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In the study of excessive measures, an important tool is the energy functional. Whenever $\mathcal{V} = (V_\alpha)_{\alpha>0}$ is a submarkovian resolvent of kernels on a measurable space $(X, \mathcal{X})$, two outstanding ordered convex cones arise. First, the convex cone $\mathcal{E}(\mathcal{V})$ of all excessive functions on X, and second, the convex cone $\mathrm{Exc}(\mathcal{V})$ of all excessive measures on $\mathcal{X}$. On the cartesian product $\mathrm{Exc}(\mathcal{V}) \times \mathcal{E}(\mathcal{V})$ is then defined the energy functional L, which generalises the notion of energy in classical Potential Theory. On the other hand, for any ordered convex cone C one consider it's dual C^*, the ordered convex cone of all positive numerical functionals on C which are: 1) additive, 2) increasing, 3) continuous in order from below, 4) finite on an increasing dense subset. Then C^* is an ordered convex cone with usual operations and pointwise order relation. Given two arbitrary ordered convex cones C_1 and C_2 we say that C_1 and C_2 are in duality with respect to a positive numerical functional $\mathcal{L}$ defined on cartesian product $C_1 \times C_2$ if:

a) for any $c_1 \in C_1$ the map $c_2 \longrightarrow \mathcal{L}(c_1, c_2)$ belongs to C_2^*

b) for any $c_2 \in C_2$ the map $c_1 \longrightarrow \mathcal{L}(c_1, c_2)$ belongs to C_1^*.

The duality functional $\mathcal{L}$ is said complete if any element from C_2^* is obtained as in a) and any element from C_1^* is obtained as in b).

From the well known properties of energy functional, the ordered convex cones $\mathrm{Exc}(\mathcal{V})$ and $\mathcal{E}(\mathcal{V})$ are then in duality with respect to L (in fact $\sphericalangle$ duality, that is we require only $\sphericalangle$ continuous in order from below in property 3). Generally, L is not complete. This happens however when $\mathcal{V}$ posseses a reference measure. The aim of this paper is to point out that whenever ξ is an excessive measure and we consider the ordered convex cones $\mathrm{Exc}_{\xi}(\mathcal{V}) = \{\eta \in \mathrm{Exc}(\mathcal{V}) : \eta \ll \xi\}$ and $\mathcal{E}(\mathcal{V})_{\xi} = \{$ the set of classes of excessive functions, finite ξ a.s., through the equivalence relation $s \sim t \Leftrightarrow s = t \; \xi$ a.s.$\}$, then we can derive from L, in a canonical manner, a functional $\mathcal{L}_{\xi}$ on the cartesian product $\mathrm{Exc}_{\xi}(\mathcal{V}) \times \mathcal{E}(\mathcal{V})_{\xi}$ that expresses the complete duality between $\mathrm{Exc}_{\xi}(\mathcal{V})$ and $\mathcal{E}(\mathcal{V})_{\xi}$ in above sense. As a consequence, we can prove directly for an arbitrary solid convex subcone M of Exc $(\mathcal{V})$, some of the results from [1] concerning duality and biduality for the ordered convex cone $\mathrm{Exc}(\mathcal{V})$, without using the description of the dual Exc $(\mathcal{V})$ given in [1] .

1. Preliminaries

In this section we recall some flacts about excessive measures, energy functional, and H-cones. Throughout this paper $(X,\mathcal{X})$ is a measurable space and $\mathcal{V}=(V_\alpha)_{\alpha>0}$ a submarkovian resolvent of kernels on $(X,\mathcal{X})$ for which the initial kernel $V=V_0=\sup_\alpha V_\alpha$ is proper (that is there exists a strictly positive measurable function f on X such that $Vf<\infty$ on X) and strict (that is $V1>0$). We denote by $\mathcal{S}(\mathcal{V})$ (resp. $\mathcal{E}(\mathcal{V})$) the convex cone of all supermedian (resp. excessive) functions on X. No condition of finitude is imposed for excessive functions in notation above.

We denote by $\mathcal{F}$ (resp. $\mathcal{F}_b$) the convex cone of all positive numerical (resp. bounded) measurable functions on X and for any measure μ on $(X,\mathcal{X})$ and for any $f\in\mathcal{F}$ we denote by $\mu(f)$ the integral of f with respect to μ.

Following [4] and [5], a positive measure ξ on $\mathcal{X}$ is called $\mathcal{V}$excessive (or simply excessive) if it is σ finite and

1. $\xi\alpha V_\alpha\leq\xi\ ,\ \forall\alpha>0$
2. $\xi\alpha V_\alpha\nearrow\xi$ when $\alpha\nearrow\infty$

It can be shown (see [4]) that in our hypothesis (V strict) if a σ finite measure ξ on $\mathcal{X}$ satisfies 1., then 2. is automatically satisfied by ξ.

We denote by Exc(V) or simply Exc the convex cone of all excessive measures on $\mathcal{X}$.

Directly from definition, it then follows quickly that for any family F in Exc, we have inf $F\in$ Exc and if F is upper directed and dominated in Exc, we have sup $F\in$ Exc, where inf and sup are taken with respect to the natural order in the boundedly complete lattice of all positive finite measures on $\mathcal{X}$.

A less trivial fact is that the Riesz decomposition property holds in Exc: if $\xi,\ \xi_1,\ \xi_2\in$ Exc, $\xi\leq\xi_1+\xi_2$, then there exists $\eta_1,\eta_2\in$ Exc

such that $\xi = \eta_1 + \eta_2$, $\eta_1 \leq \xi_1$, $\eta_2 \leq \xi_2$.
If μ is a positive measure on $\mathfrak{X}$ such that μV is σ finite, then it follows from definition that $\mu V \in \mathrm{Exc}$.

Definition ([4] or [5]).
The energy functional on the cartesian product Exc X $\mathcal{E}(\mathcal{V})$ is the map $L : \mathrm{Exc} \times \mathcal{E}(\mathcal{V}) \longrightarrow \overline{R}_+$ defined by

$$L(\xi, s) = \sup \{ \mu(s) : \mu V \leq \xi \}$$

We list the following properties of L:

1) For any $s \in \mathcal{E}(\mathcal{V})$ the map

$$\xi \longrightarrow L(\xi, s)$$

is additive, increasing, and σ continuous in order from below ($\xi_n \nearrow \xi \Rightarrow L(\xi_n, s) \nearrow L(\xi, s)$)

2) For any ξ Exc the map

$$s \longrightarrow L(\xi, s)$$

is additive, increasing, and σ continuous in order from below ($s_n \nearrow s \Rightarrow L(\xi, s_n) \nearrow L(\xi, s)$

3) For any $\xi \in \mathrm{Exc}$, $s = Vf \in \mathcal{E}(\mathcal{V})$ we have

$$L(\xi, Vf) = \xi(f)$$

4) For any $\xi = \mu V \in \mathrm{Exc}$, $s \in \mathcal{E}(\mathcal{V})$ we have

$$L(\mu V, s) = \mu(s)$$

5) For any $\xi \in \mathrm{Exc}$, there exists a strictly positive element $u \in \mathcal{E}(\mathcal{V})$ such that

$$L(\xi, u) < \infty$$

(In fact, let $g \in \mathcal{F}$, $g > 0$ such that $\xi(g) < \infty$ and put $u = Vg$).

We recall a converse of 2), which will be esential in the sequel.

