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Fine behaviour of balayages in potential theory

N. Boboc

1. Introduction

We consider a standard H-cone of functions S on a set X (i.e. S is the set of all excessive functions with respect to a submarkovian resolvent absolutely continuous on metrisable space X). For any fine open set U of X we denote by $\widetilde{U}$ the set of all points $x \in X$ for which $X \setminus U$ is thin at x and by S(U) the localization of S on U (i.e. the set of all positive functions t on U such that t is finite on a fine dense subset of U and such that there exists a sequence $(s_n)_n$ in S, s_n finite for any $n \in N$ for which the sequence $(s_n - B^{X\setminus U} s_n)_n$ increases to t on U). Any function $t \in S(U)$ may be extended to a fine continuous function on $\widetilde{U}$ which is also denoted by t. We remark that if X is a harmonic space and S is the convex cone of all superharmonic functions on X then S(U) means the set of all positive fine superharmonic functions on U.

It is proved that for any positive Borel function f on X, the function $B^{X\setminus U} f$ is fine continuous at any point $x \in U$.

Let now $(x_n)_n$ be a sequence of U which converges in X to a point $z \in U$. The sequence $(x_n)_n$ is called maximal if we have

$$\lim_{n \to \infty} B^{X\setminus U} s(x_n) = B^{X\setminus U} s \ (z)$$

for any universally continuous element s from S. It is proved that the sequence $(x_n)_n$ will be maximal iff $(x_n)_n$ converges to z with respect to the natural topology on U associated with the H-cone S(U). Therefore for any positive Borel function f on X dominated by an element $s \in S$ which is finite continuous at z then we have

$$\lim_{n \to \infty} B^{X \setminus U} f(x_n) = B^{X \setminus U} f(z).$$

Particularly we have this last relation for any positive bounded Borel function f on X.

Finally if $z \in \tilde{U}$ then it is proved that if U is a Doob set then there exists a fine neighbourhood V of z and a positive real function c on V such that

$$\varepsilon_y^{X \setminus U} \leq c(y) . \varepsilon_z^{X \setminus U} \qquad \forall \ y \in V$$

where $\varepsilon_y^{X \setminus U}$ means the balayage of the Dirac measure ε_y on the set $X \setminus U$. Such inequality extend in a more general frame the well known Harnack inequality.

These assertions extend some similar results obtained in the classical potential theory by M. Brelot (|4|), |5|) and in the frame of harmonic spaces, under various generality, by E Smyrnélis (|9|), H-Bauer (|1|) W. Hansen (|1|) W. Hansen (|6|) and I. Netuka (|8|).

2. Preliminaries

In all this paper S will be a standard H-cone of functions on a set X (|2|). We remember the following:

a) On X there are distinguished two topologies which are strongly related with S. The first, called the natural topology, is the coarsest topology on X such that any universally continuous element of S is continuous. The set X endowed with the natural topology is a metrisable space with countable basis. The set X is called saturated if any H-integral on S which is finite on the function 1 is represented as a finite measure on X; the set X is called semisaturated if any H-integral on S which is dominated by on H-measure which is universally continuous is represented as a measure on X. In fact X will be semisaturated iff the set of all H-integrals on S which are represented as measures on X (i.e. the set of all H-measures) is solid with respect to the natural order in the set S* of all H-integrals on S. The set X is called nearly saturated if any universally continuous H-integral on $\hat{S}$ is represented as a measure on X. It is known that always there exists a set $X_1 \supset X$ such that S is a standard H-cone of functions on X_1 and such that X_1 is saturated; generally the set X is fine dense in X_1.

b) If A is a subset of X we denote by B^A the map on S into S defined by

$$B^A s = \bigwedge \{ s' \in S | s' \geq s \text{ on } A \}.$$

It is known that for any $x \in X$ the map

$$s \longrightarrow B^A s(x)$$

is an H-integral on S dominated by ε_x. Hence if X is semisaturated then the above H-integral is represented as a measure on X denoted by ε_x^A. In the sequal if f is a positive Borel function on X we denote by $B^A f$ the function on X given by

$$B^A f(x) = \varepsilon_x^A(f).$$

c) If A is a subset of X then A is called polar if $B^A s = 0$ for any $s \in S$. The set A is called thin at x if there exists $s \in S$ such that $B^A s(x) < s(x)$. The set A is called totally thin if it is thin at any point $x \in X$. The set A is called semipolar if it is a countable union of totally thin sets.

