INSTITUTUL DE MATEMATICA
AL ACADEMIEI ROMANE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY
(FORMER SERIES OF THE PREPRINT SERIES IN MATH. OF INCREST)

ISSN 0250 3638

SOME NEW PROOFS IN CONNECTION WITH JORDAN OPERATORS

by

IONASCU J. EUGEN

PREPRINT No.8/1992

SOME NEW PROOFS IN CONNECTION WITH JORDAN OPERATORS

by

IONASCU J. EUGEN

April, 1992

Institute of Mathematics of the Romanian Academy, PO Box 1-764, RO-70700, Bucharest, Romania.

Some new proofs in connection with Jordan operators

by IONASCU J. EUGEN

§ 1. Introduction

In the present paper we give new proofs to some results in [5,6,7,8], where was used some techniques from the theory of decomposable operators. Our approach is different and is based on the well known theorems of factorization for nonnegative operator valued functions (see [10],[11]).

The proofs have an algebraical character with a few exceptions. Therefore, the facts can be presented in an arbitrary C^*-algebra, and we do it when it will be the case.

To be more precise, let H be a complex Hilbert space and L(H) the algebra of all bounded linear operators on H. The connection between these two types of decomposition, will be made with aid of the map $L(H)\ni T \to \Phi_T$, where $\Phi_T : \mathbb{R} \to L(H)$ is defined by

$$\Phi_T(t) = \exp(-itT^*)\exp(itT) \text{ for every } t\in\mathbb{R} .$$

This map is a very known one (see [5],[6],[7]). It is clear that $\Phi_T(t) \geq 0$ for all real t, and if $T\in L(H)$ have in addition some properties, these ones will be reflected by Φ_T in some way and reciprocally. In our case Φ_T satisfy the following theorem from [10] (th. 3.3):

<u>THEOREM A</u>. Let $P(t) = \sum_{j=0}^{2n} P_j t^j$ be a polynomial whose coefficients are operators on H and which is nonnegative on $\mathbb{R}$. Then $P = Q^*Q$ where Q is an outer functions on $\mathbb{R}$ of the form $Q(t) = \sum_0^n Q_j t^j$ for some operators

$Q_o, Q_1, \ldots, Q_n$ on H.

This must be happen if T has a Jordan decomposition, $T = N + Q$, where $N \in L(H)$ is a normal operator and Q a nilpotent operator of some order k which commute with N.

In the case $k = 2$, we give an independent elementary proof and accurate formulas for N and Q as functions of T and T^*. Unfortunately, in the general case such formulas seems to be very complicated.

§ 2. Preliminaries

The commutator $C(T,S)$ of two operators $T,S \in L(H)$ is the operator defined on $L(H)$ by $C(T,S)X = TX - XS$ for all $X \in L(H)$. Following [2] we define the relation n_k on $L(H)$ by

$$(1) \qquad Tn_kS \text{ if } C^k(T,S)(I) = 0 \qquad (k \in \mathbb{N}^*)$$

and I being the identity operator on H.

An operator $T \in L(H)$ will be called Jordan operator of order k, if has a decomposition $T = A + N$, where A is selfadjoint and $N \in L(H)$ a nilpotent of order k, commuting with A. It is easy to see that

$$(2) \qquad C^k(T,S)(I) = (T - S)^{[k]} = T^k - \binom{k}{1} T^{k-1}S + \ldots + (-1)^k S^k \quad (k \in \mathbb{N}^*)$$

and

$$(3) \qquad \exp(zT)\exp(-zS) = I + \sum_{j=1}^{\infty} (T - S)^{[j]} \frac{z^j}{j!} \text{ for all } z \in \mathbb{C}.$$

If Tn_kS for some k, then the above entire function must be polynomial. As we already saw Φ_T and Φ_{T^*} are nonnegative operator valued functions on $\mathbb{R}$, which are given by:

$$(4) \qquad \Phi_T(t) = \exp(-itT^*)\exp(itT)$$

$$(5) \qquad \Phi_{T^*}(t) = \exp(-itT)\exp(itT^*)$$

If $T^* n_k T$ for some $k \in N^*$, it is no compulsory that $T n_k T^*$, such how is showed in [6], by considering in the place of T, the multiplication with the variable on a weighted Sobolev space on $\mathbb{R}$, restricted to an invariant subspace.

