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Study of a Jet Incident on a Porous Wall in a Gravity Field

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Abstract. The problem of an inviscid jet impacting on a porous wall in a gravity field is considered. By transforming it into a minimum problem we prove existence and uniqueness theorems. Properties of the flow region and of the stream function are established. The monotonicity of the stream function with respect to the given velocities is also obtained.

1. Introduction

In recent papers:Stavre(1991), Stavre(1992) we have studied a free boundary problem concerning the impact of a jet on a porous wall, without the presence of gravity. This problem is important since it is related to the industrial process of obtaining glassy metals.

In Stavre(1991), Stavre(1992), using the techniques introduced in Alt & Caffarelli (1981), Alt *et al.*(1982), Alt *et al.*(1983) we have transformed the physical problem into a minimum problem; we have established the existence and the uniqueness of the solution (the stream function); we have also proved some important properties concerning the flow region and the stream function. Moreover, in Stavre(1991) some numerical results were given.

This problem has also been studied in Jenkins & Barton(1988). In this paper, a numerical solution procedure based on the Baiocchi transformation (see Baiocchi(1972)) was developed.

A generalized Schwarz-Christoffel transformation and a hodograph method have been used in King(1990) to reduce the problem of an inviscid jet impacting on a porous wall, when the effects of gravity are neglected, to the solution of a first-order differential equation.

In this paper we extend the results obtained in Stavre(1991) to the case where the gravity field is present. We consider only the two-dimensional symmetric flow, but the problem can be also studied for the two-dimensional asymmetric and for the axially symmetric cases using the same techniques.

Section 2 describes the physical problem and its variational formulation. Section 3 presents existence and uniqueness results. In Section 4 we establish some important

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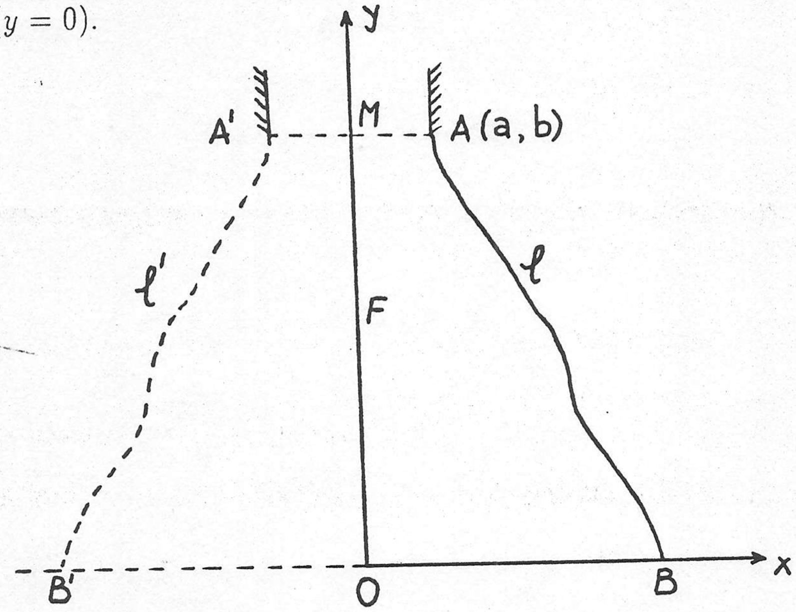
properties concerning the flow region and the stream function. Thus we prove that the flow region is contained into a bounded domain; in Jenkins & Barton(1988) this property is considered to be *a priori* known. A comparaison between the stream function in this case and the stream function for the flow without gravity is also given. The last Section contains the monotonicity of the stream function with respect to the given velocities.

2. Statement of the problem. Physical and variational formulation

We consider a steady, irrotational, incompressible flow of an inviscid fluid in a gravity field, exiting from a nozzle and impacting on a porous wall. For simplicity, we suppose that the nozzle and the porous wall have a vertical axis of symmetry, Oy; the gravity is directed in the $-y$-direction and the jet is perpendicular to the porous wall. The problem is illustrated in *Fig.*1. F denotes the flow region(which is unknown), bounded by the mouth of the nozzle($A'A$),the free boundaries(l, l')and the porous wall($y = 0$).



