INSTITUTUL DE MATEMATICA
AL ACADEMIEI ROMANE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS

OF THE ROMANIAN ACADEMY

(FORMER SERIES OF THE PREPRINT SERIES IN MATH. OF INCREST)

ISSN 0250 3638

THE DUAL OF THE CATEGORY OF TREES

by

Şerban A. Basarab

PREPRINT No.7/1992

BUCURESTI

THE DUAL OF THE CATEGORY OF TREES

by

Serban A. Basarab

April, 1992

Institute of Mathematics of the Romanian Academy, P.O.Box 1-764, RO-70700, Bucharest, Romania.

The dual of the category of trees

Şerban A. Basarab

Alexandru Brezuleanu,

in memoriam

Abstract

A set T together with a symmetric ternary operation $Y: T^3 \longrightarrow T$ is said to be a (generalized) tree if $Y(x,x,y) = x$ and $Y(Y(x,u,v), Y(y,u,v),z) = Y(Y(x,y,z),u,v)$ for all $x,y,z,u,v \in T$.

Extending suitably Stone's duality for distributive lattices it is shown that the category of trees is dual to the category having as objects the systems $(X, 0,1, \neg)$ where X is an irreducible spectral space with generic point 0,1 is the unique closed point of X and $\neg$ is a unary operation on X satisfying the following conditions:

i) $\neg\neg x = x$ for all $x \in X$,

ii) for each quasi-compact open subset U of X, the set $\neg U := \{x \in X : \neg x \notin U\}$ is quasi-compact open too, and

iii) the quasi-compact open subsets U of X satisfying $\neg U = U$ generate the topology of X.

It is also shown that the category of trees is equivalent to the category having as objects the systems $(A, \vee, \wedge, \neg)$ where $(A, \vee, \wedge)$ is a distributive lattice and $\neg$ is a unary operation on A such that the following conditions are satisfied

i) $\neg$ is a negation, i.e. $\neg\neg a = a$ and $\neg(a \vee b) = \neg a \wedge \neg b$ for $a,b \in A$, and

ii) the subset $T(A) = \{ a \in A : \neg a = a \}$ of A generates the lattice A.

./.

Introduction

According to |4|, we understand by a (generalized) tree a set together with a symmetric ternary operation $Y:T^3 \longrightarrow T$ satisfying the following equational axioms:

Absorptive law: $Y(x,x,y) = x$

Selfdistributive law: $Y(Y(x,u,v), Y(y,u,v),z) = Y(Y(x,y,z), u,v)$.

It is shown in |4| that the Λ-trees as defined in |7,1,2| (in particular, the simplicial trees), the distributive lattices and Tits' buildings are natural examples of such generalized trees.

The trees form a category Tr having as morphisms the maps $f:T \longrightarrow T'$ satisfying $f(Y(x,y,z)) = Y(f(x), f(y), f(z)$ for $x,y,z \in T$.

The main goal of the present paper is to describe the dual of the category Tr. This task is achieved by extending suitably Stone's duality for distributive lattices.

1. Stone's duality for distributive lattices

By a lattice we understand a poset A in which every non-empty finite subset F of A has both a join (a least upper bound)$\vee F$ and a meet (a greatest lower bound)$\wedge F$. This is equivalent to saying that A is equipped with two binary operations $\vee, \wedge$ such that both $(A,\vee)$ and $(A,\wedge)$ are semilattices (i.e. commutative semigroups in which every element is idempotent) satisfying the absorptive laws $a\wedge(a\vee b) = a$, $a\vee(a\wedge b) = a$.

Usually (as for instance in |5,6|) lattices are assumed to have an initial and a final element. However, from technical reasons, we are forced to consider in the following the general case.

The lattices form a category Lat having as morphisms the maps $f:A \longrightarrow B$ satisfyin $f(a\vee b) = f(a)\vee f(b)$, $f(a\wedge b) = f(a)\wedge f(b)$ for $a, b \in A$.

The lattice A is said to be distributive if the distributive law $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$ holds for all $a,b,c \in A$. Note that in a distributive lattice the dual of the identity above is satisfied too.

Denote by DLat the full subcategory of Lat having as objects the distributive lattices. The empty lattice is an initial object of DLat while the one-element lattice is a final object.

Definition. A subset I of a lattice A is called an ideal of A if I is a sub-join-semilattice of A, i.e. $a \in I$, $b \in I$ imply $a \vee b \in I$, and I is a lower set, i.e. $a \in I$, $b \leq a$ imply $b \in I$. A subset F of A satisfying axioms dual to those defining an ideal is called a filter.

An ideal I of the lattice A is said to be prime if its complement in A is a filter, i.e. $a \wedge b \in I$ implies either $a \in I$ or $b \in I$. The complement of a prime ideal is called a prime filter.

Definition. A topological space X is said to be spectral (or coherent) if

i) X is sober, i.e. every irreducible non-empty closed subset of X is the closure of a unique point of X, and

ii) the family of quasi-compact open subsets of X is closed under finite intersection (in particular, X itself is quasi-compact) and forms a base for the topology of X.

Denote by IrrSpec the category of the systems (X,0,1) where X is an irreducible spectral space with the generic point 0, having a unique closed point 1. The morphisms in IrrSpec, called coherent maps, are those continuous functions $f:X \longrightarrow Y$ for which $f(0) = 0$, $f(1) = 1$ and $f^{-1}(U)$ is quasi-compact whenever U is a quasi-compact open subset of Y.

1.1. Theorem (Stone's representation theorem for distributive lattices) The category IrrSpec is dual to the category DLat.

The duality sends an object (X,0,1) of IrrSpec to the lattice of proper quasi-compact open subsets U of X (proper means $U \neq \emptyset$, $U \neq X$, equivalently, $0 \in U$ and $1 \notin U$), and a distributive lattice A to the system (Spec A, ϕ, A), where the space Spec A is the prime spectrum of A. The points of Spec A are the prime ideals of A, while its open sets may be identified with arbitrary ideals of A, a point P being in an open set I iff $I \not\subseteq P$. The correspondence $a \longrightarrow U(a) = \{P \in \operatorname{Spec} A : a \notin P\}$ establishes a lattice isomorphism of A onto the lattice of proper quasi-compact open subsets of Spec A.

There is an alternative description of the dual of the category <u>DLat.</u>

<u>Definition.</u> By a <u>quasi-boolean lattice</u> we understand a distributive lattice A in which for arbitrary $a,b,c \in A$ such that $a \leq c \leq b$ there exists (of course unique) $d \in A$ satisfying $c \wedge d = a$ and $c \vee d = b$.

Thus the boolean algebras are those quasi-boolean lattices which have an initial and a final element.

<u>Definition</u>. By a <u>quasi-boolean space</u> we understand an object $(X,0,1)$ of <u>IrrSpec</u> such that the subspace $X \setminus \{0,1\}$ satisfies the T_1-axiom, i.e. for all $x,y \in X$, $x \longrightarrow y$ (i.e. y is contained in the closure of $\{x\}$) implies either $x = 0$ or $y = 1$ or $x = y$.

