$$

$$+[\hat{u}_2^{(t_0x_0)}]^*B_2^*\}\tilde{\lambda}^{(t_0x_0)}-[\hat{x}^{(t_0x_0)}]^*(A^*\tilde{\lambda}^{(t_0x_0)}+C_1^*\tilde{y}_1^{(t_0x_0)})\ =$$

$$=[\hat{u}_1^{(t_0x_0)}]^*[B_1^*\tilde{\lambda}^{(t_0x_0)}+D_{11}^*\tilde{y}_1^{(t_0x_0)}]-[\hat{y}_1^{(t_0x_0)}]^*\tilde{y}_1^{(t_0x_0)}\ =$$

$$=\gamma^2[\hat{u}_1^{(t_0x_0)}]^*\hat{u}_1^{(t_0x_0)}-[\hat{y}_1^{(t_0x_0)}]^*\tilde{y}_1^{(t_0x_0)}$$

It follows that

$$\gamma^2|\hat{u}_1^{(t_0x_0)}|^2-|\hat{y}_1^{(t_0x_0)}|^2-\frac{d}{dt}\{[\hat{x}^{(t_0x_0)}]^*\tilde{\lambda}^{(t_0x_0)}+[\tilde{\lambda}^{(t_0x_0)}]^*\hat{x}^{(t_0x_0)}\}\ =$$

$$=\gamma^2|\hat{u}_1^{(t_0x_0)}|^2-|\hat{y}_1^{(t_0x_0)}|^2-2\gamma^2|\hat{u}_1^{(t_0x_0)}|^2+[\hat{y}_1^{(t_0x_0)}]^*\tilde{y}_1^{(t_0x_0)}\ +$$

$$+[\tilde{y}_1^{(t_0x_0)}]^*\hat{y}_1^{(t_0x_0)}\ =-|\tilde{y}_1^{(t_0x_0)}-\hat{y}_1^{(t_0x_0)}|^2+|\tilde{y}_1^{(t_0x_0)}|^2-\gamma^2|\hat{u}_1^{(t_0x_0)}|^2$$

We deduce that

$$\int_{t_0}^{\infty}(\gamma^2|\hat{u}_1^{(t_0x_0)}|^2-|\hat{y}_1^{(t_0x_0)}|^2)\,dt+\int_{t_0}^{\infty}|\tilde{y}_1^{(t_0x_0)}-\hat{y}_1^{(t_0x_0)}|^2dt\ =$$

$$=-x_0^*\tilde{\lambda}^{(t_0x_0)}(t_0)-[\tilde{\lambda}^{(t_0x_0)}]^*x_0+\int_{t_0}^{\infty}(|\tilde{y}_1^{(t_0x_0)}|^2-\gamma^2|\hat{u}_1^{(t_0x_0)}|^2)\,dt$$

On the other hand, from (5.3)

$$\int_{t_0}^{\infty}(\gamma^2|\tilde{u}_1^{(t_0x_0)}|^2-|\tilde{y}_1^{(t_0x_0)}|^2)\,dt=-x_0^*\tilde{\lambda}^{(t_0x_0)}-[\tilde{\lambda}^{(t_0x_0)}]^*x_0+$$

$$+\gamma^2\int_{t_0}^{\infty}|\tilde{u}_1^{(t_0x_0)}-\hat{u}_1^{(t_0x_0)}|^2\,dt+\int_{t_0}^{\infty}(|\tilde{y}_1^{(t_0x_0)}|^2-\gamma^2|\hat{u}_1^{(t_0x_0)}|^2)dt$$

We have finally

$$\int_{t_0}^{\infty}(\gamma^2|\tilde{u}_1^{(t_0x_0)}|^2-|\tilde{y}_1^{(t_0x_0)}|^2)\,dt=\int_{t_0}^{\infty}(\gamma^2|\hat{u}_1^{(t_0x_0)}|^2-|\hat{y}_1^{(t_0x_0)}|^2)dt\ +$$

$$+\int_{t_0}^{\infty}\{\gamma^2|\tilde{u}_1^{(t_0x_0)}-\hat{u}_1^{(t_0x_0)}|^2+|\tilde{y}_1^{(t_0x_0)}-\hat{y}_1^{(t_0x_0)}|^2\}\,dt \qquad (5.4)$$

Remind now that $\tilde{u}_1^{(t_0x_0)}$ was the optimal solution of our second minimization problem. We shall deduce that $\tilde{u}_1^{(t_0x_0)}=\hat{u}_1^{(t_0x_0)}$, $\tilde{y}_1^{(t_0x_0)}=\hat{y}_1^{(t_0x_0)}$, hence for the optimal solution of the second optimization problem we must have $\tilde{u}_1^{(t_0x_0)}=\frac{1}{\gamma^2}[B_1^*\tilde{\lambda}^{(t_0x_0)}+D_{11}^*\tilde{y}_1^{(t_0x_0)}]$.

If we write now $\tilde{y}_1^{(t_0x_0)}=C_1^*\tilde{x}^{(t_0x_0)}+D_{11}\tilde{u}_1^{(t_0x_0)}+D_{12}\tilde{u}_2^{(t_0x_0)}$ and $\tilde{u}_2^{(t_0x_0)}=-(D_{12}^*D_{12})^{-1}[B_2^*\tilde{\lambda}^{(t_0x_0)}+D_{12}^*(C_1\tilde{x}^{(t_0x_0)}+D_{11}\tilde{u}_1^{(t_0x_0)})]$ we have

$$\tilde{y}_1^{(t_0x_0)}=C_1^*\tilde{x}^{(t_0x_0)}+D_{11}\tilde{u}_1^{(t_0x_0)}-D_{12}(D_{12}^*D_{12})^{-1}[B_2^*\tilde{\lambda}^{(t_0x_0)}+$$

$$+D_{12}^*(C_1\tilde{x}^{(t_0x_0)}+D_{11}\tilde{u}_1^{(t_0x_0)})]$$

$$\tilde{u}_1^{(t_0,x_0)} = \frac{1}{\gamma^2}\left[B_1^* - D_{11}^* D_{12}(D_{12}^* D_{12})^{-1} B_2^*\right]\tilde{\lambda}^{(t_0 x_0)} + \frac{1}{\gamma^2} D_{11}^*(I-$$

$$-D_{12}(D_{12}^* D_{12})^{-1} D_{12}^*)C_1 \tilde{x}^{(t_0,x_0)} + \frac{1}{\gamma^2} \quad D_{11}^*\left[I - D_{12}(D_{12}^* D_{12})^{-1} D_{12}^*\right] D_{11} \tilde{u}_1^{(t_0 x_0)}$$

$$(\gamma^2 I_{m_1} - D_{11}^* \Pi_1 D_{11})\tilde{u}_1^{(t_0,x_0)} = \left[B_1^* - D_{11}^* D_{12}(D_{12}^* D_{12})^{-1} B_2^*\right]\tilde{\lambda}^{(t_0 x_0)} +$$

$$+ D_{11}^* \Pi_1 C_1 \tilde{x}^{(t_0 x_0)}$$

and since $\gamma^2 I_{m_1} - D_{11}^* \Pi_1 D_{11}$ is invertible , with bounded inverse , we obtain an explicit formula for $\tilde{u}_1^{(t_0 x_0)}$ in terms of $\tilde{\lambda}^{(t_0,x_0)}$ and $\tilde{x}^{(t_0,x_0)}$. We may also write

$$\gamma^2 \tilde{u}_1^{(t_0,x_0)} = B_1^* \tilde{\lambda}^{(t_0 x_0)} + D_{11}^* C_1 \tilde{x}^{(t_0 x_0)} + D_{11}^* D_{11} \tilde{u}_1^{(t_0 x_0)} + D_{11}^* D_{12} \tilde{u}_2^{(t_0 x_0)}$$

that is

$$\gamma^2 \; \tilde{u}_1^{(t_0 x_0)} \quad - D_{11}^* D_{11} \tilde{u}_1^{(t_0 x_0)} \quad - D_{11}^* D_{12} \tilde{u}_2^{(t_0 x_0)} = B_1^* \tilde{\lambda}^{(t_0 x_0)} + D_{11}^* C_1 \tilde{x}^{(t_0 x_0)}$$

$$-D_{12}^* D_{11} \tilde{u}_1^{(t_0 x_0)} \quad - D_{12}^* D_{12} \tilde{u}_2^{(t_0 x_0)} = B_2^* \tilde{\lambda}^{(t_0 x_0)} + D_{12}^* C_1 \tilde{x}^{(t_0 x_0)}$$

hence

$$\left[\begin{pmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{pmatrix} - D_1^* D_1\right]\tilde{u}^{(t_0 x_0)} = B^* \tilde{\lambda}^{(t_0 x_0)} + D_1^* C_1 \tilde{x}^{(t_0 x_0)}$$

and we have $\tilde{u}^{(t_0 x_0)} = -K^{-1}\left[B^* \tilde{\lambda}^{(t_0 x_0)} + D_1^* C_1 \tilde{x}^{(t_0 x_0)}\right]$ with $\tilde{u} = \begin{pmatrix}\tilde{u}_1 \\ \tilde{u}_2\end{pmatrix}$

that is a complete characterization of $\tilde{u}$ in terms of $\tilde{\lambda}$ and $\tilde{x}$ .

Let us prove now

<u>Lemma 5.1</u> For every $u_1$ and every $\hat{t}$

$$\int_{\hat{t}}^{\infty}\{\gamma^2 u_1(t) - |y_1^{u_1}(t)|^2\}dt + 2\left[\tilde{\lambda}^{(t_0,x_0)}(\hat{t})\right]^*\left[x^{u_1}(\hat{t}) - \tilde{x}^{(t_0,x_0)}(\hat{t})\right] = \int_{\hat{t}}^{\infty}\{\gamma^2|\tilde{u}_1^{(t_0,x_0)}(t)|^2 - |\tilde{y}_1^{(t_0,x_0)}(t)|^2\}dt +$$

$$+ \int_{\hat{t}}^{\infty}\{\gamma^2 |u_1(t) - \tilde{u}_1^{(t_0,x_0)}(t)|^2 - |y_1^{u_1}(t) - \tilde{y}_1^{(t_0,x_0)}(t)|^2\}dt$$

<u>Proof</u> . We have $\gamma^2|u_1(t)|^2 - |y_1^{u_1}(t)|^2 = \gamma^2|\tilde{u}_1^{(t_0,x_0)}(t)|^2 - |\tilde{y}_1^{(t_0,x_0)}(t)|^2 + \gamma^2|u_1(t) - \tilde{u}_1^{(t_0,x_0)}(t)|^2 - |y_1^{u_1}(t) - \tilde{y}_1^{(t_0,x_0)}(t)|^2 +$

$$+2\gamma^2\langle\tilde{u}_1^{(t_0 x_0)}(t), u_1(t) - \tilde{u}_1^{(t_0 x_0)}(t)\rangle - 2\langle\tilde{y}_1^{(t_0,x_0)}(t), y_1^{u_1}(t) - \tilde{y}_1^{(t_0 x_0)}(t)\rangle.$$

We had also

$$\frac{d}{dt}\tilde{\lambda}^{(t_o,x_o)}(t)=-A^*(t)\,\tilde{\lambda}^{(t_o,x_o)}(t)-C_1^*(t)\tilde{y}_1^{(t_o,x_o)}(t)$$

and

$$\frac{d}{dt}\left[x^{u_1}(t)-\tilde{x}^{(t_o,x_o)}(t)\right]=A(t)\left[x^{u_1}(t)-\tilde{x}^{(t_o,x_o)}(t)\right]+$$

$$+B_1(t)\left[u_1(t)-\tilde{u}_1^{(t_o,x_o)}(t)\right]+B_2(t)\left[\hat{u}_2^{u_1}(t)-\tilde{u}_2^{(t_o,x_o)}(t)\right]$$

Hence

$$\frac{d}{dt}\left\{\left[\tilde{\lambda}^{(t_o,x_o)}(t)\right]^*\left[x^{u_1}(t)-\tilde{x}^{(t_o,x_o)}(t)\right]\right\}=$$

$$=-\left[\tilde{\lambda}^{(t_o,x_o)}(t)\right]^*A(t)\left[x^{u_1}(t)-\tilde{x}^{(t_o,x_o)}(t)\right]-\left[\tilde{y}_1^{(t_o,x_o)}(t)\right]^*C_1(t)$$

$$\left[x^{u_1}(t)-\tilde{x}^{(t_o,x_o)}(t)\right]+\left[\tilde{\lambda}^{(t_o,x_o)}(t)\right]^*\left\{A(t)\left[x^{u_1}(t)-\tilde{x}^{(t_o,x_o)}(t)\right]\right.$$

$$\left.+B_1(t)\left[u_1(t)-\tilde{u}_1^{(t_o,x_o)}(t)\right]+B_2(t)\left[\hat{u}_2^{u_1}(t)-\tilde{u}_2^{(t_o,x_o)}(t)\right]\right\}=$$

$$=\langle B_1^*(t)\,\tilde{\lambda}^{(t_o,x_o)}(t),\ u_1(t)-\tilde{u}_1^{(t_o,x_o)}(t)\rangle+$$

$$+\langle B_2^*(t)\,\tilde{\lambda}^{(t_o,x_o)}(t),\hat{u}_2^{u_1}(t)-\tilde{u}_2^{(t_o,x_o)}(t)\rangle-\langle\tilde{y}_1^{(t_o,x_o)}(t),$$

$$y_1^{u_1}(t)-\tilde{y}_1^{(t_o,x_o)}(t)-D_{11}^*(t)\left[u_1(t)-\tilde{u}_1^{(t_o,x_o)}(t)\right]-$$

$$-D_{12}(t)\left[\hat{u}_2^{u_1}(t)-\tilde{u}_1^{(t_o,x_o)}(t)\right]\rangle=-\langle\tilde{y}_1^{(t_o,x_o)},y_1^{u_1}(t)-\tilde{y}_1^{(t_o,x_o)}(t)\rangle+$$

$$+\langle B_1^*(t)\,\tilde{\lambda}^{(t_o,x_o)}(t)+D_{11}^*(t)\tilde{y}_1^{(t_o,x_o)}(t),u_1(t)-\tilde{u}_1^{(t_o,x_o)}(t)\rangle+$$

$$+\langle B_2^*(t)\,\tilde{\lambda}^{(t_o,x_o)}(t)+D_{12}(t)\tilde{y}_1^{(t_o,x_o)}(t),\hat{u}_2^{u_1}(t)-\tilde{u}_2^{(t_o,x_o)}(t)\rangle$$

We deduce that

$$-2\langle\tilde{y}_1^{(t_o,x_o)}(t),\ y_1^{u_1}(t)-\tilde{y}_1^{(t_o,x_o)}(t)\rangle=2\frac{d}{dt}\left\{\left[\tilde{\lambda}^{(t_o,x_o)}(t)\right]^*\cdot\right.$$

$$\left.\cdot\left[x^{u_1}(t)-\tilde{x}^{(t_o,x_o)}(t)\right]\right\}-\langle B_1(t)\,\tilde{\lambda}^{(t_o,x_o)}(t)+D_{11}^*(t)\tilde{y}_1^{(t_o,x_o)}(t),$$

$$u_1(t)-\tilde{u}_1^{(t_o,x_o)}(t)\rangle-\langle B_2^*(t)\,\tilde{\lambda}^{(t_o,x_o)}(t)+D_{12}^*(t)\tilde{y}_1^{(t_o,x_o)}(t),$$

$$\hat{u}_2^{u_1}(t)-\tilde{u}_2^{(t_o,x_o)}(t)\rangle,$$

Remember now that

$$B_1^*(t)\,\tilde{\lambda}^{(t_o,x_o)}(t)+D_{11}^*(t)\tilde{y}_1^{(t_o,x_o)}(t)=\gamma^2\tilde{u}_1^{(t_o,x_o)}(t)\quad\text{and}$$

$$B_2^*(t)\,\tilde{\lambda}^{(t_o,x_o)}(t)+D_{12}^*(t)\tilde{y}_1^{(t_o,x_o)}(t)=0$$

We have thus

$$-2\langle \tilde{y}_1^{(t_0x_0)}(t),\ y_1^{u_1}(t)-\tilde{y}_1^{(t_0x_0)}(t)\rangle = 2\,\frac{d}{dt}\{[\tilde{\lambda}^{(t_0x_0)}(t)]^*$$

$$[x^{u_1}(t)-\tilde{x}^{(t_0x_0)}(t)]\} - \gamma^2\langle \tilde{u}_1^{(t_0x_0)}(t), u_1(t)-\tilde{u}_1^{(t_0x_0)}(t)\rangle$$

We deduce that

$$\gamma^2|u_1(t)|^2-|y_1^{u_1}(t)|^2=\gamma^2|\tilde{u}_1^{(t_0x_0)}(t)|^2-|\tilde{y}_1^{(t_0x_0)}(t)|^2 +$$

$$+\gamma^2|u_1(t)-\tilde{u}^{(t_0x_0)}(t)|^2-|y_1^{u_1}(t)-\tilde{y}_1^{(t_0x_0)}(t)|^2 +$$

$$+2\,\frac{d}{dt}\{[\tilde{\lambda}^{(t_0x_0)}(t)]^*[x^{u_1}(t)-\tilde{x}^{(t_0x_0)}(t)]\}$$

Integrate from $\hat{t}$ to $\infty$ and use $\lim\limits_{t\to\infty}\tilde{\lambda}^{(t_0x_0)}(t)=0$,
$\lim\limits_{t\to\infty}[x^{u_1}(t)-\tilde{x}^{(t_0x_0)}(t)]=0$ to deduce the conclusion.

