INSTITUTUL DE MATEMATICA
AL ACADEMIEI ROMANE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

LEFSCHETZ TYPE RESULTS FOR PROPER INTERSECTIONS

by

LUCIAN BADESCU

PREPRINT No. 3/1993

BUCURESTI

LEFSCHETZ TYPE RESULTS FOR PROPER INTERSECTIONS

by

LUCIAN BADESCU

February, 1993

Institute of Mathematics of the Romanian Academy

P.O.Box 1-764, RO-70700 Bucharest, Romania

LEFSCHETZ TYPE RESULTS FOR PROPER INTERSECTIONS

Lucian Bădescu

Institute of Mathematics of the Romanian Academy,
P.O. Box 1-764, Ro-70700 Bucharest, Romania.

Introduction and statements of results

Let us begin by recalling two important generalizations of the classical theorems of Lefschetz on hyperplane sections.

In 1970 Barth proved the following result (see $[Ba]_1$).

Theorem A. *Let A and B be two smooth projective connected subvarieties of the complex projective space* $\mathbb{P}^n_{\mathbb{C}}$ *such that* $a+b\geq n+r$ *and* $2a\geq n+s$, *where* $a=\dim(A)$ *and* $b=\dim(B)$. *Assume that A and B have a proper intersection, i.e.* $\dim(A\cap B)=a+b-n$. *Then the natural map*

$$H^q(B,\mathbb{Q})\longrightarrow H^q(A\cap B,\mathbb{Q})$$

on singular cohomology groups with rational coefficients is an isomorphism for every q such that $q\leq\min\{r-1,s\}$.

Note. Taking $B=\mathbb{P}^n_{\mathbb{C}}$ one gets Barth's famous result asserting that the maps $H^q(\mathbb{P}^n,\mathbb{Q})\longrightarrow H^q(A,\mathbb{Q})$ are isomorphisms for every $q\leq s$, provided that $2a\geq n+s$. Later on, Barth and Larsen extended this statement to analogous assertions concerning the homotopy groups or singular homology (or cohomology) groups with coefficients in $\mathbb{Z}$. In particular, if $2a\geq n+2$ one gets (via the exponential sequence) the important fact that the natural map $\mathrm{Pic}(\mathbb{P}^n)\longrightarrow\mathrm{Pic}(A)$ is an isomorphism. Algebraic proofs of Barth's results (including the assertion about the Picard group) have been subsequently given by Ogus (see $[O]_1$ and $[O]_2$).

In 1981 Fulton and Lazarsfeld (see [F-L], §9) proved the following theorem by using a deep result conjectured by Deligne and proved by Goresky and MacPherson (whose proof involves the stratified Morse theory).

Theorem B. *Let X be a complex projective local complete intersection variety of pure dimension m, and let* $f:X\longrightarrow\mathbb{P}^n_{\mathbb{C}}\times\mathbb{P}^n_{\mathbb{C}}$ *be a finite morphism. Denote by* Δ *the diagonal of* $\mathbb{P}^n_{\mathbb{C}}\times\mathbb{P}^n_{\mathbb{C}}$ *and assume that* $m\geq n+2$. *Then* $\pi_i(X,f^{-1}(\Delta))=0$ *for* $i=1$ *and for* $3\leq i\leq m-n$, *and there is a canonical exact sequence*

$$\pi_2(f^{-1}(\Delta))\longrightarrow\pi_2(X)\longrightarrow\mathbb{Z}\longrightarrow\pi_1(f^{-1}(\Delta))\longrightarrow\pi_1(X)\longrightarrow 1,$$

where $\pi_i(X,Y)$ *(resp.* $\pi_i(X)$*) denotes the i-th homotopy group of a pair (X,Y) (resp. of X).*

As a consequence of Theorem B Fulton and Lazarsfeld find:

Corollary. *Let X be a complex projective local complete intersection of pure dimension m, let $f:X\longrightarrow\mathbb{P}^n_{\mathbb{C}}$ be a finite morphism, and let $Y\subset\mathbb{P}^n_{\mathbb{C}}$ be a closed local complete intersection of pure codimension d. Then the induced map $f_*:\pi_i(X,f^{-1}(\Delta))\longrightarrow\pi_i(\mathbb{P}^n_{\mathbb{C}},Y)$ is bijective if $i\leq m-d$ and surjective if $i=m-d+1$.*

Theorem B and its corollary generalize many previous results by Barth, Larsen and Ogus. Related results were also proved by Milnor, Lazarsfeld, Newstead, Sommese and others. We refer the reader to [F-L] §9 and its references, [So], as well as to the recent paper by Lyubeznik [Ly].

Now, let k be an arbitrary algebraically closed ground field. Then we have the following connectivity result of Fulton and Hansen [F-H] (see also $[Ba]_2$ for a special case).

