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ON DECOMPOSITION AND MANIFOLD STRUCTURE OF NONLINEAR CONTROL SYSTEMS

by

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1. Introduction

In a previous paper (see [1]) any solution for an affine control system
1)

$$\frac{dv}{dt}=f(x)+\sum_{i=1}^{m} u_i(t)\,g^i(x),\ t\in[0,T],\ x(0)=x_0\in R^n,\ u_i(.)\in L^1([0,T];R)$$

was represented using a diffeomorphism $G(p;x)$, $x\in R^n$, on R^n, which is the solution for a "gradient system".

2) $$\frac{\partial G}{\partial t_j}=X^j(p;G),\ j=1,\dots,M,\ G(0;x)=x,\ p=(t_1,\dots,t_M)\in R^M$$

and considering p as the new control guided by a controllable system $(\dim L(q^1,\dots,q^m)(p)=M,\ p\in R^M)$

3) $$\frac{dp}{dt}=\sum_{i=1}^{m} u_i(t)\ q^i(p),\ p(0)=0$$ where the smooth vector fields $X^j(p)\in C^\infty(R^n)$ depending on parameters $p\in R^M$, and $q^i\in C^\omega(R^M)$ are found such that

4) $$\sum_{j=1}^{M} X^j(p)\,q^i_j(p)=g^i,\ i=1,\dots,m.$$

The analysis was focused on the noncommuting smooth vector fields $g^1,\dots,g^m\in Vect\,(R^n)$ and the assumption that the Lie algebra $L(g^1,\dots,g^m$ is finitely generated over R provides the main tool of proofs.

It is the purpose of this paper to include f explicitely into analysis and it could be partially motivated by nonlinear control systems $\frac{dx}{dt}=f(x,u)$, $x\in R^n$, $u\in R^m$ which can be rewritten as affine control systems (1) in a larger state space $y\in R^{n+m}$

for which the ideal I_f generated in $L(f,g^1,\ldots,g^m)$ by the new drift $f(y)$ is more meaningfull than the Lie algebra $L(g^1,\ldots,g^m)$ (see application).

The main contribution can be stated as follows. We are given C^∞ vector fields $f,g^1,\ldots,g^m \in Vect(R^n)$ which are not commuting and if the ideal I_f generated by f in $L(f,g^1,\ldots,g^m)$ is finitely generated over R then new smooth vector fields $X^j(t,p) \in Vect\,(R^n)$ depending on parameters $t \in R,\ p \in R^M,\ j=0,1,\ldots,M$, and analytical vector fields $q^0,q^1,\ldots,q^m \in Vect(R^{M+1})$ are found such that

a) $$dy=f(y)\,dt+\sum_{j=1}^{M} X^j(t,p;y)\,dt_j,\quad p\in R^M \text{ is a Frobenius system}$$

b) $$\sum_{j=0}^{M} X^j(t,p)\,q_j^i(t,p)=g^i,\quad i=0,1,\ldots,m,$$

where $g^0 \triangleq f \triangleq X^0(t,p)$, $q^0=(1,0,\ldots,0)^T$.

Theorem 1 contains the above equalities and allow one to represent solutions in (1) by solving a "gradient system".

5) $$\frac{\partial G}{\partial t}=f(G),\quad \frac{\partial G}{\partial t_j}=X^j(t,p;G),\quad j=1,\ldots,M,\quad G(0,0;x_0)=x_0$$

and considering $\tilde p=(t,p)$ as the new control guided by a controllable system $(\dim I_{q^0}(q^1,\ldots,q^m)(\tilde p)=M,\ \tilde p\in R^{M+1}$.