Let now X be a nearly saturated set (with respect to S) and let X_1 be the saturated set (with respect to S) such that $X \subset X_1$. Then any Borel measurable subset of $X_1 \setminus X$ is semipolar and this property characterises the fact that X is nearly saturated. Moreover in this case a subset A of X will be semipolar iff A is semipolar as subset of X_1. If X is semisaturated then any Borel measurable subset of $X_1 \setminus X$ is polar and this property characterises that fact that X is semisaturated (|3|).

d) Suppose now that X is nearly saturated and let U be a fine open subset of X, $U \neq \emptyset$. We denote by S(U) the convex cone of all positive functions t on U such that t is finite on a fine dense subset of U and such that

$$t = \sup \{ s - B^{X \setminus U}_s \mid s \in S_b,\ s - B^{X \setminus U}_s \leq t \}.$$

It is known that (|3|) S(U) is a standard H-cone of functions on U. If X is semisaturated (with respect to S) then U is also semisaturated (with respect to S(U). The H-cone S(U) is called the localization of S on U. Moreover if $s \in S, t \in S$, $t < \infty$ and $t \leq s$ then we have

$$(s - B^{X \setminus U}_t)|_U \in S(U).$$

Particularly $s|_U \in S(U)$ for any $s \in S$. We remark that (|7|) if $t \in S(U)$ and $s \in S_b$ then the element from S(U) given by $t \wedge (s - B^{X \setminus U}_s)$ is of the form $s' - B^{X \setminus U}_{s'}$ where $s' \in S_b$, $s' \leq s$.

e) Suppose that S is a standard H-cone of functions on a set X. Then for any increasing family $(s_i)_{i \in I}$ from S the function $\sup_i s_i$ is fine continuous. Indeed for any $n \in N$ we have

$$\inf_n (\sup_{i \in I} s_i, n) = \sup_{i \in I} (\inf_n (s_i, n))$$

and the assertion follows from the fact that

$$\sup_{i \in I} (\inf_n (s_i, n)) \in S.$$

If the elements of S are called usually superharmonic on X the function of the form $\sup_{i \in I} s_i$, where $(s_i)_{i \in I}$ is an increasing family in S, is called hyperharmonic on X.

2. The standard H-cone S(U) as a standard H-cone of functions on the extension set $\widetilde{U}$.

In this paragraph if U is a fine open subset of X we denote by $\widetilde{U}$ the set of all points $z \in X$ such that $X \setminus U$ is thin at z. Obviously $\widetilde{U}$ is fine open and U is a fine dense subset of $\widetilde{U}$. We intend to represent S(U) as a standard H-cone of functions on $\widetilde{U}$. In fact any element $t \in S(U)$ is represented as the fine continuous extention on $\widetilde{U}$ of the function t.

Theorem 1. Any element $t \in S(U)$ may be extended to a fine continuous positive function $\widetilde{t}$ on $\widetilde{U}$. In this way $S(U)$ is represented as a standard H-cone of functions on the set $\widetilde{U}$ such that the subset $\widetilde{U} \setminus U$ is polar with respect to this standard H-cone of functions.