The duality between <u>DLat</u> and <u>IrrSpec</u> induces by restriction a duality between the category of quasi-boolean lattices and the category of quasi-boolean spaces. In particular, the duals of boolean algebras are those objects $(X,0,1)$ for which $X \setminus \{0,1\}$ is a Stone space.

<u>Definition</u>. By an <u>ordered quasi-boolean space</u> we understand a quasi-boolean space $(X,0,1)$ together with a partial order $\leq$ such that for all $x,y \in X$, there exists a lower quasi-compact open subset U of X satisfying $x \notin U$ and $y \in U$, whenever $x \not\leq y$. It follows that $0 \leq x \leq 1$ for all $x \in X$.

The ordered quasi-boolean spaces with order preserving coherent maps form a category <u>OQBooleSp.</u>

1.2. <u>Theorem</u>. The categories <u>IrrSpec</u> and <u>OQBooleSp</u> are canonically isomorphic.

Given an object $(X,0,1)$ of <u>IrrSpec</u>, let A be the lattice of quasi-compact open proper subsets U of X, and let $B = \left\{U \cup \bigcup_{i=1}^{n} (V_i \setminus W_i) : n \in N;\ U,V_i,W_i \in A\right\}$.

B is a quasi-boolean lattice generated by its sublattice A. Moreover $B \cup \{X\}$ is a base of the so called <u>patch topology</u> on X, with respect to which $(X,0,1)$ becomes a quasi-boolean space whose quasi-compact open proper subsets are exactly the members of B. Considering the partial order on X given by the specialization relation $\longrightarrow$ with respect to the A-topology on X, we get the ordered quasi-boolean space asociated to the object (X,0,1) of <u>IrrSpec.</u> Note that the A-open sets of X are identified with the lower (with respect to $\longrightarrow$) B - open sets of X.

Conversely, given an ordered quasi-boolean space $(X,0,1,\leq)$, the lower open subsets of X form a topology on X with respect to which (X,0,1) becomes an object of <u>IrrSpec</u>, while the specialization relation is identifed with the partial order $\leq$.

1.3. <u>Corollary.</u> The forgetful functor from the category of quasi-boolean lattices into <u>DLat</u> has a left adjoint; its value at some distributive lattice A is the quasi-boolean lattice <u>freely generated</u> by A.

2. <u>Distributive lattices and irreducible spectral spaces with negation</u>

<u>Definition.</u> By a <u>negation</u> on a distributive lattice A we understand a unary operation $\neg : A \longrightarrow A$ satisfying the following equational axioms:

<u>Double negation law:</u> $\neg\neg a = a$

<u>De Morgan law:</u> $\neg(a \vee b) = \neg a \wedge \neg b$.

Note that the equality $\neg(a \wedge b) = \neg a \vee \neg b$ also holds.

The distributive lattices with negation form a category <u>NDLat</u> having as morphisms the lattice morphisms $f: A \longrightarrow B$ satisfying $f(\neg a) = \neg f(a)$ for $a \in A$. The category of boolean algebras is identified with a non-full subcategory of <u>NDLat</u>.

<u>Definition.</u> By an <u>irreducible spectral space with negation</u> we understand an object (X,0,1) of <u>IrrSpec</u> together with a map $\neg : X \longrightarrow X$ subject to the following conditions:

i) $\neg\neg x = x$ for all $x \in X$, and

ii) for each quasi-compact open subset U of X, the set $\neg U := \{x \in X : \neg x \notin U\}$ is quasi-compact open too.

The irreducible spectral spaces with negation form a category NIrrSpec having as morphisms the coherent maps $f:X \longrightarrow \dot{Y}$ satisfying $f(\neg x) = \neg f(x)$ for $x \in X$.

The next lemma is immediate.

2.1. Lemma. Let $(X,0,1,\neg)$ be an object of NIrrSpec, and let $x,y \in X$. Then $x \longrightarrow y$ implies $\neg y \longrightarrow \neg x$. In particular, $\neg 0=1$.

2.2. Lemma. Let A be a distributive lattice and X = SpecA be its prime spectrum. There exists a canonic bijection between the negations on A and the negations on X.

Proof. Assume $\neg$ is a negation on A. Given a prime ideal P of A, the subset $\neg P := \{a \in A : \neg a \notin P\}$ is a prime ideal too, and $\neg\neg P = P$. Let D be a quasi-compact open subset of X. Then

$$\neg D = \begin{cases} X & \text{if } D = \emptyset \\ \emptyset & \text{if } D = X \\ U(\neg a) & \text{if } D = U(a) \text{ for some } a \in A, \end{cases}$$

so D is quasi-compact open too. Thus we get a negation on X.

Conversely, given a negation $\neg$ on X, define the unary operation $\neg : A \longrightarrow A$ by assigning to each $a \in A$ the unique element $\neg a \in A$ for which $\neg U(a) = U(\neg a)$ $\square$

The next theorem is an immediate consequence of Theorem 1.1. and Lemma 2.2.

2.3. Theorem. The category NIrrSpec is the dual of the category NDLat. This duality induces by restriction a duality between the category of quasi-boolean lattices with negation and the category of quasi-boolean spaces with negation.

To get an alternative description of the category NIrrSpec we need the following concept:

Definition. By an ordered quasi-boolean space with negation we understand an ordered quasi-boolean space $(X,0,1,\leqslant)$ together with a negation $\neg : X \longrightarrow X$ on the underlying quasi-boolean space $(X,0,1)$ which is compatible with the partial order $\leqslant$, i.e. $x \leqslant y$ implies $\neg y \leqslant \neg x$ for all $x,y \in X$.

The ordered quasi-boolean spaces with negation and the order preserving coherent maps which commute with negation form a category <u>NOQBooleSp</u>.

As a consequence of Theorem 1.2. we get

2.4. <u>Theorem.</u> The categories <u>NIrrSpec</u> and <u>NOQBooleSp</u> are canonically isomorphic.

2.5. <u>Corollary.</u> The forgetful functor from the category of quasi-boolean lattices with negation into <u>NDLat</u> has a left adjoint; its value at some distributive lattice with negation $(A, \neg)$ is the quasi-boolean lattice with negation <u>freely generated</u> by $(A, \neg)$

Some particularly interesting full subcategories of <u>NDLat</u> and <u>NIrrSpec</u> are defined as follows.

<u>Definition.</u> By a <u>quasi-linear lattice</u> we understand a distributive lattice with negation $(A, \neg)$ in which for each $a \in A$ either $a \leqslant \neg a$ or $\neg a \leqslant a$.

Denote by <u>QLinLat</u> the category of quasi-linear lattices.

<u>Definition.</u> An irreducible spectral space with negation $(X,0,1,\neg)$ is said to be <u>quasi-linear</u> if for all $x,y \in X$ either $x \longrightarrow y$ or $y \longrightarrow x$ or $x \longrightarrow \neg y$ or $\neg x \longrightarrow y$.

Denote by <u>QLinSpec</u> the category of quasi-linear irreducible spectral spaces.