Theorem C. *Let $f:X\longrightarrow\mathbb{P}^n\times\mathbb{P}^n$ be a finite morphism from an irreducible variety X over k such that $\dim(X)\geq n+1$. Then $f^{-1}(\Delta)$ is connected (and of positive dimension), where Δ is the diagonal of $\mathbb{P}^n\times\mathbb{P}^n$.*

If in theorem C we assume in addition that f(X) and Δ have a proper intersection (i.e. $\dim(f(X)\cap\Delta)=\dim f(X)-n$ $(=\dim(X)-n)$) then we have the following stronger conclusion involving the G_3 condition of Hironaka and Matsumura (see [Hi-Ma]):

Theorem D. *In the hypotheses of Theorem C assume in addition that f(X) and Δ have a proper intersection. Then $f^{-1}(\Delta)-W$ is G_3 in X-W for every closed subscheme W of $f^{-1}(\Delta)$ such that $\dim(W)\leq\dim(f^{-1}(\Delta))-2$, i.e. the canonical map $K(X-W)\longrightarrow K(\widehat{X-W}_{/f^{-1}(\Delta)-W})$ from the field of rational functions of X-W to the ring of formal rational functions of X-W along $f^{-1}(\Delta)-W$, is an isomorphism. Moreover, $f^{-1}(\Delta)$ meets every closed subvariety of X of dimension $\geq n$.*

Note. Theorem D is proved in [B-Ba1], and implies a result of Faltings [Fa] which assserts that $A\cap B-W$ is G_3 in A-W and in B-W for every irreducible projective subvarieties A and B of $\mathbb{P}^n$ such that $\dim(A)+\dim(B)\geq n+1$ and A and B have a proper intersection, and for every closed subscheme W of $A\cap B$ such that $\dim(W)\leq\dim(A\cap B)-2$. The proof of theorem D given in [B-Ba1] uses global methods and relies on some results of Hironaka and Matsumura [Hi-Ma], while Faltings' proof of his result requires local methods. Note also that the complex-analytic analogue of Faltings' result (in case $W=\emptyset$) was previously known (see [Ch]).

The aim of this paper is, using algebraic methods, to get results of Lefschetz type for pairs of the form $(X,f^{-1}(\Delta))$, where $f:X\longrightarrow\mathbb{P}^n\times\mathbb{P}^n$ is a finite morphism from a smooth irreducible variety X such that $\dim(X)\geq n+2$, char(k)=0 and f(X) and Δ have a proper intersection. These results will involve the Picard group and the Abhyankar-Grothendieck algebraic fundamental group π_1^{alg} (see [SGA-1]). They could be considered as complementing theorem D (in the case when $\dim(X)\geq n+2$), on the one hand, and in some sense as partial algebraic analogues of theorem B, on the other hand.

Theorem 1. *Let* $f:X\longrightarrow\mathbb{P}^n\times\mathbb{P}^n$ *be a finite morphism from a smooth irreducible variety X over k. Assume* $char(k)=0$, $\dim(X)\geq n+2$ *and* $Y:=f^{-1}(\Delta)$ *normal (where the inverse image of the diagonal is taken in the scheme theoretical sense). Assume also that* $f(X)$ *and* Δ *have a proper intersection (i.e.* $\dim(Y)=\dim(X)-n$*). Then the natural maps*

$$\underline{Pic}^0(X)\longrightarrow\underline{Pic}^0(Y) \text{ and } Alb(Y)\longrightarrow Alb(X)$$

of Picard and Albanese varieties are isomorphisms. In particular, $q(X)=q(Y)$, *where* $q(V)=h^1(V,O_V)$. *If in addition* $\dim(X)\geq n+3$ *and Y is smooth in codimension two then there is a canonical exact sequence*

$$0\longrightarrow\mathbb{Z}\longrightarrow Pic(X)\longrightarrow Pic(Y)\longrightarrow 0$$

Combining theorem 1 with some results of Barth-Larsen we get:

Corollary. *Let* $g:A\longrightarrow\mathbb{P}^n$ *be a finite morphism from a smooth irreducible variety A of dimension a over k, and let B be a smooth irreducible subvariety of dimension b of* $\mathbb{P}^n$. *Assume* $char(k)=0$, $a+b\geq n+2$ *and* $2b\geq n+2$. *Assume also that* $g(A)$ *and B have a proper intersection in* $\mathbb{P}^n$, *and that* $C:=g^{-1}(B)$ *(in the scheme theoretical sense) is normal. Then the natural maps* $\underline{Pic}^0(A)\longrightarrow\underline{Pic}^0(C)$ *and* $Alb(C)\longrightarrow Alb(A)$ *are isomorphisms. If in addition* $a+b\geq n+3$ *and C is smooth in codimension two then the natural map* $Pic(A)\longrightarrow Pic(C)$ *is an isomorphism.*

Proof of the corollary. Since $2b\geq n+2$ the results of Barth and Larsen mentioned above (which have been also proved by algebraic methods by Ogus) imply that $q(B)=0$ and $Pic(B)=\mathbb{Z}[O_B(1)]$. Set $X=A\times B$ and $f:=g\times i:X\longrightarrow\mathbb{P}^n\times\mathbb{P}^n$, with i the inclusion of B in $\mathbb{P}^n$. Then $C\cong f^{-1}(\Delta)$, and since $q(B)=0$, $Pic(X)=Pic(A\times B)\cong Pic(A)\times Pic(B)\cong Pic(A)\times\mathbb{Z}$ (by EGA III (4.6.5) and the base change theorems, or also by $[Ha]_2$, p.292, Ex. 12.6). Then the statements about $\underline{Pic}^0$ and Pic follow from theorem 1. Since the Albanese variety is the dual of the Picard variety, the statement about Alb also follows. □

Theorem 2. *Let* $f:X\longrightarrow\mathbb{P}^n\times\mathbb{P}^n$ *be a finite morphism from an irreducible variety X over k. Assume* $char(k)=0$, $\dim(X)\geq n+2$, $Y:=f^{-1}(\Delta)$ *contained in the smooth locus* $X_0:=Reg(X)$ *of X, and that* $f(X)$ *and* Δ *have a proper intersection. Then there is a canonical exact sequence of profinite groups*

$$\hat{\mathbb{Z}}\longrightarrow\pi_1^{alg}(Y)\longrightarrow\pi_1^{alg}(X_0)\longrightarrow 1,$$

where $\pi_1^{alg}(V)$ *is the algebraic fundamental group of an algebraic variety V (see [SGA-1]), the second map is induced by the inclusion* $Y\subset X_0$, *and* $\hat{\mathbb{Z}}$ *is the profinite completion of* $\mathbb{Z}$.