Theorem (see [6])
For any map $\varphi : \mathcal{E}(\mathcal{V}) \longrightarrow R_+$ which is additive, increasing, σ continuous in order from below, and finite on some strictly positive $u \in \mathcal{E}(\mathcal{V})$, $u < \infty$ $\mathcal{V}$ a.s.,

there exists a unique excessive measure ξ on $\mathcal{X}$ such that $\varphi = L(\xi, \cdot)$.

Note that generally a similar converse for property 1) is not valid (see [6] for an example)

Finally, we recall some elements from H cones theory.

Let $S = (S, \leqslant)$ be an ordered convex cone containing a null element 0 and $s \geqslant 0$ for any $s \in S$. We say that S is an H-cone if the following properties are fulfilled:

1) For any family $F \subset S$ there exists the greatest lower bound $\bigwedge F$ and $\bigwedge(s + F) = s + \bigwedge F$ for any $s \in S$.

2) For any upper directed and dominated family $F \subset S$ we have $\bigvee(s+F) = s + \bigvee F$ for any $s \in S$ ($\bigvee F$ exists from 1))

3) For any s, s_1, $s_2 \in S$ such that $s \leqslant s_1 + s_2$, there exist t_1, $t_2 \in S$, $s = t_1 + t_2$, $t_1 \leqslant s_1$, $t_2 \leqslant s_2$ (Riesz decomposition property).

On $(S, \leqslant)$ we define the specific order $\preccurlyeq$ by

$$s \preccurlyeq t \Leftrightarrow \exists\, u \in S,\ s+u = t$$

If $F \subset S$ is an upper directed family and there exists $s = \bigvee F$, we write $F \nearrow s$.

A map $\mu : S \longrightarrow \overline{R}_+$ which is additive, increasing, continuous in order from below ($F \nearrow s \Rightarrow \mu(F) \nearrow \mu(s)$) and finite increasingly dense ($\forall s \in S, \exists F \nearrow s$, $\mu(t) < \infty, \forall t \in F$) is called H integral on S. It is known that the ordered convex cone S^* of all H integrals endowed with the usual algebraic operations and the pointwise order relation is an H-cone called the dual of S.

Let now S^{**} be the dual of the H cone S^* and consider the canonical embeding of S in S^{**}

$$S \ni s \longrightarrow \tilde{s} \in S^{**} \quad , \quad \tilde{s}(\mu) = \mu(s), \forall \mu \in S^*$$

If S^* separates S, then obviously the above map is injective and S will be identified with its image through this map. S becomes a convex subcone of S^{**}

and the order "$\leqslant$" on S coincides with the trace of the order on S^{**} if S^* separates S in order too, that is

$$\mu(s_1)\leqslant\mu(s_2), \qquad \forall\mu\in S^* \Rightarrow s_1\leqslant s_2$$

Let us apply these considerations to the convex cone Exc. From previous remarks, Exc is an H cone with natural order on measures. Moreover if $F\subset$Exc we have $\bigwedge F$ = inf F and if additionally F is upper directed and dominated then $\bigvee F$ = sup F. Note that the H cone Exc posseses an important property, being a cone of measures: For any upper directed and dominated (resp. lower directed) family $F\subset$Exc, there exists an increasing (resp. decreasing) sequence $\xi_n\subset F$ such that $\bigvee F = \sup_n \xi_n$ (resp. $\bigwedge F = \inf_n \xi_n$). Therefore, it is sufficient to ask in definition of a H integral on Exc to be σ continuous in order from below.

From now on we denote by S the convex subcone of $\mathcal{E}(\mathcal{V})$ consisting of all functions $s\in\mathcal{E}(\mathcal{V})$ such that $s<\infty$ $\mathcal{V}$a.s.

Note that generally S is not an H-cone. For any $s\in S$ the map

$$\xi \longrightarrow L(\xi ,s)$$

is an H integral on Exc and if we denote by φ the map $s\longrightarrow L(\cdot,s)$, then S is embeded in Exc* as a convex subcone. Moreover φ is injective and therefore we can identify S with $\varphi(S)$.

Definition.

An element u of an ordered convex cone $(C,\leqslant)$ is called weak unit if

$$\bigvee_n (c\wedge nu) = c \quad , \qquad \forall c\in C$$

Of course, we suppose that $c\wedge nu$ exists and this happenes if C is a lower lattice for example.

It is known that in $\mathcal{E}(\mathcal{V})$, for any s_1, s_2 we have $s_1\wedge s_2 = \widehat{\inf(s_1,s_2)}$ where $\widehat{\ }$ means the excessive regularization of the supermedian function $\inf(s_1,s_2)$.

It is easy to see that for any $f>0$ such that $Vf<\infty$, $u = Vf$ is a weak unit for both S and $\mathcal{E}(\mathcal{V})$.

Generally Exc has not a weak unit. It is easy to see that Exc has a weak unit iff $\mathcal{V}$ posseses a reference measure.

Definition. Let C be an ordered set and $C_1 \subset C$ a subset. We say that C_1 is solid (resp. increasingly dense) in C ff for any $c_1 \in C_1$ and any $c \in C$, $c \leq c_1 \Rightarrow c \in C_1$ (resp. for any $c \in C$ there exists an upper directed family $F \subset C_1$ such that $c = \bigvee F$).

Proposition 1.1

Let C be an H-cone, $C_1 \subset C$ a solid increasingly dense subcone of C and $\mu \in C_1^*$. Then the functional $\bar{\mu}$ defined on C by

$$\bar{\mu}(c) = \sup\left\{\mu(c_1) \ : \ c_1 \in C_1,\ c_1 \leq c\right\}$$

is the unique extension of μ to an H integral on C and the map

$$\mu \longrightarrow \bar{\mu}$$

is an isomorphism between the ordered cones C_1^* and C^* (See [2] prop. 2.2.2.) for the proof of a more general statement)

2. Localization of the energy functional.

Throughout this section, ξ is a fixed element of Exc, and we define on $\mathcal{F}$ (cf. [1]) the equivalence relation

$$f \sim g \iff f = g \ \ \xi \text{ a.s.} \quad \forall f,g \in \mathcal{F}$$

Denote by $\mathcal{F}_\xi$ the ordered convex cone obtained by factorization of the ordered convex cone $\mathcal{F}$ relative to this equivalence relation, the algebraic operations and order being induced on $\mathcal{F}$ in the canonical manner.

Consider now $\mathrm{Exc}_\xi \stackrel{\text{def}}{=} \left\{\eta \in \mathrm{Exc},\ \eta \ll \xi\right\}$. Using the Radon Nicodym derivative it is easy to see that $\mathrm{Exc}_\xi = \left\{\eta \in \mathrm{Exc},\ \eta = \bigvee_n \eta \wedge n\xi\right\}$. Note that ξ

becomes a weak unit in Exc_ξ .

Define now $\mathcal{E}(\mathcal{V})_\xi = \{\hat{h} : h \in \mathcal{E}(\mathcal{V}),\ h < \infty\ \xi\text{ a.s.}\}$ and $S_\xi = \{\hat{s} : s \in \mathcal{E}(\mathcal{V}),\ s < \infty\ \mathcal{V}\text{ a.s.}\}$. Observing that $V1_A = 0 \Rightarrow \xi(A) = 0$ for any $A \in \mathcal{X}$ and any $\xi \in \mathrm{Exc}$, we have the following sequence of inclusions of ordered convex cones

$$S_\xi \hookrightarrow \mathcal{E}(\mathcal{V})_\xi \hookrightarrow \mathcal{F}_\xi \ .$$

Remark 2.1.