6) $$\frac{d\tilde p}{dt}=q^0(\tilde p)+\sum_{i=1}^{m} u_i(t)\,q^i(\tilde p),\quad \tilde p(0)=0\in R^{M+1}$$

As one may expect the dimension of $I_f(g^1,\ldots,g^m)(x)$ is not a constant one for $x\in R^n$ and to generate integral manifolds containing solutions in (1) with x_0 fixed is the purpose of the

Theorem 2 which states that for an arbitrary $x_0 \in R^n$ with $\dim I_f(g^1,\dots,g^m)(x_0)=k$, $k\le n$, there exists a generator system $Y^1,\dots Y^k,\dots Y^M$ for I_f such that any solution in (1) starting with $x(0)=x_0$ can be represented as

c) $x^u(t;x_0)=G(t,\hat{p}^u(t);x_0)$, $t\in[0,T]$,

where $\hat{p}^u(t)$, $t\in[0,T]$, is the solution in a controllable system and $G(t,\hat{p};x_0)$, $\hat{p}\in R^k$, generates a k-dim C^∞ manifold $M_t \ni x_0$ for each $t\in[0,T]$ and

d) $I_f(g^1,\dots,g^m)(y)=T_yM_t$ for any $y=G(t,\hat{p};x_0)$.

2. Definitions and some auxiliary results

Some definitions and auxiliary results we use here were given in [1]. Let $C^\infty(R^n)$ be the algebra of infinite differentiable functions on R^n and $C^\omega(R^n)\subseteq C^\infty(R^n)$ consisting of all analytical entire functions. Vector fields are R-linear mappings of $C^\infty(R^n)$ into itself. The Lie bracket $[X,Y]$ introduces the Lie algebra structure in the space $Vect(R^n)$ and for any $X\in Vect(R^n)$ define $ad\,X: Vect(R^n)\to Vect(R^n)$ by $ad\,X(Y)=[X,Y]$ where

$[X,Y](x)=((\partial X/\partial x)Y-(\partial Y/\partial x)X)(x)$.

The application $\exp ad\,X: Vect(R^n)\to Vect(R^n)$ is defined formally as the Taylor series $(\exp ad\,X)(Y)=Y+\frac{1}{1!}adX(Y)+\dots+\frac{1}{n!}ad^nX(Y)+\dots$ and the convergence in defined by the topology of uniform convergence of all derivatives on compact subsets of R^n.

We are given $f,g^1,\dots,g^m\in Vect(R^n)$ and denote

$L(f,g^1,\ldots,g^m)$, $I_f(g^1,\ldots,g^m)$ the Lie algebra and respectively the ideal generated by f in $L(f,g^1,\ldots,g^m)$. By definition, I_f coincides with the Lie algebra on R generated by the vector fields $ad^k f(g^i)$, $k\geq 0, i=1,\ldots,m$.

Definition 1

We say that $I_f(g^1,\ldots,g^m)$ is finitely generated over R if there exist $Y^1,\ldots,Y^M \in I_f(g^1,\ldots,g^m)$ such that any $Y\in I_f(g^1,\ldots,g^m)$ can be written $Y(x)=\sum_{j=1}^{M} a_j Y^j(x)$, $x\in R^n$, with $a_j\in R$ depending on Y . In the sequel we shall use I_f for $I_f(g^1,\ldots g^m)$. If I_f is finitely generated over R (see Lemma 1) then the exponential map $(\exp ad\, tX)(Y)$ is well defined for any $X,Y\in I_f$ and $t\in R$. Let $B=\{Y^1,\ldots,Y^M\}$ be a generator system for I_f . We define the new corresponding vector fields $X^j(t,p)\in Vect(R^n)$ depending on parameters $(t,p)\in RxR^M$,

7)

$$X_0=f, x^1(t)=(\exp adtX^0)(Y^1),$$

$$X^{j+1}(t,t_1,\ldots,t_j)=(\exp ad\, t_jX^j),\ldots,(\exp ad\, t_1X^1)(\exp ad\, tX^0)(Y^{j+1})$$

$j=1,\ldots, M-1$.

For an easier reference we restate the auxiliary results given in [1].

Lemma 1

If I_f is finitely generated over R then

c_1) $X^{j+1}(t,t_1,\ldots,t_j)=(\exp ad\, tf)(\exp ad\, t_1Y^1)\ldots(\exp ad\, t_jY^j)(Y^{j+1})$

$j=0,1,\ldots,M-1$

c_2) $X^{j+1}(t,t_1,\ldots,t_j)=(\exp ad\, t_kX^k)\ldots(\exp ad\, tX^0)(\exp ad\, t_{k+1}Y^{k+1})$.