<u>Corrolary</u>. If we take $x^{u_1}(\hat{t})=\tilde{x}^{(t_0x_0)}(t)$ we have

$$\int_{\hat{t}}^{\infty}\{\gamma^2|u_1(t)|^2-|y_1^{u_1}(t)|^2\}\,dt \geq \int_{\hat{t}}^{\infty}\{\gamma^2|\tilde{u}_1^{(t_0x_0)}(t)|^2 - |\tilde{y}_1^{(t_0x_0)}(t)|^2\}\,dt$$

We have to remark only that the second integral corresponds to the quadratic functional of the second minimization problem (5.1) for zero initial conditiond and for the input $u_1-\tilde{u}_1^{(t_0x_0)}$, hence it is positive. We deduce the very important conclusion that

$$\tilde{u}_1^{(\hat{t},\tilde{x}^{(t_0x_0)}(\hat{t}))}(t)=\tilde{u}_1^{(t_0x_0)}(t) \qquad t\geq\hat{t}>t_0$$

and this is in fact the "optimality principle" we shall use in the sequel.

Remark that from here we may deduce that $\tilde{r}^{(\hat{t},\tilde{x}^{(t_0x_0)}(\hat{t}))}(t)=\tilde{r}^{(t_0x_0)}(t)$ and next that $\tilde{x}^{(\hat{t},\tilde{x}^{(t_0x_0)}(\hat{t}))}(t)=\tilde{x}^{(t_0x_0)}(t)$
$\tilde{u}_2^{(\hat{t},\tilde{x}^{(t_0x_0)}(\hat{t}))}(t)=\tilde{u}_2^{(t_0x_0)}(t)$ hence also $\tilde{\lambda}^{(\hat{t},\tilde{x}^{(t_0x_0)}(\hat{t}))}(t)=$
$=\tilde{\lambda}^{(t_0x_0)}(t)$. Since

$$\tilde{u}^{(t_0x_0)}=-K^{-1}[B^*\tilde{\lambda}^{(t_0x_0)}+D_1^*C_1\tilde{x}^{(t_0x_0)}]$$

we may write

$$\tilde{u}^{(t_0x_0)}(t_0)=-K^{-1}(t_0)[B^*(t_0)\tilde{\lambda}^{(t_0x_0)}(t_0)+D_1^*(t_0)C_1(t_0)x_0]$$

From the formula giving the unique solution in $L^2(t_0,\infty;R^n)$ to the equation for $r$ we see that $\tilde{r}(t_0)$ is a linear function of $x_0$ and we deduce that we may write

$$\tilde{\lambda}^{(t_0 x_0)}(t_0)=R(t_0)x_0$$

We have finally

$$\tilde{u}^{(t_0 x_0)}(t_0)=-K^{-1}(t_0)[B^*(t_0)R(t_0)+D_1^*(t_0)C(t_0)]x_0$$

and from the "optimality principle" we obtained above, we deduce that

$$\tilde{u}^{(t_0 x_0)}(t)=-K^{-1}(t)[B^*(t)R(t)+D_1^*(t)C(t)]\tilde{x}^{(t_0 x_0)}(t)$$

We shall denote $F(t)\doteq -K^{-1}(t)[B^*(t)R(t)+D_1^*(t)C(t)]$

Remark that the same optimality principle leads to

$$\tilde{\lambda}^{(t_0,x_0)}(t)=R(t)\tilde{x}^{(t_0 x_0)}(t) \quad \text{for all } t\geq t_0.$$

Let now $\tau>t$ and $\tilde{x}_0=\tilde{x}^{(t_0 x_0)}(\tau)$. Then $\tilde{u}^{(\tau,\tilde{x}_0)}=P_+^{\tau}\tilde{u}^{(t_0 x_0)}$, $\tilde{x}^{(\tau,\tilde{x}_0)}=P_+^{\tau}\tilde{x}^{(t_0 x_0)}$ (optimality principle).

We have also $\tilde{\lambda}^{(\tau,\tilde{x}_0)}=P_+^{\tau}\tilde{\lambda}^{(t_0 x_0)}$ ($P_+^{\tau}$ is the projection from $L^2(-\infty,\infty)$ to $L^2(\tau,\infty)$).

Remember also that $\frac{d}{dt}\tilde{x}^{(t_0 x_0)}=A\tilde{x}^{(t_0,x_0)}+B\tilde{u}^{(t_0 x_0)}=$

$$=A\tilde{x}^{(t_0 x_0)}-BK^{-1}[B^*\tilde{\lambda}^{(t_0 x_0)}+D_1^*C_1\tilde{x}^{(t_0 x_0)}]$$

$$\frac{d}{dt}\tilde{\lambda}^{(t_0 x_0)}=-A^*\tilde{\lambda}^{(t_0 x_0)}-C_1^*\tilde{y}_1^{(t_0 x_0)}=-A^*\tilde{\lambda}^{(t_0 x_0)}-C_1^*(C_1\tilde{x}^{(t_0 x_0)}+D_1\tilde{u}^{(t_0 x_0)})=-A^*\tilde{\lambda}^{(t_0 x_0)}-C_1^*C_1\tilde{x}^{(t_0 x_0)}+C_1^*D_1K^{-1}[B^*\tilde{\lambda}^{(t_0 x_0)}+D_1^*C_1\tilde{x}^{(t_0 x_0)}]$$

hence

$$\frac{d}{dt}\tilde{x}^{(t_0 x_0)}=(A-BK^{-1}D_1^*C_1)\tilde{x}^{(t_0 x_0)}-BK^{-1}B^*\tilde{\lambda}^{(t_0 x_0)}$$

$$\frac{d}{dt}\tilde{\lambda}^{(t_0 x_0)}=[C_1^*D_1K^{-1}D_1^*C_1-C_1^*C_1]\tilde{x}^{(t_0 x_0)}-(A-BK^{-1}D_1^*C_1)^*\tilde{\lambda}^{(t_0 x_0)}$$

and $\tilde{x}^{(t_0 x_0)}(t_0)=x_0$, $\tilde{\lambda}^{(t_0,x_0)}(t_0)=R(t_0)x_0$

From here by taking into account that

$\lambda^{(t_0,x_0)}(t)=R(t)x^{(t_0,x_0)}(t)$ we deduce that R is a solution to the Riccati equation in the statement of Theorem 1.1.

We have also $\tilde{u}^{(t_0,x_0)}=F(t)\tilde{x}^{(t_0 x_0)}$ and from the properties of $\tilde{x}^{(t_0 x_0)}$ we deduce that F is stabilizing.

To the system (1.1) a dual system is associated

$$x' = A^{\#}(t)x + C_1^{\#}(t)u_1 + C_2^{\#}(t)u_2$$

$$y_1 = B_1^{\#}(t)x + D_{11}^{\#}(t)u_1 + D_{21}^{\#}(t)u_2 \qquad (\#)$$

$$y_2 = B_2^{\#}(t)x + D_{12}^{\#}(t)u_1$$

$A^{\#}(t) = A^*(-t)$ , $C_i^{\#}(t) = C_i^*(-t)$ , $B_i^{\#}(t) = B_i^*(-t)$, $D_{ij}^{\#}(t) = D_{ij}^*(-t)$ .

It is seen that if a compensator is stabilizing and $\gamma$-contracting for (1.1) the dual compensator is stabilizing and $\gamma$-contracting for ($\#$) hence we may use the above results to get a bounded stabilizing solution $R^{\#}$ for a corresponding Riccati equation. If we take $S(t) = R^{\#}(-t)$ we see that $S$ is a solution with required properties for the second Riccati equation in Theorem 1.1.

## 6. The third Riccati equation in the statement of Theorem 1.1 .

Proposition 6.1. Let R be the stabilizing solution to the Riccati equation (1.4) , F the corresponding feedback gain . Let $\gamma(x,u)$ be such that for every $(t_o, x_o)$ the solution to $x' = Ax + Bu$ , $x(t_o) = x_o$ is in $L^2(t_o, \infty; R^n)$ . Then

$$\int_{t_o}^{\infty} (\gamma^2 |u_1|^2 - |C_1 x + D_1 u|^2)dt = -x_o^* R(t_o)x_o - \int_{t_o}^{\infty} (u_1^* - x^* F^*)K(u - Fx)dt$$

Proof. A direct computation using the equation (1.4).

Remark.

$$K = \begin{pmatrix} I_{m_1} & D_{11}^* \\ 0 & D_{12}^* \end{pmatrix} \begin{pmatrix} -\gamma^2 I_{m_1} & 0 \\ 0 & I_{p_1} \end{pmatrix} \begin{pmatrix} I_{m_1} & 0 \\ D_{11} & D_{12} \end{pmatrix}$$

$$v^* K v = (v_1^* \quad v_2^*) \begin{pmatrix} I_{m_1} & D_{11}^* \\ 0 & D_{12}^* \end{pmatrix} \begin{pmatrix} -\gamma^2 I_{m_1} & 0 \\ 0 & I_{p_1} \end{pmatrix} \begin{pmatrix} I_{m_1} & 0 \\ D_{11} & D_{12} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =$$

$$= -\gamma^2 |v_1|^2 + |D_{11} v_1 + D_{12} v_2|^2$$

We deduce that the formula in the Proposition 6.1 may be written as

$$\int_{t_0}^{\infty}(\gamma^2|u_1|^2-|C_1x+D_1u|^2)dt = -x_0^*R(t_0)x_0+\int_{t_0}^{\infty}\gamma^2|u_1-F_1x|^2dt -$$
$$-\int_{t_0}^{\infty}|D_{11}(u_1-F_1x)+D_{12}(u_2-F_2x)|^2\ dt$$

We want now a new form of $\int_{t_0}^{\infty}(\gamma^2|u_1|^2-|C_1x+D_1u|^2)dt$ by using the solutions $\hat{R}$ , $\hat{r}$ .

Proposition 6.2. We have

$$\int_{t_0}^{\infty}(\gamma^2|u_1|^2-|C_1x+D_1u|^2)dt = -x_0^*\hat{R}(t_0)x_0-x_0^*\hat{r}^{u_1}(t_0)-(\hat{r}^{u_1})^*(t_0)\,x_0-$$
$$-\int_{t_0}^{\infty}[u_1^*(D_{11}^*D_{11}-\gamma^2I_{m_1})u_1+(\hat{r}^{u_1})^*B_1u_1+u_1^*B_1^*\hat{r}^{u_1}-((\hat{r}^{u_1})^*B_2+u_1^*D_{11}^*D_{12})$$
$$(D_{12}^*D_{12})^{-1}(B_2\hat{r}^{u_1}+D_{12}^*D_{11}u_1)]\,dt-\int_{t_0}^{\infty}(u_2^*-\hat{u}_2^*)(D_{12}^*D_{12})(u_2-\hat{u}_2)dt$$

$$\hat{u}_2=\hat{F}_2x+(D_{12}^*D_{12})^{-1}(B_2^*\hat{r}^{u_1}+D_{12}^*D_{11}u_1)$$

The computations have been performed when we studied the first minimization problem.

We investigate now $\hat{u}_2$ and the corresponding $\hat{x}$, if we take $u_1=F_1\tilde{x}$ with $\tilde{x}'=(A+BF)\tilde{x}$ , $\tilde{x}(t_0)=x_0$ . Denote $\tilde{p}=R\tilde{x}$ . A direct computation gives

$$\tilde{p}'=-A^*\tilde{p}-C_1^*(D_1F+C_1)\tilde{x}$$

Denote now $\tilde{r}=\tilde{p}-\hat{R}\tilde{x}$ .A computation shows that

$$\tilde{r}'=-A^*\tilde{r}-C_1^*D_1F\tilde{x}-\hat{R}BF\tilde{x}-(\hat{R}B_2+C_1^*D_{12})(D_{12}^*D_{12})^{-1}(B_2\hat{R}+D_{12}^*C_1)\tilde{x}$$

and also

$$\tilde{r}'=-A^*\tilde{r}-(\hat{R}B_1+C_1^*D_{11})F_1\tilde{x}-(\hat{R}B_2+C_1^*D_{12})[F_2\tilde{x}+(D_{12}^*D_{12})^{-1}(B_2^*\hat{R}+D_{12}^*C_1)\tilde{x}]$$

On the other hand $KF\tilde{x}=-B^*R\,\tilde{x}-D_1^*C_1\,\tilde{x}=-B^*\tilde{p}-D_1^*C_1\tilde{x}=-B^*\tilde{r}-$

$$-(B^*\hat{R}+D_1^*C_1)\,\tilde{x}$$

$$(D_{11}^*D_{11}-\gamma^2I_{m_1})F_1\tilde{x}+D_{11}^*D_{12}F_2\tilde{x}=-B_1^*\tilde{r}-(B_1^*\hat{R}+D_{11}^*C_1)\tilde{x}$$

$$D_{12}^*D_{11}F_1\tilde{x}+D_{12}^*D_{12}F_2\tilde{x}=-B_2^*\tilde{r}-(B_2^*\hat{R}+D_{12}^*C_1)\tilde{x}\quad\text{hence}$$

$$F_2\tilde{x} = -(D_{12}^*D_{12})^{-1}D_{12}^*D_{11}F_1\tilde{x} - (D_{12}^*D_{12})^{-1}B_2^*\tilde{r} - (D_{12}^*D_{12})^{-1}(B_2^*\hat{R}+D_{12}^*C_1)\tilde{x}$$

$$F_2\tilde{x} + (D_{12}^*D_{12})^{-1}(B_2^*\hat{R}+D_{12}^*C_1)\tilde{x} = -(D_{12}^*D_{12})^{-1}(B_2^*\tilde{r}+D_{12}^*D_{11}F_1\tilde{x})$$

We deduce that

$$\tilde{r}' = -A^*\tilde{r} - (\hat{R}B_1+C_1^*D_{11})F_1\tilde{x} + (\hat{R}B_2+C_1^*D_{12})(D_{12}^*D_{12})^{-1}(B_2^*\tilde{r}+D_{12}^*D_{11}F_1\tilde{x})$$

and finally

$$\tilde{r}' = -(A+B_2\hat{F}_2)^*\tilde{r} + [(\hat{R}B_2+C_1^*D_{12})(D_{12}^*D_{12})^{-1}D_{12}^*D_{11} - (\hat{R}B_1+C_1^*D_{11})]F_1\tilde{x} .$$

This last formula shows that $\tilde{r}$ defined above is just $\hat{r}_{F_1\tilde{x}}$ .
The corresponding $\hat{u}_2$ is given by the system

$$\hat{x}' = (A+B_2\hat{F}_2)\hat{x} + B_1F_1\tilde{x} - B_2(D_{12}^*D_{12})^{-1}(B_2^*\tilde{r}+D_{12}^*D_{11}F_1\tilde{x}) .$$

But $B_2^*\tilde{r}+D_{12}^*D_{11}F_1\tilde{x} = -D_{12}^*D_{12}F_2\tilde{x} - (B_2^*\hat{R}+D_{12}^*C_1)\tilde{x}$ hence

$$\hat{x}' = (A+B_2\hat{F}_2)\hat{x} + B_1F_1\tilde{x} - B_2(D_{12}^*D_{12})^{-1}D_{12}^*D_{12}F_2\tilde{x} + B_2(D_{12}^*D_{12})^{-1}(B_2^*\hat{R}+$$
$$+D_{12}^*C_1)\tilde{x} = (A+B_2\hat{F}_2)\hat{x} + B_1F_1\tilde{x} + B_2F_2\tilde{x} - B_2\hat{F}_2\hat{x}$$ and since

$\tilde{x}' = A\tilde{x} + BF\tilde{x}$ we have $(\hat{x}-\tilde{x})' = (A+B_2\hat{F}_2)(\hat{x}-\tilde{x})$ and we deduce $\hat{x} = \tilde{x}$ and again a direct computation gives $\hat{u}_2 = \tilde{u}_2$ . We obtained

<u>Theorem 6.1.</u> Let $u_1 = F_1\tilde{x}$ and $u_2$ arbitrary in $L^2(t_o, \infty; R^{m_2})$ such that $x' = Ax + B_1F_1\tilde{x} + B_2u_2$ , $x(t_o) = x_o$ gives $x \in L^2(t_o, \infty; R^n)$. Then

$$\int_{t_o}^{\infty}(\gamma^2|F_1\tilde{x}|^2 - |C_1x + D_{11}F_1\tilde{x} + D_{12}u_2|^2)\,dt \le$$

$$\le \int_{t_o}^{\infty}\{\gamma^2|F_1\tilde{x}|^2 - |C_1\tilde{x} + D_1F\tilde{x}|^2\}\,dt = -x_o^* R(t_o)x_o$$

We are now in posåtion to prove

<u>Theorem 6.2.</u> Assume that the system (1.1) has a stabilizing and $\gamma$ - contracting compensator .Then the same compensator is stabilizing and $\gamma$ -contracting ~~f~~for the system

$$\begin{aligned} &x' = (A+B_1F_1)x + B_1v_1 + B_2u_2 , \quad y_1 = -D_{12}F_2x + D_{11}v_1 + D_{12}u_2 \\ &y_2 = (C_2 + D_{21}F_1)x + D_{21}v_1 \end{aligned} \qquad (6.1)$$

We shall now construct the dual Riccati equation associated to the above transformad system.

<u>Proof.</u> The fact that a stabilizing and $\gamma$ -contracting compensator for (1.1) is $\gamma$ -contracting for the transformed system

above if it is stabilizing is proved by a simple computation. Hence to prove the above stated theorem we have to show that a stabilizing and $\gamma$ - contracting compensator for (1.1) is stabilizing for the transformed system above.