For any weak unit u of S, u is a weak unit for $\mathcal{E}(\mathcal{V})$ too. Indeed, it follows first that $u > 0$ on X. Then, using an argument from [6] there exists $f > 0$, $Vf < \infty$ and $Vf \leq u$. As we mentioned in 1, Vf is a weak unit for $\mathcal{E}(\mathcal{V})$ and therefore u is a weak unit for $\mathcal{E}(\mathcal{V})$.

Therefore S_ξ is solid and increasingly dense in $\mathcal{E}(\mathcal{V})_\xi$. Moreover, for any $h \in \mathcal{E}(\mathcal{V})$ and for any weak unit $u \in S$ we showed above that

$$h = \sup_n\ h \wedge nu$$

and therefore $\hat{h} = \bigvee_n \widehat{h \wedge nu}$ in the ordered convex cone $\mathcal{E}(\mathcal{V})_\xi$. Obviously $\widehat{h \wedge nu} \in S_\xi$ and the sequence $\widehat{h \wedge nu}$ is increasing (to $\hat{h}$).

Proposition 2.1.

For any $\eta \in \mathrm{Exc}_\xi$ and $s_1, s_2 \in \mathcal{E}(\mathcal{V})$ such that $s_1 \leq s_2$ ξ a.s. we have $L(\eta, s_1) \leq L(\eta, s_2)$.

Proof.

From [4] there exists an increasing sequence $\mu_n V \nearrow \eta$, and by properties 1) and 4) of L, it is sufficient to prove that

$$\mu(s_1) \leq \mu(s_2)$$

if $s_1 \leq s_2$ μV a.s.

Because $V_\alpha \leq V$ we have $\mu(\alpha V_\alpha s_1) \leq \mu(\alpha V_\alpha s_2), \forall \alpha > 0$ and letting $\alpha \to \infty$ we get the desidered inequality.

Definition 2,2.

For any $\xi \in \mathrm{Exc}$ let us denote by $\mathcal{L}_\xi$ the unique map on $\mathrm{Exc}_\xi \times \mathcal{E}(\mathcal{V})_\xi$ with values in $\overline{R}_+$ such that $\mathcal{L}_\xi(\eta, \hat{h}) = L(\eta, h)$ for any $\eta \in \mathrm{Exc}_\xi$ and $h \in \mathcal{E}(\mathcal{V})$ such that $h < \infty$ ξ a.s., and we call $\mathcal{L}_\xi$ the ξ localization of L.

In order to prove the properties of $\mathcal{L}_\xi$ (theorem 2.9.) we shall use essentially the considerations from [1] concerning Exc_ξ and $\mathcal{E}(\mathcal{V})$. We keep the same notations for ease of the reader interested in some details. Our interest is to emphasize the phenomenom of localization.

Following [1], the resolvent $\mathcal{V} = (V_\alpha)_{\alpha > 0}$ induces a family of morphisms $\mathcal{V}^\xi = (V_\alpha^\xi)_{\alpha > 0}$ on the ordered convex cone $\mathcal{F}_\xi$ just setting

$$V_\alpha^\xi \hat{f} = \widehat{V_\alpha f} \qquad \forall f \in \mathcal{F}, \ \forall \alpha > 0$$

by observing that $f = g$ ξ a.s. $\Rightarrow$ $Vf = Vg$ ξ a.s. Indeed if $A \in \mathcal{X}$ and $\xi(A) = 0$ then $\xi(V_\alpha 1_A) \leq \frac{1}{\alpha}\xi(A) = 0$ for any $\alpha > 0$ and letting $\alpha \to 0$ we get $\xi(V 1_A) = 0$. Then it is easy to see that the family $(V_\alpha^\xi)_{\alpha > 0}$ satisfies the resolvent equation. We call resolvent on $\mathcal{F}$ any family of morphisms $\mathcal{T} = (T_\alpha^\xi)_{\alpha > 0}$ on $\mathcal{F}_\xi$ which satisfy the resolvent equation.

It can be shown that for any $\alpha > 0$ there exists a unique morphism $V_\alpha^{\xi *}$ on $\mathcal{F}_\xi$ such that

$$\int \hat{g}\, V_\alpha^{\xi *} \hat{f}\, d\xi = \int \hat{f}\, V_\alpha^\xi \hat{g}\, d\xi \quad \forall f, g \in \mathcal{F}$$

(in the sense that there exists a kernel $V_\alpha^{\xi *}$ on $\mathcal{F}_\xi$ such that

$$\int g\, V_\alpha^{\xi *} f\, d\xi = \int f\, V_\alpha g\, d\xi \quad \forall f, g \in \mathcal{F} \,)$$

and the family $\mathcal{V}^{\xi *} = (V_\alpha^{\xi *})_{\alpha > 0}$ is a submarkovian resolvent on $\mathcal{F}_\xi$.

We note that if there exists a resolvent $\mathcal{V}^* = (V^*_\alpha)_{\alpha > 0}$ of kernels on $\mathcal{F}$ in weak duality with $\mathcal{V}$ relatively to ξ, then $V^{\xi *}_\alpha$ is obtained from V^*_α in the same manner described above.

Definition 2.4. ([1])
We denote by $E(\mathcal{V}^\xi)$ (resp. $E(\mathcal{V}^{\xi *})$) the convex cone of all elements $\hat{u} \in \mathcal{F}_\xi$ such that

$$\alpha V^\xi_\alpha \hat{u} \nearrow \hat{u} \quad (\text{resp. } \alpha V^{\xi *}_\alpha \hat{u} \nearrow \hat{u}) \qquad \text{when } \alpha \nearrow \infty$$

and $\hat{u} < \infty$ (that is $u < \infty$ ξ a.s.)

It can be shown ([3]) that $E(\mathcal{V}^\xi)$ and $E(\mathcal{V}^{\xi *})$ are H cones and we have

Theorem 2.5. ([3])

i) For any $s^* \in E(\mathcal{V}^{\xi *})$ the map $\Theta(s^*)$ from $E(\mathcal{V}^\xi)$ into $\overline{R}_+$ defined by

$$s \longrightarrow \sup\left\{\int s^* \hat{f} \, d\xi : V^\xi \hat{f} \leq s\right\}$$

belongs to $(E(\mathcal{V}^\xi))^*$ and the map

$$s^* \longrightarrow \Theta(s^*)$$

is an isomorphism between the H cones $E(\mathcal{V}^{\xi *})$ and $(E(\mathcal{V}^\xi))^*$

ii) For any $s \in E(\mathcal{V}^\xi)$ the map $\Theta^*(s)$ from $E(\mathcal{V}^{\xi *})$ into $\overline{R}_+$ defined by

$$s^* \longrightarrow \sup\left\{\int s \hat{f} \, d\xi : V^{\xi *} \hat{f} \leq s^*\right\}$$

belongs to $(E(\mathcal{V}^{\xi *}))^*$ and the map

$$s \longrightarrow \Theta^*(s)$$

is an isomorphism between the H cones $E(\mathcal{V}^\xi)$ and $(E(\mathcal{V}^{\xi *}))^*$, and we have

$$\Theta(s^*)(s) = \Theta^*(s)(s^*), \quad \forall s \in E(\mathcal{V}^\xi),\ s^* \in E(\mathcal{V}^{\xi *}).$$

The next result gives a characterization of the elements of Exc_ξ .