$\ldots(\exp ad\, t_j Y^j)(Y^{j+1})$

$0 \le k \le j-1, \quad j=1,2,\ldots, \quad M-1$

In addition (C_1) and (C_2) hold for $T^{j+1}(t,t_1,\ldots,t_j)(Y)$ and $T^{j+1}(t,t_1,\ldots,t_j)(Y)\in I_f$ for any $Y\in I_f$, $j=0,1,\ldots,M-1$, where $T^{j+1}(t,t_1,\ldots,t_j): I_f \to I_f$ is the linear application obtained from $X^{j+1}(t,t_1,\ldots,t_j)$ replacing Y^{j+1} by Y. Using Lemma 1 the vector fields defined in (7) meet the following commuting property.

8) $\quad [X^j, X^i] = \partial_j X^i, \; j=0,1,\ldots,i-1, \; i=1,\ldots,M$

where $\partial_j X^i = \partial X^i/\partial t_j$, $t_0 = t$, and the Lie bracket is taken with respect to $x\in R^n$.

Lemma 2

Assume that I_f is finitely generated over R. Then the vector fields $X^0, X^1, \ldots, X^M$ in (7) meet the Frobenius commuting property (8).

Write $\tilde{p}=(t,p)$ and denote $\partial^r a/\partial\tilde{p}^r$ the multiindex partial derivative with $r=r_0+r_1+\ldots+r_M$; $r\in N$, $\partial\tilde{p}^r \triangleq (\partial^{r_0}t)(\partial^{r_1}t_1),\ldots,(\partial^{r_M}t_M)$.

Lemma 3

Assume that I_f is finitely generated over R and let $\{Y^1,\ldots,Y^M\}$ be a generator system for I_f.

Let $X^j(t,p)$, $j=0,1,\ldots,M$, $t\in R$, $p\in R^M$ be the vector fields defined in (7). Then each $X^j(t,p)$, $j=1,\ldots,M$, can be written

$X^j(t,p)=\sum_{k=1}^{M} a_k^j(t,p)Y^k$ with $a_k^j \in C^\infty(R^{M+1})$ fulfilling

$c_1)$ $\quad (\det A(t,p))^{-1}$

is in $C^\infty(R^{M+1})$, where $A(t,p)=(a^1(t,p),\ldots,a^M(t,p))$

c_2) $|(\partial^r a_k^j/\partial\tilde{p}^r)(0)|\le C^{r+1}$, $(\forall)$ $r=r_0+r_1+\ldots+r_M\ge 0, r\in N$,

for some $C>0$.

3. The decomposition and manifold structure for affine control systems

Let $B=\{Y^1,\ldots,Y^M\}\subset I_f$ be a generator system of I_f over R and without loosing generality we assume that the first vector fields in B are the original ones $g^1,\ldots,g^m$. Define new vector fields $X^j(t,p)$ as in (7) (via Lemma 1).

9) $X^0=f$, $X^1(t)=(\exp adtf)(g^1)$,

$$X^m(t,t_1,\ldots,t_{m-1})=(\exp adtf)(\exp adt_1g^1)\ldots(\exp adt_{m-1}g^{m-1})(g^m)$$

$$X^{j+1}(t,t_1,\ldots,t_j)=(\exp adtf)(\exp adt_1g^1)\ldots(\exp adt_jY^j)(Y^{j+1})$$

$j=m,\ldots,M-1$.

Using Lemma 2, the following "gradient system"

10) $$\frac{\partial G}{\partial t}=f(G),\ \frac{\partial G}{\partial t_1}=X^1(t;G),\ldots,\frac{\partial G}{\partial t_M}=X^M(t,t_1,\ldots,t_{M-1};G),$$

meet the Frobenius commuting conditions in (8).

Theorem 1. Let $f,g^1,\ldots,g^m\in Vect(R^n)$ be given. Let I_f be finitely generated over R and $\{Y^1,\ldots,Y^M\}\supset\{g^1,\ldots,g^m\}$ a generator system for I_f . Let $X^j(t,p)$, $j=0,1,\ldots,M$, be the vector fields defined in (9). Then there exist analytical vector fields $q^i\in Vect(R^{M+1})$, $i=0,1,\ldots,m$, such that

a_1) $\sum_{j=0}^{M}X^j(t,p)q_j^i(t,p)=g^i$, $i=0,1,\ldots,m$, where $X^0=f=g^0$ and

$q^0=(1,0,\ldots,0)^T\in R^{M+1}$

a_2) $\dim I_{q^0}(q^1,\ldots,q^m)(t,p)=M$ $(\forall)(t,p)\in R^{M+1}$.