2.6. <u>Proposition.</u> The duality <u>NDLat</u> $\longrightarrow$ <u>NIrrSpec</u> induces by restriction a duality <u>QLinLat</u> $\longrightarrow$ <u>QLinSpec</u>.

<u>Proof.</u> Let $(A, \neg)$ be a distributive lattice with negation. We have to show that the necessary and sufficient condition for $(A, \neg)$ to be quasi-linear is that its dual $(\text{Spec } A, \phi, A, \neg)$ is quasi-linear.

First assume that $(A, \neg)$ is quasi-linear, and let $P, Q \in \text{Spec } A$ be such that $P \not\subseteq Q$, $Q \not\subseteq P$ and $P \not\subseteq \neg Q$. Let $d \in \neg P$, i.e. $\neg d \notin P$. We have to whow that $d \in Q$. By hypothesis there exist $a \in P \setminus Q$, $b \in Q \setminus P$ and $c \in P$ such that $\neg c \in Q$. Let $e: = $ $= (a \wedge d) \vee (\neg b \wedge c)$. As $(A, \neg)$ is quasi-linear, we distinguish two cases:

Case 1: $e \leq \neg e$. Then $a \wedge d \leq b \vee \neg c \in Q$, whence $a \wedge d \in Q$. Since by assumption $a \notin Q$ it follows $d \in Q$.

Case 2: $\neg e \leq e$. Then $b \wedge \neg d \leq a \vee c \in P$, whence $b \wedge \neg d \in P$, contrary to the assumption that $b \notin P$ and $\neg d \notin P$.

Consequently, $e \leq \neg e$ and hence $d \in Q$ as contended.

Next assume that $(\mathrm{Spec}A, \phi, A, \neg)$ is quasi-linear, and let $a \in A$ be such that $a \nleq \neg a$. To conclude that $\neg a \leq a$ we have to show that for each $Q \in \mathrm{Spec}A$, $a \in Q$ implies $\neg a \in Q$. Let $Q \in \mathrm{Spec}A$ be such that $a \in Q$, so $\neg a \notin \neg Q$. By hypothesis there exists $P \in \mathrm{Spec}A$ such that $\neg a \in P$ and $a \notin P$, so $\neg a \in P \cap \neg P$. As $a \in Q \setminus P$ and $\neg a \in P \setminus \neg Q$ it follows that either $P \subseteq Q$ or $\neg P \subseteq Q$, whence $\neg a \in P \cap \neg P \subseteq Q$. □

3. Some basic properties of trees

Let T be a tree with the ternary operation Y.

Definition. A subset I of T is said to be an ideal (or a convex subset) of T if for all $a,b,c \in T$, $a \in I$ and $b \in I$ imply $Y(a,b,c) \in I$.

As the intersection of a family of ideals of T is also an ideal, we may speak on the ideal generated by a subset S of T and denote it by $|S|$. Nothe that $|\phi| = \phi$ $|\{a\}| = \{a\}$ for $a \in T$, and $|\{a,b\}| := |a,b| = \{Y(a,b,c) : c \in T\} = \{c \in T : Y(a,b,c) = c\}$ for $a,b \in T$, cf. |4| Lemma 2.5.

Definition. By a cell (or a simplex) of the tree T we understand an ideal I of T of the form $I = |a,b|$ with $a,b \in T$. Given a cell I, any $a \in T$ for which there exists $b \in T$ such that $I = |a,b|$ is called an end of the cell I. The (non-empty) subset of all ends of the cell I, denoted by ∂I, is called the boundary of the cell I.

According to |4| Lemma 2.5., the boundary ∂I of a cell I is a subtree of I and there exists a canonic map $\partial I \longrightarrow \partial I$, $a \longmapsto \bar{a}$ such that $I = |a,\bar{a}|$, $\bar{\bar{a}} = a$ for $a \in \partial I$ and $\overline{Y(a,b,c)} = Y(\bar{a},\bar{b},\bar{c})$ for $a,b,c \in \partial I$. Given $a \in \partial I$, the cell I becomes a distributive lattice with respect to the partial order $b \subset_a c$ iff $b \in |a,c|$, with the initial element a and the final element $\bar{a}$. The boundary ∂I is identified with the boolean subalgebra of the distributive lalttice $(I, \subset_a)$ consisting of those elements which have (unique) complements.

Some useful elementary facts proved in |4|§2 are collected in the next proposition.

3.1. Proposition Let T be a tree.

a) $|a,b| \cap |a,c| = |a, Y(a,b,c)|$ and $|a,b| \cap |b,c| \cap |c,a| = \{Y(a,b,c)\}$ for $a,b,c \in T$.

b) $c \in |a,b|$ iff $|a,c| \cap |b,c| = \{c\}$.

c) Given $a \in T$, T becomes a meet-semilattice with respect to the partial order $\underset{a}{\subset}$ given by $b \underset{a}{\subset} c$ if $b \in |a,c|$, with the meet $b \underset{a}{\cap} c = Y(a,b,c)$ and the initial element a.

d) For $a, b_1, \ldots, b_n \in T$, $n \geq 1$, $\bigcap_{i=1}^{n} |a,b_i| = |a,b|$, where $b = \underset{a}{\cap}\{b_1,..,b_n\}$ is the meet of the family $\{b_1,..,b_n\}$ with respect to the order $\underset{a}{\subset}$.

e) For $a_1,\ldots,a_n \in T$, $n \geq 1$ the ideal $|a_1,\ldots,a_n|$ generated by the finite subset $\{a_1,\ldots,a_n\}$ equals $\{\underset{a}{\cap}\{a_1,\ldots,a_n\} : a \in T\} = \{b \in T : \underset{b}{\cap}\{a_1,\ldots,a_n\} = b\}$

f) Let $a_1,\ldots,a_n, b_1,\ldots,b_m \in T$ be such that $|a_1,\ldots,a_n| \cap |b_1,\ldots,b_m|$ is non-empty. Then $|a_1,\ldots,a_n| \cap |b_1,\ldots,b_m| = |\underset{a_1}{\cap}\{b_1,\ldots b_m\},\ldots,\underset{a_n}{\cap}\{b_1,\ldots,b_m\}| = |\underset{b_1}{\cap}\{a_1,\ldots,a_n\},\ldots, \underset{b_m}{\cap}\{a_1,\ldots,a_n\}|$. In particular for $a,b,c,d \in T$, either $|a,b| \cap |c,d|$ is empty or $|a,b| \cap |c,d| = |Y(a,b,c), Y(a,b,d)| = |Y(a,c,d), Y(b,c,d)|$.

g) For each subset S of T, the ideal |S| is the union $\bigcup_F |F|$, where F ranges over the family of all finite subsets of S.

Definition. A tree T is said to be boolean if for every cell I of T, $\partial I = I$, respectively linear if every cell of T has at most two ends.

Let T be a non-empty tree and let $a \in T$. Then, according to |4| Lemma 2.6., T is boolean iff T is a quasi-boolean lattice with respect to the order $\underset{a}{\subset}$.