After coupling the compensator to the transformed system we obtain $\xi' = (\mathcal{A}_0 + \mathcal{B}_0 \mathcal{F}_0)\xi$ , $\xi = \begin{pmatrix} x \\ x_c \end{pmatrix}$

$$\mathcal{A}_0 = \begin{pmatrix} A+B_2D_cC_2 & B_2C_2 \\ B_cC_2 & A_c \end{pmatrix}, \quad \mathcal{B}_0 = \begin{pmatrix} B_1+B_2D_cD_{21} \\ B_cD_{21} \end{pmatrix}$$

$\mathcal{F}_0 = (F_1 \quad 0)$ . Since $\mathcal{A}_0$ is obtained by coupling the compensator to (1.1) it defines an exponentially stable evolution. If there exists $c > 0$ such that for all $(t_0, \xi_0)$

$$\int_{t_0}^{\infty} |X_{\mathcal{A}_0+\mathcal{B}_0\mathcal{F}_0}(t,t_0)\xi_0|^2 dt \leq c|\xi_0|^2$$

the evolution defined by $\mathcal{A}_0 + \mathcal{B}_0\mathcal{F}_0$ is exponentially stable. Hence if the claim in the statement were not true $\xi_0^k$, $t_k$, $\tau_k$ would exist with $|\xi_0^k| = 1$ , $t_k < \tau_k$,

$$\int_{t_k}^{\tau_k} |X_{\mathcal{A}_0+\mathcal{B}_0\mathcal{F}_0}(t,t_k)\xi_0^k|^2 dt > k$$

. Denote

$$\xi^k(t) = X_{\mathcal{A}_0+\mathcal{B}_0\mathcal{F}_0}(t,t_k)\xi_0^k \quad , \quad u_1^k = \mathcal{F}_0(t)\xi^k(t) =$$

$= F_1(t)x^k(t)$ . We may write $\xi^k(t) = X_{\mathcal{A}_0}(t,t_k)\xi_0^k +$

$$+ \int_{t_k}^{\tau_k} X_{\mathcal{A}_0}(t,s)\mathcal{B}_0(s)u_1^k(s)ds$$

hence by standard arguments

$$k < \int_{t_k}^{\tau_k} |\xi^k(t)|^2 dt \leq \frac{\mathcal{B}^2}{2\alpha} + \frac{\mathcal{B}^2}{\alpha^2}\int_{t_k}^{\tau_k} |u_1^k|^2 dt$$

and we deduce that $\lim_{k\to\infty} \int_{t_k}^{\tau_k} |u_1^k|^2 dt = \infty$ . For $\tau_k$ as above we consider the problem

$$\xi' = \mathcal{A}_0\xi + \mathcal{B}_0 u_1 \quad , \quad \xi(\tau_k) = \xi^k(\tau_k)$$

$$\min\left\{\int_{\tau_k}^{\infty} [\gamma^2|u_1|^2 - |\mathcal{C}\xi + Du_1|^2] dt\right\}$$

$\mathscr{C} = ( C_1+D_{12}D_cC_2 \quad D_{12}C_c) \qquad D=D_{11}+D_{12}D_cD_{21}$ .

Since for every $u_1$, $\xi$ is a solution to the system obtained after compensation in (1.1) we deduce that for zero initial conditions $\int_{\tau_k}^{\infty}\{\gamma^2|u_1|^2 - |\mathscr{C}\xi +Du_1|^2\}dt \geq \delta^2\int_{\tau_k}^{\infty}|u_1|^2dt$ and such inequality implies existence and uniqueness of the optimum. Denote such optimum by $\hat{u}_1^k$ , $\hat{\xi}^k$ and define $\hat{u}_2^k=C_c\hat{x}_c^k+D_c\hat{y}_2^k$ , $\hat{y}_2=C_2\hat{x}^k+D_{21}\hat{u}_1^k$ ,$\hat{y}_1^k=C_1\hat{x}^k+D_{11}\hat{u}_1^k+D_{12}\hat{u}_2^k$ .

We obtain an admissible triple ($\hat{u}_1^k$ , $\hat{u}_2^k$ , $\hat{x}^k$ ) ; we consider also the triple obtained if we take $u_1=F_1\tilde{x}$ and use Theorem 6.1 to obtain

$$\int_{\tau_k}^{\infty}\{\gamma^2|F_1x|^2-|\mathscr{C}\tilde{\xi}^k+DF_1\tilde{x}|^2\}dt \leq -(x^k(\tau_k))^*R(\tau_k)x^k(\tau_k)$$

hence from the optimality property

$$\int_{\tau_k}^{\infty}\{\gamma^2|\hat{u}_1^k|^2-|\mathscr{C}\hat{\xi}^k+D\hat{u}_1^k|^2\}dt \leq -(x^k(\tau_k))^* \quad R(\tau_k)x^k(\tau_k)$$

On the other hand we had the general formula

$$\int_{t_k}^{\tau_k}(\gamma^2|u_1|^2-|C_1x+D_1u_1|^2)\,dt = x^*(\tau_k)R(\tau_k)x(\tau_k)-$$
$$-x^*(t_k)R(t_k)x(t_k)-\int_{t_k}^{\tau_k}\gamma^2|u_1-F_1x|^2dt - \int_{t_k}^{\tau_k}|D_{11}(u_1-F_1x)+D_{12}(u_2-$$
$$-F_2x)|^2dt$$

From here taking into account that $u_1^k=F_1x^k$ we deduce that

$$\int_{t_k}^{\tau_k}(\gamma^2|u_1^k|^2-|C_1x^k+D_1u_1^k|^2)\,dt = (x^k(\tau_k))^*R(\tau_k)x^k(\tau_k)-$$
$$-[x^k(t_k)]^*R(t_k)x^k(\tau_k) - \int_{t_k}^{\tau_k}|D_{12}(u_2^k-F_2x^k)|^2dt$$

and if we define our triple $u_1^k$ ,$x^k$ , $u_2^k$ for $t \geq \tau_k$ to be $\hat{u}_1^k$ , $\hat{x}^k$, $\hat{u}_2^k$ we shall have

$$\int_{t_k}^{\infty}\{\gamma^2|u_1^k|^2-|\mathscr{C}\xi^k+Du_1^k|^2\} \leq 0$$

For the same input , but for zero values at $t_k$ we have

$$\int_{t_k}^{\infty}\{\gamma^2|u_1^k|^2-|\mathscr{C}\hat{\xi}_1^k+Du_1^k|^2\}dt \geq \delta\int_{t_k}^{\infty}|u_1^k|^2dt \geq \delta^2\int_{t_k}^{\tau_k}|u_1^k|^2dt \to \infty$$

Write $\quad \xi^k = \hat{\xi}_0^k + \hat{\xi}_1^k \qquad \hat{\xi}_0^k(t)=X_{\mathscr{A}_0}(t,t_k)\xi_0^k$

$$\hat{\xi}_1^k(t)=\int_{t_k}^{\tau_k} X_{\mathcal{A}_0}(t,s)\mathcal{B}_0(s)u_1^k(s)ds+\int_{\tau_k}^{t} X_{\mathcal{A}_0}(t,s)\mathcal{B}_0(s)\hat{u}_1^k(s)ds$$

In the same way we decompose $u_2^k$, $y_1^k$, $y_2^k$ and since $\mathcal{A}_0$ defines an exponentially stable evolution

$$\int_{t_k}^{\infty}\{|(y_2^k)_0|^2+|(u_2^k)_0|^2+|(y_1^k)_0|^2\}dt\le c<\infty$$

$$\int_{t_k}^{\infty}|(y_1^k)_1|^2dt\le\theta^2\int_{t_k}^{\infty}|u_1^k|^2dt$$

We have further

$$\int_{t_k}^{\infty}(\gamma^2|u_1^k|^2-|y_1^k|^2)dt=\int_{t_k}^{\infty}\{\gamma^2|u_1^k|^2-|(y_1^k)_0+(y_1^k)_1|^2\}dt=$$

$$=\int_{t_k}^{\infty}\{\gamma^2|u_1^k|^2-|(y_1^k)_1|^2\}dt-2\int_{t_k}^{\infty}\langle(y_1^k)_0,(y_1^k)_1\rangle dt-$$

$$-\int_{t_k}^{\infty}|(y_1^k)_0|^2dt$$ and we deduce that

$$\left|\left[\int_{t_k}^{\infty}(\gamma^2|u_1^k|^2-|y_1^k|^2)dt\right]\Big/\left[\int_{t_k}^{\infty}(\gamma^2|u_1^k|^2-|(y_1^k)_1|^2)dt\right]-1\right|\le$$

$$\le\left[2\int_{t_k}^{\infty}|(y_1^k)_0|\cdot|(y_1^k)_1|dt+\int_{t_k}^{\infty}|(y_1^k)_0|^2dt\right]\Big/\left[\int_{t_k}^{\infty}(\gamma^2|u_1^k|^2-|(y_1^k)_1|^2)dt\right]\le$$

$$\le\left[2c\theta\left(\int_{t_k}^{\infty}|u_1^k|^2dt\right)^{1/2}+c^2\right]\Big/\left[\delta^2\int_{t_k}^{\infty}|u_1^k|^2dt\right]\longrightarrow 0$$

for $k\to\infty$. We deduce that for $k$ large enough

$$\int_{t_k}^{\infty}(\gamma^2|u_1^k|^2-|y_1^k|^2)dt=(1+\alpha_k)\int_{t_k}^{\infty}(\gamma^2|u_1^k|^2-|(y_1^k)_1|^2)dt\ge$$

$$\ge\frac{1}{2}\int_{t_k}^{\infty}(\gamma^2|u_1^k|^2-|(y_1^k)_1|^2)^2dt\ge\frac{\delta^2}{2}\int_{t_k}^{\tau_k}|u_1^k|^2dt>0$$

a contradiction with $\int_{t_k}^{\infty}(\gamma^2|u_1^k|^2-|\mathcal{C}\xi^k+\mathcal{D}u_1^k|^2)dt\le 0$.

The theorem is thus proved. Let us check for completeness that the compensator is $\gamma$-contracting for the modified system.

Recall that the compensated system (1.1) reads

$$\begin{aligned}
x'&=(A+B_2D_cC_2)x+B_2C_cx_c+(B_1+B_2D_cD_{21})u_1\\
x_c'&=B_cC_2x+A_cx_c+B_cD_{21}u_1\\
y_1&=C_1x+D_{11}u_1+D_{12}(C_cx_c+D_cC_2x+D_cD_{21}u_1)
\end{aligned}\qquad(6.2)$$

while the compensated modified system is

$$x' = [A+B_2D_cC_2+(B_1+B_2D_cD_{21})F_1]\,x + B_2C_cx_c+(B_1+B_2D_cD_{21})v_1$$

$$x_c' = B_c(C_2+D_{21}F_1)x + A_cx_c + B_cD_{21}v_1$$

$$y_1 = -D_{12}F_2x + D_{11}v_1 + D_{12}(C_cx_c+D_cD_{21}v_1)+D_{12}D_c(C_2+D_{21}F_1)\,x$$

Let $v_1 \in L^2(R,R^{m_1})$ ; denote the corresponding response in $L^2$ by $(\tilde{x}\ ,\tilde{x}_c)$ . Take in the compensated system (6.2) the input $u_1=v_1+F_1\tilde{x}$ to get

$$x' = (A+B_2D_cC_2)x + B_2C_cx_c+(B_1+B_2D_cD_{21})v_1+(B_1+B_2D_cD_{21})F_1\tilde{x}$$

$$x_c' = B_cC_2x+A_cx_c+B_cD_{21}v_1+B_cD_{21}F_1\tilde{x}$$

We deduce $(\tilde{x}-x)' = (A+B_2D_cC_2)(\tilde{x}-x) + B_2C_c(\tilde{x}_c-x_c)$

$(\tilde{x}_c-x_c)' = B_cC_2(\tilde{x}-x) + A_c(\tilde{x}_c-x_c)$ that is

$$\begin{pmatrix}\tilde{x}-x\\ \tilde{x}_c-x_c\end{pmatrix}' = \mathcal{A}_0\begin{pmatrix}\tilde{x}-x\\ \tilde{x}_c-x_c\end{pmatrix}$$

and since $\mathcal{A}_0$ defines an exponentially stable evolution , we must have $\tilde{x}=x$ , $\tilde{x}_c=x_c$.

From our basic relation we have further

$$\int_{-\infty}^{\infty}(\gamma^2|u_1|^2 - |y_1|^2)\,dt = \int_{-\infty}^{\infty}\gamma^2|u_1-F_1x|^2dt - \int_{-\infty}^{\infty}|D_{11}(u_1-F_1x)+$$

$+D_{12}(u_2-F_2x)|^2\ dt$ . But $u_1-F_1x=v_1$, $\tilde{y}_1=-D_{12}F_2x+D_{11}v_1+$ $+D_{12}CC_cx_c+D_cD_{21}v_1)+D_{12}D_c(C_2+D_{21}F_1)x$ , $u_2=C_cx_c+D_cy_2=$ $=C_cx_c+D_c(C_2x+D_{21}u_1)=C_cx_c+D_cC_2x+D_cD_{21}(v_1+F_1x)$ ,

$\tilde{y}_1=D_{11}v_1+D_{12}(u_2-F_2x)$

hence $\int_{-\infty}^{\infty}(\gamma^2|u_1|^2-|y_1|^2)dt = \int_{-\infty}^{\infty}(\gamma^2|v_1|^2-|\tilde{y}_1|^2)dt$

We know that $\int_{-\infty}^{\infty}(\gamma^2|u_1|^2-|y_1|^2)dt \geq \delta^2\int_{-\infty}^{\infty}|u_1|^2dt$ hence

$$\int_{-\infty}^{\infty}(\gamma^2|v_1|^2-|\tilde{y}_1|^2)dt \geq \delta^2\int_{-\infty}^{\infty}|u_1|^2dt$$

But from the exponential stability of the compensated system we have $\int_{-\infty}^{\infty}|x|^2\,dt \leq c^2\int_{-\infty}^{\infty}|u_1|^2dt$ hence

$\int_{-\infty}^{\infty}|u_1-F_1x|^2dt \leq q^2\int_{-\infty}^{\infty}|u_1|^2dt$ and from here $\int_{-\infty}^{\infty}|u_1|^2dt \geq \frac{1}{q^2}\int_{-\infty}^{\infty}|v_1|^2dt$

We obtained finally $\int_{-\infty}^{\infty}|\tilde{y}_1|^2dt \leq \left(\gamma^2-\frac{\delta^2}{q^2}\right)\int_{-\infty}^{\infty}|v_1|^2dt$

and since $v_1$ was arbitrary the compensator results $\gamma$-contracting . Let us end this section by writing explicitly the third Riccati equation for which existence of a bounded on $\mathbb{R}$ stabilizing solution is obtained from the existence of a stabilizing , $\gamma$ - contracting compensator for the dual of the modified system (6.1) .This equation reads

$$\hat{S}'=(A+B_1F_1)\hat{S}+\hat{S}(A+B_1F_1)^*+B_1B_1^*-\left\{\begin{pmatrix}-D_{12}F_2\\ C_2+D_{21}F_1\end{pmatrix}\hat{S}+\right.$$

$$\left.+\begin{pmatrix}D_{11}\\ D_{21}\end{pmatrix}B_1^*\right\}^*\hat{K}^{-1}\left\{\begin{pmatrix}-D_{12}F_2\\ C_2+D_{21}F_1\end{pmatrix}\hat{S}+\begin{pmatrix}D_{11}\\ D_{21}\end{pmatrix}B_1^*\right\}$$

The solutions R and $\hat{S}$ are the only used to construct a stabilizing and $\gamma$-contracting compensator as it can be seen from [6]. To obtain the existence of $\hat{S}$ we still have to check the detectability assumption which has been used in the proof for existence of R . The modified system is associated to

$A+B_1F_1$ , $B_1$, $B_2$,$-D_{12}F_2$ , $C_2+D_{21}F_1$ , $D_{11}$, $D_{12}$, $D_{21}$ ;

We deduce first that the projections $\Pi_1$, $\Pi_2$ are not changed . We shall need the property of stabilizability for $(A+B_1F_1-B_1D_{21}^*(D_{21}D_{21}^*)^{-1}(C_2+D_{21}F_1)$ , $B_1\ \Pi_2$ ) and such property follows from the stabilizability property in the statement of Theorem 1.1 .

<u>Remark.</u> If we write the first Riccati equation associated to the above considered modified system (6.1) we see that it is in fact a " Bernoulli equation " and the corresponding global positive semidefinite ,stabilizing solution is just the zero one.