Theorem 2.6 ([4])

For any $\eta \in \mathrm{Exc}_\xi$ there exists an unique element $\hat{\eta} \in E(\mathcal{V}^{\xi *})$ such that

$$\eta = \hat{\eta} \cdot \xi$$

and the map

$$\eta \xrightarrow{\Psi} \hat{\eta}$$

is an isomorphism between the H cones Exc_ξ and $E(\mathcal{V}^{\xi *})$.

(In fact, the proof imitates the case of two resolvents in weak duality with respect to ξ, by using the Randon Nicodym derivative and the relation of duality.

Proposition 2.7. ([1] corollary 2.5)

The cones $E(\mathcal{V}^{\xi})$ and $\mathcal{E}(\mathcal{V})_{\xi}$ coincide.

Recall that $\mathcal{E}(\mathcal{V})_{\xi} = \{\hat{h} : h \in \mathcal{E}(\mathcal{V}),\ h < \infty\ \xi\ \text{a.s.}\}$

Proposition 2.8.

For any $\eta \in \mathrm{Exc}_{\xi}$ and for any $s \in \mathcal{E}(\mathcal{V})$, $s < \infty\ \xi$ a.s. we have

$$\mathcal{L}_{\xi}(\eta, \hat{s}) = L(\eta, s) = \Theta(\hat{\eta})(\hat{s}) = \Theta^{*}(\hat{s})(\hat{\eta}) \tag{1}$$

where Θ, Θ^{*} are the isomorphisms defined in theorem 2.5 and $\hat{\eta}$ is the unique element of $E(\mathcal{V}^{\xi *})$ such that $\eta = \hat{\eta} \cdot \xi$, given by theorem 2.6.

Proof.

Of course we have to prove only the second equality in (1) because the first is just definition of $\tilde{\mathcal{L}}_{\xi}$ and the third is the last statement of theorem 2.5., in relation with proposition 2.7.

Therefore, from definition of Θ, we have to prove the relation

$$L(\eta, s) = \sup\{\eta(f) : Vf \leq s \quad \xi \quad \text{a.s.}\}$$

(Of course, $f \in \mathcal{F}$ in above expression).

Indeed, from Hunt's theorem, there exists a sequence $Vf_n \nearrow s$ and therefore

$$L(\eta, s) = \sup_n L(\eta, Vf_n) = \sup_n \eta(f_n) \leq \sup\{\eta(f) : Vf \leq s\ \xi\ \text{a.s.}\}$$

For the converse inequality, we use proposition 2.1. We have

$$Vf \leq s\ \xi\ \text{a.s.} \Rightarrow \eta(f) = L(\eta, Vf) \leq L(\eta, s)$$

and the proof is finished.

Theorem 2.9.

1) The map

$$\hat{h} \longrightarrow \mathcal{L}_{\xi}(\cdot, \hat{h})$$

is an isomorphism between the H cones $\mathcal{E}(\mathcal{V})_{\xi}$ and Exc^{*}_{ξ} .

2) The map

$$\eta \longrightarrow \mathcal{L}_{\xi}(\eta,\cdot)$$

is an isomorphism between the H-cones Exc_{ξ} and $\mathcal{E}(\mathcal{V})^{*}_{\xi}$.

Proof.

We recall that $\mathcal{L}_{\xi}$ is the unique map on $\mathrm{Exc}_{\xi} X.\ \mathcal{E}(\mathcal{V})_{\xi}$ such that

$$\mathcal{L}_{\xi}(\eta,\hat{h})=L(\eta,h)$$

for any $\eta\in\mathrm{Exc}_{\xi}$ and for any $h\in\mathcal{E}(\mathcal{V})$, $h<\infty$ ξ a.s. From theorems 2.5, 2.6 we keep that Θ^{*} is an isomorphism between $\mathcal{E}(\mathcal{V})_{\xi}$ and $E(\mathcal{V}\xi^{*})^{*}$; Θ is an isomorphism between $E(\mathcal{V}\xi^{*})$ and $\mathcal{E}(\mathcal{V})^{*}_{\xi}$,and finally the map $\eta \xrightarrow{\Psi} \hat{\eta}$ is an isomorphism between Exc_{ξ} and $E(\mathcal{V}\xi^{*})$. Proposition 2.8 shows that

1) $\qquad \mathcal{L}_{\xi}(\cdot,\hat{h}) = \Theta^{*}(\hat{h})(\Psi(\cdot))\quad , \quad \forall \hat{h}\in\mathcal{E}(\mathcal{V})_{\xi}$

2) $\qquad \mathcal{L}_{\xi}(\eta,\cdot)=\Theta(\Psi(\eta))(\cdot)\quad , \quad \forall \eta\in\mathrm{Exc}_{\xi}$

and both statements of theorem are clear now.

Remark that the H cone $E(\mathcal{V}\xi^{*})$ played an evanescent role for our purpose. We produced from L, a map $\mathcal{L}_{\xi}$ on $\mathrm{Exc}_{\xi} X \mathcal{E}(\mathcal{V})_{\xi}$ that expresses the complete duality between Exc_{ξ} and $\mathcal{E}(\mathcal{V})_{\xi}$ although L fails to have this property relative to Exc and $\mathcal{E}(\mathcal{V})$. Therefore we shall identify $\mathcal{E}(\mathcal{V})_{\xi}$ with the dual of Exc_{ξ} , that is the same with the $\measuredangle$ dual of Exc_{ξ} as we remarked in section 1. However, not the description of Exc^{*}_{ξ} itself is essential in what follows, as we shall see next section. We shall use essentially the fact that S_{ξ} is embeded, in this way, solid and increasingly dense in Exc^{*}_{ξ} in the strong sense stated in remark 2.1., by using weak units of S, that is in a manner not depending on ξ.

3. Duality and biduality for solid convex subcones of Exc.

Throughout this section M is a fixed solid convex subcone of Exc. It then follows that M is an H-cone (with the order induced by Exc) which posseses a dual M^{*}. The purpose of this section are to establish the position of S relatively to M^{*} and then to establish the position of M as a subcone of M^{**}.

Proposition 3.1.

Let us consider the set $\overline{M} = \bigcup_{\xi \in M} \mathrm{Exc}_\xi$. Then $\overline{M}$ is a solid convex subcone of Exc, and M is solid and increasingly dense in $\overline{M}$ as a convex subcone.

Proof.

We remarked in section 2 that for any $\xi \in \mathrm{Exc}$ we have $\mathrm{Exc}_\xi = \{\eta \in \mathrm{Exc};\ \eta = \bigvee_n \eta \wedge n\xi\}$.

For the first assertion, observe first that each Exc_ξ is a solid convex subcone of Exc and the family $(\mathrm{Exc}_\xi)_{\xi\in M}$ is upper directed relatively to inclusion (for any $\xi_1, \xi_2 \in M$ let $\xi = \xi_1 + \xi_2 \in M$; then $\mathrm{Exc}_{\xi_1} \subset \mathrm{Exc}_\xi$ and $\mathrm{Exc}_{\xi_2} \subset \mathrm{Exc}_\xi$). For the second assertion, let $\eta \in \overline{M}$ and choose $\xi \in M$ such that $\eta \in \mathrm{Exc}_\xi$. It then follows $\eta = \bigvee_n \eta \wedge n\xi$ and consider the sequence $\xi_n = \eta \wedge n\xi$. We have $n\xi \in M$, M being convex cone and therefore $\xi_n \in M$ because $\xi_n \leq n\xi$ and M is solid in Exc. Therefore M is even $\preccurlyeq$ increasingly dense in $\overline{M}$ since $\bigvee_n \xi_n = \xi$. The solidity of M in $\overline{M}$ is obvious, since M is solid in Exc.