Note also that a tree T is linear iff the following equivalent statements hold:

i) For all $a,b,c \in T$, $c \in |a,b|$ implies $|a,b| = |a,c| \cup |c,b|$.

ii) For all $a,b \in T$, the partial order $\underset{a}{\subset}$ induces on the cell $|a,b|$ a total (linear) order with the initial element a and the final element b.

4. From distributive lattices with negation to trees

The category DLat of distributive lattices is naturally identified with a non-full subcategory of the category Tr of trees. Given a distributive lattice A, the ternary operation $Y: A^3 \longrightarrow A$, $(a,b,c) \longmapsto Y(a,b,c) = (a\wedge b)\vee(b\wedge c)\vee(c\wedge a) = (a\vee b)\wedge(b\vee c)\wedge(c\vee a)$ is a tree operation on A. Obviously, any lattice morphism is a tree morphism.

If $\neg$ is a negation on a distributive lattice A then this one is an automorphism of the underlying tree of A. Thus the category NDLat of distributive lattices with negation is identified with a non-full subcategory of the category ITr of trees with involution; the objects of ITr are pairs (T,s) consisting of a tree T and of an automorphism s of T subject to $s^2 = id_T$, while the morphism $(T,s) \longrightarrow (T',s')$ are tree morphisms $f: T \longrightarrow T'$ satisfying the equality $f\circ s = s'\circ f$.

By composing the forgetful functor NDLat$\longrightarrow$ITr with the functor ITr $\longrightarrow$Tr, $(T,s) \longmapsto T^S = \{x \in T : sx=x\}$, we get a functor $\mathcal{T}$: NDLat$\longrightarrow$Tr, assigning to a distributive lattice with negation $(A,\neg)$ the subtree of A with universe $\{a \in A : \neg a = a\}$.

4.1. Lemma. The functor $\mathcal{T}$:NDLat $\longrightarrow$ Tr induces by restriction a functor from the category NQBooleLat of quasi-boolean lattices with negation to the category BooleTr of boolean trees, respectively a functor from the category QLinLat of quasi-linear lattices to the category LinTr of linear trees.

Proof. Let $(A,\neg)$ be a quasi-boolean lattice with negation, and let $a,b,c \in T := \mathcal{T}(A)$ be such that c belongs to the cell $|a,b|$ of T. In particular, $a\wedge b \leq c \leq a\vee b$. By assumption there exists a unique $d \in A$ such that $c\wedge d = a\wedge b$ and $c\vee d = a\vee b$. Applying the negation, we get $c\vee \neg d = a\vee b$ and $c\wedge \neg d = a\wedge b$, whence $d = \neg d \in T$. Moreover, it follows easily that the cells $|a,b|$ and $|c,d|$ of the tree T coincide, concluding that the tree T is boolean.

Next let $(A,\neg)$ be a quasi-linear lattice, and $a,b,c,d \in T$ be such that c,d belong to the cell $|a,b|$ of T, whence $a\wedge b \leq c \leq a\vee b$ and $a\wedge b \leq d \leq a\vee b$. We have to show that $d \in |a,c| \cup |c,b|$. Set $e = (a\wedge c)\vee(b\wedge d)$. By hypothesis we distinguish two cases.

Case 1: $e \leq \neg e$. Then $a\wedge c \leq b\vee d$, whence $a\wedge c \leq a\wedge(b\vee d) = (a\wedge b)\vee(a\wedge d) = a\wedge d \leq d$.

Applying the negation we get also $d \leq a \vee c$, and hence $d \in |a,c|$.

Case 2: $\neg e \leq e$. It follows $a \wedge d \leq b \vee c$. Proceeding as in the case 1, we get $d \in |b,c|$. $\square$

5. From trees to irreducible spectral spaces with negation

The aim of this section is to construct a contravariant functor Spec: Tr $\longrightarrow$ NIrrSpec from the category of trees to the category of irreducible spectral spaces with negation as defined in §2.

5.1. Shadows in trees

Definition. Given two subsets A and B of a tree T, let $Sh_A(B)$ be the subset of T consisting of those $x \in T$ for which there exists $a \in A$ such that the intersection $|a,x| \cap B$ is non-empty. Call $Sh_A(B)$ the shadow of B with respect to A.

In particular, for $A = \{a\}$ and $B = \{b\}$, $Sh_a(b) := Sh_A(B) = \{x \in T : b \in |a,x|\} = \{x \in T : b \subset_a x\}$.

The basic properties of the sets $Sh_A(B)$ for $A,B \subseteq T$ are collected in the following lemma.

5.1.1. Lemma. Let A and B be subsets of a tree T.

a) If A is non-empty then $B \subseteq Sh_A(B)$.

b) $Sh_A(B) = \bigcup_{a \in A,\, b \in B} Sh_a(b)$.

c) $Sh_a(Sh_b(c)) = Sh_{Y(a,b,c)}(c)$ for $a,b,c \in T$.

d) If A is an ideal then $Sh_A(Sh_A(B)) = Sh_A(B)$.

e) If A is an ideal then A and $Sh_A(B)$ are disjoint iff A and B are disjoint.

f) If A and B are ideals then $Sh_A(B)$ is an ideal.

Proof. The statements a) and b) are immediate.

c) Let $x \in Sh_a(Sh_b(c))$. Then there exists $y \in |a,x|$ such that $c \in |b,y|$. It follows $Y(x,Y(a,b,c),c) = Y(x,Y(a,b,c), Y(y,b,c)) = Y(Y(x,a,y), b,c) = Y(y,b,c) =$

$= c$, i.e. $c \in |x, Y(a,b,c)|$, whence $x \in Sh_{Y(a,b,c)}(c)$.

Conversely, assuming $x \in Sh_{Y(a,b,c)}(c)$, we get

$$c = Y(x, Y(a,b,c),c) = Y(Y(x,a,c), Y(x,c,c), b) = Y(y(x,a,c), b,c).$$

Seting $y = Y(x,a,c)$, it follows $y \in |a,x|$ and $c \in |b,y|$, whence $x \in Sh_a(Sh_b(c))$.