## 7. <u>A spectral radius property.</u>

The Riccati equations for which existence of global, bounded ,positive semidefinite ,stabilizing solutions follows from existence of a stabilizing and $\gamma$ - contracting compensator may be written as

$$R'+(A-BK^{-1}D_1^*C_1)^*R+R(A-BK^{-1}D_1^*C_1)-RBK^{-1}B^*R+C_1^*(I_{p_1}-D_1K^{-1}D_1^*)C_1=0$$

$$S'=(A-B_1D_2^*\hat{K}^{-1}C)S+S(A-B_1D_2^*\hat{K}^{-1}C)^*-SC^*\hat{K}^{-1}CS+B_1(I_{m_1}-D_2^*\hat{K}^{-1}D_2)B_1^*$$

$$\hat{S}'=(A+B_1F_1-B_1D_2^*\hat{K}^{-1}\hat{C})\hat{S}+\hat{S}(A+B_1F_1-B_1D_2^*\hat{K}^{-1}\hat{C})^*S-$$
$$-\hat{S}\hat{C}^*\hat{K}^{-1}\hat{C}\hat{S}+B_1(I_{m_1}-D_2^*\hat{K}^{-1}D_2)B_1^*$$

$\hat{C}=\begin{pmatrix} -D_{12}F_2 \\ C_2+D_{21}F_1 \end{pmatrix}$ . Let $Z_1$ be a matrix solution to

$Z_1'=-(A-B_1D_2^*\hat{K}^{-1}C-SC^*\hat{K}^{-1}C)^*Z_1$ , $Z_1(t_0)=I_n$ . Since $S$ is stabilizing $A-B_1D_2^*\hat{K}^{-1}C-SC^*\hat{K}^{-1}C$ defines an exponentially stable evolution and $Z_1$ corresponds to an antistable evolution , $|Z_1(t)|\le\beta e^{-\alpha(t_0-t)}$ , $t\le t_0$ .Define $Z_2=SZ_1$ and check that $t\to\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ is a matrix- solution for the Hamiltonian system

$$Z_1'=-(A+B_1D_2^*\hat{K}^{-1}C)^*Z_1+C^*\hat{K}^{-1}CZ_2$$

$$Z_2'=B_1(I_{m_1}-D_2^*\hat{K}^{-1}D_2)B_1^*Z_1+(A-B_1D_2^*\hat{K}^{-1}C)Z_2$$

In the same way to the third Riccati equation a canonical system is associated of the form

$$\hat{Z}_1'=-(A-B_1D_2^*K^{-1}C+B_1F_1)\ Z_1+C\ K^{-1}C\ Z_2$$

$$Z_2'=B_1(I_{m_1}-D_2^*\hat{K}^{-1}D_2)B_1^*\hat{Z}_1+(A-B_1D_2^*\hat{K}^{-1}\hat{C}+B_1F_1)\hat{Z}_2$$

we have $|Z_1(t)|+|Z_2(t)|\le\beta e^{-\alpha(t_0-t)}$ $t\le t_0$

and also $|\hat{Z}_1(t)|+|\hat{Z}_2(t)|\le\hat{\beta}e^{-\hat{\alpha}(t_0-t)}$ $t\le t_0$

An essential remark is that every matrix solution for the canonical systems above with antistable evolution will be a linear combination of the special solutions considered above.

Consider now the Liapunov transformation defined by

$$\mathcal{T}(t)=\begin{pmatrix} I_n & -\frac{1}{\gamma^2}R(t) \\ 0 & I_n \end{pmatrix}$$ . We shall prove that the

two Hamiltonian systems above are equivalent under this transformation , that is $\hat{\mathcal{H}} = \mathcal{T}'\mathcal{T}^{-1} + \mathcal{T}\mathcal{H}\mathcal{T}^{-1}$
We shall deduce from here that $t \rightarrow \mathcal{T}(t)\begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix}$
is a matrix solution to the Hamiltonian system defined by $\hat{\mathcal{H}}$ ,having antistable evolution, hence

$$\mathcal{T}(t)\begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} = \begin{pmatrix} \hat{Z}_1(t) \\ \hat{Z}_2(t) \end{pmatrix} M$$

Since $\mathcal{T}$ is nonsingular the solutions above are linearly independent , hence M is nonsingular. Since $Z_1 - \frac{1}{\gamma^2}RZ_2 = \hat{Z}_1M$ , $Z_2 = \hat{Z}_2M$ if we set $t=t_0$ we deduce that $I_n - \frac{1}{\gamma^2}R(t_0)S(t_0) = M$ hence $I_n - \frac{1}{\gamma^2}R(t)S(t)$ is nonsingular for all t , and

$$\hat{S}(t) = S(t)[I_n - \frac{1}{\gamma^2}R(t)S(t)]^{-1}$$

This formula gives an important connection between the global stabilizing , bounded ,positive semidefinite solutions of the three Riccati equations associated to our problem. Simple algebraic manipulations show now that the spectral radius of RS is strictly less than $\gamma^2$ for all t.

<u>Appendix</u>. Equivalence of the Hamiltonian systems. We have

$$\mathcal{H} = \begin{pmatrix} -(A-B_1D_2^*\hat{K}^{-1}C)^* & C^*\hat{K}^{-1}C \\ \tilde{B}_1 & A-B_1D_2^*\hat{K}^{-1}C \end{pmatrix}$$

$$\hat{\mathcal{H}} = \begin{pmatrix} -(A+B_1F_1-B_1D_2^*\hat{K}^{-1}\hat{C})^* & \hat{C}^*\hat{K}^{-1}\hat{C} \\ \tilde{B}_1 & A+B_1F_1-B_1D_2^*\hat{K}^{-1}\hat{C} \end{pmatrix}$$

Denote $\tilde{\mathcal{H}} = \begin{pmatrix} \tilde{\mathcal{H}}_{11} & \tilde{\mathcal{H}}_{12} \\ \tilde{\mathcal{H}}_{21} & \tilde{\mathcal{H}}_{22} \end{pmatrix} = \mathcal{T}'\mathcal{T}^{-1} + \mathcal{T}\mathcal{H}\mathcal{T}^{-1}$ hence

$$\tilde{\mathcal{H}}\mathcal{T} = \begin{pmatrix} \tilde{\mathcal{H}}_{11} & \tilde{\mathcal{H}}_{12} \\ \tilde{\mathcal{H}}_{21} & \tilde{\mathcal{H}}_{22} \end{pmatrix}\begin{pmatrix} I & -\frac{1}{\gamma^2}R \\ 0 & I \end{pmatrix} = \begin{pmatrix} \tilde{\mathcal{H}}_{11} & \tilde{\mathcal{H}}_{12} - \frac{1}{\gamma^2}\tilde{\mathcal{H}}_{11}R \\ \tilde{\mathcal{H}}_{21} & \tilde{\mathcal{H}}_{22} - \frac{1}{\gamma^2}\tilde{\mathcal{H}}_{21}R \end{pmatrix} =$$

$$\begin{pmatrix} 0 & -\frac{1}{\gamma^2}R \\ 0 & 0 \end{pmatrix}' + \begin{pmatrix} I & -\frac{1}{\gamma^2}R \\ 0 & I \end{pmatrix}\begin{pmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ \mathcal{H}_{21} & \mathcal{H}_{22} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\gamma^2}R' \\ 0 & 0 \end{pmatrix} +$$

$$+ \begin{pmatrix} \mathcal{H}_{11} - \frac{1}{\gamma^2}R\mathcal{H}_{21} & \mathcal{H}_{12} + \frac{1}{\gamma^2}R\mathcal{H}_{11}^* \\ \mathcal{H}_{21} & -\mathcal{H}_{11}^* \end{pmatrix}$$

We deduce that $\tilde{\mathcal{H}}_{11} = \mathcal{H}_{11} - \frac{1}{\gamma^2}R\mathcal{H}_{21}$, $\tilde{\mathcal{H}}_{12} - \frac{1}{\gamma^2}\tilde{\mathcal{H}}_{11}R =$
$\mathcal{H}_{12} + \frac{1}{\gamma^2}R\tilde{\mathcal{H}}_{11} - \frac{1}{\gamma^2}R'$, $\tilde{\mathcal{H}}_{21} = \mathcal{H}_{21}$, $\tilde{\mathcal{H}}_{22} - \frac{1}{\gamma^2}\tilde{\mathcal{H}}_{21}R = -\mathcal{H}_{11}^*$;

we see that $\tilde{\mathcal{H}}_{21} = \mathcal{H}_{21} = \hat{\mathcal{H}}_{21}$ and we have to check the other blocks. We have

$\tilde{\mathcal{H}}_{22} = -\mathcal{H}_{11}^* + \frac{1}{\gamma^2}\tilde{\mathcal{H}}_{21}R = A - B_1D_2^*\hat{K}^{-1}\hat{C} + \frac{1}{\gamma^2}B_1(I - D_2^*\hat{K}^{-1}D_2)B_1^*R +$
$+ B_1D_2^*\hat{K}^{-1}(\hat{C} - C)$ and we have to check that

$\frac{1}{\gamma^2}B_1(I - D_2^*\hat{K}^{-1}D_2)B_1^*R = B_1F_1 + B_1D_2^*\hat{K}^{-1}(C - \hat{C})$. Remember that

$$F = -K^{-1}(D_1^*C_1 + B^*R) = -\begin{pmatrix} D_{11}^*D_{11} - \gamma^2 I & D_{11}^*D_{12} \\ D_{12}^*D_{11} & D_{12}^*D_{12} \end{pmatrix}^{-1}\left[\begin{pmatrix} D_{11}^*C_1 \\ D_{12}^*C_1 \end{pmatrix} + \begin{pmatrix} B_1^*R \\ B_2^*R \end{pmatrix}\right]$$

. Denote $\hat{F}_1 = D_2^*\hat{K}^{-1}(\hat{C} - C) + \frac{1}{\gamma^2}(I - D_2^*\hat{K}^{-1}D_2)B_1^*R =$

$$= D_2^*\hat{K}^{-1}\left[\begin{pmatrix} -D_{12}F_2 - C_1 \\ D_{21}F_1 \end{pmatrix} - \frac{1}{\gamma^2}D_2B_1^*R\right] + \frac{1}{\gamma^2}B_1^*R$$

$$KF = \begin{pmatrix} D_{11}^*D_{11} - \gamma^2 I & D_{11}^*D_{12} \\ D_{12}^*D_{11} & D_{12}^*D_{12} \end{pmatrix} \cdot \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = -\begin{pmatrix} D_{11}^*C_1 \\ D_{12}^*C_1 \end{pmatrix} - \begin{pmatrix} B_1^*R \\ B_2^*R \end{pmatrix}$$

$$(D_{11}^*D_{11}-\gamma^2 I)F_1+D_{11}^*D_{12}F_2=-D_{11}^*C_1-B_1^*R$$

$$D_{12}^*D_{11}F_1+D_{12}^*D_{12}F_2=-D_{12}^*C_1-B_2^*R$$

$$D_{11}^*(D_{11}F_1+D_{12}F_2+C_1)=\gamma^2 F_1-B_1^*R$$

$$F_1=\frac{1}{\gamma^2}B_1^*R+\frac{1}{\gamma^2}D_{11}^*(D_{11}F_1+D_{12}F_2+C_1)$$

$$\check{F}_1=D_{11}F_1+D_{12}F_2+C_1$$

$$\hat{F}_1=D_2^*\hat{K}^{-1}\left[\begin{pmatrix}-D_{12}F_2-C_1\\ D_{21}F_1\end{pmatrix}-D_2F_1+\frac{1}{\gamma^2}D_2D_{11}^*\check{F}_1\right]+F_1-\frac{1}{\gamma^2}D_{11}^*\check{F}_1=$$

$$=D_2^*\hat{K}^{-1}\left[\begin{pmatrix}-\check{F}_1+D_{11}F_1-D_{11}F_1\\ D_{21}F_1-D_{21}F_1\end{pmatrix}+\frac{1}{\gamma^2}D_2D_{11}^*\check{F}_1\right]+F_1-\frac{1}{\gamma^2}D_{21}^*\check{F}_1=$$

$$=D_2^*\hat{K}^{-1}\begin{pmatrix}(\frac{1}{\gamma^2}D_{11}D_{11}^*-I_{p_1})\check{F}_1\\ \frac{1}{\gamma^2}D_{21}D_{11}^*\check{F}_1\end{pmatrix}+F_1-\frac{1}{\gamma^2}D_{11}^*\check{F}_1=\frac{1}{\gamma^2}(D_{11}^*\quad D_{21}^*)\hat{K}^{-1}$$

$$\begin{pmatrix}(D_{11}D_{11}^*-\gamma^2 I_{p_1})\check{F}_1\\ D_{21}D_{11}^*\check{F}_1\end{pmatrix}-\frac{1}{\gamma^2}D_{11}^*\check{F}_1+F_1$$

Taking into account that $\begin{pmatrix}D_{11}D_{11}^*-\gamma^2 I_{p_1}\\ D_{21}D_{11}^*\end{pmatrix}$ id the first column of $\hat{K}$ we have $\hat{K}^{-1}\begin{pmatrix}D_{11}D_{11}^*-\gamma^2 I_{p_1}\\ D_{21}D_{11}^*\end{pmatrix}=\begin{pmatrix}I_{p_1}\\ 0\end{pmatrix}$

and $\hat{F}_1=\frac{1}{\gamma^2}(D_{11}^*\quad D_{21}^*)\begin{pmatrix}I_{p_1}\\ 0\end{pmatrix}\check{F}_1-\frac{1}{\gamma^2}D_{11}^*\check{F}_1+F_1=F_1$

We deduce finally that $\tilde{\mathcal{H}}_{22}=A-B_1D_2^*\hat{K}^{-1}\hat{C}+B_1F_1=\hat{\mathcal{H}}_{22}$ ,

$$\tilde{\mathcal{H}}_{11}=\mathcal{H}_{11}\quad-\frac{1}{\gamma^2}R\mathcal{H}_{21}=-\tilde{\mathcal{H}}_{22}^*=-\hat{\mathcal{H}}_{22}^*=\hat{\mathcal{H}}_{11}$$

We have only to check that $\tilde{\mathcal{H}}_{12}=\hat{\mathcal{H}}_{12}$

$$\tilde{\mathcal{H}}_{12}=\frac{1}{\gamma^2}\tilde{\mathcal{H}}_{11}R+\mathcal{H}_{12}+\frac{1}{\gamma^2}R\mathcal{H}_{11}^*-\frac{1}{\gamma^2}R'=$$

$$=\frac{1}{\gamma^2}\mathcal{H}_{11}R-\frac{1}{\gamma^4}R\mathcal{H}_{21}R+\mathcal{H}_{12}+\frac{1}{\gamma^2}R\mathcal{H}_{11}^*-\frac{1}{\gamma^2}R'=$$

$$=-\frac{1}{\gamma^2}(A+B_1D_2^*\hat{K}^{-1}C)^*R-\frac{1}{\gamma^2}R(A-B_1D_2^*\hat{K}^{-1}C)-\frac{1}{2}R'+C^*\hat{K}^{-1}C-$$

$$-\frac{1}{\gamma^2}RB_1R=-\frac{1}{\gamma^2}(R+A^*R+RA)+\frac{1}{\gamma^2}\left[C^*\hat{K}^{-1}D_2B_1^*R+RB_1D_2^*\hat{K}^{-1}C\right)+$$

$$+C^*\hat{K}^{-1}C-\frac{1}{\gamma^4}RB_1R\,.$$

We may write

$$R'+A^*R+RA=-C_1^*C_1-\gamma^2 F_1^*F_1+F^*D_1^*D_1F \qquad \text{hence}$$

$$\tilde{\mathcal{H}}_{12}=\frac{1}{\gamma^2}C_1^*C_1+F_1^*F_1-\frac{1}{\gamma^2}F^*D_1^*D_1F-\frac{1}{\gamma^4}RB_1(I_{m_1}-D_2^*\hat{K}^{-1}D_2)B_1^*R+$$

$$+\frac{1}{\gamma^2}C^*\hat{K}^{-1}D_2B_1^*R+\frac{1}{\gamma^2}RB_1D_2^*\hat{K}^{-1}C+C^*\hat{K}^{-1}C=\frac{1}{\gamma^2}C_1^*C_1+F_1^*F_1-$$

$$-\frac{1}{\gamma^2}F^*D_1^*D_1F+(C+\frac{1}{\gamma^2}D_2B_1^*R)^*\hat{K}^{-1}(C+\frac{1}{\gamma^2}D_2B_1^*R)-\frac{1}{\gamma^4}RB_1B_1^*R=$$

$$=\frac{1}{\gamma^2}C_1^*C_1+F_1^*F_1-\frac{1}{\gamma^2}F^*D_1^*D_1F-\frac{1}{\gamma^4}RB_1B_1^*R+(C-\hat{C}+\frac{1}{\gamma^2}D_2B_1^*R)^*\hat{K}^{-1}(C-\hat{C}+$$

$$+\frac{1}{\gamma^2}D_2B_1^*R)+\hat{C}^*\hat{K}^{-1}(C-\hat{C}+\frac{1}{\gamma^2}D_2B_1^*R)+(C-\hat{C}+\frac{1}{\gamma^2}D_2B_1^*R)^*\hat{K}^{-1}\hat{C}+\hat{C}^*\hat{K}^{-1}\hat{C}$$

We have as above

$$C-\hat{C}+\frac{1}{\gamma^2}D_2B_1^*R=-\frac{1}{\gamma^2}\begin{pmatrix} D_{11}D_{11}^*-\gamma^2 I_{p_1} \\ D_{21}D_{11}^* \end{pmatrix}\check{F}_1 \qquad \text{and we deduce}$$

$$\tilde{\mathcal{H}}_{12}=\frac{1}{\gamma^2}C_1^*C_1+F_1^*F_1-\frac{1}{\gamma^2}F^*D_1^*D_1F-\frac{1}{\gamma^4}RB_1B_1^*R+\frac{1}{\gamma^2}\check{F}_1^*(D_{11}D_{11}^*-\gamma^2 I_{p_1})\check{F}_1+$$

$$+\frac{1}{\gamma^2}\check{F}_1^*D_{12}F_2+\frac{1}{\gamma^2}F_2^*D_{12}^*\check{F}_1+\hat{\mathcal{H}}_{12}=\frac{1}{\gamma^2}C_1^*C_1+F_1^*F_1-\frac{1}{\gamma^2}F^*D_1^*D_1F-(F_1-$$

$$-\frac{1}{\gamma^2}D_{11}^*\check{F}_1)^*(F_1-\frac{1}{\gamma^2}D_{11}\check{F}_1)+\frac{1}{\gamma^4}\check{F}_1^*D_{11}D_{11}^*\check{F}_1-\frac{1}{\gamma^2}\check{F}_1^*\check{F}_1+\frac{1}{\gamma^2}\check{F}_1^*D_{12}F_2+$$

$$+\frac{1}{\gamma^2}F_2^*D_{12}^*\check{F}_1+\hat{\mathcal{H}}_{12}=\frac{1}{\gamma^2}C_1^*C_1-\frac{1}{\gamma^2}F^*D_1^*D_1F-\frac{1}{\gamma^2}\check{F}_1^*\check{F}_1+\frac{1}{\gamma^2}\check{F}_1^*D_1F+$$

$$+\frac{1}{\gamma^2}F^*D_1^*\check{F}_1+\hat{\mathcal{H}}_{12}=\frac{1}{\gamma^2}C_1^*C_1-\frac{1}{\gamma^2}(D_1F-\check{F}_1)^*(D_1F-\check{F}_1)+\hat{\mathcal{H}}_{12}$$

and finally $\tilde{\mathcal{H}}_{12}=\hat{\mathcal{H}}_{12}$.