In addition, assume that each $X^j(t,p)$ generates a flow and denote $G_j(\tau)(x)$, $\tau \in R$, $x \in R^n$ the flow generated by

Y^j, $j=0,1,\ldots,M(Y^0=f)$. Then the solution in (10) with

$G(0,0;x)=x$ can be written

b_1) $G(t,p;x)=G_0(t) \circ G_1(t_1) \circ \ldots \circ G_M(t_M)(x)$ and

b_2) $g \in I_f(g^1,\ldots,g^m)$ iff $q \in I_{q^0}(q^1,\ldots,q^m)$ will exist such

that $\sum_{j=0}^{M} X^j(t,p)\, q_j(t,p)=g \quad (t,p)\in R^{M+1}$.

Proof

The proof is similar to that of Theorem 1 in [1].

By hypothesis the conclusions in Lemma 3 hold and each $X^j(t,p)$ can be written

11) $X^j(t,p)=\sum_{k=1}^{M} a_k^j(t,p)\,Y^k$, $j=1,\ldots,M$ where $a_k^j \in C^\omega(R^{M+1})$ and

12) $|(\partial^r a_k^j/\partial \tilde{p}^r)(0)| \le C^{r+1}$, $(\forall)$ $r=r_0+\ldots+r_m$, $r \in N$ for some fixed $C>0$.

Since $X^0=f=g^0$ and $Y_i=g_i$, $i=1,\ldots,m$ it is obvious that solving

$\sum_{j=1}^{M} X^j(t,p)\; q_j^i(t,p)=g^i$ is equivalent to finding $b^i \in R^M$ such that

13) $\sum_{j=1}^{M} a^j(t,p)\, b_j^i(t,p)=e_i$, $i=1,\ldots,$ where $e_1,\ldots,e_M$ is the

canonical base in R^M, $a^j(t,p)=(a_1^j(t,p),\ldots,a_M^j(t,p))^T$ and a_k^j are

defined in (11). Denote $A(t,p)=(a^1(t,p),\ldots,a^M(t,p))$ and using [1] (see application) the analytical scalar function $(\det A(t,p))^{-1}$ can be excplicitely computed which allow one to

find analytical functions $b^1(t,p),\ldots, b^m(t,p)$ fulfilling (13).

Define $q^0(t,p)=(1,0,\ldots,0)^T \in R^{M+1}$, $q^i(t,p)=\begin{pmatrix} 0 \\ b^i(t,p)\end{pmatrix} \in R^{M+1}$, $i=1,\ldots,m$

and using (13) we get

14) $\sum_{j=0}^{M} X^j(t,p)\, q_j^i(t,p) = g^i,\ i=0,1,\ldots,m,\ \ g^0 \triangleq f.$

The second part in Theorem 1 is based on the existence of solutions in (10) which allow one to write

15) $\frac{\partial G}{\partial t_j} = X^j(t,p;G(t,p)),\ j=0,1,\ldots,M,\ t_0 \triangleq t$

$$\sum_{j=0}^{M} \frac{\partial G}{\partial t_j}(t,p)\, q_j^i(t,p) = g^i(G(t,p)),\ i=0,1,\ldots,m$$

Taking directional derivatives in (15) we obtain

16) $\sum_{j=0}^{M} \frac{\partial G}{\partial t_j}(t,p)\, [q^{i_1},q^{i_2}]_j(t,p) = [g^{i_1},g^{i_2}]\,(G(t,p))$

for any $i_1,i_2 \in \{0,1,\ldots,m\}$.

Since the solution $G(s,r;x)$, $(s,r) \in V(t,p)$ in (10) was defined such that $G(t,p;x)=x$, from (16) we get

17) $\sum_{j=0}^{M} X^j(t,p;x)\, [q^{i_1},q^{i_2}]_j(t,p) = [g^{i_1},g^{i_2}]\,(x)$

for any $x \in R^n$ and $i_1,i_2 \in \{0,1,\ldots,m\}$.