Using proposition 1.1 any H integral $\mu \in M^*$ can be extended uniquely to an element $\overline{\mu}$ of $\overline{M}^*$ and therefore $M^* \simeq \overline{M}^*$.

So, we can study $\overline{M}^*$ instead of M^* and we will work with $\overline{M}^*$ in the sequel.

Let now $\mu \in \overline{M}^*$ and denote $\mu_\xi = \mu | \mathrm{Exc}_\xi$ for $\xi \in M$.

Proposition 3.2.

For any $\mu \in \overline{M}^*$ and $\xi \in M$ the restriction μ_ξ of μ to Exc_ξ belongs to Exc_ξ^*. Conversely, for any functional $\mu : \overline{M} \longrightarrow \overline{R}_+$ such that $\mu | \mathrm{Exc}_\xi \in \mathrm{Exc}_\xi^*$ for any $\xi \in M$, it follows $\mu \in \overline{M}^*$. Therefore we can identify $\overline{M}^*$ with the family $\{(\mu_\xi)_{\xi \in M} : \mu_\xi \in \mathrm{Exc}_\xi^*, \forall \xi \in M$ and such that for any $\xi_1, \xi_2 \in M, \xi_1 \leq \xi_2$ we have $\mu_{\xi_2} | \mathrm{Exc}_{\xi_1} \in \mathrm{Exc}_{\xi_1}^*\}$.

Proof.

The first assertion follows from definition of H integrals, and the fact that Exc_ξ is a solid subcone of Exc, for any $\xi \in Exc$. The second follows since $\overline{M} = \bigcup_{\xi \in M} Exc_\xi$ and the family $(Exc_\xi)_{\xi \in M}$ is upper directed.

We wish to use the results from section 2 to study $\overline{M}^*$. For any $\mu \in \overline{M}^*$, the net $(\mu_\xi)_{\xi \in M}$ increases intuitively to μ and we know the description of μ_ξ, for any ξ. MOre precisely, for any $m \in \overline{M}$ and for any $\xi \in M$, let us consider the projection of m on the cone Exc_ξ, that is take $B_\xi(m) \overset{def}{=} \bigvee\{\eta \in Exc_\xi ; \eta \leq m\}$. From definition, $B_\xi(m) \in Exc_\xi$, and the net $B_\xi(m)_{\xi \in M}$ increases to m only because $m \in \overline{M} = \bigcup_{\xi \in M} Exc_\xi$, and then it seems natural to approximate μ with the functionals $B^*_\xi \mu = \mu \circ B_\xi$. However, we don't know yet neither if $B^*_\xi \mu$ belongs to $\overline{M}^*$. Suggested by [1], the following proposition makes more exact above discussion.

Proposition 3.3.

a) For any $\xi \in Exc$ and $m \in Exc$, we have

$$B_\xi(m) = \bigvee_n m \wedge n\xi \tag{1}$$

and the operation B_ξ defined on Exc is additive, increasing, continuous in order from below, contractive ($B_\xi \leq 1$) and idempotent ($B_\xi = B^2_\xi$)

b) The operation $m \longrightarrow m - B_\xi(m)$ defined on Exc is increasing.

Proof.

For relation (1) we have to prove only that

$$B_\xi(m) \leq \bigvee_n m \wedge n\xi$$

(since $m \wedge n\xi \leq m$ and belongs to the solid cone Exc_ξ)

Let $\eta \in Exc_\xi$, $\eta \leq m$. We have

$$\eta = \bigvee_n \eta \wedge n\xi \leq \bigvee_n m \wedge n\xi$$

and therefore

$$B_\xi(m) = \vee \{\eta \in Exc; \eta \leq m\} \leq \bigvee_n m \wedge n.\xi .$$

It follows now from general theory of H cones that the operation B_ξ posseses all the properties listed in a) (see [2], th.2.2.9). As to b) it suffices to show that the operation $I-B$ is increasing. Let $m_1 \leq m_2$. We have

$$m_1+B_\xi(m_2)=m_1+\bigvee_n m_2\wedge n\xi=\bigvee_n (m_1+m_2\wedge n\xi) =$$
$$=\bigvee_n (m_1+m_2)\wedge(m_1+n\xi)\leq\bigvee_n (m_1+m_2)\wedge(m_2+n\xi)=m_2+B_\xi(m_1)$$

and the proof is finished.

Corollary 3.4.

For any $\mu\in M^*$ and for any $\xi\in M$ we have

a) $B^*_\xi\mu\in\overline{M}^*$ (and we remarked that $\sup\limits_M B^*_\xi\mu=\mu$)

b) $\mu-B^*_\xi\mu\in\overline{M}^*$.

Let us consider now $L|\overline{M}\times S$. It then follows exactly as in the case $M=\mathrm{Exc}$ that the map

$$s \xrightarrow{\varphi} L(\cdot,s)$$

is a map from S into $\overline{M}^*$, which is not generally injective (For example take $M=\overline{M}=\mathrm{Exc}_\xi$). However denote $\varphi(S)$ the image of S in $\overline{M}^*$ through φ. $\varphi(S)$ is an convex subcone of $\overline{M}^*$, and for any $s\in S$ the action of $\varphi(s)$ on $\overline{M}$ is given by the formula

$$\varphi(s)(m) = L(m,s) \qquad ,\forall\, m\in\overline{M} \tag{1}$$

On the other hand note that $\overline{M}^*$ separates $\overline{M}$ since $\overline{M}^*\supset\varphi(S)$ and $\varphi(S)$ separates $\overline{M}$ (in order too) by above relation (1). Therefore we can identify $\overline{M}$ with it's image in $\overline{M}^{**}$ through the map

$$m\longrightarrow\tilde{m} \quad , \quad \tilde{m}(\mu)=\mu(m) \quad ,\forall\mu\in M^*$$

and M becomes an ordered convex subcone of M^{**}.

Proposition 3.5.

For any weak unit $u\in S$, the element $\varphi(u)\in\varphi(S)\subset\overline{M}^*$ is a weak unit for $\overline{M}^*$.

Proof.

First, we show that for any $\nu, \mu \in \overline{M}^*$ we have

$$(\nu \wedge \mu)_\xi = \nu_\xi \wedge \mu_\xi$$

(Recall that $\mu_\xi \overset{\text{def}}{=} \mu|\mathrm{Exc}_\xi$ for any $\mu \in \overline{M}^*$ and $\xi \in M$). The first $\wedge$ is taken in the H cone $\overline{M}^*$ and the second in the H cone Exc^*_ξ. The inequality $\leqslant$ is then obvious. For the converse, from proposition 3.2 it suffices to show that the map

$$\overline{M} \ni m \longrightarrow \nu_\xi \wedge \mu_\xi(m) \quad \text{for } m \in \mathrm{Exc}_\xi \subset \overline{M}$$

is well defined on $\overline{M}$ that is we have

$$\nu_{\xi_1} \wedge \mu_{\xi_1} = \nu_{\xi_2} \wedge \mu_{\xi_2} \,|\, \mathrm{Exc}\,\xi_1 \qquad \text{if } \xi_1 \leqslant \xi_2.$$