Corollary. *Let* $g:A\longrightarrow\mathbb{P}^n$ *be a finite morphism from a smooth irreducible variety A of dimension a over k, and let B be a smooth irreducu-*

cible subvariety of dimension b of $\mathbb{P}^n$. Assume char(k)=0, $a+b\geq n+2$ and $2b>n$. Assume also that g(A) and B have a proper intersection in $\mathbb{P}^n$. Then there is a canonical exact sequence of profinite groups

$$\hat{\mathbb{Z}}\longrightarrow\pi_1^{alg}(g^{-1}(B))\longrightarrow\pi_1^{alg}(A)\longrightarrow 1.$$

Proof of the corollary. We proceed similarly as in the proof of the corollary of theorem 1. Since $2b>n$, B is algebraically simply-connected (this fact is already a consequence of theorem C, see [F-L]). Then use the isomorphism $\pi_1^{alg}(A\times B)\cong\pi_1^{alg}(A)\times\pi_1^{alg}(B)$ (see [SGA-1]), and the right-hand side is isomorphic to $\pi_1^{alg}(A)$ because B is algebraically simply-connected. □

Proofs of theorems 1 and 2

The proofs of theorems 1 and 2 rely on the Grothendieck-Lefschetz theory as developed in [SGA-1] (see also $[Ha]_1$), and on the idea of Deligne's (see [F-L]) consisting in using the following elementary and remarkable geometric construction (which was also used in other classical or mordern problems, see e.g. [La-Sc]).

In the projective space $\mathbb{P}^{2n+1}$ of coordinates $(x_0,\dots,x_{2n+1})$ consider the linear subspaces

$L_1=\{x=(x_0,\dots,x_{2n+1})\in\mathbb{P}^{2n+1}/x_0=\dots=x_n=0\}$ and

$L_2=\{x=(x_0,\dots,x_{2n+1})\in\mathbb{P}^{2n+1}/x_{n+1}=\dots=x_{2n+1}=0\}$,

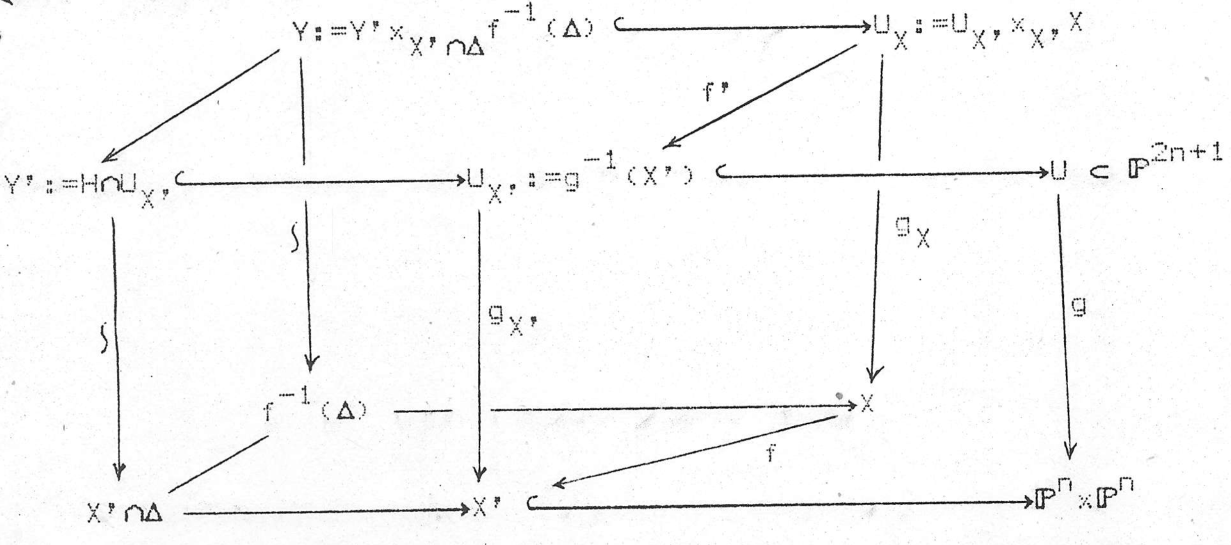
and set $U=\mathbb{P}^{2n+1}-(L_1\cup L_2)$. Consider the morphism $g:U\longrightarrow\mathbb{P}^n\times\mathbb{P}^n$ defined by

$g(x_0,\dots,x_{2n+1})=((x_0,\dots,x_n),(x_{n+1},\dots,x_{2n+1}))$.