If Φ_T is polynomial, since Φ_T is nonnegative, it must have even degree. Indeed if $\Phi_T(t) = \sum_{j=0}^{2n+1} P_j t^j$ then $P_{2n+1} = \lim_{t\to\infty} \frac{\Phi_T(t)}{t^{2n+1}}$ implies $P_{2n+1} \geq 0$ and also $P_{2n+1} = \lim_{t\to-\infty} \Phi_T(t)/t^{2n+1} \leq 0$, hence $P_{2n+1} = 0$. By definition $S, T \in L(H)$ are quasinilpotent equivalent and write $S \overset{q}{\sim} T$ if $\lim_{n\to\infty} ||(S - T)^{[n]}||^{1/n} = \lim_{n\to\infty} ||(T - S)^{[n]}||^{1/n} = 0$.

§ 3. The goal of this section is to prove the following theorem which characterizes the Jordan operators in terms of T and T^*.

THEOREM 3.1

Let $T \in L(H)$ and $k \in N^*$. The following two conditions are equivalent:

(i) T is an Jordan operator of order k

(ii) $T^* n_{2k-1} T$ and $T n_{2k-1} T^*$ (see (1))

If (i) holds, $T = A + N$ where $A = A^*$ and N is nilpotent of order k commuting with A. Then we have

$$C^{2k-1}(T^*, T)(I) = C^{2k-1}(N^*, N)(I) = 0$$

corresponding to (2) and the fact that $N^k = N^{*k} = 0$.

Hence (ii) follows. Obviously $k = 1$ is a trivial case. For the converse we give first the proof of the case $k = 2$, since as we say, the treatment is elementary and we get exact formulas for A and N as functions of T and T^*.

According to the observation which was made in § 2, (ii) implies that there are equalities:

$$\Phi_T(t) = I + A_1 t + A_2 t^2 \text{ and } \Phi_{T^*}(t) = I + B_1 t + B_2 t^2 \text{ for all } t \in \mathbb{R}$$

where

$$A_1 = - B_1 = i(T^* - T),\quad A_2 = - \tfrac{1}{2}(T^2 - 2TT^* + T^{*2}),\quad B_2 = -\tfrac{1}{2}(T^{*2} - 2T^*T + T^2)$$

From the fact that $\Phi_T(t)\Phi_{T^*}(t) = \Phi_{T^*}(t)\Phi_T(t) = I$ for every real t, it follows

(6) $$A_2B_2 = B_2A_2 = 0,\quad A_1B_2 + A_2B_1 = 0,\quad A_2 + A_1B_1 + B_2 = 0,$$

so that

(7) $$A_2 + B_2 = A_1^2 = B_1^2 .$$

Let $C = iB_1(A_2 - B_2)$, which is selfadjoint since from (6)

$$C^* = - i(A_2 - B_2)B_1 = iA_1B_2 + iB_1A_2 = C$$

Then there exists $R = C^{\frac{1}{3}}$, the selfadjoint cubic root, obtained by the continous functional calculus for normal operators.

Let show that the operator

(8) $$N = \tfrac{1}{2}(iA_1 + R)$$

is the desired nilpotent of order 2.

First of all, let's check the next two properties of R

(9) $$R^2 = A_1^2 \quad \text{and} \quad RA_1 + A_1R = 0.$$

Indeed we may write the sequence

$$R^2 = C^{\frac{2}{3}} = (C^2)^{\frac{1}{3}} = [B_1(A_2 - B_2)(A_2 - B_2)B_1]^{\frac{1}{3}} = [B_1(A_2^2 + B_2^2)B_1]^{\frac{1}{3}} =$$

$$= [B_1(A_2 + B_2)^2B_1]^{\frac{1}{3}} = [B_1^6]^{\frac{1}{3}} = A_1^2.$$

Certainly we have used again relations (6).