Proof. The first part of the theorem follows using the fact that the convex cone of elements of the form $(s-B^{X\setminus U}s)/U$ where $s \in S_b$ is a solid subcone of $S(U)$, which is increasingly dense in $S(U)$ and contains a strictly positive element namely $(s_o-B^{X\setminus U}s_o)/U$, where s_o is a bounded continuous generator of S. Indeed any element of the form $(s-B^{X\setminus U}s)/U$ where $s \in S_b$ has a fine continuous extention on $\widetilde{U}$ (i.e. $(s-B^{X\setminus U}s)/\widetilde{U}$) and the function $(s_o - B^{X\setminus U}s_o)/\widetilde{U}$ is a strictly positive function on $\widetilde{U}$. Now if $t \in S(U)$ we have

$$t(x) = \sup_n (\inf (t,n(s_o-B^{X\setminus U}s_o)(x))) \qquad \forall \ x \in U.$$

Since there exists $s_n \in S_b$, $s_n \leqslant n\, s_o$ such that

$$\inf (t, n(s_o - B^{X\setminus U}s_o)) (x) = (s_n - B^{X\setminus U}s_n) (x) \qquad \forall \ x \in U$$

then the element

$$t_n = \inf (t, n(s_o-B^{X|U}s_o))$$

has a fine continuous extention $\widetilde{t}_n$ on $\widetilde{U}$. The function

$$\widetilde{t} = \sup_n \widetilde{t}_n$$

is a fine continuous extention of t because

$$\inf (\widetilde{t}, n(s_o - B^{X\setminus U}s_o)) = \widetilde{t}_n \text{ on } \widetilde{U}.$$

From the above definition of $\widetilde{t}$ it follows immediately that we have

$$\widetilde{t_1 + t_2} = \widetilde{t}_1 + \widetilde{t}_2$$

$$\widetilde{\inf (t_1,t_2)} = \inf (\widetilde{t}_1, \widetilde{t}_2)$$

$$t_i \uparrow t \implies \widetilde{t}_i \uparrow \widetilde{t}$$

and therefore replaceing t by $\widetilde{t}$ the H-cone $S(U)$ is represented as a standard H-cone of functions on the set $\widetilde{U}$.

Let now s_o be a bounded continuous generator of S. To prove that $\tilde{U}\setminus U$ is polar in $\tilde{U}$ with respect to the standard H-cone of functions S(U) on $\tilde{U}$ it will be sufficient to whow that

$$\tilde{U}_B{}^{\tilde{U}\setminus U}(s-B^{X\setminus U}s_o) = 0$$

where $\tilde{U}_B A$ means the balayage on the subset A of $\tilde{U}$ with respect to the H-cone of functions S(U) on $\tilde{U}$.

Let $(G_n)_n$ be a sequence of fine open subset of X such that $G_n \supset X\setminus U$ and such that

$$\bigwedge_n B^{G_n} s_o = B^{X\setminus U} s_o .$$

We have

$$B^{G_n} s_o - B^{X\setminus U} s_o = B^{G_n} s_o - B^{X\setminus U}(B^{G_n} s_o) \in S(U)$$

$$B^{G_n} s_o - B^{X\setminus U} s_o \geq s_o - B^{X\setminus U} s_o \quad \text{on } G_n \cap \tilde{U}$$

and therefore

$$\tilde{U}_B{}^{\tilde{U}\setminus U}(s_o - B^{X\setminus U} s_o) \leq B^{G_n} s_o - B^{X\setminus U} s_o \qquad \forall\ n\in N,$$

$$\tilde{U}_B{}^{\tilde{U}\setminus U}(s_o - B^{X\setminus U} s_c) = 0.$$

Theorem 2. Suppose that X is semisaturated and let f be a positive Borel function on X. Then for any fine open subset U of X the function $B^{X\setminus U}f$ is fine continuous at any point of $\tilde{U}$. If moreover f is dominated by an element $s\in S$ then we have

$$B^{X\setminus U}f|_{\tilde{U}} \in S(U), \qquad B^{X\setminus U}f|_{\tilde{U}} \preccurlyeq_{S(U)} s|_{\tilde{U}}$$