## REFERENCES


1. T.Başar and P.Bernhard -$H_\infty$ Optimal Control and Related Minimax Design Problems, A Dynamic Game Approach, Birkhauser, 1991.

2. G.DaPrato ,A.Ichikawa-Quadratic control for linear time varying systems,SIAM J.Control Optimiz. 28 (1990),359-381.

3. J.C.Doyle,K.Glover,P.P.Khargonekaar, B.A.Francis- State space solutions to standard $H_2$ and $H_\infty$ control problems-

IEEE Trans.Automatic Control,34,8,(1989),831-847

4. V.Drăgan ,A.Halanay -Global solutions for matrix Riccati Differential Equations via input-output operators - "Recent Trends in Ordinary Differential Equations" World Scientific Series in Applicable Analysis (Editor,R.P. Agarval , World Scientific -Singapore ). in press.

5. V.Drăgan,A.Halanay -Stabilizing Compensators ensuaring atenuation of perturbations for linear time-varying dystems- (submitted for publication ).

6. K.Glover,J.C.Doyle- A state space approach to $H_\infty$ Optimal Control. From Model to Data -J.C.Willems,Ed.1989,179-218.

7. G.Tadmor -Worst-Case Design in the Time Domain: The maximum principle and the standard $H_\infty$ problem - Math. of Control Signals and Systems ,1990 Springer Verlag New-York,3,301-324.

# STABILIZING COMPENSATORS WITH DISTURBANCE ATTENUATION

by Vasile Drăgan

Abstract. Extending and adapting recent constructions in $H^\infty$ control, stabilizing compensators are constructed for time-varying control systems, such that after compensation the operator from perturbations to the quality output is strictly contracting.

## 1. INTRODUCTION. MAIN RESULT.

Consider the system

$$\begin{aligned} x' &= A(t)x + B_1(t)u_1 + B_2(t)u_2 \\ y_1 &= C_1(t)x + D_{11}(t)u_1 + D_{12}(t)u_2 \\ y_2 &= C_2(t)x + D_{21}(t)u_1 \end{aligned} \tag{1.1}$$

where $A:\mathbb{R} \to \mathcal{M}_{n\times n}(\mathbb{R})$, $B_i:\mathbb{R} \to \mathcal{M}_{n\times m_i}$, $C_i:\mathbb{R} \to \mathcal{M}_{p_i\times n}$ $i=1,2$, $D_{ij}:\mathbb{R} \to \mathcal{M}_{p_i\times m_j}$ $i,j=1,2$ (matrices of corresponding dimensions with real entires) are continuous and bounded.

A compensator for (1.1) is a system

$$\begin{aligned} x_c' &= A_c(t)x_c + B_c(t)u_c \\ y_c &= C_c(t)x_c + D_c(t)u_c \end{aligned} \tag{1.2}$$

$A_c, B_c, C_c, D_c$ being continuous and bounded on $\mathbb{R}$. The compensator is coupled to the system (1.1) by taking $u_c = y_2$, $u_2 = y_c$.

A compensator (1.2) is stabilizing for (1.1) if the system obtained by such coupling defines an exponentially stable evolution in the absence of disturbances $u_1$. If the compensator is stabilizing , for every disturbance $u_1$ in $L^2(\mathbb{R},\mathbb{R}^{m_1})$

there exists a unique output $y_1$ in $L^2(\mathbb{R},\mathbb{R}^{p_1})$ and $T_{y_1u_1}$ thus defined is a linear operator. The compensator is $\gamma$-disturbance attenuating if $\| T_{y_1u_1} \| < \gamma$.

In the first part of this study ( [1] ) we have proved that under the structure assumptions : 1) $D_{12}^*(t)D_{12}(t)$ and $D_{21}(t)D_{21}^*(t)$ are invertible with bounded inverses;

2) with

$$\Pi_1(t)=I_{p_1}-D_{12}(t)(D_{12}^*(t)D_{12}(t))^{-1}D_{12}^*(t)$$

$$\Pi_2(t)=I_{m_1}-D_{21}^*(t)(D_{21}(t)D_{21}^*(t))^{-1}D_{21}(t)$$

$(\Pi_1C_1,\ A-B_2(D_{12}^*D_{12})^{-1}D_{12}^*C_1)$ is detectable and $(A-B_1D_{21}^*(D_{21}D_{21}^*)^{-1}C_2,\ B_1\Pi_2)$ is stabilizable, existence of a stabilizing with $\gamma$-disturbance attenuation compensator (1.2) for (1.1) implies existence of positive semidefinite, bounded , stabilizing solutions for three Riccati equations associated to the problem. The main result of this paper consists in the construction of a family of stabilizing compensators with $\gamma$- disturbance attenuation by using these solutions of the Riccati equations.

## 2. SOME PRELIMINARY RESULTS.

Denote $B=(B_1 \quad B_2)$, $C=\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ , $D_1=(D_{11} \quad D_{12})$

$K=D_1^*D_1-\gamma^2\begin{pmatrix} I_{m_1} & 0 \\ 0 & 0 \end{pmatrix}$ .. The first Riccati equation reads

$$R'+A^*(t)R+RA(t)-[RB(t)+C_1^*(t)D_1(t)]K^{-1}(t)[B^*(t)R+D_1^*(t)C_1(t)]+$$
$$+C_1^*(t)C_1(t)=0 \qquad (2.1)$$

and our assumption is existence of a bounded , positive semidefinite solution such that with $F=-K^{-1}(B^*R+D_1^*C_1)$ , $A+BF$ defines an exponentially stable evolution.

Let $F=\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ and associate the system

$$x'=[A(t)+B_1(t)F_1(t)]x+B_1(t)u_1+B_2(t)u_2$$
$$y_1=-D_{12}(t)F_2(t)x+D_{11}(t)u_1+D_{12}(t)u_2 \tag{2.2}$$
$$y_2=[C_2(t)+D_{21}(t)F_1(t)]\,x+D_{21}(t)u_1$$

We have proved in [1] that a stabilizing compensator with $\gamma$-disturbance attenuation for (2.1) is stabilizing with $\gamma$-disturbance attenuation for (2.1). We shall prove now the converse of this result and we shall construct a family of stabilizing compensators with $\gamma$-disturbance attenuation for (2.2). We shall use the following result (see for example [4]).

<u>Proposition 2.1</u> Consider the system

$$x'=\hat{A}(t)x+\hat{B}(t)u$$
$$y=\hat{C}(t)x+\hat{D}(t)u$$

with continuous and bounded on $\mathbb{R}$ coefficients. Assume that $\hat{A}$ defines an exponentially stable evolution. Then the following properties are equivalent

a) The input-output operator $T_{yu}$ from $L^2(\mathbb{R},\mathbb{R}^m)$ to $L^2(\mathbb{R},\mathbb{R}^p)$ has the norm strictly less than $\gamma$.

b) The Riccati equation

$$P' +\hat{A}(t)P+P\hat{A}(t)+[P\hat{B}(t)+\hat{C}^*(t)\hat{D}(t)]\,[\gamma^2 I-\hat{D}^*(t)\hat{D}(t)]^{-1}[\hat{B}_1^*(t)P+$$
$$+\hat{D}^*(t)\hat{C}(t)] +\hat{C}^*(t)\hat{C}(t) =0$$

has a positive definite, bounded on $\mathbb{R}$ solution $\tilde{P}$ such that $\hat{A}-\hat{B}(\gamma^2 I-\hat{D}^*\hat{D})^{-1}(\hat{B}^*\tilde{P}+\hat{D}^*\hat{C})$ defines an exponentially stable evolution.

Let us remark now that if (1.2) is stabilizing with $\gamma$-disturbance attenuation for (1.1), then $x_c'=A_c x_c+\frac{1}{\sqrt{\gamma}} B_c u_c$, $y_c=\frac{1}{\sqrt{\gamma}} C_c x_c + \frac{1}{\gamma} D_c u_c$ is stabilizing with 1-disturbance attenuation (shortly contracting ) for the system

$$x' = Ax + \frac{1}{\sqrt{\gamma}} B_1 u_1 + \sqrt{\gamma} B_2 u_2$$
$$y_1 = \frac{1}{\sqrt{\gamma}} C_1 x + \frac{1}{\sqrt{\gamma}} D_{11} u_1 + D_{12} u_2$$
$$y_2 = \sqrt{\gamma} C_2 x + D_{21} u_1$$

Conversely , if $x_c' = \hat{A}_c x_c + \hat{B}_c u_c$ , $y_c = \hat{C}_c x_c + \hat{D}_c u_c$ id stabilizing and contracting for the above system, then by taking $A_c = \hat{A}_c$ $B_c = \sqrt{\gamma}\, \hat{B}_c$ , $C_c = \sqrt{\gamma}\, \hat{C}_c$ , $D_c = \gamma \hat{D}_c$ we obtain a stabilizing compensator for (1.1) with $\gamma$ -disturbance attenuation. It is why in the following we shall restrict ourselves to the case $\gamma = 1$ .

Proposition 2.2 Consider the system (1.1) and a compensator (1.2) .Assume: 1) A and $A_c$ define exponentially stable evolutions 2) If $\tilde{P}$ is the unique bounded on $\mathbb{R}$ solution to the Liapunov equation $P' + A^* P + C^* C = 0$ , then

$$C^* \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{pmatrix} + \tilde{P} B = 0 \qquad \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{pmatrix}^* \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{pmatrix} = I$$

$((I - D_{12} D_{12}^*) C_1 , A + B_2 B_2^* \tilde{P} )$ is detectable.

3) $D_{21}(t)$ is invertible for all $t \in \mathbb{R}$ and $A - B_1 D_{21}^{-1} C_2$ defines an exponentially stable evolution

4) Let $G_c$ the input-output operator from $L^2(\mathbb{R}, \mathbb{R}^{m_2})$ to $L^2(\mathbb{R}, \mathbb{R}^{m_2})$ associated to the compensator (1.2). Then $\| G_c \| < 1$ .

Then (1.2) stabilizes (1.1) and is contracting .

Proof. Although this result is essentialle contained in [4] we shall scketch here a proof.

a) Apply Proposition 2.1 to (1.2) under assumption 4) to obtain a bounded ,positive semidefinite stabilizing solution $\tilde{Z}$ to the corresponding Riccati equation .The evolution after coupling the compensator id defined by

$$x'=Ax+B_c(C_cx_c+D_cC_2x)$$
$$x_c'=B_cC_2x+A_cx_c \qquad (2.3)$$

Let $\begin{pmatrix} x \\ x_c \end{pmatrix}$ be a solution of this system and let

$$V(t)=x^*(t)\tilde{P}(t)x(t)+x_c^*(t)\tilde{Z}(t)x_c(t)$$

A direct calculation shows that

$$V'(t)=-|C_1(t)x(t)|^2-|C_2(t)x(t)|^2+x^*(t)\tilde{P}(t)B_2(t)[C_c(t)x_c(t)+$$
$$+D_c(t)C_2(t)x(t)]+[C_c(t)x_c(t)+D_c(t)C_2(t)x(t)]^*B_2(t)\tilde{P}(t)x(t)-$$
$$-x_c^*(t)[\tilde{Z}(t)B_c(t)+C_c^*(t)D_c(t)][I-D_c^*(t)D_c(t)]^{-1}[B_c^*(t)\tilde{Z}(t)+$$
$$+D_c^*(t)C_c(t)]x_c(t)-|C_c(t)x_c(t)|^2+x_c^*(t)\tilde{Z}(t)B_c(t)C_2(t)x(t)+$$
$$+x^*(t)C_2^*(t)B_c^*(t)\tilde{Z}(t)x_c(t)$$

The first condition in 2) gives $C_1^*D_{12}=-\tilde{P}B_2$ and we deduce with

$$\tilde{L}_c=-(I-D_c^*D_c)^{-1}(B_c^*\tilde{Z}+D_c^*C_c)$$

$$V'(t)=-|(I-D_{12}D_{12}^*(t))C_1(t)x(t)|^2-|C_c(t)x_c(t)+D_c(t)C_2(t)x(t)-$$
$$-B_2^*(t)\tilde{P}(t)x(t)|^2-|(I-D_c^*(t)D_c(t))^{1/2}[\tilde{L}_c(t)x_c(t)+C_2(t)x(t)]|^2$$

It follows that $t\mapsto V(t)$ is decreasing. We deduce next

$$\int_{t_0}^{T}|(I-D_{12}D_{12}^*)C_1x|^2dt+\int_{t_0}^{T}|C_cx_c+(D_c^*C_2-B_2^*\tilde{P})x|^2dt+$$
$$+\int_{t_0}^{T}|(I-D_c^*D_c)^{1/2}(\tilde{L}_cx_c-C_2x)|^2dt=V(t_0)-V(T)\le V(t_0)\le \qquad (2.4)$$
$$\le c(|x(t_0)|^2+|x_c(t_0)|^2)$$

with $c$ not depending upon $t_0$ and $T$; and we may take $T=\infty$.

The detectability assumption in 3) implies existence of $\tilde{L}$ such that $\hat{A}\overset{def}{=}A+B_2B_2^*\tilde{P}+\tilde{L}(I-D_{12}D_{12}^*)C_1$ defines an exponentially stable evolution. We may write further the representation

$$x(t)=X_{\hat{A}}(t,t_0)x_c+\int_{t_0}^{t}X_{\hat{A}}(t,s)B_2(s)[C_c(s)x_c(s)+(D_c^*(s)C_2(s)-$$
$$-B_2^*(s)\tilde{P}(s)x(s)]ds+\int_{t_0}^{t}X_{\hat{A}}(t,s)\tilde{L}(s)(I-D_{12}(s)D_{12}^*(s))C_1(s)x(s)ds$$

been shown in [1] to be a necessary condition for existence of a stabilizing and contracting compensator).

Take now $T_{22}(t)=[D_{12}^*(t)D_{12}(t)]^{1/2}$

$$T_{21}(t)=[D_{12}^*(t)D_{12}(t)]^{-1/2}D_{12}^*(t)D_{11}(t)$$

and define $T_{11}$ from

$$T_{11}^*(t)T_{11}(t)=I_{m_1}-D_{11}^*(t)\Pi_1(t)D_{11}(t) \quad .$$

Then

$$K(t)=\begin{pmatrix} T_{11}(t) & 0 \\ T_{21}(t) & T_{22}(t) \end{pmatrix}^* \cdot \begin{pmatrix} -I_{m_1} & 0 \\ 0 & I_{m_2} \end{pmatrix} \cdot \begin{pmatrix} T_{11}(t) & 0 \\ T_{21}(t) & T_{22}(t) \end{pmatrix}$$

$$-T_{22}^*\,(T_{21} \quad T_{22})\,F=B_2^*R+D_{12}^*C_1$$

Denote
$$A_F=A+B_2T_{22}^{-1}(T_{21} \quad T_{22})F \qquad (2.6)$$
$$C_{1,F}=C_1+D_{12}T_{22}^{-1}(T_{21} \quad T_{22})F$$

We may write then

$$R'(t)+A_F^*(t)R(t)+R(t)A_F(t)+F_1^*(t)T_{11}^*(t)T_{11}(t)F_1(t)+ C_{1,F}^*(t)C_{1,F}(t)=0 \qquad (2.7)$$

A rather standard Liapunov equation argument, based on (2.7) leads to the fact that $A_F$ defines an exponentially stable evolution ([2]).

We have only to remark that with $\hat{A}=A_F$ , $\hat{C}=\begin{pmatrix} -T_{11}F_1 \\ C_{1,F} \end{pmatrix}$

$\hat{L}=((B_1-B_2 \quad T_{22}^{-1}T_{21})T_{11}^{-1} \qquad 0\ )$ $\hat{A}+\hat{L}\hat{C}=A+BF$ ,hence $(\hat{A},\hat{C})$ is detectable. Consider now the system:

$$x'=(A+B_1F_1)x+B_1T_{11}^{-1}v_1+B_2v_2$$
$$y_1=-T_{22}F_2x+T_{21}T_{11}^{-1}v_1+T_{22}v_2 \qquad (2.2')$$
$$y_2=(C_2+D_{21}F_1)x+D_{21}T_{11}^{-1}v_1$$

Proposition 2.3 The compensator (1.2) is stabilizing and contracting for (2.2) if and only if it is stabilizing and contracting for (2.2').