The normal velocities on $A'A(V_0)$ and on the porous wall(V_f) are two given constants, with $V_0 > V_f > 0$. Denoting by u the stream function, the velocity components are given by:

$$v_1 = -\frac{\partial u}{\partial y};\ v_2 = \frac{\partial u}{\partial x} \tag{2.1}$$

Using the symmetry of the flow,we shall study the problem in $\{x > 0\}$, Oy axis becoming a stream line.

The point of intersection of the free boundary l with the porous wall has the coordinates: $x = \frac{V_0}{V_f}a\,, y = 0$.

On the free stream line l, from the Bernoulli equation, we obtain:

$$|\frac{\partial u}{\partial n}| = \sqrt{V_0^2 + 2g(b - y)} \tag{2.2}$$

Hence, the equations and boundary conditions which describe the physical problem are:

$$\begin{cases} \Delta u = 0, \ \ u > 0 \ \ in\ F \cap \{x > 0\}, \\ u = V_0(a - x) \quad on\ AA' \cap \{x > 0\}, \\ u = V_0 a \qquad on\ Oy \cap F, \\ u = V_0 a - V_f x \quad on\ OB, \\ u = 0, \ |\frac{\partial u}{\partial n}| = \sqrt{V_0^2 + 2g(b - y)} \ \ on\ l. \end{cases} \tag{2.3}$$

In order to obtain the variational formulation of the problem, we shall extend it on a known but unbounded domain:

$$D = (0, \infty) \times (0, b). \tag{2.4}$$

In Section 4 we shall prove that the flow region is contained in $(0, \frac{V_0}{V_f}a) \times (0, b)$. Let Γ be the boundary of D and $f : \Gamma \to \mathbf{R}$ the following function:

$$f = \begin{cases} V_0(a - x)^+ \qquad on\ \{(x, b)/x > 0\}, \\ V_0 a \qquad on\ \{(0, y)/0 < y < b\}, \\ (V_0 a - V_f x)^+ \qquad on\ \{(x, 0)/x > 0\}, \end{cases} \tag{2.5}$$

where $v^+ = max(v, 0)$.

We define the nonempty, closed, convex set:

$$K = \{v \in H^1(D \cap B_R), (\forall) R > 0/v = f\ on\ \Gamma, v \geq 0\ a.e.\ in\ D\} \tag{2.6}$$

where $B_R = \{(x, y) \in \mathbf{R}^2/|x|^2 + |y|^2 < R^2\}$

If H is the Heaviside function:

$$H(x) = \begin{cases} 1 \qquad if\ x > 0, \\ 0 \qquad if\ x \leq 0, \end{cases} \tag{2.7}$$

we obtain for the physical problem the following variational formulation:

$$\begin{cases} Find\ u \in K, \\ J(u) \leq J(v)\ for\ all\ v \in K, \end{cases} \tag{2.8}$$

where:

$$J(u) = \int_D \{| \nabla u|^2 + [V_0^2 + 2g(b - y)]H(u)\} dx dy. \tag{2.9}$$

We can prove that if u is a solution for the minimum problem (2.8) then u satisfies also the physical problem (2.3) by using the following results of Alt & Caffarelli(1981):

(i) if u is a minimum of J then $u \in C^{0,1}(D)$ and u is harmonic in $\{u > 0\}$,

(ii) the free boundary $\partial\{u > 0\} \cap D$ is analytic.

3. Existence and uniqueness results

Theorem 3.1 *The minimum problem (2.8) has a unique solution u.*

Proof. Since the domain D is unbounded we must show first that there exists $v \in K$ such that $J(v) < \infty$. We choose $w \in H^1((0, \frac{V_0}{V_f}a) \times (0,b))$ with $w = V_0 a - V_f x$ on $\{(x,0)/0 < x < \frac{V_0}{V_f}a\}$, $w = V_0 a$ on $\{(0,y)/0 < y < b\}$, $w = V_0(a-x)^+$ on $\{(x,b)/0 < x < \frac{V_0}{V_f}a\}$, $w = 0$ on $\{(\frac{V_0}{V_f}a, y)/0 < y < b\}$, we define:

$$v = \begin{cases} w^+ & in\ (0, \frac{V_0}{V_f}a) \times (0,b), \\ \\ 0 & in\ (\frac{V_0}{V_f}a, \infty) \times (0,b) \end{cases}$$

and we get $v \in K$ and $J(v) < \infty$.