For the second equality from (9), let us consider a sequence of polynomials (P_n) with $P_n(0) = 0$, uniformly convergent on the spectrum of C, to the map $t \to t^{\frac{1}{3}}$.

Hence, it is easy to see that $P_n(C)B_1 + B_1P_n(C) = 0$ for all nonnegative integers n. Then, after passing to the limit, we get $RA_1 + A_1R = 0$.

Therefore, equalities (9) imply that

$$N^2 = \tfrac{1}{4}[R^2 - A_1^2 + i(RA_1 + A_1R)] = 0.$$

To finish the proof, it is enough to show that the operator $A = T - N$ is selfadjoint and commutes with N.

Let us notice that N can be written in the form

$$(10) \qquad N = \tfrac{1}{2}\{T - T^* - [(T - T^*)(TT^* - T^*T)]^{\frac{1}{3}}\}$$

and then A has the form

$$(11) \qquad A = \tfrac{1}{2}\{T + T^* + [(T - T^*)(TT^* - T^*T)]^{\frac{1}{3}}\}$$

For the commutativity we use the following lemma which has, in some way, an independent character

<u>LEMMA 3.2</u>

Let $\mathcal{A} \subset L(H)$ be a C^*-algebra, $\mathcal{N}(\mathcal{A}) = \{N / N \in \mathcal{A},\ N^2 = 0\}$ and the maps σ_+, σ_- defined by

$$(12) \qquad \sigma_{\pm}(N) = \tfrac{1}{2}[N - N^* \pm (N^*N)^{\frac{1}{2}} \mp (NN^*)^{\frac{1}{2}}] \text{ for all } N$$

Then we have

(i) $\quad \sigma_{\pm}(\mathcal{N}(\mathcal{A})) \subset \mathcal{N}(\mathcal{A})$

(ii) $\quad \sigma_+ \circ \sigma_- = \sigma_- \circ \sigma_+ = id$

(iii) $\quad N \in \mathcal{N}$ commutes with a selfadjoint operator $A \in L(H)$ if and only if A commutes with $N - N^*$ and NN^* (or N^*N).

<u>Proof</u>. Let us denote by $R_o = (N^*N)^{\frac{1}{2}} - (NN^*)^{\frac{1}{2}}$. Then

$$4.\sigma_+^2(N) = (N - N^*)^2 + R_o^2 + (N - N^*)R_o + R_o(N - N^*) = 0$$

if we show that $R_o^2 + (N - N^*)^2 = 0$ and $(N - N^*)R_o + R_o(N - N^*) = 0$.

We choose analogously a sequence of polynomials $(Q_n)_{n \geq 0}$, with $Q_n(0) = 0$

uniformly convergent on the interval $[0, ||N||^2]$ to the function $t \to t^{\frac{1}{2}}$.

Hence, for every nonnegative integer n, is not difficult to see that we have the equalities:

$$Q_n(N^*N)Q_n(NN^*) = 0, \; NQ_n(N^*N) = Q_n(NN^*)N \text{ and}$$

$$NQ_n(NN^*) = Q_n(N^*N)N = 0.$$

Then, after passing to the limit, it follows that $(N^*N)^{\frac{1}{2}}(NN^*)^{\frac{1}{2}}=0$ $N(N^*N)^{\frac{1}{2}} = (NN^*)^{\frac{1}{2}}N$ and $N(NN^*)^{\frac{1}{2}} = 0$.

Finally, these relations imply the desired ones:

$$(13) \quad R_o^2 = NN^* + N^*N = -(N - N^*)^2, \; R_o(N - N^*) + (N - N^*)R_o = 0.$$

Analogously $\sigma_-(N)^2 = 0$. This shows (i).