Obviously, d) is a consequence of c).

e) Assume A is an ideal and let $a \in A \cap Sh_A(B)$, i.e. $b \in |a',a|$ for some $a' \in A$, $b \in B$. Thus $b \in |a',a| \cap B \subseteq A \cap B$, whence $A \cap B$ is non-empty.

f) Assuming that A and B are ideals, let $x,y \in Sh_A(B)$ and $z \in |x,y|$. Thus $b_1 \in |a_1,x|$ and $b_2 \in |a_2,y|$ for some $a_1,a_2 \in A$, $b_1,b_2 \in B$. To conclude that $z \in Sh_A(B)$ it suffices to show that $Y(b_1,b_2,z) \in |Y(a_1,a_2,Y(b_1,b_2,z)), z|$ since $Y(b_1,b_2,z) \in$ $\in |b_1,b_2| \subseteq B$ and $Y(a_1,a_2, Y(b_1,b_2,z)) \in |a_1,a_2| \subseteq A$. Taking the point z as a root of the tree T and using the notation $\subset, \cap$ and $\cup$ instead of $\underset{z}{\subset}, \underset{z}{\cap}, \underset{z}{\cup}$, we get

$$x \cap y = Y(x,y,z) = z,\ Y(b_1,b_2,z) = b_1 \cap b_2 = Y(a_1,b_1,x) \cap Y(a_2,b_2,y) = (a_1 \cap a_2 \cap b_1 \cap b_2) \cup (a_1 \cap y \cap b_1 \cap b_2) \cup (a_1 \cap a_2 \cap b_1 \cap y) \cup (x \cap a_2 \cap b_1 \cap b_2) \cup (a_1 \cap x \cap a_2 \cap b_2) \subset (a_1 \cap a_2) \cup$$
$$\cup (a_1 \cap b_1 \cap b_2) \cup (a_2 \cap b_1 \cap b_2) = Y(a_1, a_2, b_1 \cap b_2) = Y(a_1, a_2, Y(b_1, b_2, z)),$$ as required. $\square$

5.2. The fundamental existence theorem for prime ideals in a tree.

An ideal P of a tree T is said to be *prime* if its complement in T is also an ideal. Thus the complement $\neg P := T \setminus P$ of a prime ideal P of T is a prime ideal too. In particular, the empty set $\emptyset$ and the whole T are prime ideals.

Denote by Spec T the non-empty set of all prime ideals of the tree T. Given a subset A of T, let $V(A) = \{P \in \mathrm{Spec}T : A \subseteq P\}$ and $U(A) = \{P \in \mathrm{Spec}T : P \cap A = \emptyset\} = \{\neg P : P \in V(A)\}$. Obviously, $V(A) = V(|A|)$ and $U(A) = U(|A|)$ for each $A \subseteq T$.

5.2.1. *Theorem.* Let A and B be subsets of a tree T. The necessary and sufficient condition for $V(A) \cap U(B)$ to be non-empty is that $|A| \cap |B|$ is empty.

Proof. Assuming that $V(A) \cap U(B)$ is non-empty, let $P \in V(A) \cap U(B)$. Then $|A| \subseteq P$ and $|B| \subseteq \neg P$, whence $|A| \cap |B|$ is empty.

Conversely, assume $|A| \cap |B|$ is empty. We may assume that A is non-empty since otherwise $\phi \in V(A) \cap U(B)$. By Zorn's lemma there exists an ideal P of T which is maximal amongst those containing the ideal $|A|$ and disjoint from the ideal $|B|$. According to Lemma 5.1.1.- the statements e) and f), $P = Sh_{|B|}(P)$. It remains to show that the ideal P is prime. Let $x,y \in \neg P$ and $z \in |x,y|$. We have to show that $z \in \neg P$. Let $Q = |P \cup \{x\}|$. As P is an ideal, it follows by Proposition 3.1.- the statements e) and g)- that $Q = \bigcup_{p \in P} |x,p|$. By the maximality of P there exists $b \in |B| \cap Q$, i.e. $b \in |B| \cap |x,p|$ for some $p \in P$. On the other hand, since $z \in |x,y| \cap |p,z|$ and $b \in |x,p| \cap |b,y|$ it follows by Proposition 3.1.-the statement f)-that $Y(x,y,p) \in \in |p,z| \cap |b,y|$, whence $|p,z| \cap |b,y|$ is non-empty. Assuming $z \in P$, we get $|p,z| \subseteq P$ and hence $|b,y| \cap P$ is non-empty. Consequently, $y \in Sh_{|B|}(P) = P$, contrary to our assumption. Thus $z \in \neg P$, as contended. □

5.2.2. <u>Corollary.</u> For every subset A of the tree T, $|A| = \bigcap_{P \in V(A)} P$.

5.3. <u>The prime spectrum of a tree</u>

Let T be a tree and X = Spec T be the set of all prime ideals of T. The family of the subsets $U(A) = \{P \in X : P \cap A = \emptyset\}$, for A ranging over the finite subsets of T, contains $X = U(\emptyset)$ and is closed under finite intersection, and hence is the base of a topology on X; call it the <u>spectral topology</u> on X. By Theorem 5.2.1 the map $I \mapsto U(I)$ induces a bijection of the set of all finitely generated ideals of T onto the base above.

Note that the subfamily of basic open sets $U(a) = \{P \in X : a \notin P\}$ for $a \in T$ generates the spectral topology on X.

For each $P \in X$, the closure of $\{P\}$ is V(P); i.e. the specialization relation on X coincides with the inclusion of prime ideals. In particular, $X = V(\emptyset)$, i.e. X is irreducible with the unique generic point $\emptyset$. On the other hand, as $\{T\} = V(T)$, T is the unique closed point of X. Note also that X is quasi-compact since $U(\emptyset) = X$ is the unique basic open set containing T.

5.3.1. <u>Lemma.</u> The space X = Spec T is sober.

<u>Proof.</u> We have to show that the function $P \mapsto V(P)$ maps bijectively X onto the set of all non-empty irreducible closed subsets of X. The injectivity is obvious.

Assuming that C is a non-empty irreducible closed subset of X, let $P = \bigcap_{Q \in C} Q$. We have to show that the ideal P is prime and $P \in C$. Let $a,b \in \neg P$ and assume that there exists some $c \in |a,b| \cap P$. It follows $C \subseteq V(c) \subseteq V(a) \cup V(b)$, whence, by the irreducibilit of C, either $C \subseteq V(a)$ or $C \subseteq V(b)$, contrary to the assumption $a,b \in \neg P$. Therefore the ideal P is prime.

It remains to show that $P \in C$. Assuming $P \notin C$, there exist $a_1,\dots,a_n \in T$, $n \geq 1$, such that $P \in \bigcap_{i=1}^{n} U(a_i)$ and $C \subseteq \bigcup_{i=1}^{n} V(a_i)$. Since C is irreducible, it follows $C \subseteq V(a_{i_o})$ for some $i_o \in \{1,\dots,n\}$, whence $a_{i_o} \in P$, a contradiction. □

5.3.2. Proposition. The necessary and sufficient condition for an open subset D of X = SpecT to be quasi-compact is that D is a finite union of basic open subsets of X.