Then g is the projection of a locally trivial $\mathbb{G}_m$-bundle (in the Zariski topology), where $\mathbb{G}_m$ is the multiplicative group. Let H be the linear subspace of $\mathbb{P}^{2n+1}$ defined by the equations

$$x_0=x_{n+1},\ x_1=x_{n+2},\dots,\ x_n=x_{2n+1}.$$

Then $H\subset U$ and g/H defines an isomorphism between H and Δ. In particular, if we set $X':=f(X)$, we get the commutative diagram



with g_X, and g_X also projections of locally trivial $\mathbb{G}_m$-bundles. Let Z' be the closure of $U_{X'}$ in $\mathbb{P}^{2n+1}$. Then Y' is the zero locus of the section $s'=(s'_0,\dots,s'_n)\in H^0(Z',O_{Z'}(1)^{\oplus(n+1)})$, with $s'_i=(x_i-x_{n+i+1})/Z'$, $i=0,\dots,n$. Since by the hypotheses of theorems 1 and 2 X' and Δ have a proper intersection and since $Y'\cong X'\cap\Delta$ we get $\mathrm{codim}_{Z'}(Y')=\mathrm{codim}_{U_{X'}}(Y')=\mathrm{codim}_{X'}(X'\cap\Delta)+1=n+1$. In other words, Y' is a complete intersection of Z' with the $n+1$ hyperplanes of $\mathbb{P}^{2n+1}$ given by $x_i=x_{n+i+1}$, $i=0,1,\dots,n$.

Lemma 1. *In the above notations the Grothendieck-Lefschetz condition $Lef(U_X,Y)$ holds for the pair (U_X,Y), i.e. for every open subset V of U_X such that $Y\subset V$ and for every locally free sheaf F on V of finite rank, there is an open subset V' with $Y\subset V'\subseteq V$ such that the natural map $H^0(V',F)\longrightarrow H^0(\hat{U}_{X/Y},\hat{F})$ is an isomorphism, where $\hat{U}_{X/Y}$ is the formal completion of U_X along Y.*

Proof of lemma 1. We shall give two proofs.

First proof. Since $f:X\longrightarrow X'$ is finite and surjective, the morphism $f':U_X\longrightarrow U_{X'}$ is also finite and surjective. In particular, it makes sense to speak about the extension $K(Z')=K(U_{X'})\longrightarrow K(U_X)$ of fields of rational functions. Let then Z the normalization of Z' in $K(U_X)$, and set $O_Z(1):=u^*(O_{Z'}(1))$, where $u:Z\longrightarrow Z'$ is the canonical morphism. Since u is finite and $O_{Z'}(1)$ ample, $O_Z(1)$ is also ample.

On the other hand, in both theorems 1 and 2 $f^{-1}(\Delta)$ is contained in $\mathrm{Reg}(X)$. Because the morphism g_X is the projection of a $\mathbb{G}_m$-bundle and $Y\cong f^{-1}(\Delta)$ (via g_X), it follows that $Y\subset\mathrm{Reg}(U_X)$. Therefore $\mathrm{Reg}(U_X)\subset Z$. In other words, the embeddings $Y\longrightarrow U_X$ and $Y\longrightarrow Z$ are equivalent in the sense that both U_X and Z contain a common open neighbourhood of Y (namely $\mathrm{Reg}(U_X)$). Set $s_i=u^*(s'_i)$ and $s=(s_0,s_1,\dots,s_n)$. Then s is a global section of the ample vector bundle $O_Z(1)^{\oplus(n+1)}$ and the zero locus of s coincides with Y (by the above commutative diagram). Then $Z-Y$ is covered by the $n+1$ affine open subsets $D_+(s_0),\dots,D_+(s_n)$. In particular, computing the cohomology of $Z-Y$ by Cech we get that $cd(Z-Y)\leq n$, where $cd(Z-Y)$ is Hartshorne's cohomological dimension defined by $cd(Z-Y)=\max\{q\geq 0/\exists G\in Coh(Z-Y)$ such that $H^q(Z-Y,G)\neq 0\}$ (see e. g. [Ha]$_1$). Recalling that $\dim(Z)=\dim(X)+1\geq n+3$, we infer that $cd(Z-Y)\leq\dim(Z)-2$. If Z were smooth then by a result of Hartshorne (see [Ha]$_1$ p.96, (3.4)) we would get that $Lef(Z,Y)$ holds. But in general Z is not smooth. However, Z is normal, $Y\subset\mathrm{Reg}(Z)$, and Y is a local complete intersection in Z (since Y is even a global complete intersection in Z). Then, by an appropriate change of Hartshorne's arguments (see [B-Ba1], theorem 1.3), the inequality $cd(Z-Y)\leq\dim(Z)-2$ still implies $Lef(Z,Y)$ in our situation. □

Second proof. Since Y' is the intersection of Z' with a linear subspace of $\mathbb{P}^{2n+1}$ of codimension n+1 in $\mathbb{P}^{2n+1}$ and $\mathrm{codim}_{Z'}(Y')=n+1$, by [Hi-Ma], lemma 4.3, Y' is G_3 in Z', or else, Y' is G_3 in $U_{X'}$. Because $f':U_X \longrightarrow U_{X'}$ is proper and surjective, by [Hi-Ma], theorem 2.7 we infer that $Y=f'^{-1}(Y')$ is also G_3 in U_X, and hence Y is G_3 in Z because $Y \subset \mathrm{Reg}(U_X) \subset Z$. Since Y is the zero locus of a section of an ample vector bundle of rank $n+1<\dim(Z)$, it follows that Y meets every hypersurface of Z. Then essentially applying a result of Speiser (see [Sp], theorem 1, or also $[\mathrm{Ha}]_1$, p. 202, cor.2.2) we deduce that Lef(Z,Y) holds. In [Sp] and in $[\mathrm{Ha}]_1$ the variety Z was assumed to be smooth. However, Speiser's proof still works if Z is only smooth along Y. □