For the point (ii), let us compute $\sigma_-(\sigma_+(N))$:

$$\sigma_-(\sigma_+(N)) = \tfrac{1}{2}(\sigma_+(N) - \sigma_+(N)^* - [\sigma_+(N)^* \cdot \sigma_+(N)]^{\frac{1}{2}} + [\sigma_+(N)\cdot\sigma_+(N)^*]^{\frac{1}{2}} = \tfrac{1}{2}(N - N^* - [\sigma_+^*\sigma_+]^{\frac{1}{2}} + [\sigma_+\sigma_+^*]^{\frac{1}{2}})$$

This time, from (13) we have

$$\sigma_+^*\cdot\sigma_+ = \tfrac{1}{4}[R_o^2 - (N - N^*)^2 - (N - N^*)R_o + R_o(N - N^*)] = $$

$$= \tfrac{1}{2}[NN^* + N^*N - (N - N^*)R_o] = \tfrac{1}{4}[|N + N^*| - (N + N^*)]^2$$

where $|N + N^*| = (NN^* + N^*N)^{\frac{1}{2}}$.

Analogously

$\sigma_+\sigma_+^* = \tfrac{1}{4}[|N + N^*| + N + N^*]^2$ and by the uniquenes of the nonnegative square root, we, infer that

$$(\sigma_+^*\sigma_+)^{\frac{1}{2}} = \tfrac{1}{2}(|N + N^*| - N - N^*), \; (\sigma_+\sigma_+^*)^{\frac{1}{2}} = \tfrac{1}{2}(|N + N^*| + N + N^*)$$

Therefore $\sigma_-(\sigma_+(N)) = \tfrac{1}{2}(N - N^* + N + N^*) = N$ which proves (ii).

The equivalence from (iii) is easy to prove, in one way. Let us suppose that the selfadjoint operator A commutes with $N - N^*$ and N^*N.

Then A commute with NN^*, since $NN^* + N^*N + (N - N^*)^2 = 0$. Hence, A commutes with $(N^*N)^{\frac{1}{2}}$, $(NN^*)^{\frac{1}{2}}$ and then with $\sigma_+(N)$ and $\sigma_+^*(N)$. Finally, A commutes with $\sigma_-(\sigma_+(N)) = N$ as required.

Returning to the proof of theorem 3.1, on the account of the above lemma, we must show that A commutes with $N - N^* = T - T^*$ and NN^*.

Using (11), the commutation with $T - T^*$ becomes $2(TT^* - T^*T) = i(RA_1 - A_1R)$ which, by equalities (9), this is the same with:

(14) $$iRA_1 = T^*T - TT^*.$$

From this we obtain:

$$- iRA_1RA_1RA_1 = iR^3A_1 = (T^*T - TT^*)^3.$$

But $R^3 = iB_1(A_2 - B_2)$ and then

$iR^3A_1^3 = - B_1(A_2 - B_2)A_1^3 = (B_2 - A_2)A_1^4 = (B_2 - A_2)(B_2 + A_2)^2 =$
$= (B_2 - A_2)^3$ and here again by (9), $B_2 - A_2 = T^*T - TT^*$.
Hence, (14) holds.

To prove the commutation with NN^* we use the identities fulfilled by T:

$$C^3(T^*,T)(I) = C^3(T,T^*)(I) = 0.$$

More precisely we have

$$C^3(T,T^*)(I) = T^2(T - T^*) - 2T(T - T^*)T^* + (T - T^*)T^{*2} = 0$$

and replacing $T = A + N$ and using the fact already proved, that A commute with $N - N^*$ we obtain the desired equality: $ANN^* = NN^*A$.

Let observe that the formulas (10) and (11) which give the operators A and N as functions of T and T^*, are specific to one case of noncommutig functional calculus.

Now, we consider the general case of the Theorem 3.1. From (ii), we get that the maps Φ_T and Φ_{T^*} are polynomials of degree at most $2k - 2$.

. // .

Hence, as we have said, we apply the Theorem A of factorization from [11] due to Rosenblum and Rovnyak, which generalizes the analogous classical result of Fejer and Riesz.
Moreover, we need the following variant of Theorem 2.4 from [10], concerning the uniquenes of such factorization.

THEOREM B

Let G_1 and G_2 be two operator valued outer functions on $\mathbb{R}$. (in the sense of [11]).

(i) We have $G_1^*G_1 = G_2^*G_2$ a.e. if and only if $G_2 = UG_1$ $G_1 = U^*G_2$ a.e. where U is a (constant) partial isometric operator on H.