By theorem 2.9 the map

$$\hat{h}_\xi \longrightarrow \mathcal{L}_\xi(\cdot, \hat{h}_\xi)$$

is an isomorphism of H cones between $\mathcal{E}(\mathcal{V})_\xi$ and Exc^*_ξ. For $\xi = \xi_2$, choose $h, g \in \mathcal{E}(\mathcal{V})$, h, g finite ξ_2 a.s., such that

$$\nu_{\xi_2}(m) = \mathcal{L}_{\xi_2}(m, \hat{h}_{\xi_2}) = L(m,h)$$

and

$$\mu_{\xi_2}(m) = \mathcal{L}_{\xi_2}(m, \hat{g}_{\xi_2}) = L(m,g)$$

for any $m \in \mathrm{Exc}\,\xi_2$. If we take $m \in \mathrm{Exc}\,\xi_1 \subseteq \mathrm{Exc}\,\xi_2$ arbitrary, we get

$$\nu_{\xi_1}(m) = \mathcal{L}_{\xi_1}(m, \hat{h}_{\xi_1}) = L(m,h)$$

and

$$\mu_{\xi_1}(m) = \mathcal{L}_{\xi_1}(m, \hat{g}_{\xi_1}) = L(m,g)$$

On the other hand, it is easy to see that for any $g, h \in \mathcal{E}(\mathcal{V})$ and for any $\xi \in M$ we have $\widehat{(g \wedge h)}_\xi = \hat{g}_\xi \wedge \hat{h}_\xi$ and therefore, for any $m \in \mathrm{Exc}\,\xi_1 \subseteq \mathrm{Exc}\,\xi_2$ we have

$$\nu_{\xi_1} \wedge \mu_{\xi_1}(m) = \mathcal{L}_{\xi_1}(m, \hat{h}_{\xi_1} \wedge \hat{g}_{\xi_1}) = \mathcal{L}_{\xi_1}(m, \widehat{h \wedge g}_{\xi_1}) =$$
$$= L(m, h \wedge g) = \mathcal{L}_{\xi_2}(m, \widehat{h \wedge g}_{\xi_2}) = \mathcal{L}_{\xi_2}(m, \hat{h}_{\xi_2} \wedge \hat{g}_{\xi_2}) =$$
$$= \nu_{\xi_2} \wedge \mu_{\xi_2}(m)$$

and the above assertion is proved.

Take now $u \in S$ weak unit and $\mu \in \overline{M}^*$. For any $\xi \in M$ we have

$$(\mu \wedge n\varphi(u))_\xi = \mu_\xi \wedge n\varphi(u)_\xi = \mathcal{L}_\xi(\cdot, \hat{h}_\xi \wedge n\hat{u}_\xi) =$$

$$= L(\cdot, h \wedge nu)$$

Hence

$$(\bigvee_n \mu \wedge n\varphi(u))_\xi = \bigvee_n (\mu \wedge n\varphi(u))_\xi = \sup_n L(\cdot, h \wedge nu) =$$

$$= L(\cdot, h) = \mu_\xi$$

and therefore

$$\bigvee_n \mu \wedge n\varphi(u) = \mu \qquad , \quad \forall \mu \in M^*$$

that is $\varphi(u)$ is a weak unit.

Let now $\mu \in \overline{M}^*$. By theorem 2.9, for any $\xi \in M$ there exist an unique $\hat{h}_\xi \in \mathcal{E}(\mathcal{V})_\xi$ such that

$$\mu_\xi = \mathcal{L}_\xi(\cdot, \hat{h}_\xi)$$

If there exists $t \in S$ such that $\mu \leq \varphi(t)$, then for any $\xi \in M$, there exists an $s_\xi \in S$ such that $\hat{h}_\xi = \hat{s}_\xi$. Indeed, we have

$$\hat{h}_\xi \leq \hat{t}_\xi \Rightarrow \hat{h}_\xi = \hat{h}_\xi \wedge \hat{t}_\xi = \widehat{h_\xi \wedge t_\xi}$$

where $h_\xi \in \mathcal{E}(\mathcal{V})$ is an representant of $\hat{h}_\xi$, and we can take $s_\xi = h_\xi \wedge t$.

Proposition 3.6.

Let $\mu \in \overline{M}^*$ dominated by an element of $\varphi(S)$ and for $\xi \in M$ choose an element $s_\xi \in S$ such that $\hat{h}_\xi = \hat{s}_\xi$. We have then

$$B^*_\xi(\mu) = B^*_\xi(\varphi(s_\xi)).$$

Proof.

For any $m \in \overline{M}$ we have

$$B^*_\xi \mu(m) = \mu(B_\xi(m)) = \mu_\xi(B_\xi(m)) = \mathcal{L}_\xi(B_\xi(m), \hat{h}_\xi) =$$

$$= \mathcal{L}_\xi(B_\xi(m), \hat{s}_\xi) = L(B_\xi(m), s_\xi) = B^*_\xi(\varphi(s_\xi))(m).$$

Corollary 3.7

The set of all specific minorants of elements from $\varphi(S)$ in $\overline{M}^*$ is increasingly dense in $\overline{M}^*$.

Proof.

From proposition 3.5 it suffices to show that for any $\mu \in \overline{M}^*$ dominated by an element of $\varphi(S)$ (like $n \cdot \varphi(u)$, where $u \in S$ is a weak unit), there exists an F upper directed family consisting of specific minorants of elements from $\varphi(S)$, such that $\mu = \vee F$ in $\overline{M}^*$. We take the family

$$F = \{B^*_\xi(\mu)\}_{\xi \in M}$$

Indeed, for any $\xi \in M$, let s_ξ be an element of S such that

$$\mu_\xi = \mathscr{L}_\xi(\cdot, s_\xi)$$

From proposition 3.6 we have then

$$B^*_\xi(\mu) = B^*_\xi(\varphi(s_\xi))$$

and from corollary 3.4 we have

$$B^*_\xi(\varphi(s_\xi)) \lessdot \varphi(s_\xi) \in \varphi(S)$$

and

$$\bigvee_{\xi \in M} B^*_\xi \mu = \mu$$

Theorem 3.8

The convex cone $\overline{M}$ is a solid subcone of its bidual $\overline{M}^{**}$.

Proof.

Let $m \in \overline{M}$ and let $\Gamma \in \overline{M}^{**}$ such that

$$\Gamma \leqslant \tilde{m}$$

that is $\Gamma : M^* \longrightarrow \overline{R}_+$ is additive, increasing, continuous in order from below (and finite increasingly dense) such that

$$\Gamma(\mu) \leqslant \tilde{m}(\mu) = \mu(m), \quad \forall \mu \in \overline{M}^*$$

Consider the functional $\Gamma \circ \varphi : S \longrightarrow \overline{R}_+$.

Obviously $\Gamma \circ \varphi$ is additive, increasing and continuous in order from below.

Let $u \in S$ be a weak unit such that

$$m(\varphi(u)) = \varphi(u)(m) = L(m, u)$$

(We can choose $u = Vf$, where $f \in \mathcal{F}$ is such that $f > 0$, $m(f) < \infty$ and $V(f) < \infty$. This choice is possible since m is $\lessdot$ finite and V is proper).