Proof. Suppose that f is of the form $f = t'-t''$ where $t',t'' \in S$, $t'' \leq t'$, t'' finite. We want to show that $B^{X\setminus U}f/_{\tilde{U}} \in S(U)$. Indeed let $s\in S$ be such that $f\leq s$. For any $s'\in S$ such that $s'\geq t'$ on $X\setminus U$ we have

$$s'\geq s'\wedge t'', \quad B^{X\setminus U}(s'\wedge t'') = B^{X\setminus U}t''$$

and therefore

$$(s'-B^{X\setminus U}t'')|_U \in S(U).$$

Since s' is arbitrary we get

$$B^{X\setminus U}t''/_U \preccurlyeq_{S(U)} \wedge\{ s'|_U \,|s' \in S,\ s' \geq t' \text{ on } X\setminus U\} =$$

$$= B^{X\setminus U}t'|_U.$$

Hence $B^{X\setminus U}f|_U \in S(U)$. From

$$B^{X\setminus U}f = B^{X\setminus U}t' - B^{X\setminus U}t''$$

it follows that $B^{X\setminus U}$ f is fine continuous on X and therefore on $\tilde{U}$.

Let now F_o be the set of all bounded functions f on X such that there exists an increasing sequence $(f_n)_n$ in $(S_b-S_b)_+$ with $f = \sup\limits_n f_n$. If $f \in F_o$ and $(f_n)_n$ is as above we have

$$B^{X\setminus U}f_n \uparrow B^{X\setminus U}f .$$

From the previous considerations we have

$$B^{X\setminus U}f_n|_{\tilde{U}} \in S(U)$$

and therefore, since $B^{X\setminus U}f$ is bounded,

$$B^{X\setminus U}f|_{\tilde{U}} \in S(U).$$

Particularly $B^{X\setminus U}f$ is fine continuous on $\tilde{U}$. We remark also that if $f \in F_o$, $g \in F_o$, $f \leq g$ then we have

$$B^{X\setminus U}(g-f)|_{\tilde{U}} \in S(U)$$

Indeed let $(f_n)_n$ be an increasing sequence in $(S_b-S_b)_+$ such that $\sup\limits_n f_n = f$. Since $g-f_n \in F_o$ we have

$$B^{X\setminus U}(g-f_n)|_{\tilde{U}} \in S(U)$$

and $(B^{X\setminus U}(g-f_n)|_{\tilde{U}})_n$ is a specifically decreasing in S(U). Hence

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$$B^{X\setminus U}(g-f)|_{\widetilde{U}} = \inf_n B^{X\setminus U}(g-f_n)|_{\widetilde{U}} \in S(U).$$

Suppose now that f is a positive bounded Borel function on X.

We have

$$B^{X\setminus U}f = \inf\{B^{X\setminus U}g \mid g \in F_o,\ g \geq f\}.$$

Since the family $(B^{X\setminus U}g|_{\widetilde{U}})_{g\in F_o,\ g\geq f}$ is specifically decreasing in S(U)

we get

$$B^{X\setminus U}f|_{\widetilde{U}} = \inf\{B^{X\setminus U}g|_{\widetilde{U}} \mid g\in F_o,\ g\geq f\} \in S(U).$$

Particularly $B^{X\setminus U}f$ is fine continuous on $\widetilde{U}$.

Suppose that f is a positive Borel function on X. We have

$$B^{X\setminus U}f = \sup_n B^{X\setminus U}(\inf(f,n)).$$

From the previous considerations it follows that $B^{X\setminus U}f|_{\widetilde{U}}$ is hyperharmonic on U (with respect to the H-cone S(U)) and therefore it is fine continuous on U. Moreover if f is dominated by an element s of S then $B^{X\setminus U}f|_{\widetilde{U}}$ is dominated by $\acute{s}|_U$ and therefore belongs to S(U).. In this case we have also