Lemma 2. *In the above situation and notations (including the proof of lemma 1), for every locally free sheaf G of finite rank on $\hat{Z}:=\hat{Z}_{/Y}$, the sheaf $G(m):=G\otimes O_{\hat{Z}}(m)$ is generated by its global sections.*

Proof of lemma 2. Lemma 2 is a direct consequence of Proposition 1.3, page 168 of $[\mathrm{Ha}]_1$ (which is the key point in proving the effectivity part of Leff(Z,Y)). In fact, proposition 1.3 of Hartshorne's book assumes that Z is smooth. Since in our situation $Y \subset \mathrm{Reg}(Z)$, one can replace Z by a resolution of singularities $Z^{\#}$ in such a way that Reg(Z) remains unchanged, and then we can apply proposition 1.3 from Hartshorne's book quoted above to $(Z^{\#},Y)$ (the existence of such a resolution is assured at least in characteristic zero by Hironaka). However, we do not really need the resolution of singularities, because if we look carefully at the proof of proposition 1.3 in Hartshorne's book we see that the hypothesis that $Y \subset \mathrm{Reg}(Z)$ is actually sufficient to apply this result directly to (Z,Y). □

Corollary (of lemmas 1 and 2). *In the situation of lemma 1 there is a commutative diagram*

$$(*) \qquad \begin{array}{ccc} Y & \hookrightarrow & U_X \\ \wr\downarrow & & \downarrow g_X \\ f^{-1}(\Delta) & \hookrightarrow & X \end{array}$$

such that g_X is the projection of a locally trivial $\mathbb{G}_m$-bundle (in the Zariski topology), $Y \subset \mathrm{Reg}(U_X)$ and the effective Grothendieck-Lefschetz condition $\mathrm{Leff}(U_X,Y)$ holds, i.e. $\mathrm{Lef}(U_X,Y)$ holds and for every locally free sheaf G of finite rank on $\hat{U}_{X/Y}$ there is an open neighbourhood V of Y in U_X and a locally free sheaf F of finite rank on V with the property that $\hat{F}_{/Y} \cong G$.

Proof of the corollary. The existence of diagram (*) was proved above. Moreover, by lemma 1 we know that $\mathrm{Lef}(U_X,Y)$ holds. It remains to prove the effective part of $\mathrm{Leff}(U_X,Y)$. We proceed exactly as in the proof of theorem 1.5 of [Ha]$_1$,pp.172-173. For the convenience of the reader we repeat the argument here. Let G be a locally free sheaf on $\hat{Z}=\hat{Z}_{/Y}=\hat{U}_{/Y}$ of finite rank. Using lemma 2 twice one gets an exact sequence of the form

$$O_{\hat{Z}}(-p)^{\oplus b} \xrightarrow{\ \phi\ } O_{\hat{Z}}(-m)^{\oplus a} \longrightarrow G \longrightarrow 0$$

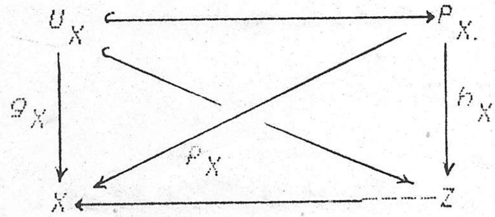
for some positive integers m, p, a and b. Considering the locally free sheaf $E=\underline{Hom}_Z(O_Z(-p)^{\oplus b},O_Z(-m)^{\oplus a})$, from Lef(Z,Y) we infer that there is an open neighbourhood V' of Y in U_X such that the natural map $H^0(V,E) \longrightarrow H^0(\hat{Z},\hat{E})$ is an isomorphism. This implies that the homomorphism $\phi \in H^0(\hat{Z},\hat{E})$ comes from a homomorphism

$$\mu \in Hom_V(O_V(-p)^{\oplus b},O_V(-m)^{\oplus a})=H^0(V,E),$$

i.e. $\hat{\mu}=\phi$. Then it is clear that $F':=Coker(\mu)$ is a coherent sheaf on V' such that $\hat{F'}\cong G$. Since G is locally free, F' is also locally free along Y. Therefore, shrinking V' if necessary, we get the desired neighbourhood V of Y in U_X such that $F:=F'/V$ is locally free and $H^0(V,F)\cong H^0(V',F')\cong H^0(\hat{Z},\hat{F})$. □

Note. Using theorem D and arguments similar to those of the second proof of lemma 1, one can prove that, under the hypotheses of theorems 1 or 2, Lef(X,$f^{-1}(\Delta)$) also holds.

Lemma 3. *In the hypotheses of theorem 1 and in the above notations the morphism g_X of diagram (*) can be included in a commutative diagram of the form*



where $P_X=\mathbb{P}(O_X(1,0)\oplus O_X(0,1))$ is the projective bundle associated to the pull-back of the vector bundle $O(1,0)\oplus O(0,1)$ on $\mathbb{P}^n\times\mathbb{P}^n$ via the morphism f, p_X is the canonical projection of P_X, the complement of U_X in P_X is the union of two mutually disjoint irreducible divisors E_1 and E_2 with the property that p/E_i defines an isomorphism between E_i and X, i=1,2, and h_X is birational morphism which blows down E_1 and E_2 to subvarieties of Z of codimension ≥2.