Repeating what is done in (17) we obtain a homomorphism between the two algebras $L(g^0,g^1,\ldots,g^m)$, $L(q^0,q^1,\ldots,q^m)$ such that $g \in L(g^0,\ldots,g^m)$ iff $q \in L(q^0,\ldots,q^m)$ will exist fulfilling

18) $\sum_{j=0}^{M} X^j(t,p)\, q_j(t,p) = g,\ (t,p) \in R^{M+1}$

Starting in (10) with $G(0,0;x)=x$ and noticing that $X^{j+1}(t,t_1,\ldots,t_j;x) = Y^{j+1}(x)$ for $t=t_1=\ldots=t_j=0,\ j=0,1,\ldots,M-1$ we get the solution in (10) as is defined in (b_1) and the proof is complete.

The following theorem states the manifold structure of the solution in (10) with $x_0 \triangleq G(0,x_0) \in R^n$ fixed which provides the support of all solutions in (1) with $x(0)=x_0$.

Remark. If I_f is finitely generated over R with $\{Z^1,\dots,Z^M\}$ as a generator system and $\dim I_f(g^1,\dots,g^m)(x_0)=k\le n$ then there exists a generator system $\{Y^1,\dots,Y^M\}$ for I_f such that

c_1) $Y^1(x_0)\dots,\ Y^k(x_0)$ are linearly independent in R^n.

c_2) $Y^j(x_0)=0,\ j=k+1,\dots,M$,

c_3) $\{Y^1,\dots,Y^M\}=\{Z^1,\dots,Z^M\}T$, with a nonsingular $T\in L(R^M;R^M)$.

Theorem 2

Let $x_0\in R^n$ be fixed and $\dim I_f(g^1,\dots,g^m)(x_0)=k\le n$.

Let I_f be finitely generated over R and $\{Y^1,\dots,Y^M\}$ a generator system for I_f which meets $(c_1),(c_2)$ in the Remark. Assume that each Y^i generates a flow

$G_i(\zeta)(x),\ \zeta\in R,\ x\in R^n,\ i=0,1,\dots,k$, where $Y^0\triangleq f$. Write

$\hat{p}=(t_1,\dots,t_k)$ and define $G(t,\hat{p};x_0)=G_0(t)\circ G_1(t_1)\circ\dots\circ G_k(t_k)(x_0)$.

Then

α) i) $S_t\triangleq\{G(t,\hat{p};x_0)\ \hat{p}\in R^k\}\subseteq R^n$ is a k-dim C^∞manifold,

ii) $I_f(g^1,\dots,g^m)(y)=T_yS_t,\ (\forall)\ y\in S_t$ where " T_y " means tangent space

iii) $I_f(g^1,\dots,g^m)(G(t,\hat{p};x_0))=\mathrm{span}\{\dfrac{\partial G}{\partial t_i}(t,\hat{p};x_0),i=1,\dots,k\}$

β) there exist analytical entire functions $b_i(t,\hat{p})\in R^k$ such that any solution $x_t^u(x_0),\ t\in[0,T]$, in (1) can be rewritten as

$x_t^u(x_0)=G_0(t)(y^u(t))$, where

$y^u(t)\in S_0,\ y^u(t)=G(0,\hat{p}(t);x_0),\ t\in[0,T]$, and

$$\frac{d\hat{p}(t)}{dt}=\sum_{i=1}^{m} u_i(t) b_i(t,\hat{p}(t)), \quad \hat{p}(0)=0, \quad t\in[0,T] .$$

Proof

By hypothesis the conclusions in Lemmas 1,2 and 3 hold. Let $\{Y^1,\ldots,Y^M\}$ be the generator system given by hypothesis and $X^j(t,p)$, $j=1,\ldots,M$, defined as in Lemma 1. Since $Y^j(x_0)=0, j=k+1,\ldots,M$ we get that the solution in the following Frobenius system