$$B^{X\setminus U}(\inf(f,n))|_{\widetilde{U}} \preccurlyeq_{S(U)} B^{X\setminus U}(\inf(s,n))|_{\widetilde{U}} \preccurlyeq \inf(s,n)|_{\widetilde{U}}$$

and therefore

$$B^{X\setminus U}f|_{\widetilde{U}} \preccurlyeq_{S(U)} \acute{s}|U.$$

3. The natural topoogy in the standard H-cone of functions S(U) on the set U and the maximal sequences in U.

If U is an open subset of X then it is known that the natural topology on U induced by S(U) coincides with the restriction to U of the natural topology on X induced by S. This assertion is not true if U is fine open. Also even if U is open the natural topology induced by S(U) on the fine open set $\widetilde{U}$ does not coincides with the restruction to $\widetilde{U}$ of the natural topology on X induced by the H-cone S.

Proposition 3. Suppose that U is a fine open subset of X. Then the natural topology on U induced by the standard H-cone $S(U)$ on U is the coarsest topology on U for which any function of the form $s-B^{X\setminus U}s$ is continuous where s runs the set of all universally continuous elements from S.

Proof. Let S_0 be the set of all universally continuous elements from S. We show that for any $s\in S_0$ the element $s-B^{X\setminus U}s$ is 1-continuous with respect to the standard H-cone $S(U)$. Indeed let $(t_n)_n$ be an increasing sequence from $S(U)$ such that $\sup\limits_n t_n = s - B^{X\setminus U}s$ on U. We have

$$t_n + B^{X\setminus U}s \in S$$

and

$$t_n + B^{X\setminus U}s \uparrow s$$

Hence the sequence $(t_n + B^{X\setminus U}s)_n$ converges uniformely to s on X and therefore the sequene $(t_n)_n$ converges uniformely to $s-B^{X\setminus U}s$ on U.

Conversely suppose that $t \in S(U)$ is an universally continuous element from $S(U)$ and let s_0 be a bounded continuous generator of S. We want to show that t is of the form $t = s-B^{X\setminus U}s$ where $s\in S$ is s_0-continuous. Indeed since $s_0-B^{X\setminus U}s_0$ is strictly positive on U it follows that there exists $\alpha > 0$ such that $t\leq \alpha(s_0-B^{X\setminus U}s_0)$ and therefore there exists $s \in S$ such that $t=s-B^{X\setminus U}s$. Moreover we may suppose that s is such that if $s'\in S$ satisfies the property

$$s-B^{X\setminus U}s \leq s'-B^{X\setminus U}s'$$

then $s\leq s'$. Particularly we deduce that $s\leq \alpha s_0$. We prove now that s is s_0 continuous. Indeed let $(s_n)_n$ be an increasing sequence in S such that $s = \sup\limits_n s_n$ on X. We have

$$B^{X\setminus U}s_n \uparrow B^{X\setminus U}s$$

and therefore

$$\lim_{n\to\infty} (s_n-B^{X\setminus U}s_n) = s-B^{X\setminus U}s,$$

$$\sup_n \left(\bigwedge_{k\geq n} (s_k-B^{X\setminus U}s_k)\right) = s-B^{X\setminus U}s$$

Since $t = s - B^{X\setminus U} s$ is universally continuous in $S(U)$ then for any $\varepsilon > 0$ there exists $n_\varepsilon \in N$ such that

$$n \geq n_\varepsilon \implies s - B^{X\setminus U} s \leq \bigwedge_{k\geq n} (s_k - B^{X\setminus U} s_k) + \varepsilon (s_0 - B^{X\setminus U} s_0)$$

$$n \geq n_\varepsilon \implies s - B^{X\setminus U} s \leq s_n - B^{X\setminus U} s_n + \varepsilon (s_0 - B^{X\setminus U} s_0)$$

and therefore

$$n \geq n_\varepsilon \implies s < s_n + \varepsilon s_0$$

which means that s is s_0 - continuous.