Proof. It suffices to show that for each finite non-empty subset A of T, the basic open set U(A) is quasi-compact. Assume $U(A) = \bigcup_{i \in I} D_i$, where the D_i's are open. Without loss of generality we may assume that for each $i \in I$, $D_i = U(B_i)$ for some finite non-empty subset B_i of T. Suppose that for each finite subset F of I, $U(A) \not\subseteq \bigcup_{i \in F} D_i$. Let M_F be the set of those functions $f: F \longrightarrow \bigcup_{i \in F} B_i$ satisfying $f(i) \in B_i$ for $i \in F$ and $U(A) \not\subseteq \bigcup_{i \in F} U(f(i))$. By hypothesis, the finite sets M_F are non-empty. The sets M_F together with the restriction maps $M_{F_2} \longrightarrow M_{F_1}$ for $F_1 \subseteq F_2$ form a directed projective system, and hence the projective limit $M = \varprojlim M_F$ is non-empty. Consequently, there exists a function $f: I \longrightarrow \bigcup_{i \in I} B_i$ such that $f(i) \in B_i$ for $i \in I$ and for each finite subset F of I, $U(A) \cap V(f(F))$ is non-empty. According to Theorem 5.2.1., $|A| \cap |f(F)|$ is empty for each finite subset F of I. As $f(I) = \bigcup_F f(F)$, we get $|f(I)| = \bigcup_F |f(F)|$ by Proposition 3.1. - the statement g) -, whence $|A| \cap |f(I)| = \emptyset$. By Theorem 5.2.1. again it follows $U(A) \not\subseteq \bigcup_{i \in I} U(f(i))$, contrary to the assumption that $U(A) \subseteq \bigcup_{i \in I} D_i$. □

5.3.3. Lemma. The necessary and sufficient condition for a quasi-compact open proper subset D of $X = \operatorname{Spec} T$ to satisfy the equality $D = \neg D := \{P \in X : \neg P \notin D\}$ is that $D = U(a)$ for some (unique) $a \in T$.

Proof. Obviously $\neg U(a) = U(a)$ for all $a \in T$.

Let D be a quasi-compact open proper subset of X and assume $D = \neg D$. By Proposition 5.3.2., D has the form $\bigcup_{i=1}^{n} U(A_i)$ where $n \geq 1$ and the A_i's are non-empty finite subsets of T.

First let us show that $\bigcap_{i=1}^{n} |A_i|$ is non-empty. The case $n = 1$ is trivial so we may assume $n \geq 2$. Let $k \in \{1,\dots,n\}$ be maximal with the property $\bigcap_{i=1}^{k} |A_i| \neq \emptyset$ and suppose $k<n$. By Theorem 5.2.1. there exists $P \in U(A_{k+1}) \cap V(\bigcap_{i=1}^{k} |A_i|)$. Since $P \in U(A_{k+1}) \subseteq D = \neg D$, it follows $\neg P \notin \bigcup_{i=1}^{k} U(A_i)$, i.e. $a_i \in \neg P$ for some $a_i \in A_i$, $1 \leq i \leq k$. Pick some b in $\bigcap_{i=1}^{k} |A_i|$. Then $c := \bigcap_b \{a_1,\dots,a_k\} \subseteq \neg P$. On the other hand, $c \in \bigcap_{i=1}^{k} |b,a_i| \subseteq \bigcap_{i=1}^{k} |A_i| \subseteq P$, a contradiction.

Next let us show that $\bigcap_{i=1}^{n} |A_i|$ is a singleton. Assuming the contrary, let $a,b \in \bigcap_{i=1}^{n} |A_i|$ be such that $a \neq b$. By Theorem 5.2.1. there exists a prime ideal P such that $a \in P$ and $b \in \neg P$. As $a \in P \cap \bigcap_{i=1}^{n} |A_i|$ it follows $P \notin D$, and hence $\neg P \in \neg D = D$. Consequently, $b \notin \neg P$ since $b \in \bigcap_{i=1}^{n} |A_i|$, a contradiction.

Let a be the unique element of the ideal $\bigcap_{i=1}^{n} |A_i|$. Obviously, $D \subseteq U(a)$, whence $U(a) = \neg U(a) \subseteq \neg D = D$, so $D = U(a)$ as contended. $\square$

According to Lemma 5.3.1. and Theorem 5.3.2., Spec $T := (\operatorname{Spec} T, \emptyset, T, \neg)$ is an object of the category NIrrSpec. If $f: T \longrightarrow T'$ is a tree morphism then the map $\operatorname{Spec} T' \longrightarrow \operatorname{Spec} T$, $P' \longmapsto f^{-1}(P')$ is a morphism in NIrrSpec, so we get a contravariant functor Spec : Tr $\longrightarrow$ NIrrSpec. By Lemma 5.3.3., the spectral topology on Spec T for any tree T is generated by those quasi-compact open proper subsets D of Spec T satisfying $\neg D = D$.

5.3.4. Lemma. The functor Spec: Tr $\longrightarrow$ NIrrSpec induces by restriction a functor from the category BooleTr of boolean trees (respectively from the category LinTr of linear trees) to the category NQBooleSp of quasi-boolean spaces with negation (respectively to the category QLinSpec of quasi-linear irreducible spectral spaces).

Proof. First assume T is a boolean tree and let P,Q be proper prime ideals of T such that $P \subseteq Q$. We have to show that $P = Q$. Assuming the contrary, there exist $a,b,c \in T$ such that $a \in P$, $c \in Q \setminus P$ and $b \notin Q$. It follows $Y(a,b,c) \in Q \setminus P$ so we may assume from the beginning that $c \in |a,b|$. By hypothesis there exists $d \in T$ such that $|a,b| = |c,d|$. As $a \in P$, $c \notin P$ we get $d \in P \subseteq Q$, whence $b \in |c,d| \subseteq Q$, a contradiction.

Next assume T is a linear tree and let P, Q be prime ideals of T such that $P \nsubseteq Q$, $Q \nsubseteq P$ and $P \nsubseteq \neg Q$. We have to show that $\neg P \subseteq Q$. By assumption there exist $a,b,c \in T$ such that $a \in P \cap \neg Q$, $b \in Q \cap \neg P$ and $c \in P \cap Q$. As $Y6a,b,c) \in |a,c| \cap |b,c| \subseteq P \cap Q$, we may assume $c \in |a,b|$. Let $d \in \neg P$. By hypothesis we distinguish two cases:

Case 1: $Y(a,b,d) \in |a,c|$. Then $Y(a,b,d) \in |a,c| \cap |b,d| \subseteq P \cap \neg P = \emptyset$, a contradiction.

Case 2: $Y(a,b,d) \in |c,b|$. Then $Y(a,b,d) \in Q \cap |a,d|$, whence $d \in Q$ since $a \notin Q$. $\square$

6. The distributive lattice with negation freely generated by a tree.

By composing the contravariant functor Spec : Tr $\longrightarrow$ NIrrSpec as defined in §5 with the duality NIrrSpec $\longrightarrow$ NDLat we get a covariant functor $\mathcal{L}$: Tr $\longrightarrow$ NDLat which assigns to a tree T the distributive lattice of quasi-compact open proper subsets of Spec T together with the negation $D \mapsto \neg D = \{P \in \operatorname{Spec} T : \neg P \notin D\}$.

According to Theorem 2.3., Proposition 2.6. and Lemma 5.3.4., the functor $\mathcal{L}$ induces by restriction the functors Boole Tr $\longrightarrow$ NQBooleLat, LinTr $\longrightarrow$ QLinLat.

6.1. Lemma. There exists a canonic natural transformation $\eta : \mathrm{id}_{\mathrm{Tr}} \longrightarrow \mathcal{T} \circ \mathcal{L}$ Moreover η is an isomorphism.