19) $$\frac{\partial G}{\partial t}=f(G), \quad \frac{\partial G}{\partial t_j}=X^j(t,p;G), \quad j=1,\ldots,M, p\in R^M, G(0,0;x_0)=x_0$$

can be written as

20) $$G(t,p;x_0)=G_0(t)\, o G_1(t_1)\, o\ldots o G_k(t_k)(x_0) \triangleq G(t,\hat{p};x_0)$$

for any $t\in R, p\in R^M$, where $\hat{p}=(t_1,\ldots,t_k)$ .On the other hand the matrix $\frac{\partial G}{\partial x}(t,\hat{p};x_0)$ is a nonsingular one and

21) $$[\frac{\partial G}{\partial x}(t,\hat{p};x_0)]^{-1}=[\frac{\partial G_k}{\partial x}(t_k;x_0)]^{-1}\ldots[\frac{\partial G_1}{\partial x}(t_1;x_{k-1})]^{-1}[\frac{\partial G_0}{\partial x}(t;x_k)]^{-1}$$

where $x_1=G_k(t_k)(x_0),\ldots,x_{k-1}=G_1(t_1)\, o\ldots o G_k(t_k)(x_0), x_k=G(t,\hat{p};x_0)$. By definition (see (20))

22) $$\frac{\partial G}{\partial t_j}(t,p;x_0)=X^j(t,p;G(t,\hat{p};x_0))=0 \quad j=k+1,\ldots,M$$ for

any $t\in R, p\in R^M, \hat{p}\in R^k$. We shall show that $\frac{\partial G}{\partial t_j}(t,\hat{p};x_0)$, $j=1,\ldots,k$, are linearly independent in R^n by proving that

23) $$\hat{X}^j(t,\hat{p}) \triangleq [\frac{\partial G}{\partial x}(t,\hat{p};x_0)]^{-1}X^j(t,\hat{p};G(t,\hat{p};x_0)), \quad j=1,\ldots k$$

are linearly independent.

A straight computation shows that we have the following representations

24) $$\hat{X}^j(t,\hat{p})=\{(\exp ad{-t_k}Y^k)\ldots(\exp ad{-t_{j+1}}Y^{j+1})Y^j\}(x_0),$$

$$\hat{X}^k(t,\hat{p})=Y^k(x_0), \quad j\le k-1 .$$

Denote B^j a (MxM) matrix associated to Y^j using the matrix representation of $[Y^j,Y^1],\dots,[Y^j,Y^k],\dots[Y^j,Y^M]$ according to the fixed generator system $\{Y^1,\dots,Y^M\}$. Using (24), the equations (23) can be rewritten

25)

$$\hat{X}^j(t,\hat{p})=\{Y^1(x_0),\dots,Y^k(x_0),0,\dots,0\}(\exp-t_kB^k)\dots(\exp-t_{j+1}B^{j+1})\,l_j$$

$$\hat{X}^k(t,\hat{p})=\{Y^1(x_0),\dots,Y^k(x_0),0,\dots,0\}l_k,\ j=1,\dots,k-1$$

where $l_i\in R^M$, $i=1,\dots,M$, is the canonical base.

Define a (Mxk) matrix $A(\hat{p})$ by

26) $A(\hat{p})=(\hat{a}(\hat{p}),\dots,a^k(\hat{p}))$, where $a^j(\hat{p})\in R^M$, meets

27) $a^j(\hat{p})=(\exp-t_kB^k)\dots(\exp-t_{j+1}B^{j+1})\,l_j,\ j=1,\dots k-1,\quad a^k(\hat{p})=l_k$

(see (24)).

From each $a^j(\hat{p})$ eliminate the last $M-k$ components and denote it by $\hat{a}^j(\hat{p})\in R^k$. Write $\hat{A}(\hat{p})=(\hat{a}^1(\hat{p}),\dots,\hat{a}^k(\hat{p}))$ and (25) becomes

28) $\hat{X}(t,\hat{p})\triangleq\hat{X}(\hat{p})=\{Y^1(x_0),\dots,\ Y^k(x_0)\}\hat{A}(\hat{p})$

where $\hat{X}(t,\hat{p})=\{\hat{X}^1(t,\hat{p}),\dots,\ \hat{X}^k(t,\hat{p})\}$ and $\hat{A}(\hat{p})$ a nonsingular matrix for which $(\det\hat{A}(\hat{p}))^{-1}$ can be computed as an analytical entire function (see application in [1]). Using (28) in (23) we get that $\frac{\partial G}{\partial t_j}(t,\hat{p};x_0),j=1,\dots k$, are linearly independent and therefore S_t meets the first conclusion in the theorem. Further we represent $g^i(G(t,\hat{p};x_0)),\ i=1,\dots,m$, using $X^j(t,\hat{p};G(t,\hat{p};x_0))$, $j=1,\dots,k$. From Lemma 3 and application in [1] we get the representation for $X^j(t,p),\ j=1,\dots,M$, in (19), as