The assertion from Proposition follows now using the fact that any $s \in S$ which is s_0-continuous is the uniform limit of an increasing of universally continuous elements of S.

Definition. The natural topology on $\tilde{U}$ (induced by the standard H-cone of function $S(U)$) is denoted by $\mathcal{T}_0(U,\tilde{U})$. The topology $\mathcal{T}_1 = \mathcal{T}_1(U,\tilde{U})$ which is the coarsest topology on $\tilde{U}$ which is finer that $\mathcal{T}_0(U\,\tilde{U})$ and $\mathcal{T}_0(X)/_{\tilde{U}}$ is called the maximal topology on $\tilde{U}$ associated with the standard H-cone $S(U)$.

Corollary 4. The maximal topology on $\tilde{U}$ is the coarsest topology on $\tilde{U}$ for which any function on $\tilde{U}$ of the form $s/_{\tilde{U}}$ and $B^{X\setminus U} s/_{\tilde{U}}$ is continuous where s runs the set of all universally continuous elements of S.

Corollary 5. Let $z \in U$ and let $(x_n)_n$ be a sequence in $\tilde{U}$. Then $(x_n)_n \longrightarrow z$ in the maximal topology iff $(x_n)_n \longrightarrow z$ in $\mathcal{T}_0(X)$ and for any universally continuous element s of S we have

$$B^{X\setminus U} s(z) = \lim_{n\to\infty} B^{X\setminus U} s(x_n).$$

Remark. If $z \in \tilde{U} \setminus U$ then a sequence $(x_n)_n$ in U which converges to z in the maximal topology on U is called "maximal with respect to z". This terminology was introduced by Brelot (|5|) and was used by Smyrnelis (|9|), Bauer(|1|), Hansen (|6|) and Netuka (|8|) in the case of the theory of harmonic spaces.

Theorem 6. Let U be a fine open subset of X and let $z \in \tilde{U}$. Then

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for any $t \in S(U)$ for which there exists $s \in S$ finite continuous at z such that

$$t \preccurlyeq_{S(U)} s|_U$$

we have

$$\text{fine} \lim_{U \ni x \to z} t(x) = \mathcal{T}_1 - \lim_{U \ni x \to z} t(x)$$

or equivalent

$$\text{fine} \lim_{U \ni x \to z} t(x) = \lim_{n \to \infty} t(x_n)$$

for any sequence $(x_n)_n$ in U which is "maximal with respect to z".

Proof. Any element $t' \in S(U)$ is considered as a function on $\tilde{U}$. By hypothesis there exists $t' \in S(U)$ such that

$$t + t' = s \quad \text{on } \tilde{U}$$

By the definition of the topology $\mathcal{T}_1(U,\tilde{U})$ the functions t, t' and $s/_{\tilde{U}}$ are lower semicontinuous with respect to the topology $\mathcal{T}_0(U,\tilde{U})$ and therefore with respect to $\mathcal{T}_1(U,\tilde{U})$. By hypothesis s is continuous at z with respect to $\mathcal{T}_0(X)|_{\tilde{U}}$ and therefore with respect to $\mathcal{T}_1(U,\tilde{U})$. Hence t and t' are also continuous at z with respect to $\mathcal{T}_1(U,\tilde{U})$. Hence we have

$$t(z) = \text{fine} \lim_{U \ni x \to z} t(x) = \mathcal{T}_1\text{-}\lim_{U \ni x \to z} t(x) .$$

Proposition 7. Let U be a fine open subset of X and let $z \in \tilde{U}$. Then, for any positive Borel function f on X for which there exists $s \in S$ finite continuous at z such that $f \leq s$, the function $B^{X \setminus U} f|_{\tilde{U}}$ is $\mathcal{T}_1$ - continuous at z.

Proof. The assertion follows from Theorem 6 since from Theorem 2 we deduce

$$B^{X \setminus U} f/_{\tilde{U}} \preccurlyeq_{S(U)} s/_{\tilde{U}} .$$

Corollary 8. Let U be a fine open subset of X. Then for any positive bounded function f on X the function $B^{X \setminus U} f/_{\tilde{U}}$ is $\mathcal{T}_1$ - continuous.