Proof. Given a tree T, $\mathcal{T}(\mathcal{L}(T)) = \{U(a) \; : a \in T\}$ by Lemma 5.3.3. The canonic map $\eta(T) : T \longrightarrow \mathcal{T}(\mathcal{L}(T))$, $a \mapsto U(a)$ is a tree isomorphism by Theorem 5.2.1. $\square$

6.2. Lemma. There exists a canonic natural transformation $\varepsilon : \mathcal{L}\circ\mathcal{T} \longrightarrow id_{\underline{NDLat}}$. Moreover $\varepsilon(A)$ is injective for any distributive lattice with negation A.

Proof. Let $(A,\neg)$ be a distributive lattice with negation, and $T := \mathcal{T}(A,\neg)$ be the subtree of A with universe $\{a\in A : \neg a = a\}$. The map $2^A \longrightarrow 2^T$, $P \longrightarrow P\cap T$ induces a morphism Spec $(A,\neg) \longrightarrow$ Spec (T) in the category NIrrSpec. Indeed, let P be a prime ideal of A, and let $a,b,c\in T$. Assuming $a,b\in P$ it follows $Y(a,b,c)\in P$ since $Y(a,b,c)\leq a\vee b\in P$. Assuming $a\notin P$, $b\notin P$ we get $Y(a,b,c)\notin P$ since $a\wedge b\leq Y(a,b,c)$ and $a\wedge b\notin P$. Consequently, $P\cap T$ is a prime ideal of the tree T. Note that $\neg P\cap T = T\setminus P = \neg(P\cap T)$ for $P\in$ Spec A; recall that $\neg P = \{a\in A : \neg a\notin P\}$ for $P\in$ Spec A. As $\{P\in \text{Spec } A : P\cap T\in U_T(a)\} = U_A(a)$ for all $a\in T$, the map above is coherent.

By duality (Theorem 2.3), we get a morphism $\varepsilon(A) : \mathcal{L}(T)\longrightarrow A$ of distributive lattices with negation. To conclude that $\varepsilon(A)$ is injective it suffices to show that the canonic map SpecA$\longrightarrow$SpecT is onto. Let Q be a prime ideal of the tree T. Denote by I the ideal of the lattice A generated by Q, and by F the filter of A generated by $\neg Q = T\setminus Q$.

Claim: The ideal I is disjoint from the filter F.

Assuming the contrary, we get some $a_1,\dots,a_n\in\neg Q$, $b_1,\dots,b_m\in Q$, $n\geq 1$, $m\geq 1$, such that $\bigwedge_{i=1}^{n} a_i \leq \bigvee_{j=1}^{m} b_j$. Set $a = \bigwedge_{i=1}^{n} a_i$, $b = \bigvee_{j=1}^{m} b_j$, $c = \bigcap_{b_1}\{a_1,\dots,a_n\}$. We get $c = a\vee \bigvee_{i=1}^{n}(a_i\wedge b_1)$ and $\bigcap_{c}\{b_1,\dots,b_m\} = (\bigwedge_{j=1}^{m} b_j)\vee \bigvee_{j=1}^{m}(b_j\wedge c) = (\bigwedge_{j=1}^{m} b_j)\vee(b\wedge c) =$

$$= (\bigwedge_{j=1}^{m} b_j)\vee(a\wedge b)\vee \bigvee_{i=1}^{n}(a_i\wedge b_1) = c, \text{ whence } c\in|a_1,\dots,a_n|\cap|b_1,\dots,b_m|\subseteq\neg Q\cap Q = \emptyset,$$

a contradiction.

Consequently, I is disjoint from F as claimed. According to the fundamental existence theorem for prime ideals in distributive lattices, there exists a prime ideal P of A containing I and disjoint from F, whence $P\cap T = Q$. $\square$

6.3. Theorem. The functor $\mathcal{L} : \underline{Tr}\longrightarrow\underline{NDLat}$ is a left adjoint of the functor $\mathcal{T} : \underline{NDLat}\longrightarrow\underline{Tr}$. In other words, $\mathcal{L}(T)$ is the distributive lattice with negation freely generated by a tree T.

Proof. Since the natural transformation $\eta : \mathrm{id}_{\underline{Tr}} \longrightarrow \mathcal{T}\circ\mathcal{L}$ and $\varepsilon : \mathcal{L}\circ\mathcal{T} \longrightarrow \mathrm{id}_{\underline{NDLat}}$ satisfy obviously the trangular identities $\mathcal{T}(\varepsilon)\circ\eta(\mathcal{T}) = \mathrm{id}_{\mathcal{T}}$ and $\varepsilon(\mathcal{L})\circ\mathcal{L}(\eta) = \mathrm{id}_{\mathcal{L}}$, $\mathcal{L}$ is a left adjoint of $\mathcal{T}$ having unit η and counit ε.

As a consequence of Theorem 6.3. and Lemmata 6.1. and 6.2. we get:

6.4. Theorem a) The functor $\mathcal{L}$: Tr $\longrightarrow$ NDLat induces an equivalence between the category Tr of trees and the full subcategory of NDLat consisting of those distributive lattices with negation $(A, \neg)$ which are generated as lattices by their subtrees $\mathcal{T}(A) = \{a \in A : \neg a = a\}$.

b) The contravariant functor Spec : Tr $\longrightarrow$ NIrrSpec induces a duality between the category Tr and the full subcategory of NIrrSpec consisting of those irreducible spectral spaces with negation $(X, 0, 1, \neg)$ whose topology is generated by the quasi-compact open subsets D satisfying $\neg D = D$.

6.5. Examples

i) Let $(A, \vee, \wedge)$ be a distributive lattice. The product $A \times A$ becomes a distributive lattice with negation with respect to the operations

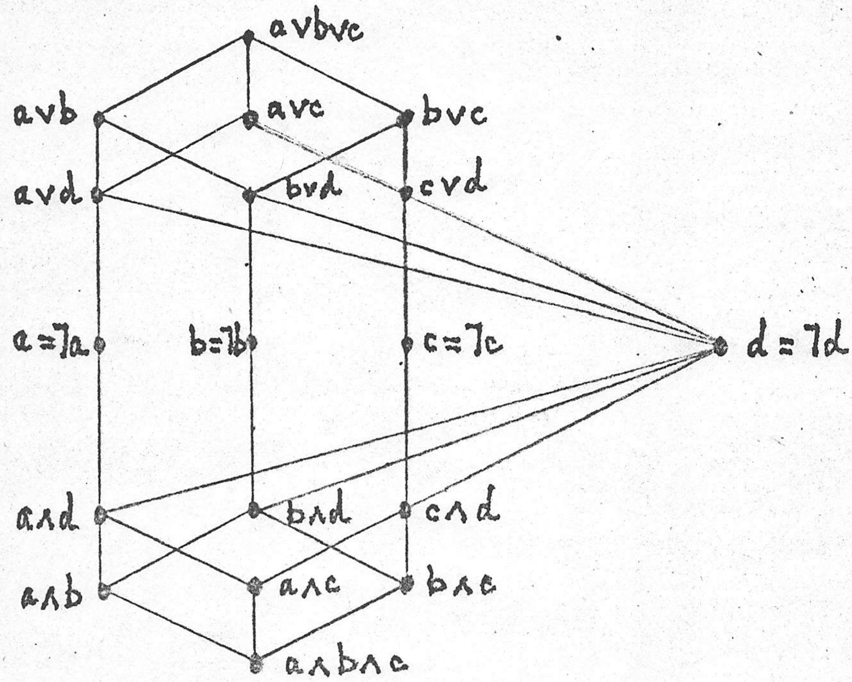
$$(a,b) \vee (a',b') = (a \vee a', b \wedge b')$$

$$(a,b) \wedge (a',b') = (a \wedge a', b \vee b')$$

$$\neg(a,b) = (b,a).$$

The diagonal embedding $a \mapsto (a,a)$ identifies the underlying tree of A with the subtree of the lattice above consisting of the elements which are invariant under the negation $\neg$. Thus we get the distributive lattice with negation freely generated by the underlying tree of A. Note also that the prime ideals of the underlying tree of A correspond bijectively to the pairs (P,F) consisting of a prime ideal P and of a prime filter F of the distributive lattice A.