Proof. Consider the blowing up $h:P\longrightarrow\mathbb{P}^{2n+1}$ of $\mathbb{P}^{2n+1}$ along the union $L_1\cup L_2$ of the linear subspaces considered above. Then it is well known (and easy to prove, see e.g. [La-Sc]) that P is isomorphic to the projective bundle $\mathbb{P}(O(1,0)\oplus O(0,1))$, $g\circ h$ is a morphism which coincides to the canonical projection p of $\mathbb{P}(O(1,0)\oplus O(0,1))$, and p/L'_i defines an isomorphism beteen L'_i and $\mathbb{P}^n\times\mathbb{P}^n$, with $L'_i:=h^{-1}(L_i)$. Consider the cartesian diagram

$$\begin{array}{ccccc} P_X & \longrightarrow & P_{X'} & \longrightarrow & P \\ \downarrow p_X & & \downarrow p_{X'} & & \downarrow p \\ X & \longrightarrow & X' & \longrightarrow & \mathbb{P}^n\times\mathbb{P}^n \end{array}$$

The existence of the open immersion $U_X \longrightarrow P_X$ comes from the fact that U_X is the pull-back of $U=\mathbb{P}^n-(L_1\cup L_2)$, while the existence of $U_X \longrightarrow Z$ from the fact that Z is the normalization of Z' in $K(U_X)$ and from the smoothness of U_X (recall that X is smooth by hypotheses and $g_X:U_X\longrightarrow X$ is a $\mathbb{G}_m$-bundle). Set $E_i = L_i'\times_{\mathbb{P}^n\times\mathbb{P}^n}X$, $i=1,2$. Then it is clear that $E_1\cup E_2$ is the complement of U_X in P_X. Since p/L_i' defines an isomorphism between L_i' and $\mathbb{P}^n\times\mathbb{P}^n$ it follows that p_X/E_i defines an isomorphism between E_i and X, $i=1,2$. In particular, E_1 and E_2 are irreducible divisors whose intersection is empty.

Since $P_{X'}$ and Z' are both irreducible and contain $U_{X'}=U\cap Z'$ as an open subset, we infer that $h(P_{X'})=Z'$. Recalling that P_X is smooth (as a $\mathbb{P}^1$-bundle over the smooth variety X) and the definition of Z, $h(P_{X'})=Z'$ implies that P_X dominates Z (and this yields the morphism h_X).

Finally, it remains to prove that $h_X(E_i)$ are subvarieties of Z of codimension ≥ 2, $i=1,2$. This follows from $h_X(E_i)\cap Y=\emptyset$ together with the observation that Y is the zero locus of a section of an ample vector bundle on Z whose rank is $<\dim(Z)$. □

Now we are ready to prove theorems 1 and 2.

Proof of theorem 1. Because X is smooth and $p_X:P_X\longrightarrow X$ is a $\mathbb{P}^1$-bundle, P_X (and hence also U_X) is smooth. Since $Y = f^{-1}(\Delta)$ is the zero locus of the section $s=(s_0,\dots,s_n)\in H^0(Z,O_Z(1)^{\oplus(n+1)})$ and $\mathrm{codim}_Z(Y)= n+1$, the normal bundle $N_{Y,Z}$ of Y in Z is isomorphic to $O_Y(1)^{\oplus(n+1)} := O_Z(1)^{\oplus(n+1)}/Y$, and hence $I_Y/I_Y^2\cong O_Y(-1)^{\oplus(n+1)}$, where I_Y denotes the sheaf of ideals of Y in Z. Therefore for every $m\geq 1$ we get:

$$(1)\qquad I_Y^m/I_Y^{m+1} \cong S^m(I_Y/I_Y^2) \quad \text{(with } S^m \text{ the m-th symmetric power)}$$
$$\cong S^m(O_Y(-1)^{\oplus(n+1)}).$$

In particular, for every $m\geq 1$ I_Y^m/I_Y^{m+1} is a direct sum of line bundles of the form $O_Y(-j)$ with $j>0$. Recalling that $Y=f^{-1}(\Delta)$ is normal of dimension ≥ 2 and $\mathrm{char}(k)=0$ by hypotheses, a well known vanishing theorem of Kodaira-Mumford implies that

$$(2)\qquad H^1(Y,I_Y^m/I_Y^{m+1}) = 0 \qquad \text{for all } m\geq 1.$$

According to lemma 3 we get the commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(P_X) & \xrightarrow{\;a\;} & \mathrm{Pic}(\hat{U}_{X/Y})=\mathrm{Pic}(\hat{Z}_{/Y}) \\ & {\scriptstyle b}\searrow \quad \nearrow{\scriptstyle c} & \\ & \mathrm{Pic}(U_X) & \end{array}$$

in which all arrows are the natural ones. By the corollary of lemmas 1 and 2 one gets that the effective Grothendieck-Lefschetz condition $\mathrm{Leff}(U_X,Y)$ holds. Taking into account that U_X is smooth and Y meets every divisor of U_X (in fact, Y meets every hypersurface of Z), then $\mathrm{Leff}(U_X,Y)$ implies that the map c is an isomorphism.