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We remember that an element t of an H-cone T is called <u>subtractible</u> if we have

$$t' \in T,\ t \leq t' \Longrightarrow t \precsim t'.$$

For instance if $(X, {}^*H)$ is a harmonic space and U is an open subset of X then any positive harmonic function is subtractible in the H-cone $S(U)$ of all positive superharmonic functions on U. Moreover any positive superharmonic function on U which is harmonic outside a polar set is also subtractible.

<u>Proposition 9.</u> Let U be a fine open subset of X and let $z \in \tilde{U}$. Then for any subtractible element $t \in S(U)$ for which there exists $s \in S$ finite continuous at z such that $t \leq s/_U$ we have

$$\text{fine-}\lim_{U \ni x \to z} t(x) = \tau_1 - \lim_{U \ni x \to z} t(x) .$$

<u>Proof.</u> The assertion follows from Theorem 6 since we have

$$t \precsim_{S(U)} s/U.$$

<u>Corollary 10.</u> Let U be a fine open subset of X. Then for any bounded substractible element $t \in S(U)$ we have that t is a τ_1-continuous function on U.

4. <u>Doob sets and Harnack generalised inequality</u>

In this section U will be a fine open subset of X and z a point in X such that $X \setminus U$ is thin at z. We denote as in the preceding sections by $\tilde{U}$ the set of all points $y \in X$ such that $X \setminus U$ is thin at y. We suppose that X is semisaturated we respect to the H-cone S.

<u>Lemma 11</u>. Let F_z be the set of all positive Borel functions f on X such that $\varepsilon_z^{X \setminus U}(f) < \infty$. Then there exists $f_o \in F_z$ such that for any $f \in F_z$ we have

$$\left\{ y \in \tilde{U} \mid \varepsilon_y^{X \setminus U}(f_o) < \infty \right\} \subset \overline{\left\{ y \in \tilde{U} \mid \varepsilon_y^{X \setminus U}(f) < \infty \right\}}^{f} .$$

where $\overline{A}^f$ means the fine closure of A.

<u>Proof.</u> For any $f \in F_Z$ we denote

$$A_f = \{ y \in \tilde{U} \mid \varepsilon_y^{X\setminus U}(f) < \infty \}.$$

Obviously A_f is fine open and $z \in A_f$. Moreover if $(f_n)_n$ is a sequence in F_z then there exists $f \in F_z$ such that

$$A_f = \bigcap_n A_{f_n} .$$

Indeed let $(a_n)_n$ be a sequence of strictly positive real numbers such that

$$\sum_n a_n \varepsilon_z^{X\setminus U}(f_n) < \infty .$$

Then

$$f := \sum_n a_n f_n \in F_z$$

and we have

$$\varepsilon_y^{X\setminus U}(f) < \infty \Longrightarrow \varepsilon_y^{X\setminus U}(f_n) < \infty \qquad \forall\ n \in N$$

and therefore

$$A_f \subset \bigcap_n A_{f_n} .$$

Now we remark that for any $f \in F_z$ the set $\overline{A}_f^{fine}$ is absorbent in $\tilde{U}$ with respect to $S(U)$. From this fact we deduce that there exists a sequence $(f_n)_n$ in F_z such that

$$\bigcap_{f \in F_z} \overline{A}_f^{fine} = \bigcap_n \overline{A}_{f_n}^{fine} .$$

<u>Definition.</u> A fine open subset U of X is called a <u>Doob set</u> (with respect to S) if for any fine open subset V of U there exists a fine open subset V_o of V such that $V \setminus V_o$ is totally thin and such that for any positive Borel function f on X for which $B^{X\setminus U}f$ is finite dense on V then $B^{X\setminus U}f$ is finite on V_o.