(ii) If $G_1^*G_1 = G_2^*G_2$, for continuously G_1, G_2, and $G_1(t_o) = G_2(t_o)$ for some fixed $t_o \in \mathbb{R}$, then $G_1 = G_2$.

The proof of this variant is obvious, via the original result from [11].

By these theorems, there are unique polynomials u and v whose coefficients are operators on H, such that

(15) $$\Phi_T(t) = u^*(t)u(t), \quad \Phi_{T^*}(t) = v(t)v^*(t) \text{ for all } t \in \mathbb{R}.$$

(16) $$u(0) = v(0) = I, \quad u(t) = \sum_{j=0}^{k-1} U_j t^j \qquad v(t) = \sum_{j=0}^{k-1} V_j t^j$$

and u, v are outer functions on R (in sense of [7]).

Let us show first that $u(t)v(t) = v(t)u(t) = I$ for every real t.
Since $\Phi_T(t) \cdot \Phi_{T^*}(t) = \Phi_{T^*}(t)\Phi_T(t) = I$, we have

$$u^*(t)u(t)v(t)v^*(t) = v(t)v^*(t)u^*(t)u(t) = I \qquad (t \quad R)$$

But for small $t > 0$, $u(t)$ and $v(t)$ are invertible operators. That implies next equalities:

(17) $$(u(t)v(t))^* \cdot u(t)v(t) = u(t)v(t)(u(t)v(t))^* = I$$

We know that $u(t)v(t) = \sum_{j=0}^{s} W_j t^j$ for some integer $s \geq 0$ and $W_j \in L(H)$.

From (17) we see that $W_s^* W_s = 0$ if $s > 0$. This implies the fact that $W_j = 0$ for $j > 0$ and then

$$u(t)v(t) = v(t)u(t) = I$$

in fact for all $t \in \mathbb{R}$.

Now, for t and s in $\mathbb{R}$ we can write, by (15)

$$u^*(t + s)u(t + s) = \exp(-isT^*)u^*(t)u(t)\exp(isT) \text{ or}$$

(18) $u^*(t+s)u(t+s) = [u(s)\exp(-isT)u(t)\exp isT]^* u(s)\exp(-isT)u(t)\exp isT$

Let us observe that the map $t \to u(s)\exp(-isT)u(t)\exp isT$, for every fixed $s \in \mathbb{R}$, is the nontangential limit of some outer function on the half-plane $y > 0$ to $\mathbb{R}$ (in sense of [11]).

This happens since if G is outer on R and $X \in L(H)$ then it is also true for the maps $t \to XG(t)$ and $t \to G(t)X$. Therefore, (18) and Theorem 3.3(ii) give us the following

(19) $u(t + s) = u(s)\exp(-isT)u(t)\exp isT$

If we differentiate (19) with respect to s, and evaluate it at $s = 0$ we get

(20) $u'(t) = u'(0)u(t) + i[u(t), T]$

where $[A, B] = AB - BA$ for $A, B \in L(H)$.

From (15), an identication of coefficients gives $U_1^* + U_1 = iT - iT^*$ or $T + iU_1 = (T + iU_1)^*$. Then we can define the selfadjoint operator $A = T + iU_1$ and $N = -iU_1$.

Solving the equation (20) with these new notations we have

(21) $u(t) = \exp(-itA)\exp(itT)$.

Then if we invert this, we get

(22) $v(t) = \exp(-itT)\exp itA$.

Now, let us write that u and v are polynomials of degree at most $k - 1$:

(23) $C^k(A, T)(I) = C^k(T, A)(I) = 0$.

For $k = 2$ it is easy to see that (23) implies the commutation of A and T. This is true in general and we sketch the proof for $k = 3$, since it goes analogous for every k. Therefore (23) becomes

(24) $$A^3 - 3A^2T + 3AT^2 - T^3 = T^3 - 3T^2A + 3TA^2 - A^3 = 0$$

Since A is selfadjoint, there exist a spectral measure E concentrated on R (see [6]) which gives a spectral resolution of the identity $E(t) = E((-\infty, t])$ and we have the representation

$$A = \int_R t\,dE(t).$$

Let $\sigma = [a, b]$ and $\omega = [c, d]$ two closed and disjoint intervals and γ_1, γ_2 two closed Jordan curves, which surround respectively σ and ω and have no intersection. Then we get

$$E(\omega) = \frac{1}{2\pi i}\int_{\gamma_2} f(\xi)\,d\xi$$

where $f(\xi) = \int_{\mathbb{R}} \frac{1}{\xi - t}\cdot\chi_\omega(t)\,dE(t)$ is analytic on $C|\omega$.