Using lemma 3 one gets an exact sequence of the form

$$0\longrightarrow \mathbb{Z}[E_1]\oplus\mathbb{Z}[E_2]\longrightarrow \mathrm{Pic}(P_X)\xrightarrow{\ b\ }\mathrm{Pic}(U_X)\longrightarrow 0.$$

The injectivity of the map $\mathbb{Z}[E_1]\oplus\mathbb{Z}[E_2]\longrightarrow \mathrm{Pic}(P_X)$ comes from the following facts: $E_1\cap E_2=\emptyset$, h_X blows down E_1 and E_2 to subvarieties of Z of codimension ≥ 2, and from the projectivity of P_X. Recalling that c is an isomorphism the above exact sequence translates into

$$(3)\qquad 0\longrightarrow \mathbb{Z}[E_1]\oplus\mathbb{Z}[E_2]\longrightarrow \mathrm{Pic}(P_X)\xrightarrow{\ a\ }\mathrm{Pic}(\hat{U}_{X/Y})\longrightarrow 0.$$

The exact sequence (3) yields in particular the isomorphism

$$(4)\qquad \underline{\mathrm{Pic}}^0(P_X)\cong \underline{\mathrm{Pic}}^0(\hat{U}_{X/Y}).$$

Now, for all $m\geq 1$ consider the m-th infinitesimal neighbourhood $Y_m=(Y,O_{U_X}/I_Y^m)$ of Y in U_X and the standard exact sequence

$$0\overset{(1)}{=}H^1(Y,I_Y^m/I_Y^{m+1})\longrightarrow \mathrm{Pic}(Y_{m+1})\longrightarrow \mathrm{Pic}(Y_m)\longrightarrow H^2(Y,I_Y^m/I_Y^{m+1}).$$

Since char(k)=0, by Lefschetz's principle we may assume that k is the field $\mathbb{C}$ of complex numbers. Then $NS(Y_m):=\mathrm{Pic}(Y_m)/\mathrm{Pic}^0(Y_m)\subseteq H^2(Y,\mathbb{Z})$ for every $m\geq 1$. (Alternatively, instead of using Lefschetz's principle we could use analogous facts in terms of etale cohomology.) This implies that for every $m\geq 1$ the maps $NS(Y_{m+1})\longrightarrow NS(Y_m)$ are injective. Therefore one gets the exact sequence of algebraic groups

$$(5)\qquad 0\longrightarrow \underline{\mathrm{Pic}}^0(Y_{m+1})\xrightarrow{\ a_m\ }\underline{\mathrm{Pic}}^0(Y_m)\xrightarrow{\ b_m\ }H_m=H^2(Y,I_Y^m/I_Y^{m+1}),$$

with H_m a linear algebraic group. By a result of Grothendieck [FGA] the normality of $Y=Y_1$ implies that $\underline{\mathrm{Pic}}^0(Y_1)$ is an abelian variety. Since all the maps a_m are injective we infer that $\underline{\mathrm{Pic}}^0(Y_m)$ is an abelian variety for all $m\geq 1$. Therefore the maps b_m are all zero because H_m is a linear algebraic group. This implies that the map a_m is an isomorphism for every $m\geq 1$, whence

$$(6)\qquad \underline{\mathrm{Pic}}^0(\hat{U}_{X/Y})\cong \mathrm{inv}_m\lim \underline{\mathrm{Pic}}^0(Y_m)\cong \underline{\mathrm{Pic}}^0(Y).$$

From (4) and (6) we get that the natural map $\underline{\mathrm{Pic}}^0(P_X)\longrightarrow \underline{\mathrm{Pic}}^0(Y)$ is an isomorphism. Finally, since P_X is a $\mathbb{P}^1$-bundle over X we get the first statement of theorem 1. The second statement follows from the first one because the Albanese variety is the dual of the Picard variety (in the sense of the theory of abelian varieties). The equality $q(X)=q(Y)$ is a consequence of the first assertion and the fact that in characteristic zero the irregularity is the dimension of the Picard variety (see [FGA]).

Assume now that $\dim(X)\geq n+3$ and $\dim(\mathrm{Sing}(Y))\leq \dim(Y)-3$, i.e. $\dim(Y)\geq 3$ and Y smooth in codimension two. Notice that Y is a Cohen-Macaulay variety because Y is a complete intersection in U_X and U_X is smooth. Since Y is a Cohen-Macaulay projective variety which is smooth in codimension two and I_Y^m/I_Y^{m+1} $(m\geq 1)$ is a direct sum of line bundles of the form $O_Y(-j)$ with $j>0$, by a generalization of the Kodaira vanishing theorem (see [Sh-So], §7.80) we get

$$H^2(Y,I_Y^m/I_Y^{m+1})=0 \quad \text{for every } m\geq 1.$$

Having these vanishings, the above arguments (and especially (5)) show that the natural maps $\mathrm{Pic}(U_X)\longrightarrow \mathrm{Pic}(\hat{U}_{X/Y})\longrightarrow \mathrm{Pic}(Y)$ are both isomorphisms. Now look at the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{Z}[E_1] & \xrightarrow{\ id\ } & \mathbb{Z}[E_1] & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathbb{Z}[E_1]\oplus\mathbb{Z}[E_2] & \longrightarrow & \mathrm{Pic}(P_X) & \longrightarrow & \mathrm{Pic}(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & \mathbb{Z}[E_2] & \longrightarrow & \mathrm{Pic}(P_X-E_1) & \longrightarrow & \mathrm{Pic}(Y) & \longrightarrow & 0 \\
 & & | & & | & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The middle row is exact by the exact sequence (3) and the isomorphism $\mathrm{Pic}(\hat{U}_{X/Y})\cong\mathrm{Pic}(Y)$. Since the first two column are obviously exact we infer that the third row is also exact. Finally, since E_1 is a section of p_X we have an isomorphism $\mathrm{Pic}(P_X)\cong\mathrm{Pic}(X)$. In other words, we get the exact sequence