We have then

$$\Gamma \circ \varphi(u) \leq \tilde{m}(\varphi(u)) < \infty$$

and hence, from [6] there exists an unique $m_1 \in$ Exc such that

$$\Gamma \circ \varphi(s) = L(m_1, s) \quad , \quad \forall s \in S$$

But we know that $\Gamma \leq \tilde{m}$ on $\bar{M}^*$ and hence on $\varphi(S)$. Therefore

$$\Gamma \circ \varphi(s) = L(m_1, s) \leq L(m, s) = \tilde{m}(\varphi(s))$$

for any $s \in S$. If we put $s = Vf$ we get

$$m_1(f) \leq m_2(f) \quad , \quad \forall f \in F$$

that is $m_1 \leq m$, and hence $m_1 \in \bar{M}$, which is a solid subcone of Exc. We show that

$$\Gamma = \tilde{m}_1 \qquad \text{on} \quad \bar{M}^*$$

We known that $\Gamma = m_1$ on $\varphi(S)$ by construction of m_1. Let $\mu \in M^*$ dominated by an element of $\varphi(S)$ and for any $\xi \in M$ choose an element $s_\xi \in S$ such that

$$\mu_\xi = \mathcal{L}_\xi(.,\hat{s}_\xi)$$

From proposition 3.6 we have then

$$B^*_\xi(\mu) = B^*_\xi(\varphi(s_\xi)) \ , \qquad \forall \xi \in M$$

We can write then

$$\Gamma(B^*_\xi(\mu)) = \Gamma(B^*_\xi(\varphi(s_\xi)) \leq \Gamma(\varphi(s_\xi)) =$$
$$= \tilde{m}_1(\varphi(s_\xi)) = L(m_1, s_\xi) \tag{1}$$

If we take ξ large, such that $m_1 \in \text{Exc}_\xi$, we have

$$L(m_1, s_\xi) = \mathcal{L}_\xi(m_1, \hat{s}_\xi) = \mu_\xi(m_1) = \mu(m_1) = \tilde{m}_1(\mu)$$

and recalling (1) for ξ large we get

$$\Gamma(B^*_\xi(\mu)) \leq \tilde{m}_1(\mu) \tag{2}$$

and therefore, Γ being continuous in order from below, we get

$$\Gamma(\mu) = \sup_{\xi \in M} \Gamma(B^*_\xi(\mu)) \leq \tilde{m}_1(\mu) \tag{3}$$

Now, if $\mu \in \bar{M}^*$ is an specific minorant of an element $\varphi(s) \in \varphi(S)$ such that

$$\Gamma(\varphi(s)) = \tilde{m}_1(\varphi(s)) < \infty \tag{4}$$

we get from (3):

$$\Gamma(\mu) \leq \tilde{m}_1(\mu)$$

and

$$\Gamma(\varphi(s)-\mu)\leq\tilde{m}_1(\varphi(s)-\mu)$$

and by (4) these inequalities must be equalities and therefore

$$\Gamma(\mu)=\tilde{m}_1(\mu) \qquad (5)$$

Using now corollary 3.4. and proposition 3.5, from (5) we get

$$\Gamma(\mu)=\tilde{m}_1(\mu) \quad , \forall\mu\in\overline{M}^*$$

both functionals being continuous in order from below.

Corollary 3.9

In the case M_o= Exc, the above proof shows that any $\Gamma\in \mathrm{Exc}^{**}$, finite on some $\varphi(u)$, where u is a weak unit of S, belongs necessary to Exc. In this case we noted that S may be identified with $\varphi(S)$ since φ is injective.

We recall one more element from H-cones theory.

Proposition 3.10 ([1])

Let T be an H cone which posseses a weak unit u. Then the set $\{\Gamma\in T^*: \Gamma(u)<\infty\}$ is increasingly dense in T^*.

For a proof see ([1] , proposition 4.1.)

Theorem 3.11

The convex cone $\overline{M}$ is an increasingly dense subcone of its bidual $\overline{M}^{**}$.

Proof.

From proposition 3.10 applied to the H cone $\overline{M}^*$, it suffices to show that for any $\Gamma\in \overline{M}^{**}$, $\Gamma(\varphi(u))<\infty$ for some weak unit $u\in S$, there exists an upper directed family $F\subseteq\overline{M}$, such that $VF=\Gamma$ where V is the supremum in $\overline{M}^{**}$.

Fix Γ as above. For any $\xi\in M$, define the element $B_\xi\Gamma$ of $\overline{M}^{**}$ by the formula

$$B_\xi\Gamma(\mu)=\Gamma(B^*_\xi\mu), \qquad M^*$$

It is then easy to check that $B_\xi\Gamma\in\overline{M}^{**}$ for any $\xi\in M$, the family $F=\{B_\xi\Gamma\}_{\xi\in M}$ is obviously upper directed and from corollary 3.4 we get $VF=\Gamma$

We claim that $B_\xi\Gamma\in \mathrm{Exc}_\xi\subseteq\overline{M}$, for any $\xi\in M$. For this, it suffices to find, for each $\xi\in M$, an element $m_\xi\in \mathrm{Exc}_\xi$ such that

$$B_\xi \Gamma = \tilde{m}_\xi \qquad \text{on } \varphi(S)$$

since then, the proof of theorem 3.8 shows that the above equality holds on $\overline{M}^*$.

In order to find m_ξ let us consider the map

$$s \longrightarrow B_\xi \Gamma(\varphi(s))$$

from S into $\overline{R}_+$, which is additive, increasing, continuous in order from below and finite for $s = u$.

We remark that if $s_1, s_2 \in S$, $s_1 \leqslant s_2$ ξ a.s. we have

$$B_\xi \Gamma(\varphi(s_1)) \leqslant B_\xi \Gamma(\varphi(s_2)) .$$

Indeed, it suffices to show that

$$B^*_\xi \varphi(s_1) \leqslant B^*_\xi \varphi(s_2) \tag{2}$$

Let $\eta \in \overline{M}$ arbitrary. We have

$$B^*_\xi \varphi(s_1)(\eta) = \varphi(s_1)(B_\xi \eta) = L(B_\xi \eta, s_1)$$

$$B^*_\xi \varphi(s_2)(\eta) = \varphi(s_2)(B_\xi \eta) = L(B_\xi \eta, s_2)$$

Hence (2) follows using proposition 2.1 since $B_\xi \eta \in \mathrm{Exc}_\xi$, for any $\eta \in \overline{M}$.

Now, we can define the map γ_ξ on S_ξ by

$$\gamma_\xi(\hat{s}) = B_\xi' \Gamma(\varphi(s)) , \forall s \in S$$

From (2), γ_ξ is well defined, increasing and then it is additive, and finite for $\hat{s} = \hat{u}$. We show that γ_ξ is continuous in order from below. Let $\hat{s} \in S_\xi$ and $\{s_i\}_{i \in I}$ an upper directed family such that $\hat{s_i} \nearrow \hat{s}$. In order to show that

$$\gamma_\xi(\hat{s_i}) \nearrow \gamma_\xi(\hat{s})$$

it suffices to show that

$$B^*_\xi \varphi(s_i) \nearrow B^*_\xi \varphi(s) \quad \text{in } \overline{M}^*$$

For this, let $\eta \in \overline{M}$ arbitrary. Using theorem 2.9, we have

$$B^*_\xi \varphi(s_i)(\eta) = \varphi(s_i)(B_\xi \eta) = L(B_\xi \eta, s_i) = \mathcal{L}_\xi(B_\xi \eta, \hat{s_i})$$

$$\nearrow \mathcal{L}_\xi(B_\xi \eta, \hat{s}) = B^*_\xi \varphi(s)(\eta).$$

Therefore γ_ξ is an H integral on the H cone S_ξ considered as a solid and increasingly dense convex subcone of $\mathcal{E}(\mathcal{V})_\xi$.