For obtaining the existence of a solution of problem (2.8), we can prove as in Alt *et al.*(1983) that J is coercive and lower semicontinuous with respect to the weak topology in $H^1(D \cap B_R), (\forall)\, R > 0$ and, by using a Weierstrass theorem the existence follows.

For proving the uniqueness of the solution of (2.8) we consider u_1 and u_2 two solutions of (2.8). Since the proof is similar to the one in Stavre(1991)(see Appendix) we shall outline only the main steps.

(i) there are no connected components C of $\{u_1 > 0\} \cap \{u_2 > 0\}$ with $\partial C \cap \Gamma^+ = \emptyset$, where $\Gamma^+ = \{(x,0)/0 < x < \frac{V_0}{V_f}a\} \cup \{(0,y)/0 < y < b\} \cup \{(x,b)/0 < x < a\}$,

(ii) the set $\{u_1 > 0\} \cap \{u_2 > 0\}$ is connected,

(iii) $u_i \le V_0 a$ in D, for $i = 1,2$,

(iv) there exists $(x_0, y_0) \in \{u_1 > 0\} \cap \{u_2 > 0\}$, with $u_1(x_0,y_0) = u_2(x_0,y_0)$,

(v) $u_1 = u_2$ in $\{u_1 > 0\} \cap \{u_2 > 0\}$ and hence, $u_1 = u_2$ in D.

4. Study of the flow region, of the stream function and of the free boundary

In the sequel, we shall compare the solution of problem (2.8) with the stream function of the flow without gravity. Then we shall prove an important result concerning the flow region.

By using monotone rearrangements(see Kawohl(1985)) we shall establish the monotonicity of the stream function with respect to x and y and, as a consequence, some properties of the free boundary.

Theorem 4.1 *If u is the solution of problem (2.8) and U is the stream function in the case where the gravity field is neglected, then $u \le U$ in D .*

Proof. Denoting by U the stream function of the flow when the gravity is absent, we can obtain U as the solution of the variational problem:

$$\begin{cases} Find\ U \in K, \\ \mathcal{J}(U) \leq \mathcal{J}(v)\ for\ all\ v \in K, \end{cases} \tag{4.1}$$

where:

$$\mathcal{J}(v) = \int_D \{|\nabla v|^2 + V_0^2 H(v)\} dxdy. \tag{4.2}$$

By taking $v = min(u, U) \in K$ in (2.8) and $v = max(u, U) \in K$ in (4.1) we obtain the following inequalities:

$$\int_D \{|\nabla u|^2 + [V_0^2 + 2g(b-y)]H(u)\} dxdy \leq$$

$$\int_D \{|\nabla min(u,U)|^2 + [V_0^2 + 2g(b-y)]H(min(u,U))\} dxdy,$$

$$\int_D [|\nabla U|^2 + V_0^2 H(U)] dxdy \leq \int_D [|\nabla max(u,U)|^2 + V_0^2 H(max(u,U))] dxdy,$$

and, by adding them, it follows:

$$\int_{D\cap\{U<u\}} 2g(b-y)[H(u) - H(U)] dxdy \leq 0.$$

Hence $mes(D \cap \{U < u\}) = 0$ or $H(u) = H(U)$ in $D \cap \{U < u\}$.

In the first case it is obvious that $U \geq u$ in D .

In the second case, since $D \cap \{U < u\} \subset D \cap \{u > 0\}$ we get $H(u) = H(U) = 1$ in $D \cap \{U < u\}$. Hence $D \cap \{U < u\} \subset D \cap \{U > 0\}$.

We consider $(x,y) \in D$ with $U(x,y) = 0$ and we obtain $(x,y) \in D \cap \{U \geq u\}$ which leads to $u(x,y) = 0$. It follows that $\{u > 0\} \subset \{U > 0\}$. The function $U - u$ is harmonic in the open set $\{u > 0\}$ and $U - u \geq 0$ on $\partial\{u > 0\}$. By applying the maximum principle we get $U > u$ in $\{u > 0\}$ and hence $U \geq u$ in D.