$$0\longrightarrow\mathbb{Z}[E_2]\longrightarrow\mathrm{Pic}(X)\longrightarrow\mathrm{Pic}(Y)\longrightarrow 0,$$

which completes the proof of theorem 1. □

Proof of theorem 2. In the diagram (*) the morphism $g_X:U_X\longrightarrow X$ is a locally trivial $\mathbb{G}_m$-bundle. Restricting it to X_0 we get the locally trivial $\mathbb{G}_m$-bundle $g_0:U_0=U_{X_0}\longrightarrow X_0$. Then considering the homotopy sequence of g_0 in the algebraic version given by Grothendieck (see [SGA-1], Exp.XIII, Exemple 4.4 and Prop. 4.1):

$$(7)\qquad \pi_1^{alg}(\mathbb{G}_m)\longrightarrow\pi_1^{alg}(U_0)\longrightarrow\pi_1^{alg}(X_0)\longrightarrow 1,$$

in which the first map is induced by the inclusion $\mathbb{G}_m\longrightarrow U_0$ (with $\mathbb{G}_m$ regarded as a fibre of g_0), and the second map by g_0. Since X_0 is smooth, U_0 is also smooth. By the corollary of lemmas 1 and 2 we get that the effective condition Leff(U_0,Y) holds. Then proceeding exactly as in the proof of theorem 2.1 page 175 of $[Ha]_1$ we infer that $\pi_1^{alg}(U_0)\cong\pi_1^{alg}(Y)$. Therefore the exact sequence (7) becomes

$$\pi_1^{alg}(\mathbb{G}_m)\longrightarrow\pi_1^{alg}(Y)\longrightarrow\pi_1^{alg}(X_0)\longrightarrow 1$$

Finally, since $\pi_1^{alg}(\mathbb{G}_m)\cong\hat{\mathbb{Z}}$ we get the result. □

Acknowledgement. A preliminary version of this paper was written during my stay at the University Paris VII. I want to thank this Institution and especially the members of "Equipe de Physique Mathematique et Geometrie" for hospitality and stimulating atmosphere, and the CNRS for financial support. I also want to thank the organizers of the L'Aquila Conference for their invitation and support.

References

[B-Ba] L. Bădescu & E. Ballico, Formal rational functions along zero loci of sections of ample vector bundles, Preprint 1991.

$[Ba]_1$ W. Barth, Transplanting cohomology classes in complex projective space, Amer.J.Math. 92 (1970), 951-967.

$[Ba]_2$ W. Barth, Verallgemeinerung des Bertinischen Theorems in Abelschen Mannigfaltigkeiten, Ann. Sc. Norm. Sup. Pisa 23 (1969), 317-330.

[Ch] W.-L. Chow, On meromorphic maps of algebraic varieties, Annals of Math. 89 (1969), 391-403.

[Fa] G. Faltings, Some theorems about formal functions, Publ. RIMS, Kyoto Univ. 16 (1980), 721-732.

[F-H] W. Fulton & J. Hansen, A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings, Annals of Math. 110 (1979), 159-166.

[F-L] W. Fulton & J. Lazarsfeld, Connectivity and its applications in algebraic geometry, Springer Lect. Notes Math. No. 862 (1981), 26-92.

[EGA-III] A. Grothendieck & J. Dieudonne, Éléments de Géometrie Algébrique, Chap. III, Publ. Math. IHES No.8 (1961).

[FGA] A. Grothendieck, Fondements de la Géometrie Algébrique, Seminaire Bourbaki 1957-62, Secretariat Math. Paris. (Exposés No. 232 & 236 sur le Schemas de Picard.)

[SGA-1] A. Grothendieck, Revêtements étales et groupe fondamental, Springer Lect. Notes Math. No. 224 (1971).

[SGA-2] A. Grothendieck, Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux, North. Holland, Amsterdam (1968).

[Ha]$_1$ R. Hartshorne, Ample subvarieties of algebraic varieties, Springer Lect. Notes Math. No. 156 (1970).

[Ha]$_2$ R. Hartshorne, Algebraic Geometry, Springer Verlag, 1977.

[Hi-Ma] H. Hironaka & H. Matsumura, Formal functions and formal embeddings, J. Math. Soc. Japan 20 (1968), 52-82.

[Lar] M. Larsen, On the topology of complex projective manifolds, Invent. Math. 19 (1973), 327-365.

[La-Sc] A. T. Lascu & D. B. Scott, An algebraic correspondence with applications to blowing up Chern classes, Ann. Mat. Pura Appl. 102 (1975), 1-36.

[Ly] G. Lyubeznik, Étale cohomological dimension and the topology of algebraic varieties, To appear in Annals Math.

[O]$_1$ A. Ogus, Local cohomological dimension of algebraic varieties, Annals Math 98 (1973),327-365.

[O]$_2$ A. Ogus, On the formal neighbourhood of a subvariety of a projective space, Amer. J. Math. 97 (1975), 1085-1107.

[Sh-So] B Shiffman & A. J. Sommese, Vanishing theorems on complex manifolds, Progress in Math. No. 56, Birkhauser (1985).

[So] A. J. Sommese, Submanifolds of abelian varieties, Math. Ann. 233 (1978), 229-256.

[Sp] R. Speiser, Cohomological dimension of abelian varieties, Amer. J. Math. 95 (1973), 1-34.