Now, we use some ideas from [3] (where we can find a generalization of the present result):

<u>Proposition 3.4</u>

Let T L(H) and A L(H) selfadjoint such that $T \overset{q}{\sim} A$. Then T and A commute.

By computations from (24) the function

$$\tilde{f}(\xi) = f(\xi) - \frac{T - A}{1} f'(\xi) + \frac{(T - A)^{[2]}}{2!} f''(\xi)$$

satisfy the equality

$$(\lambda - T)\tilde{f}(\xi) = E(\omega) \quad \text{for } \xi \in C|\omega$$

and then the map

$$g(\xi) = T\tilde{f}(\xi) - \frac{A - T}{1!} T\tilde{f}'(\xi) + \frac{(A - T)^{[2]}}{2!} T\tilde{f}''(\xi)$$

satisfy the equality

$$(\lambda - A)g(\xi) = TE(\omega) \quad \text{for all } \xi \in C|\omega.$$

Then we have

$$E(\sigma)TE(\omega) = \frac{1}{2\pi i}\int_{\gamma_1} (\xi - A)^{-1}E(\sigma)TE(\omega)d\xi =$$

$$= \frac{1}{2\pi i}\int_{\gamma_1} E(\sigma)g(\xi)d\xi = 0. \qquad (\sigma \subset C|\omega)$$

Since, ω is arbitrary in $R\setminus\sigma$, this implies that $E(\sigma)T(I - E(\sigma)) = 0$. The same is true for T^* and then $TE(\sigma) = E(\sigma)T$ or equivalently $TA = AT$.

The proof of Proposition 3.4 can be made in the same fashion.

Returnig to the proof of Theorem 3.1, since A and T commute, (23) sais that $N = T - A$ is nilpotent of order k and that finishes the implication (ii) (i).

The lemma 3.2 is in fact inspired from the work of Helton [6].

Finally, I should like to take this opportunity and I express my gratitude to Professor Stefan Frunza which was my first teacher in the field of Operators Theory and have suggested me these problems.

Institute of Mathematics of the Romanian Academy

P.O. Box 1 - 764

70700 Bucharest, R o m a n i a

REFERENCES

1. Agler, J., Sub-Jordan operators, Thesis, Indiana University (1980)

2. Agler, J., Sub-Jordan operators: Bishop's theorem, spectral inclusion and spectral sets, J. Operator Theory, 7(1982) 373-395

3. Colojoară, I; Foias, C., Theory of generalized spectral operators Gordon and Breach, New York 1968

4. Dunford, N; Schwartz, J.T., Linear Operators III Spectral Operators, Interscience Publishers, New York, 1971

5. Frunză, Ş t., Jordan Operators on Hilbert Spaces, J. Operator Theory 18(1987), 201-212

6. Helton, J.W., Infinite dimensional Jordan operators and Sturm-Liouville conjugate point theory, Trans. Amer. Math. Soc. 170(1972), 305-331

7. Joseph A. Ball and J. William Helton, Nonnormal dilations, disconjugacy and constrained spectral factorization, Integral Eq. and Op. Theory, vol. 3/2 1980

8. Joseph A. Ball and Thomas R. Fanney, Closability of differential operators and real subjordan operators, Operator Theory: Advances and Applications, vol. 48, 1990 Birkhauser Verlag Basel

9. Martin Schechter, Principles of functional analysis, Acad. Press New York and London

10. Rosenblum, M; Rovnyak, J., The factorization problem for nonnegative operator valued functions, Bulletin of A.M.S. vol. 77, nr. 3, 1971

11. Rosenblum, M & Rovnyak, J., Hardy Classes and Operator Theory, New York: Oxford University Press, 1985