ii) Let T be the linear tree with 4 distinct elements a,b,c,d such that $Y(a,b,c) = d$. The prime spectrum of T has 8 points namely the prime ideals $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$ and their complements in T. The distributive lattice with negation $\mathcal{L}(T)$ freely generated by the tree T has 18 distinct elements and is described by the diagram



7. The distributive lattice freely generated by a tree

Let t: DLat ⟶ Tr be the forgetful functor which identifies the category of distributive lattices with a non-full subcategory of the category of trees. Denote by l: Tr ⟶ DLat the functor obtained by composing the functor $\mathcal{L}$: Tr ⟶ NDLat as defined in §6 with the forgetful functor NDLat ⟶ DLat. The functor l assigns to a tree T the distributive lattice l(T) of quasi-compact open proper subsets of the spectral space Spec T. Note that T is identified with a subtree of l(T) which generates l(T) as a lattice.

7.1. Proposition. The functor l: Tr ⟶ DLat is the left adjoint of the forgetful functor t: DLat ⟶ Tr. In other words, l(T) is the distributive lattice freely generated by a tree T.

Proof. Let T and A be a tree and respectively a distributive lattice. Given a tree morphism f: T ⟶ A, we have to extend it uniquely to a lattice morphism $\bar{f}$: l(T) ⟶ A.

As for each prime ideal P of the distributive lattice A, $f^{-1}(P)$ is a prime ideal of the tree T, we get a map f_*: Spec A ⟶ Spec T. One checks easily that f_* is a morphism in the category IrrSpec. By Stone's duality, we get the required lattice morphism $\bar{f}: l(T) \longrightarrow A$. ◻

8. The boolean tree freely generated by a tree

Given a tree T, let Spec T = (Spec T, ϕ , T, $\neg$) be the irreducible spectral space with negation associated to T. Denote by $\mathcal{B}(T)$ the subtree of the power set $2^{\text{Spec } T}$ consisting of those proper subsets D of Spec T which are quasi-compact and open with respect to the patch topology on Spec T (cf. §1) and satisfy the condition $\neg D = D$. The tree T is identified with a subtree of $\mathcal{B}(T)$, and $\mathcal{B}(T)$ is boolean according to Lemma 4.1. Thus we get a functor $\mathcal{B}$: Tr$\longrightarrow$ BooleTr.

8.1. Proposition. The functor $\mathcal{B}$ is a left adjoint of the forgetful functor BooleTr$\longrightarrow$ Tr.

Proof. Immediate by Theorems 2.4. and 6.4. Indeed, we get a duality between the category Tr of trees and the full subcategory of NOQBooleSp consisting of those ordered quasi-boolean spaces with negation $(X,0,1,\leqslant,\neg)$ which satisfy the following condition: the lattice of lower quasi-compact open proper subsets of X is generated by its members D for which $\neg D = D$. ◻

8.2. Remark. Let $(X,0,1,\neg)$ be the dual of a boolean tree, and $\leqslant$ be a partial order on X making $(X,0,1,\leqslant,\neg)$ an ordered quasi-boolean space with negation. The lattice of lower quasi-compact open proper subsets of X is not necessarily generated by its members D satisfying $\neg D = D$. For instance, let T be the linear, boolean tree consisting of two distinct points x,y. Then X = Spec T has four points, namely ϕ, T, $\{x\}$, $\{y\}$, while the open proper subsets of X are $U(x) = \{\phi, \{y\}\}$, $U(y) = \{\phi, \{x\}\}$, $U(x) \cap U(y) = \{\phi\}$ and $U(x) \cup U(y) = \{\phi, \{x\}, \{y\}\}$; they form a boolean lattice $\mathcal{L}(T)$ with 4 elements together with the negation $\neg$ given by $\neg U(x) = U(x)$, $\neg U(y) = U(y)$, $\neg(U(x) \cap U(y)) = U(x) \cup U(y)$. Consider the linear order $\phi \leqslant \{x\} \leqslant \{y\} \leqslant T$ on X, making $(X, \phi, T, \leqslant, \neg)$ an object of the category NOQBooleSp. The lower open proper subsets of X form the chain $U(x) \cap U(y) \subsetneq U(x) \subsetneq U(x) \cup U(y)$, whose unique member D satisfying $\neg D = D$ is $U(x)$.

8.3. Example. Let T be the linear tree with four distinct elements a,b,c,d such that Y(a,b,c) = d. The embedding $a \mapsto \{a\}, b \mapsto \{b\}, c \mapsto \{c\}, d \mapsto \phi$ identifies T with a subtree of the power set of the set with three elements $\{a,b,c\}$, whose underlying tree is the boolean tree freely generated by the linear tree T.

References

1. R.C. Alperin and H.Bass, Length functions of group actions on Λ - trees. In "Combinatorial group theory and topology", ed. S.M.Gersten and J.R. Stallings, Annals of Mathematics Studies 111, 265-378, Princeton University Press 1987.

2. Ş.A. Basarab, On a problem raised by Alperin and Bass. In "Arboreal Group Theory", ed. R.C. Alperin, Mathematical Sciences Research Institute Publications 19, 35-68, Springer 1991.

3. Ş.A. Basarab, On a problem raised by Alperin and Bass I: Group actions on groupoids, Journal of Pure and Applied Algebra 73 (1991), 1-12.

4. Ş.A. Basarab, On a problem raised by Alperin and Bass II: Metric and order theoretic aspects, submitted to Journal of Pure and Applied Algebra.

5. A. Brezuleanu et R.Diaconescu, Sur la duale de la catégorie des treillis, Revue Roumaine de Mathématiques pures et appliquées 14:3(1969), 311-323.

6. P.T. Johnstone, Stone Spaces, Cambridge University Press, 1982.

7. J.Morgan and P. Shalen, Valuations, trees and degenerations of hyperbolic structures. I, Annals of Mathematics 120 (1984), 401-476.

Şerban A. Basarab
Institute of Mathematics
Romanian Academy
P.O.Box 1-764
RO-70700 Bucharest
ROMANIA