<u>Proof.</u> We have only to study the input output operator since stabilization is not affected when we replace (2.2) by (2.2′). Take $D_\perp$ such that $D_\perp(t)D_\perp^*(t)=\Pi_1(t)$ to have for $U(t)=(D_\perp(t) \quad D_{12}(t)(D_{12}^*(t)D_{12}(t))^{-1/2})$ $\quad U(t)U^*(t)=I_{p_1}$.

Denote

$$A_o=\begin{pmatrix} A+B_1F_1+B_2D_c(C_2+D_{21}F_1 & B_2C_c \\ B_c(C_2+D_{21}F_1) & A_c \end{pmatrix}$$

$$B_o=\begin{pmatrix} B_1+B_2D_cD_{21} \\ B_cD_{21} \end{pmatrix} \qquad C_o=\quad(-F_2+D_c(C_2+D_{21}F_1) \quad C_c)\ .$$

After coupling (1.2) to (2.2) we obtain

$$\frac{d}{dt}\begin{pmatrix} x \\ x_c \end{pmatrix}=A_o\begin{pmatrix} x \\ x_c \end{pmatrix}+B_ou_1 \qquad y_1=D_{12}C_o\begin{pmatrix} x \\ x_c \end{pmatrix}+(D_{11}+D_{12}D_cD_{21})\,u_1 \quad (2.8)$$

and after coupling (1.2) to (2.2′) we obtain

$$\frac{d}{dt}\begin{pmatrix} x \\ x_c \end{pmatrix}=A_o\begin{pmatrix} x \\ x_c \end{pmatrix}+B_oT_{11}^{-1}v_1$$

$$\tilde{y}_1=T_{22}C_o\begin{pmatrix} x \\ x_c \end{pmatrix}+(T_{21}+T_{22}D_cD_{21})T_{11}^{-1}v_1 \qquad (2.8')$$

Denote by $G$ the input-output operator associated to (2.8) and by $\tilde{G}$ the input-output operator associated to (2.8′).

A direct calculations shows that

$$G=\ U\begin{pmatrix} D_\perp^*D_{11}\ T_{11}^{-1} \\ \tilde{G} \end{pmatrix}\cdot T_{11} \qquad (2.9)$$

where $U, D_\perp^*D_{11}T_{11}^{-1}$ , $T_{11}$ are operators defined by multiplication with corresponding matrices. We have

$\|y_1\|^2=\langle Gu_1,Gu_1\rangle=\langle u_1,G^*Gu_1\rangle$ and by (2.9) , using the fact that $U$ is isometric we deduce

$$\|y_1\|^2=\langle u_1,D_{11}^*D_\perp D_\perp^*D_{11}u_1\rangle+\langle u_1,T_{11}^*\tilde{G}^*\tilde{G}\,T_{11}u_1\rangle =$$

$$=\langle u_1,D_{11}^*\Pi_1D_{11}u_1\rangle+\langle u_1,T_{11}^*\tilde{G}^*\tilde{G}T_{11}u_1\rangle$$

and taking into account the definition of $T_{11}$

$$\|y_1\|^2 = \langle u_1, u_1 \rangle - \langle u_1, T_{11}^*(I-\tilde{G}^*\tilde{G})T_{11}u_1 \rangle$$

and finally $\langle u_1, (I-G^*G)u_1 \rangle = \langle u_1 T_{11}^*(I-\tilde{G}^*\tilde{G})T_{11}u_1 \rangle$ .

From here it follows that $\|G\|<1$ iff $\|\tilde{G}\|<1$ and the proposition id proved.

We are now in position to prove the main result of this section

<u>Theorem 2.1</u> If (1.2) is stabilizing and contracting for (2.2) then it is stabilizing and contracting for (1.1).

<u>Proof.</u> From Proposition 2.3 we may assume that (1.2) is stabilizing and contracting for (2.2′). With notations (2.6) consider the system

$$x' = A_F x + (B_1 - B_2 T_{22}^{-1} T_{21}) u_1 + B_2 T_{22}^{-1} \tilde{u}_2$$

$$y_1 = C_{1,F} x + \Pi_1 D_{11} u_1 + D_{12} T_{22}^{-1} \tilde{u}_2 \qquad (2.10)$$

$$\tilde{y}_2 = -T_{11} F_1 x + T_{11} u_1$$

By coupling (1.2) to (2.2′) we have seen that we get (2.8′).

If we take $v_1 = \tilde{y}_2$ and $\tilde{u}_2 = \tilde{y}_1$ we may see that when coupling (2.8′) to (2.10) we obtain the same input-output operator as by coupling (1.2) to (1.1). The conclusion of Theorem 2.1 is obtained by applying Proposition 2.2 to systems(2.10) and (2.8′).

In fact if (1.2) is stabilizing and contracting for (2.2′) we have (2.8′) with exponentially stable evolution and contracting input-output operator properties required for the compensator in Proposition 2.2.

A direct checking shows that the corresponding $D_{21}$ is in (2.10) $T_{11}$ and is invertible; the corresponding matrix for condition 3) in Proposition 2.2 is

$$A_F + (B_1 - B_2 T_{22}^{-1} T_{21}) T_{11}^{-1} T_{11} F_1 = A_F + B_1 F_1 - B_2 T_{22}^{-1} T_{21} F_1 = A + B_2 T_{22}^{-1} T_{21} F_1 +$$

$$+ B_2 F_2 + B_1 F_1 - B_2 T_{22}^{-1} T_{21} F_1 = A + BF$$ and defines an exponentially stable evolution. Condition 1) is also clearly satisfied .

The Liapunov equation in 2) reads

$$P'+A_F^*P+PA_F+C_{1,F}^*C_{1,F}'+F_1^*T_{11}^*T_{11}F_1=0$$

and we see that $\widetilde{P}=R$ ; we have further

$$(C_{1,F}^* \quad -F_1^*T_{11}^*)\begin{pmatrix} \Pi_1 D_{11} & D_{12}T_{22}^{-1} \\ T_{11} & 0 \end{pmatrix} +R(B_1-B_2T_{22}^{-1}T_{21} \quad B_2T_{22}^{-1}) = (C_{1,F}^*\Pi_1 D_{11}-F_1^*T_{11}^*T_{11}+R(B_1-B_2T_{22}^{-1}T_{21}) \quad C_{1,F}^*D_{12}T_{22}^{-1}+$$

$$+RB_2T_{22}^{-1})$$ . We had $RB_2+C_1^*D_{12}+F^*\begin{pmatrix} T_{21}^* \\ T_{22}^* \end{pmatrix} T_{22}=0$ and

$$C_{1,F}^*=C_1^*+F^*\begin{pmatrix} T_{21}^* \\ T_{22}^* \end{pmatrix} T_{22}^{-1}D_{12}^* ,$$

$$C_{1,F}^*D_{12}=C_1^*D_{12}+F^*\begin{pmatrix} T_{21}^* \\ T_{22}^* \end{pmatrix} T_{22}^{-1}D_{12}^*D_{12}=-RB_2 \text{ , hence}$$

$$C_{1,F}^*D_{12}T_{22}^{-1}+RB_2T_{22}^{-1}=0$$

$$\Pi_1 D_{11}=D_{11}-D_{12}(D_{12}^*D_{12})^{-1}D_{12}^*D_{11}=D_{11}-D_{12}T_{22}^{-2}T_{22}T_{21}=$$

$$=D_{11}-D_{12}T_{22}^{-1}T_{21}$$ hence

$$C_{1,F}^*\Pi_1 D_{11}-RB_2T_{22}^{-1}T_{21}=C_{1,F}^*D_{11}-(C_{1,F}^*D_{12}+RB_2)T_{22}^{-1}T_{21}=C_{1,F}^*D_{11}$$

$$C_{1,F}^*D_{11}=C_1^*D_{11}+F^*\begin{pmatrix} T_{21}^* \\ T_{22} \end{pmatrix} T_{22}^{-1}D_{12}^*D_{11}=C_1^*D_{11}+F^*\begin{pmatrix} T_{21}^* \\ T_{22} \end{pmatrix} T_{21} =$$

$$=C_1^*D_{11}+(F_1^*T_{21}^*+F_2^*T_{22})T_{21}=F_1^*T_{11}^*T_{11}-RB_1$$

and the first part of condition 2) is checked.

What concerns the detectability , let us see that we have to consider the matrices $A_F+B_2T_{22}^{-2}B_2^*R= A+B_2T_{22}^{-1}T_{21}F_1+B_2F_2+$
$+B_2(D_{12}^*D_{12})^{-1}B_2^*R=A+B_2F_2+B_2T_{22}^{-1}T_{21}F_1-B_2T_{22}^{-2}(T_{22}T_{21}F_1+T_{22}^2F_2+$
$+D_{12}^*C_1)=A-B_2(D_{12}^*D_{22})^{-1}D_{12}^*C_1$ and

$$(I-D_{12}T_{22}^{-1}T_{22}^{-1}D_{12}^*)C_{1,F}=(I-D_{12}(D_{12}^*D_{12})^{-1}D_{12}^*)C_{1,F}=$$

$$=\Pi_1 C_{1,F}=\Pi_1 C_1+\Pi_1 D_{12}T_{22}^{-1}(T_{21} \quad T_{22})F=\Pi_1 C_1$$

since $\Pi_1 D_{12}=0$ and the detectability condition is the one in the statement. It remains to check condition 2)

$$\begin{pmatrix} D_{11}^{*}\Pi_1 & T_{11}^{*} \\ T_{22}^{-1}D_{12} & 0 \end{pmatrix}\begin{pmatrix} \Pi_1 D_{11} & D_{12}T_{22}^{-1} \\ T_{11} & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} D_{11}^{*}\Pi_1 D_{11}+T_{11}^{*}T_{11} & 0 \\ 0 & T_{22}^{-1}D_{12}^{*}D_{12}T_{22}^{-1} \end{pmatrix}$$

But $T_{11}^{*}T_{11}=I-D_{11}^{*}\Pi_1 D_{11}$ and the condition is verified.

Let us check the system obtained by coupling (2.8′) to (2.10)

$$\frac{d}{dt}\begin{pmatrix} x \\ x_c \end{pmatrix} = A_o\begin{pmatrix} x \\ x_c \end{pmatrix} + B_o T_{11}^{-1}(-T_{11}F_1\tilde{x}+T_{11}u_1)$$

$$\frac{d}{dt}\tilde{x} = A_F\tilde{x}+(B_1-B_2T_{22}^{-1}T_{21})u_1+B_2T_{22}^{-1}\Big[T_{22}C_o\begin{pmatrix} x \\ x_c \end{pmatrix}+(T_{21}+$$

$$+T_{22}D_cD_{21})T_{11}^{-1}(-T_{11}F_1\tilde{x}+T_{11}u_1)\Big]$$

$$y_1=C_{1,F}\tilde{x}+\Pi_1D_{11}u_1+D_{12}T_{22}^{-1}\Big[T_{22}C_o\begin{pmatrix} x \\ x_c \end{pmatrix}+(T_{21}+T_{22}D_cD_{21})T_{11}^{-1}$$

$$(-T_{11}F_1\tilde{x}+T_{11}u_1)\Big]$$

$$\frac{d}{dt}\begin{pmatrix} x \\ x_c \end{pmatrix} = \begin{pmatrix} A+B_1F_1+B_2D_c(C_2+D_{21}F_1) & B_2C_c \\ B_c(C_2+D_{21}F_1) & A_c \end{pmatrix}\begin{pmatrix} x \\ x_c \end{pmatrix} -$$

$$-\begin{pmatrix} B_1+B_2D_cD_{21} \\ B_cD_{21} \end{pmatrix}\left(F_1\tilde{x}-u_1\right) = \begin{pmatrix} (A+B_2D_cC_2)x+B_2C_cx_c \\ B_cC_2x+A_cx_c \end{pmatrix} +$$

$$+\begin{pmatrix} (B_1+B_2D_cD_{21})F_1(x-\tilde{x}) \\ B_cD_1F_1(x-\tilde{x}) \end{pmatrix} + \begin{pmatrix} (B_1+B_2D_cD_{21})u_1 \\ B_cD_{21}u_1 \end{pmatrix}$$

$$\frac{d\tilde{x}}{dt} = \left[A+B_2(T_{22}^{-1}T_{21}F_1+F_2)\right]\tilde{x}+B_2(-F_2x+D_c(C_2+D_{21}F_1)x+C_cx_c\Big]-$$

$$-B_2T_{22}^{-1}T_{21}(F_1\tilde{x}-u_1)-B_2D_cD_{21}(F_1\tilde{x}-u_1)+B_1u_1-B_2T_{22}^{-1}T_{21}u_1 =$$

$$=A\tilde{x}+B_2F_2(\tilde{x}-x)+B_2D_cC_2x+B_2D_cD_{21}F_1(x-\tilde{x})+B_2C_cx_c+$$

$$+B_2D_cD_{21}u_1+B_1u_1$$

$$\frac{d\tilde{x}}{dt} = A\tilde{x}+B_2C_cx_c+B_2D_cC_2x+(B_2F_2-B_2D_cD_{21}F_1)(\tilde{x}-x)+B_2D_cD_{21}u_1+B_1u_1$$

and $\frac{dx}{dt} = Ax+B_2D_cC_2x+B_2C_cx_c+(B_1+B_2D_cD_{21})F_1(x-\tilde{x})+B_1u_1+B_cD_cD_{21}u_1$

We deduce that $\frac{d}{dt}(\tilde{x}-x)=A(\tilde{x}-x)+B_2F_2(\tilde{x}-x)+B_1F_1(\tilde{x}-x)$ and since $A+BF$ defines an exponentially stable evolution. We deduce that $\tilde{x}=x$ and the evolution for $\begin{pmatrix} x \\ x_c \end{pmatrix}$ is the same as when coupling (1.2) to (1.1) . We see further that

$$y_1=C_1x+D_{12}T_{22}^{-1}T_{21}F_1x+D_{12}F_2x+\Pi_1D_{11}u_1+D_{12}(-F_2x+D_cC_2x+D_cD_{21}F_1x+$$
$$+C_cx_c)-D_{12}(T_{22}^{-1}T_{21}+D_cD_{21})(F_1x-u_1)=D_{12}D_cD_{21}u_1+D_{12}T_{22}^{-1}T_{21}u_1+$$
$$+(C_1+D_{12}D_cC_2)x+\Pi_1D_{11}u_1$$

$$\Pi_1D_{11}+D_{12}T_{22}^{-1}T_{21}=D_{11}-D_{12}(D_{12}^*D_{12})^{-1}D_{12}^*D_{11}+D_{12}(D_{12}^*D_{12})^{-1}D_{12}^*D_{11}=D_{11}$$

and finally $y_1=(C_1+D_{12}D_cC_2)x+(D_{11}+D_{12}D_cD_{21})u_1$ and the claim is proved, the result of the coupling of (2.8′) to (2.10) is the same as when coupling (1.2) to (1.1). The theorem is completely proved.

## 3. THE CASE $B_1\Pi_2=0$

A class of systems for which this structure holds is of course the one for which $D_{21}$ is invertible since for such systems $\Pi_2=0$ . For such systems the stabilizybility assumption implies that $A-B_1D_{21}^*(D_{21}D_{21}^*)^{-1}C_2$ defines an exponentially stable evolution.

Let us recall now that we had $D_2=\begin{pmatrix} D_{11} \\ D_{21} \end{pmatrix}$, $\hat{K}=D_2D_2^*-\mu^2\begin{pmatrix} I_{p_1} & 0 \\ 0 & 0 \end{pmatrix}$

$\Pi_2=I_{m_1}-D_{21}^*(D_{21}D_{21}^*)^{-1}D_{21}$ and $I_{p_1}-D_{11}\Pi_2D_{11}^* \geq \nu\, I_{p_1}$ .

Let us compute $\hat{K}^{-1}D_2B_1^*x = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ .

We have

$$(D_{11}D_{11}^*-I_{p_1})u_1+D_{11}D_{21}^*u_2=D_{11}B_1^*x$$
$$D_{21}D_{11}^*u_1+D_{21}D_{21}^*u_2=D_{21}B_1^*x$$

Since $D_{21}D_{21}^*$ is invertible, we deduce that

$$u_2=(D_{21}D_{21}^*)^{-1}D_{21}B_1^*x-(D_{21}D_{21}^*)^{-1}D_{21}D_{11}^*u_1$$

and the first relation leads to

$$(D_{11}D_{11}^*-I_{p_1})u_1-D_{11}D_{21}^*(D_{21}D_{21}^*)^{-1}D_{21}D_{11}^*u_1=D_{11}B_1^*x -$$

$$-D_{11}D_{21}^*(D_{21}D_{21}^*)^{-1}D_{21}B_1^*x=D_{11}\Pi_2B_1^*x=0$$

that is $(D_{11}\Pi_2D_{11}^*-I_{p_1})u_1=0$ hence $u_1=0$, $u_2=(D_{21}D_{21}^*)^{-1}D_{21}B_1^*x$

We deduce that

$$\hat{K}^{-1}D_2B_1^*=\begin{pmatrix}0\\(D_{21}D_{21}^*)^{-1}D_{21}B_1^*\end{pmatrix}$$

The second equation reads

$$S'=AS+SA^*-SC^*\hat{K}^{-1}C\,S-B_1D_2^*\hat{K}^{-1}CS-SC^*\hat{K}^{-1}D_2B_1^*-B_1D_2^*\hat{K}^{-1}D_2B_1^*+B_1B_1^*$$

we have $C^*\hat{K}^{-1}D_2B_1^*=(C_1^* \quad C_2)\begin{pmatrix}0\\(D_{21}D_{21}^*)^{-1}D_{21}B_1^*\end{pmatrix}=$

$$=C_2(D_{21}D_{21}^*)^{-1}D_{21}B_1^*$$

$$B_1D_2^*\hat{K}^{-1}D_2B_1^*=(B_1D_{11}^* \quad B_1D_{21}^*)\begin{pmatrix}0\\(D_{21}D_{21}^*)^{-1}D_{21}B_1^*\end{pmatrix}=$$

$$=B_1D_{21}^*(D_{21}D_{21}^*)^{-1}D_{21}B_1^*$$

and the Riccati equation is

$$S'=[A-B_1D_{21}^*(D_{21}D_{21}^*)^{-1}C_2^*]S-S[A^*-C_2(D_{21}D_{21}^*)^{-1}D_{21}B_1^*] -$$

$$-SC^*\hat{K}^{-1}CS+B_1[I-D_{21}^*(D_{21}D_{21}^*)^{-1}D_{21}]B_1$$

and if $\Pi_2B_1^*=0$ we deduce that the stabilizing solution is $S=0$.