29) $\{X^1(t,p),\dots,\ X^M(t,p)\}=\{Y^1,\dots,Y^M\}A(t,p)$

where $A(t,p)\in\mathcal{L}(R^M,R^M)$ is a nonsingular matrix for which

$(\det A(t,p))^{-1}$ is an analytical entire function which provides that $\{X^1(t,p),\dots,X^M(t,p)\}$ is a generator system for I_f and therefore (ii) and (iii) in (α) hold. On the other hand there exists a nonsingular $T\in\mathcal{L}(R^M,R^M)$ such that $B\triangleq\{g^1,\dots,g^m,$ $\hat{Y}^{m+1},\dots,\hat{Y}^M\}=\{Y^1,\dots,Y^M\}T$ and B is a generator system for I_f.
To solve

30) $\sum_{j=1}^{M} X^j(t,p)q^j(t,p)=g$ where $g\in\{g^1,\dots,g^m\}$ is enough to find the solution $q(t,p)$ for

31) $T^{-1}A(t,p)q(t,p)=e$, where $e\in\{1_1,\dots,1_m\}$. Therefore, we get

32) $q(t,p)=A^{-1}(t,p)Te$, $e\in\{1_1,\dots,1_m\}$ and using (32) in (30)

33) $\sum_{j=1}^{M} X^j(t,p;G(t,\hat{p};x_0))q_j(t,p)=g(G(t,\hat{p};x_0),g\in\{g^1,\dots,g^m\}$.

Since $X^j(t,p;G(t,\hat{p};x_0))=0$, $j=k+1,\dots,M$, (see (22)), from (33) follows easily

34) $\sum_{j=1}^{k} X^j(t,\hat{p};G(t,\hat{p};x_0))\hat{q}_j(t,\hat{p})=g(G(t,\hat{p};x_0))$

for any $g\in\{g^1,\dots,g^m\}$, where $\hat{q}(t,\hat{p})=pr_{R^k}q(t,\hat{p})$. Write $\hat{q}^i(t,\hat{p})$ for the solution in (34) corresponding to g^i, $i=1,\dots,m$, and define the system

35) $\frac{d\hat{p}}{dt}=\sum_{i=1}^{m} u_i(t)\hat{q}^i(t,\hat{p})$, $\hat{p}(0)=0$, $t\in[0,T]$

whose solutions meets the conclusion (β).

The proof is complete.

<u>Application</u>

Nonlinear control systems could be viewed as affine control systems and the corresponding ideal I_f is defined as follows.

Let $f(x,u):R^n x U \to R^n$ be a C^∞ function and consider the corresponding system,

1) $\frac{dx}{dt}=f(x,u(t)), x(0)=x_0, u(t)\in U\subseteq R^m,\ t\in[0,T]$

For any admissible control $u(t)=u_0+\int_0^t v(s)\,ds, u(t)\in U$ with

$v(\cdot)\in L_1([0,T];R^m)$, the system (1) is rewritten

2) $\frac{dy}{dt}=f(y)+\sum_{i=1}^{m} v_i(t)g^i,\ y=\begin{pmatrix}x\\u\end{pmatrix},\ y(0)=\begin{pmatrix}x_0\\u_0\end{pmatrix},\ v(\cdot)\in L^1([0,T];R^m)$,

where $f(y)=\begin{pmatrix}f(x,u)\\0\end{pmatrix},\ g^i=\begin{pmatrix}0\\l_i\end{pmatrix}\in R^{n+m}$, and $l_1,\ldots,l_m\in R^m$ is the canonical base.