Remark. Suppose that $(R^{n+1}, {}^*H)$, $n\geq 0$ is the harmonic space associated with the heat equation on R^{n+1}. Then any open subset of R^{n+1} is a Doob set with respect to the H-cone S of all positive superharmonic functions s on this harmonic space. Indeed let U be an open subset of R^{n+1}, V be a fine open subset of U and let f be a positive Borel function on X such that $B^{X\setminus U}f$ is finite dense on V. If $x_0=(x',t_0)\in V$ and DxI is a rectangle with the center in x such that $D\times I\subset U$ then if $y_0=(y_0',t)\in (D\times I)\cap V$ is such that

$$B^{X\setminus U}f(y_0)<\infty$$

we have

$$B^{X\setminus U}f(y',t')<\infty$$

for any $y'\in D$ and any $t'<t$. Hence the set

$$\{y\in V\cap(D\times I)\,|\,B^{X\setminus U}f\ (y)=\infty\}$$

is a subset of a set of the form

$$\{y=(y',t')/\ t'=t_0'\}$$

for a suitable t_0' and therefore the set

$$\{y\in V/B^{X\setminus U}f(y)=+\infty\}$$

is totally thin.

Theorem 12. Suppose that U is a Doob set with respect to S. Then there exists a fine neighbourhood V of z such that for any positive Borel function f or X we have

$$\varepsilon_z^{X\setminus U}(f)<\infty \implies \varepsilon_y^{X\setminus U}(f)<\infty \qquad \forall\, y\in V.$$

Proof. Let F_z be the set of all positive Borel function f on X with $\varepsilon_z^{X\setminus U}(f)<\infty$ and let for any $f\in F_z$,

$$A_f=\{y\in U/\ \varepsilon_y^{X\setminus U}(f)<\infty\}.$$

From Lemma 11 there exists $f_0\in F_z$ such that

$$A_{f_0}\subset \overline{A_f}^{\,\text{fine}} \qquad \forall\, f\in F_z\ ,$$

From this relation and from the fact that U is a Doob set it follows that there exists a fine open subset A_o of A_{f_o} such that $A_{f_o} \setminus A_o$ is totally thin and such that $B^{X\setminus U}f$ is finite on A_o whenever $f \in F_z$. The set $V_o = A_o \cup \{z\}$ is a fine neighbourhood of z for which we have

$$y \in V_o,\ \varepsilon_z^{X\setminus U}(f) < \infty \implies \varepsilon_y^{X\setminus U}(f) < \infty .$$

Corollary 13. (Harnack inequality). Suppose that U is a Doob set with respect to S. Then there exists a fine neighbourhood V of z and a positive real function C on V such that

$$y \in V \implies \varepsilon_y^{X\setminus U} < C(y)\, \varepsilon_z^{X\setminus U} .$$

Proof. From the preceding theorem there exists a fine neighbourhood V of z such that for any positive Borel function f on X we have

$$y \in V,\ \varepsilon_z^{X\setminus U}(f) < \infty \implies \varepsilon_y^{X\setminus U}(f) < \infty .$$

Hence we have

$$y \in V,\ \varepsilon_z^{X\setminus U}(f) = 0 \implies \varepsilon_y^{X\setminus U}(f) = 0$$

and therefore $\varepsilon_y^{X\setminus U}$ is absolutely continuous with respect to $\varepsilon_z^{X\setminus U}$ for any $y \in V$. Hence the Radom derivative g_y of $\varepsilon_y^{X\setminus U}$ with respect to $\varepsilon_z^{X\setminus U}$ is a positive bounded function. If we put, for any $y \in V$,

$$c(y) = \inf \left\{ \alpha > 0 /\ g_y \leq \alpha \ ,\ \varepsilon_z^{X\setminus U}\text{-a.s} \right\}$$

then we have

$$\varepsilon_y^{X\setminus U} \leq c(y)\, \varepsilon_z^{X\setminus U} .$$

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