We shall perform now some algebraic manipulations. Take $\tilde{D}_\perp(t)$ a $(m_1-p_2)\times m_1$ matrix such that

$$V(t)=\begin{pmatrix}\tilde{D}_\perp(t)\\(D_{21}(t)D_{21}^*(t))^{-1/2}D_{21}(t)\end{pmatrix}$$

is unitary. Recall also that we defined another unitary matrix $U(t)$. Let us write

$$U^*(t)D_{11}(t)V^*(t)=\begin{pmatrix}D_{11}^{11}(t) & D_{11}^{12}(t)\\D_{11}^{21}(t) & D_{11}^{22}(t)\end{pmatrix}$$

<u>Lemma 3.1</u> We have $V(t)T_{11}(t)V^*(t)=\begin{pmatrix} T^{11}_{11}(t) & T^{12}_{11}(t) \\ 0 & T^{22}_{11}(t)\end{pmatrix}$

with $T^{11}_{11}(t)$ and $T^{22}_{11}(t)$ invertible.

<u>Proof.</u> Since $D_\perp(t)D_\perp^*(t)=\Pi_1(t)$ , $U(t)=(D_\perp(t) \quad D_{12}(t)\cdot (D^*_{12}(t)D_{12}(t))^{-1/2})$ we have

$$V(t)D^*_{11}(t)\mathcal{U}(t)=V(t)(D^*_{11}(t)D_\perp(t) \quad D^*_{11}(t)D_{12}(t)(D^*_{12}(t)D_{12}(t))^{-1/2})=$$

$$=\begin{pmatrix} (D^{11}_{11}(t))^* & (D^{21}_{11}(t))^* \\ (D^{12}_{11}(t))^* & (D^{22}_{11}(t))^* \end{pmatrix}$$

$$V(t)[I_{m_1}-D^*_{11}(t)\Pi_1(t)D_{11}(t)]V^*(t)=V(t)V^*(t)-V(t)D^*_{11}(t)D_\perp(t)\cdot$$

$$\cdot D^*_\perp(t)D_{11}(t)V^*(t) = \begin{pmatrix} I_{m_1-p_2} & 0 \\ 0 & I_{p_2}\end{pmatrix} - \begin{pmatrix} (D^{11}_{11}(t))^* \\ (D^{12}_{11}(t))^* \end{pmatrix}\cdot$$

$$(D^{11}_{11}(t) \quad D^{12}_{11}(t))=\begin{pmatrix} I_{m_1-p_2}-(D^{11}_{11}(t))^*D^{11}_{11}(t) & -(D^{11}_{11}(t))^*D^{12}_{11}(t) \\ -(D^{12}_{11}(t))^*D^{11}_{11}(t) & I_{p_2}-(D^{12}_{11}(t))^*D^{12}_{11}(t)\end{pmatrix}$$

Since $I_{m_1}-D^*_{11}(t)\Pi_1(t)D_{11}(t)\geq \nu I_{m_1}>0$ we have

$I_{m_1-p_2}-(D^{11}_{11}(t))^*D^{11}_{11}(t)\geqslant \nu_1 I_{m_1-p_2}$ and also

$$I_{p_2}-(D^{12}_{11}(t))^*D^{12}_{11}(t)-(D^{12}_{11}(t))^*D^{11}_{11}(t)[I_{m_1-p_2}-(D^{11}_{11}(t))^*D^{11}_{11}(t)]^{-1}$$
$$(D^{11}_{11}(t))^*D^{12}_{11}(t)\geqslant \nu_2 I_{p_2}$$

Recall also that

$T^*_{11}(t)T_{11}(t)=I_{m_1}-D^*_{11}(t)\Pi_1(t)D_{11}(t)$, hence

$$V(t)[I_{m_1}-D^*_{11}(t)\Pi_1(t)D_{11}(t)]V^*(t)=V(t)T^*_{11}(t)T_{11}(t)V^*(t)=$$
$$=V(t)T^*_{11}(t)V^*(t)V(t)T_{11}(t)V^*(t)$$

If we write

$V(t)T_{11}(t)V^*(t)=\begin{pmatrix} T^{11}_{11}(t) & T^{12}_{11}(t) \\ 0 & T^{22}_{11}(t)\end{pmatrix}$ we must compute

$T^{ij}_{11}$ such that $\begin{pmatrix} (T^{11}_{11}(t))^* & 0 \\ (T^{12}_{11}(t))^* & (T^{22}_{11}(t))^*\end{pmatrix}\begin{pmatrix} T^{11}_{11}(t) & T^{12}_{11}(t) \\ 0 & T^{22}_{11}(t)\end{pmatrix}=$

$$=\begin{pmatrix} I_{m_1-p_2}-(D_{11}^{11}(t))^* D_{11}^{11}(t) & -(D_{11}^{11}(t))^* D_{11}^{12}(t) \\ -(D_{11}^{12}(t))^* D_{11}^{11}(t) & I_{p_2}-(D_{11}^{12}(t))^* D_{11}^{12}(t) \end{pmatrix}$$

Reading to
$$(T_{11}^{11}(t))^* T_{11}^{11}(t)=I_{m_1-p_2}-(D_{11}^{11}(t))^* D_{11}^{11}(t)$$
$$(T_{11}^{11}(t))^* T_{11}^{12}(t) =-(D_{11}^{11}(t))^* D_{11}^{12}(t) \tag{3.1}$$
$$(T_{11}^{12}(t))^* T_{11}^{12}(t)+(T_{11}^{22}(t))^* T_{11}^{22}(t)=I_{p_2}- -(D_{11}^{12}(t))^* D_{11}^{12}(t)$$

For the first equation in (3.1) we may take $T_{11}=[ I_{m_1-p_2}- -(D_{11}^{11})^* D_{11}^{11} ]^{1/2}$ and $T_{11}^{11}$ is invertible ; the second equation gives $T_{11}^{12}$ and the last eqmation allows to define $T_{11}^{22}$ ,invertible. Recall that

$$I=V^*V=(\tilde{D}_\perp^* \quad D_{21}^*(D_{21}D_{21}^*)^{-1/2}) \begin{pmatrix} \tilde{D}_\perp \\ (D_{21}D_{21}^*)^{-1/2}D_{21} \end{pmatrix} =$$

$$=\tilde{D}_\perp^* \tilde{D}_\perp +D_{21}^*(D_{21}D_{21}^*)^{-1}D_{21} \quad \text{hence } \tilde{D}_\perp^*\tilde{D}_\perp =I-D_{21}^*(D_{21}D_{21}^*)^{-1}D_{21}=\Pi_2$$

and $\quad U^*(I_{p_1}-D_{11}\Pi_2 D_{11}^*)U=I-U^*D_{11}\tilde{D}_\perp^*\tilde{D}_\perp D_{11}^*U$ .

On the other hand $U^*D_{11}V^*=U^*D_{11}(\tilde{D}_\perp^* \quad D_{21}^*(D_{21}D_{21}^*)^{-1/2})$ hence

$U^*D_{11}\tilde{D}_\perp^* = \begin{pmatrix} D_{11}^{11} \\ D_{11}^{21} \end{pmatrix}$ and we deduce that

$$U^*(I-D_{11}\Pi_2 D_{11}^*)U = \begin{pmatrix} I_{p_1-m_2}-D_{11}^{11}(D_{11}^{11})^* & -D_{11}^{11}(D_{11}^{21})^* \\ -D_{11}^{21}(D_{11}^{11})^* & I_{m_2}-D_{11}^{21}(D_{11}^{21})^* \end{pmatrix}$$

The condition $I_{p_1}-D_{11}\Pi_2 D_{11}^* > \gamma I_{p_1}$ is equivalent to

$I_{p_1-m_2}-D_{11}^{11}(D_{11}^{11})^* > \gamma_1 I_{p_1-m_2}$ , $I_{m_2}-D_{11}^{21}(D_{11}^{21})^*-D_{11}^{21}(D_{11}^{11})^*[I_{p_1-m_2}- -D_{11}^{11}(D_{11}^{11})^*]^{-1}D_{11}^{11}(D_{11}^{21})^* > \gamma_2 I_{m_2}$

Use all these formulae to write

$$I_{p_2}-(D_{11}^{12})^* D_{11}^{12} -(D_{11}^{12})^* D_{11}^{11}[I_{m_1-p_2}-(D_{11}^{11})^* D_{11}^{11}]^{-1}(D_{11}^{11})^* D_{11}^{12}=$$
$$=I_{p_2}-(D_{11}^{12})^*\{ I_{p_1-m_2}+D_{11}^{11}[I_{m_1-p_2}-(D_{11}^{11})^* D_{11}^{11}]^{-1}(D_{11}^{11})^*\}D_{11}^{12} =$$
$$=I_{p_2}-(D_{11}^{12})^*[I_{p_1-m_2}-D_{11}^{11}(D_{11}^{11})^*]^{-1}D_{11}^{12} =I_{p_2}-(D_{21}D_{21}^*)^{-1/2}D_{21}D_{11}^*\Pi_1(I_{p_1}-$$

$-D_{11}\Pi_2 D_{11}^*)^{-1}\Pi_1 D_{11}D_{21}^*(D_{21}D_{21}^*)^{-1/2} \overset{\text{def}}{=} \hat{D}_{21}^*\hat{D}_{21}$

$$I_{m_2}-D_{11}^{21}(D_{11}^{21})^* - D_{11}^{21}(D_{11}^{11})^*[I_{p_1-m_2}-D_{11}^{11}(D_{11}^{11})^*]^{-1}D_{11}^{11}(D_{11}^{21})^* =$$

$$= I_{m_2}-(D_{12}^*D_{12})^{-1/2}D_{12}^*D_{11}\Pi_2[I_{m_1}-D_{11}^*\Pi_1 D_{11}]^{-1}\Pi_2 D_{11}^* D_{12}(D_{12}^*D_{12})^{-1/2} =$$

$$= \hat{D}_{12}\hat{D}_{12}^*$$

<u>Lemma 3.2</u> Consider the system

$$x_\ell' = A_\ell x_\ell + B_\ell u_\ell$$
$$y_\ell = C_\ell x_\ell + D_\ell u_\ell$$

and assume: $\alpha$) $A_\ell$ defines an exponentially stable evolution $\beta$) $\| G_\ell \| < 1$, where $G_\ell$ is the corresponding input-output operator. Asociate the system

$$x_\ell' = A_\ell x_\ell + (0 \quad B_\ell) V \tilde{u}_\ell$$
$$y_\ell = \hat{D}_{12}C_\ell x_\ell + (D_{11}^{21}(T_{11}^{11})^{-1} \quad \hat{D}_{12}D_\ell) V \tilde{u}_\ell .$$

Then the corresponding input-output operator $\tilde{G}_\ell$ satisfies $\|\tilde{G}_\ell\| < 1$.

<u>Proof-</u> We have $\|\tilde{G}_\ell\| = \|\tilde{G}_\ell^*\|$ and $\tilde{G}_\ell^*$ is defined by

$$y(t) = V^*(t)\begin{pmatrix} [(T_{11}^{11}(t))^*]^{-1}(D_{11}^{21})^*(t) \\ D_\ell^*(t)\hat{D}_{12}^*(t) \end{pmatrix} u(t) - V^*(t)\begin{pmatrix} 0 \\ B_\ell^*(t) \end{pmatrix} \cdot \int_t^\infty X_{A_\ell}^*(s,t)C_\ell^*(s)\hat{D}_{12}^*(s)u(s)\,ds$$

$$\|y\|^2 = \int_{-\infty}^\infty |y(t)|^2 dt = \int_{-\infty}^\infty u^*(t)D_{11}^{21}(t)(T_{11}^{11}(t))^{-1}[(T_{11}^{11}(t))^*]^{-1}(D_{11}^{21})^* u(t)dt + \int_{-\infty}^\infty | D_\ell^*(t)\hat{D}_{12}^*(t)u(t) - B_\ell^*(t)\int_t^\infty X_{A_\ell}^*(s,t)C_\ell^*(s)\hat{D}_{12}^*(s)u(s)ds|^2 dt = \langle u, D_{11}^{21}[(T_{11}^{11})^* T_{11}^{11}]^{-1}(D_{11}^{21})^* u\rangle +$$

$$+\langle \hat{D}_{12}u, G_\ell G_\ell^* \hat{D}_{12}u\rangle .$$ Since

$I-D_{11}^{21}[(T_{11}^{11})^* T_{11}^{11}]^{-1}(D_{11}^{21})^* = \hat{D}_{12}\hat{D}_{12}^*$ we have further

$$\|y\|^2 = \langle u,u\rangle + \langle \hat{D}_{12}^*u, (G_\ell G_\ell^* - I)\hat{D}_{12}^*u\rangle \leq \|u\|^2 + (\|G_\ell\|^2 - 1)\|\hat{D}_{12}^*u\|^2 \leq$$

$$\leq [1-(1-\|G_\ell\|^2)\mu]\|u\|^2$$ where $\mu I_{m_2} \leq \hat{D}_{12}\hat{D}_{12}^*$

We shall use the lemma above to prove

<u>Proposition 3.1</u> Assume that a) $D_{12}^{*}D_{12}$ , $D_{21}D_{21}^{*}$ are invertible with bounded inverses b) $D_{11}^{*}\Pi_1 D_{11} \leq r I_{m_1} < I_{m_1}$ $D_{11}\Pi_2 D_{11}^{*} \leq rI_{p_1} < I_{p_1}$ for all $t \in \mathbb{R}$ c) $(\Pi_1 C_1, A-B_2(D_{12}^{*}D_{12})^{-1}D_{12}^{*}C_1)$ is detectable d) The Riccati eqaution $R'+A^{*}R+RA-(RB+C_1^{*}D_1)K^{-1}(B^{*}R+D_1^{*}C_1)+C_1^{*}C_1=0$ has a bounded on $\mathbb{R}$, positive semidefinite, stabilizing solution.

Consider the system family

$$x'=Ax+B_1u_1+B_2u_2$$

$$y_1=C_1x+D_{11}u_1+D_{12}u_2$$

For this system a family of stabilizing compensators with disturbance attenuation (contracting) is

$$x_\ell'=A_\ell x_\ell+(0 \quad B_\ell)V u_\ell$$

$$y_\ell=\hat{D}_{12}C_\ell x_\ell+(D_{11}^{21}(T_{11}^{11})^{-1} \quad \hat{D}_{12}D_\ell)V u_\ell$$

$$u_\ell=T_{11}(u_1-F_1x)$$

$$u_2=T_{22}^{-1}y_\ell-T_{22}^{-1}T_{21}u_1+T_{22}^{-1}(T_{21} \ldots T_{22})Fx$$

where $A_\ell$, $B_\ell$, $C_\ell$, $D_\ell$ are arbitrary, with $A_\ell$ defining an exponentially stable evolution and strictly contracting input-output operator.

<u>Proof.</u> After coupling the compensator one has

$$x'=A_Fx+(B_1-B_2T_{22}^{-1}T_{21})u_1+B_2T_{22}^{-1}y_\ell$$

$$y_1=C_Fx+\Pi_1D_{11}u_1+D_{12}T_{22}^{-1}y_\ell$$

$$y_2=-T_{11}F_1x+T_{11}u_1$$

$$x_\ell'=A_\ell x_\ell+(0 \quad B_\ell)V y_2$$

$$y_\ell=\hat{D}_{12}C_\ell x_\ell+(D_{11}^{21}(T_{11}^{11})^{-1} \quad \hat{D}_{12}D_\ell)Vy_2$$

and Proposition 2.2 and Lemma 3.2 are applied.

In fact we have seen that the last relations correspond to exponentially stable evolution and strictly contracting input-output operator, while the first relations corresponding to a systems satisfying the conditions in Proposition 2.2. In this way we have only to check the claim concerning the result of coupling and we see it directly.