From proposition 1.1 and theorem 2.9, there exists then, an unique element $m_\xi \in Exc_\xi$ such that

$$\gamma_\xi(\hat{s}) = \mathcal{L}_\xi(m_\xi, \hat{s}), \quad \forall \hat{s} \in S_\xi$$

and hence we get

$$B_\xi \Gamma(\varphi(s)) = \gamma_\xi(\hat{s}) = \mathcal{L}_\xi(m_\xi, \hat{s}) = L(m_\xi, s) =$$
$$= \tilde{m}_\xi(\varphi(s)),$$

for any $s \in S$, hence $B_\xi \Gamma \in Exc_\xi \subseteq \overline{M}$ for any $\xi \in M$. Considering $F = (B_\xi \Gamma)_{\xi \in M}$ we have $F \subseteq \overline{M}$ and $F \nearrow \Gamma$, and the theorem is proved.

Theorem 3.12

M is solid and increasingly dense in M^{**}.

Proof.

From proposition 1.1, any $\mu \in M^*$ extends uniquely to an element $\overline{\mu} \in \overline{M}^*$ and denote by ω the map

$$\mu \longrightarrow \overline{\mu}$$

that is an isomorphism between the H cones M^* and $\overline{M}^*$. Now, take $\Gamma \in M^{**}$ dominated by an element $\tilde{m}$ of M^{**}, where $m \in M$. It then follows

$$\Gamma o \omega^{-1} \leq \tilde{m} o \omega^{-1} = \tilde{m} \qquad \text{on } \overline{M}^*$$

where m is regarded now as element of $\overline{M}$.

From theorem 3.8, there exists $m_1 \in \overline{M}$, hence $m_1 \in M$ since M is solid, such that

$$\Gamma o \omega^{-1} = \tilde{m}_1 = \tilde{m}_1 \, o \, \omega^{-1} \qquad \text{on } \overline{M}^*$$

and therefore $\Gamma = \tilde{m}$ on M^* and the firsr assertion of theorem is proved.

For the second take $\Gamma \in M^{**}$. Then $\Gamma o \omega^{-1} \in M^{**}$ and from theorem 3.11, there exists a family $\overline{F} \subset \overline{M}$ such that $\overline{F} \nearrow \Gamma o \omega^{-1}$ in $\overline{M}^{**}$. For each element $t \in \overline{F}$, consider the set $F_t = \{\eta \in M; \eta \leq t\}$.

Then F_t is upper directed and $\vee F_t = t$ in $\overline{M}$ since M is increasingly dense in $\overline{M}$. Hence $F_t \nearrow t$ in $\overline{M}^{**}$ for any $t \in \overline{F}$. Let $F = \bigcup_t F_t \subset M$. Then F is upper directed since $\overline{F}$ and each F_t are upper directed. We also have

$$VF = V_t (VF_t) = V\overline{F} = \Gamma \circ \overline{\omega}^{1} \qquad \text{in } \overline{M}^{**}$$

Now, obviuosly $F \circ \overline{\omega}^{1} = F$ and hence

$$VF = \Gamma \qquad \text{in } M^{*}$$

and the second assertion of theorem is proved.

Remark.

In the particular case M = Exc, the fact that Exc is increasingly dense in Exc^{**} results directly from proposition 3.10 and the proof of theorem 3.8.

4. Exc and reference measure.

In this section we characterize those H-cones which are isomorphic with some Exc(V), where the resolvent $\mathcal{V}=(V_\alpha)_{\alpha>0}$ posseses in addition a reference measure. Of course this is a strong restriction, but this case covers however sufficient examples. We need some additional elements from H-cones theory.

Definition ([2])

Let S be an H cone with a weak unit u. An element $s\in S$ is called u continuous if for any family $F\subset S$, $F\nearrow s$ and for any $\varepsilon>0$, there exists $t\in F$ such that

$$s\leqslant t+\varepsilon u$$

Denote by Su the set of all u continuous elements of S. It can be shown that Su is a specifically solid convex subcone of S. An element $s\in S$ is said to be universally continuous if it is u continuous for any weak unit u and denote by S_0 the set of all such elements of S. It then follows that S_0 is an specifically solid convex subcone of S since $S_0 = \bigcap_{\substack{u\in S\\ \text{weak unit}}} Su$

Definition ([2])

An H cone S is called a standard H cone if:

1) There exists a weak unit in S.

2) There exists a countable subset D of S_0 such that for any $s\in S$, there exists an upper directed family F of elements from D such that $F\nearrow s$.

We can state now the announced characterization. If a resolvent $\mathcal{V}$ posseses a reference measure we say that $\mathcal{V}$ is absolutely continuous.

Theorem 4.1.

An H cone M is isomorphic with the H cone Exc V of all excessive measures with respect to an absolutely continuous submarkovian resolvent $\mathcal{V}=(V_\alpha)_{\alpha>0}$ on a measurable space $(X,\mathcal{X})$ for which the initial kernel V is proper and strict, if and only if M is the dual of an standard H cone.

Proof.

If we have a resolvent $\mathcal{V}$ as above, it is known that $S = \{s \in \mathcal{E}(\mathcal{V}) : s < \infty\ \mathcal{V}\text{a.s.}\}$ is a standard H cone ([2]) and Exc V is isomorphic with S^* through the energy functional (see remark following this theorem).

It follows also that Exc(V) is an standard H cone. Conversely, let S be a standard H cone such that $M = S^*$. If we pick a weak unit $u \in S$ and consider $Ku = \{\mu \in M : \mu(u) \leqslant 1\}$ then it can be shown that Ku is a simplex in the cone M and a compact metrizable space with respect to the coarsest topology rendering continuous on M the elements of S_0 through the duality relation between S and M. Using then Choquet's representation theorem S is identified with the standard H cone of functions Su on the set Xu of all extreme points of Ku less 0, through the duality relation

$$s \longrightarrow \tilde{s} \qquad \tilde{s}(\mu) = \mu(s), \quad \forall \mu \in Xu$$

(see [2] , theorem 4.2.12). It follows then (see [2] , theorem 4.4.4.) that there exists an absolutely continuous submarkovian resolvent $\mathcal{V} = (V_\alpha)_{\alpha > 0}$ (for which the initial kernel V is proper and strict) on $(Xu, \mathcal{B}(Xu))$ such that S is a solid and increasingly dense subcone of $\overline{S} = \{s \in \mathcal{E}(\mathcal{V}) : s < \infty\ \mathcal{V}\text{a.s.}\}$.

Using then proposition 1.1. it follows that we have

$$M = S^* \simeq \overline{S}^* \simeq \text{Exc } V .$$

Remark

We have already noted in section 1 that $\mathcal{V}$ posseses a reference measure iff Exc posseses a weak unit. In fact, if ν is a finite reference measure for $\mathcal{V}$ then $\xi_0 = \nu V$ is an excessive measure of reference too and it follows easily that any other excessive measure is absolutely continuous with respect to ξ_0, that is Exc = Exc$_{\xi_0}$ and we have that ξ_0 is a weak unit for Exc = Exc$_{\xi_0}$.

Therefore, in this case, if we consider this (reference) measure ξ_0 in theorem 2.9., we get that every equivalence class in $\mathcal{E}(\mathcal{V})_{\xi_0}$ contains a single representant from $\mathcal{E}(\mathcal{V})$, hence $\mathcal{E}(\mathcal{V})_{\xi_0} \simeq S$, $\mathrm{Exc}_{\xi_0} = \mathrm{Exc}$, $\mathcal{L}_{\xi_0} = L$, and therefore the first assertion of above theorem becomes a sp corollary of theorem 2.9.

Moreover, we get $\mathrm{Exc}^* \simeq S$ through the energy functional, that expresses the complete duality between the H cones S and Exc in this case.

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