It is easily seen that the "gradient" and "controllable" systems (see th.1) as well as the description of the manifold supporting "gradient" system (see th.2) for (2) projected on R^n define the corresponding decomposition and supporting manifold for (1). The definition of the ideal generated by f in $L(f,g^1,\ldots,g^m)$ is based on

$adf(g^i)(y)=\begin{pmatrix}\frac{\partial f}{\partial u}(x,u)\,l_i\\0\end{pmatrix}=\begin{pmatrix}\frac{\partial f}{\partial u_i}(x,u)\\0\end{pmatrix}\triangleq\begin{pmatrix}h_i(x,u)\\0\end{pmatrix}$ and

$ad^k f(g^i)(y)=\begin{pmatrix}ad_x^{k-1}f(h_i)(x,u)\\0\end{pmatrix}$ for any $k\geq 1, i=1,\ldots,m$, where

$ad_x f(h_i)(x,u)=\frac{\partial f}{\partial x}(x,u)\,h_i(x,u)-\frac{\partial h_i}{\partial x}(x,u)\,f(x,u)$. We get that

$I_f(g^1,\ldots,g^m)$ can be written as a Lie algebra

$L(b_1,\ldots,b_m, ad^k f(h_i), k\geq 0,\ i=(1,\ldots,m))$ over R , where

$b_j,\ ad^k f(h_i)\in Vect(R^{n+m})$ are given by $b_j(\varphi)\triangleq\partial\varphi/\partial u_j,\ ad^k f(h_i)(\varphi)=$

$$\sum_{p=1}^{n} ad_x^k\, f(h_i)_p \,\partial\varphi/\partial x_p \quad, \; j=1,\ldots,m, \; i=1,\ldots,m, \text{ for any } \varphi \in C^\infty(R^{n+m})$$

Definition 1

We say that $I_f(g^1,\ldots,g^m)$ is finitely generated over real polynomial $P(u_1,\ldots,u_m)$ if there exist $Y^1(x),\ldots, Y^M(x) \in I_f$ depending only on $x \in R^n$ such that any $Y \in I_f$ can be written

$$Y(x,u) = \sum_{i=1}^{M} c_i(u)\, Y^i(x), \text{ where } c_i(u),\; i=1,\ldots,M,$$

are real polynomials of $(u_1,\ldots,u_m)$ depending on Y.
To rewrite an arbitrary solution in (1) and to describe the corresponding supporting manifold we need to assume that $I_f(g^1,\ldots,g^m)$ is finitely generated over real polynomials $P(u_1,\ldots,u_m)$ which can be accomplished if we impose that the Lie algebra $L(ad_x^k\, f(h_j),\; j=1,\ldots,\tilde{N},\; k \geq 0)$ is finitely generated over $P(u_1,\ldots,u_m)$, where $h_1(x,u),\ldots,h_{\tilde{N}}(x,u)$ are all partial derivatives of $f(x,u)$ with respect to $u_1,\ldots,u_m$.
The computations and results in Theorems 1 and 2 do not change essentially.
Even more, working with affine control systems

$$\frac{dx}{dt} = f(x) + \sum_{i=1}^{m} u_i g^i(x), \; x \in R^n, \; f, g^i \in Vect(R^n)$$

the concept of finite generation over R can be replaced by the following one

Definition 2

Let $x_o \in R^n$ and $G(t_1,\ldots,t_k)(x_o)$ be an orbit of $I_f(g^1,\ldots,g^m)$ starting from x_o. We say that $I_f(g^1,\ldots,g^m)$ is finitely generated with respect to orbits starting from x_o if there exist

$Y^1(.),\ldots,Y^M(.) \in I_f(g^1,\ldots,g^m)$ such that any $Y \in I_f(g^1,\ldots,g^m)$ along to $G(t_1,\ldots,t_k)(x_o)$ can be written $Y(G(t_1,\ldots,t_k)(x_o)) =$

$$= \sum_{j=1}^{M} a_j(t_1,\ldots,t_k)\, Y^j(G(t_1,\ldots,t_k)(x_o)) \text{ with } a_j \in C^\infty(R^k) \text{ depending on } Y,$$

and $G(t_1, \ldots, t_k)(x_o)$.

It is my believe that using new concept in definition 2 the decomposition and manifold structure stated in theorems 1 and 2 will preserve the content.

References

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