<u>Remark.</u> If we denote $Vu_1=\begin{pmatrix} u_1^1 \\ u_1^2 \end{pmatrix}$ then $u_2$ above does not depend upon $u_1^1$. We have indeed $u_2=T_{22}^{-1}\hat{D}_{12}C_\ell x_\ell +$

$+T_{22}^{-1}(D_{11}^{21}(T_{11}^{11})^{-1} \quad \hat{D}_{12}D_\ell )VT_{11}V^*\begin{pmatrix} u_1^1-F_1^1x \\ u_1^2-F_1^2x \end{pmatrix} -T_{22}^{-1}T_{21}V^*\begin{pmatrix} u_1^1-F_1^1x \\ u_1^2-F_1^2x \end{pmatrix}+$

$+T_{22}^{-1}(T_{21} \quad T_{22})Fx$. On the other hand $T_{21}V^*=$

$=(D_{12}^*D_{12})^{-1/2}D_{12}^*D_{11}(\tilde{D}_\perp^* \; D_{21}^*(D_{21}D_{21}^*)^{-1/2})=(D_{11}^{21} \quad D_{11}^{22})$

and we deduce $u_2=T_{22}^{-1}\hat{D}_{12}C_\ell x_\ell+T_{22}^{-1}(\hat{D}_{11}+\hat{D}_{12}D_\ell\hat{D}_{21})(u_1^2-F_1^2x)+F_2x$

where $\hat{D}_{11}=D_{11}^{21}(T_{11}^{11})^{-1}T_{11}^{12}-D_{11}^{22}=-D_{11}^{22}(I-(D_{11}^{11})^*D_{11}^{11})^{-1}(D_{11}^{11})^*D_{11}^{11}-$

$-D_{11}^{22}=-(D_{12}^*D_{12})^{-1/2}D_{12}^*D_{11}[I+\Pi_2(I-D_{11}^*\Pi_1D_{11})^{-1}\Pi_2D_{11}^*\Pi_1D_{11}]\cdot$

$D_{21}^*(D_{21}D_{21}^*)^{-1/2}$.

Since $y_2=C_2x+D_{21}u_1$ we deduce $u_1^2=-(D_{21}D_{21}^*)^{-1/2}C_2x+(D_{21}D_{21}^*)^{-1/2}y_2$

and since $F_1^2=(D_{21}D_{21}^*)^{-1/2}D_{21}F_1$ we shall have

$u_2=T_{22}^{-1}\hat{D}_{12}C_\ell x_\ell-T_{22}^{-1}(\hat{D}_{11}+\hat{D}_{12}D_\ell\hat{D}_{21})(D_{21}D_{21}^*)^{-1/2}(C_2+D_{21}F_1)x+F_2x+$

$+T_{22}^{-1}(\hat{D}_{11}+\hat{D}_{12}D_\ell\hat{D}_{21})(D_{21}D_{21}^*)^{-1/2}y_2$

We have also $x_\ell'=A_\ell x_\ell-B_\ell\hat{D}_{21}(D_{21}D_{21}^*)^{-1/2}(C_2+D_{21}F_1)x$

The main result in this section is

<u>Theorem 3.1</u> Add to the assumptions in Proposition 3.1 the ones we mentioned at the start of the section. Then a family of stabilizing and contracting compensators is

$\hat{x}'=[A-B_1D_{21}^*(D_{21}D_{21}^*)^{-1}C_2+B_2F_2-B_2T_{22}^{-1}(\hat{D}_{11}+\hat{D}_{12}D_\ell\hat{D}_{21})(D_{21}D_{21}^*)^{-1/2}(C_2+$

$+D_{21}F_1)]\hat{x}+B_2T_{22}^{-1}\hat{D}_{12}C_\ell x_\ell+[B_1D_{21}^*(D_{21}D_{21}^*)^{-1}+B_2T_{22}^{-1}(\hat{D}_{11}+$

$+\hat{D}_{12}D_\ell\hat{D}_{21})(D_{21}D_{21}^*)^{-1/2}]y_2$

$$x_\ell' = -B_\ell \hat{D}_{21}(D_{21}D_{21}^*)^{-1/2}(C_2+D_{21}F_1)\hat{x} + A_\ell x_\ell + B_\ell \hat{D}_{21}(D_{21}D_{21}^*)^{-1/2} y_2$$

$$u_2 = T_{22}^{-1}\hat{D}_{12}C_\ell x_\ell - T_{22}^{-1}(\hat{D}_{11}+\hat{D}_{12}D_\ell \hat{D}_{21})(D_{21}D_{21}^*)^{-1/2}(C_2+D_{21}F_1)\hat{x} +$$

$$+F_2\hat{x} + T_{22}^{-1}(\hat{D}_{11}+\hat{D}_{12}D_\ell\hat{D}_{21})(D_{21}D_{21}^*)^{-1/2}y_2$$

where $A_\ell, B_\ell, C_\ell, D_\ell$ is an arbitrary system with $A_\ell$ defines an exponentially stable evolution and the input-output operator ia strictly contractimg .

Proof. The condition $B_1\Pi_2=0$ implies $B_1\tilde{D}_1^*=0$ and

$$B_1u_1 = B_1V^*Vu_1 = (B_1\tilde{D}_1^* \quad B_1D_{21}^*(D_{21}D_{21}^*)^{-1/2})\begin{pmatrix} u_1^1 \\ u_1^2 \end{pmatrix} = B_1D_{21}^*(D_{21}D_{21}^*)^{-1/2}u_1^2$$

and in the system obtained after coupling the compensator one gets $\frac{d}{dt}(x-\hat{x}) = [A - B_1D_{21}^*(D_{21}D_{21}^*)^{-1}C_2\ (x-\hat{x}) + B_1D_{21}^*(D_{21}D_{21}^*)^{-1/2}\cdot$ $\cdot[u_1^2 - (D_{21}D_{21}^*)^{-1/2}(y_2 - C_2x)]$ leading finally to $x=\hat{x}$ and to the system

$$x' = Ax + B_1u_1 + B_2u_2$$
$$y_1 = C_1x + D_{11}u_1 + D_{12}u_2$$
$$x_\ell' = A_\ell x_\ell - B_\ell\hat{D}_{21}(u_1^2 - F_1^2x\cdot)$$
$$u_2 = T_{22}^{-1}\hat{D}_{12}C_\ell x_\ell + T_{22}^{-1}(\hat{D}_{11}+\hat{D}_{12}D_\ell\hat{D}_{21})(u_1^2 - F_1^2x) + F_2x$$

which is stable and with contracting input-output operator according to Proposition 3.1

## 4. THE CASE $\Pi_1 C_1 = 0$

We shall use the duality defined by $\#$ . The dual of (1.1) is obtained as

$$x' = A^\#(t)x + C_1^\#(t)u_1 + C_2^\#(t)u_2$$
$$y_1 = B_1^\#(t)x + D_{11}^\#(t)u_1 + D_{21}^\#(t)u_2$$
$$y_2 = B_2^\#(t)x + D_{12}^\#(t)u_1 .$$

We see that $\Pi_2^\#(t) = I - (D_{21}^\#(t))^*[D_{21}^\#(t)(D_{21}^\#(t))^*]^{-1}D_{21}^\#(t) =$

$$= I - D_{12}(-t)[D_{12}^*(-t)D_{12}(-t)]^{-1}[D_{12}(-t)]^* = \Pi_1(-t)$$

and the structure assumption is equivalent to $\Pi_1(-t)C_1(-t)=0$ hence $\Pi^{\#}_2(t)(C^{\#}_1(t))^*=0$, that is $C^{\#}_1(t)\Pi^{\#}_2(t)=0$ and the dual system has the structure of the preceding section.

We have still to check that the matrix

$$A^{\#}(t)-C^{\#}_1(t)(D^{\#}_{12}(t))^*[D^{\#}_{12}(t)(D^{\#}_{12}(t))^*]^{-1}B^{\#}_2(t)$$

defines an exponentially stable evolution ; the above matrix reads

$$A^*(-t)-C^*_1(-t)D_{12}(-t)\,[D^*_{12}(-t)D_{12}(-t)]^{-1}B^*_2(-t)=$$
$$=[A(-t)-B_2(t)\,(D^*_{12}(-t)D_{12}(-t))^{-1}D^*_{12}(-t)C_1(-t)]^*=$$
$$=(A-B_2(D^*_{12}D_{12})^{-1}D^*_{12}C_1)^{\#}(t)\ .$$

Hence , in our results we have to assume that $A-B_2(D^*_{12}D_{12})^{-1}$ $D^*_{12}C_1$ defines an exponentially stable evolution in order to be able to apply directly to the dual system the results in the precedeing section.

<u>Theorem 4.1</u> Assume a) $D^*_{12}D_{12}$ and $D_{21}D^*_{21}$ are invertible with bounded inverses; b) $D^*_{11}\Pi_1D_{11}\le r\,I_{m_1}< I_{m_1}$ and $D_{11}\Pi_2D^*_{11}\le rI_{p_1}< I_{p_1}$ for all $t\in\mathbb{R}$

c) $\Pi_1C_1\equiv 0$ and $A-B_2(D^*_{12}D_{12})^{-1}D^*_{12}C_1$ defines an exponentially stable evolution

d) $(A-B_1D^*_{21}(D_{21}D^*_{21})^{-1}C_2\quad B_1\Pi_2)$ is stabilizable.

e) The Riccati equation $S'=A(t)S+SA^*(t)-[SC^*(t)+B_1(t)D^*_2(t)]\hat{K}^{-1}(t)[C(t)S+D_2(t)B^*_1(t)]+B_1(t)B^*_1(t)$ has a bounded ,positive semidefinite ,stabilizing solution (that is $A+HC$ defines an exponentially stable evolution, with $H=(H_1\quad H_2)=-[\tilde{S}C^*+B_1D^*_2]\hat{K}^{-1}$ ).Then a parametrized family of stabilizing and contracting compensators is

$$\hat{x}'=[A-B_2(D^*_{12}D_{12})^{-1}C_1+HC_2-(B_2+H_1D_{12})(D^*_{12}D_{12})^{-1/2}(\tilde{D}_{11}+$$
$$+\tilde{D}_{12}D_\ell\tilde{D}_{21})(D_{21}D^*_{21})^{-1/2}C_2]\hat{x}-(B_2+H_1D_{12})(D^*_{12}D_{12})^{-1/2}\hat{D}_{12}C_\ell x_\ell-$$

$$- [H_2-(B_2+H_1D_{22})(D_{12}^*D_{12})^{-1/2}(\tilde{D}_{11}+\tilde{D}_{12}D_\ell\tilde{D}_{21})(D_{21}D_{21}^*)^{-1/2}]\,y_2$$

$$x_\ell'=B_\ell\tilde{D}_{21}(D_{21}D_{21}^*)^{-1/2}C_2\hat{x}+A_\ell x_\ell-B_\ell\tilde{D}_{21}(D_{21}D_{21}^*)^{-1/2}y_2$$

$$u_2=-[(D_{12}^*D_{12})^{-1}D_{12}^*C_1+(D_{12}^*D_{12})^{-1/2}(\tilde{D}_{11}+\tilde{D}_{12}D_\ell\tilde{D}_{21})(D_{21}D_{21}^*)^{-1/2}C_2]\hat{x}+$$

$$-(D_{12}^*D_{12})^{-1/2}\tilde{D}_{12}C_\ell x_\ell+(D_{12}^*D_{12})^{-1/2}(\tilde{D}_{11}+\tilde{D}_{12}D_\ell\tilde{D}_{21})(D_{21}(t)D_{21}^*(t))^{-1/2}y_2$$

with $\tilde{D}_{11}=-(D_{12}^*D_{12})^{-1/2}D_{12}^*[I_{p_1}+D_{11}\Pi_2D_{11}^*\Pi_1(I_{p_1}-D_{11}\Pi_2D_{11}^*)^{-1}\Pi_1]\cdot$

$\cdot D_{11}D_{21}^*(D_{21}D_{21}^*)^{-1/2}$

$\tilde{D}_{12}=\{I_{m_2}-(D_{12}^*D_{12})^{-1/2}D_{12}^*D_{11}\Pi_2(I_{m_1}-D_{11}^*\Pi_1D_{11})^{-1}\Pi_2D_{11}^*\cdot$

$\cdot D_{12}(D_{12}^*D_{12})^{-1/2}\}^{1/2}$

$\tilde{D}_{21}=\{I_{p_2}-(D_{21}D_{21}^*)^{-1/2}D_{21}D_{11}^*\Pi_1(I_{p_1}-D_{11}\Pi_2D_{11}^*)^{-1}\Pi_1D_{11}\cdot$

$\cdot D_{21}^*(D_{21}D_{21}^*)^{-1/2}\}^{1/2}$

## 5. A PARAMETRIZED FAMILY OF STABILIZING AND CONTRACTING COMPENSATORS

For the system (2.2) it is seen that $\Pi_1C_1=0$ and that the corresponding matrix that has to define an exponentially stable evolution is $A+B_1F_1+B_2F_2$ .

The stabilizability assumption is required for the pair $(A+B_1F_1-B_1D_{21}^*(D_{21}D_{21}^*)^{-1}(C_2+D_{21}F_1)\,,\,B_1\Pi_2)$ which is writtem as $(A-B_1D_{21}^*(D_{21}D_{21}^*)^{-1}C_2+B_1\Pi_2F_1,\,B_1\Pi_2)$ and this pair is stabilizable if $(A-B_1D_{21}^*(D_{21}D_{21}^*)^{-1}C_2,\,B_1\Pi_2)$ is stabilizable.

As we have seen in the first part of the study the Riccati equation is the one obtained as a necessary condition.

The final result , obtained by writing the compensator in Theorem 4.1 for the system (2.2) is

Theorem 5.1 Assume a) $D_{12}^*D_{12}$ and $D_{21}D_{21}^*$ are invertible with bounded inverses

b) $D_{11}^{*}\Pi_1 D_{11} \leq rI_{m_1} < I_{m_1}$ and $D_{11}\Pi_2 D_{11}^{*} \leq rI_{p_1} < I_{p_1}$

for all $t \in \mathbb{R}$

c) $(\Pi_1 C_1,\ A-B_2(D_{12}^{*}D_{12})^{-1}D_{12}^{*}C_1)$ is detectable

and $(A-B_1D_{21}^{*}(D_{21}D_{21}^{*})^{-1}C_2,\ B_1\Pi_2)$ is stabilizable.

d) The Riccati equation (2.1) has a bounded, positive semidefinite, stabilizing solution (that is $A+BF$ defines an exponentially stable evolution where $F=\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = -K^{-1}(B^{*}R + D_1^{*}C_1)$.

e) The Riccati equation

$$\hat{S}' = (A+B_1F_1)\hat{S} + \hat{S}(A+B_1F_1)^{*} - (\hat{S}\hat{C}^{*} + B_1D_2^{*})\hat{K}^{-1}(\hat{C}\hat{S} + D_2B_1^{*}) + B_1B_1^{*}$$

with $\hat{C} = \begin{pmatrix} -D_{12}F_2 \\ C_2+D_{21}F_1 \end{pmatrix}$ has a bounded, positive semidefinite solution (that is $A+B_1F_1+H\hat{C}$ defines an exponentially stable evolution where $H=(H_1 \quad H_2) = -(\hat{S}\hat{C}^{*} + B_1D_2^{*})\hat{K}^{-1}$ ).

Under these conditions a family of stabilizing and contracting compensators for (1.1) is

$$\hat{x}' = [A+BF+H_2(C_2+D_{21}F_1) - (B_2+H_1D_{12})(D_{12}^{*}D_{12})^{-1/2}(\tilde{D}_{11}+\tilde{D}_{12}D_\ell\tilde{D}_{21})$$

$$\cdot (D_{21}D_{21}^{*})^{-1/2}(C_2+D_{21}F_1)]\hat{x} - (B_2+H_1D_{12})(D_{12}^{*}D_{12})^{-1/2}\tilde{D}_{12}C_\ell x_\ell -$$

$$-[H_2 - (B_2+H_1D_{12})(D_{12}^{*}D_{12})^{-1/2}(\tilde{D}_{11}+\tilde{D}_{12}D_\ell\tilde{D}_{21})(D_{21}D_{21}^{*})^{-1/2}]y_2$$

$$x_\ell' = B_\ell\tilde{D}_{21}(D_{21}D_{21}^{*})^{-1/2}(C_2+D_{21}F_1)\hat{x} + A_\ell x_\ell + B_\ell\tilde{D}_{21}(D_{21}D_{21}^{*})^{-1/2}y_2$$

$$u_2 = [F_2 - (D_{12}^{*}D_{12})^{-1/2}(\tilde{D}_{11}+\tilde{D}_{12}D_\ell\tilde{D}_{21})(D_{21}D_{21}^{*})^{-1/2}(C_2+D_{21}F_1)]\hat{x} -$$

$$-(D_{12}^{*}D_{12})^{-1/2}\tilde{D}_{12}C_\ell x_\ell + (D_{12}^{*}D_{12})^{-1/2}(\tilde{D}_{11}+\tilde{D}_{12}D_\ell\tilde{D}_{21})(D_{21}D_{21}^{*})^{-1/2}y_2$$

where $\tilde{D}_{11}$, $\tilde{D}_{12}$, $\tilde{D}_{21}$ and $A_\ell, B_\ell, C_\ell, D_\ell$ are the same as in Theorem 4.1.

Remarks 1. The Riccati equation from e) of Theorem 5.1 may be obtained from the two dual to each other Riccati equations under a spectral radius property as it has been seen in [1].

2. The main lines of the proof of the results of this study follow the ones in [3].

3. A less systematic version for the case $D_{11}=0$ is contained in [2].

## REFERENCES

1. V.Drăgan - Global stabilizing solutions to Riccati equations via stabilizing compensators with disturbance attenuation - (submitted for publication )

2. V.Drăgan, A.Halanay -Stabilizing compensators which ensure attenuation of perturbations -submitted for publication).

3. K.Glover,J.C.Doyle-A State Space Approach to H Optimal Control - From Data to Model, J.C.Willems Editor Springer Verlag 1989,179-217.

4. A.Halanay -Advances in Linear Control Theory and Riccati equations - Rendiconti Sem.Mat. Univ.Polit.Torino (to appear ).