Theorem 4.2 *The flow region $\{u > 0\}$ is contained into the bounded rectangle $(0, \frac{V_0}{V_f}a) \times (0, b)$, denoted D_0 .*

Proof. We obtain this Theorem as a consequence of Theorem 4.1 Indeed, since $u \leq U$ in D , it follows that $\{u > 0\} \subset \{U > 0\}$; but from Stavre(1991) (see Theorem 3.1.2) we have $\{U > 0\} \subset D_0$.

Lemma 4.3 *a)The solution u of (2.8) is monotone decreasing with respect to x.*

b)If the constants: a, b, V_0, V_f satisfy: $\frac{a^2}{b^2} > \frac{V_0^2+2gb}{V_0^2} \cdot \frac{V_0^2+2gb-V_f^2}{(V_0-V_f)^2}$, then u is monotone decreasing with respect to y.

Proof. Let u^* be the monotone decreasing rearrangement of u with respect to x(see Kawohl(1985)).

By taking in (2.8) $v = u - \varepsilon(u - V_0 a)^+, \varepsilon \in (0,1)$ we obtain $u \leq V_0 a$ in D. Since the values of u on $\{(x,0)/x > 0\}$ and $\{(x,b)/x > 0\}$ are monotone decreasing in x

and $0 \leq u \leq V_0 a$ in D_0, it follows that u^* is an element of K. For such a function, we have:

$$\int_{D_0} |\nabla u^*|^2 dxdy \leq \int_{D_0} |\nabla u|^2 dxdy \tag{4.3}$$

and, hence:

$$J(u^*) \leq J(u). \tag{4.4}$$

By using the uniqueness of the solution of problem (2.8), it follows from (4.4) that $u = u^*$ and the proof of a) is complete.

For proving the second part of Lemma 4.3 we have to compare first the solution u with its boundary values $(V_0 a - V_f x)^+$ and $V_0(a-x)^+$.

We can prove as in Stavre(1991) that $U \leq (V_0 a - V_f x)^+$ in D, where U is the function introduced in Theorem 4.1 . Hence Theorem 4.1 yields:

$$u \leq (V_0 a - V_f x)^+. \tag{4.5}$$

In order to establish the other inequality we take in (2.8):

$$v = max\{u, [V_0 a - x(\frac{V_0 - V_f}{b} y + V_f)]^+\}$$

and we obtain:

$$\int_{D\cap\{u<[V_0 a - x(\frac{V_0-V_f}{b}y+V_f)]^+\}} \{|\nabla u|^2 - |\nabla [V_0 a - x(\frac{V_0 - V_f}{b} y + V_f)]^+|^2 +$$

$$+[V_0^2 + 2g(b-y)][H(u) - 1]\} dxdy \Leftrightarrow$$

$$\int_D |\nabla \{[V_0 a - x(\frac{V_0 - V_f}{b} y + V_f)]^+ - u\}^+|^2 dxdy -$$

$$-2\int_D \nabla [V_0 a - x(\frac{V_0 - V_f}{b} y + V_f)]^+ \cdot \nabla \{[V_0 a - x(\frac{V_0 - V_f}{b} y + V_f)]^+ - u\}^+ dxdy +$$

$$+\int_{D\cap\{u<[V_0 a - x(\frac{V_0-V_f}{b}y+V_f)]^+\}} [V_0^2 + 2g(b-y)][H(u)-1] dxdy \leq 0.$$

Since the second integral is equal to zero, it follows:

$$\int_{D\cap\{u>0\}\cap\{u<[V_0 a - x(\frac{V_0-V_f}{b}y+V_f)]^+\}} |\nabla \{[V_0 a - x(\frac{V_0 - V_f}{b} y + V_f)]^+ - u\}^+|^2 dxdy +$$

$$\int_{D\cap\{u=0\}\cap\{u<[V_0 a - x(\frac{V_0-V_f}{b}y+V_f)]^+\}} \{|\nabla [V_0 a - x(\frac{V_0 - V_f}{b} y + V_f)]^+|^2 - V_0^2 - 2g(b-y)\} dxdy \leq 0.$$

The first integral being positive, we get:

$$\int_{D\cap\{0=u<[V_0 a - x(\frac{V_0-V_f}{b}y+V_f)]^+\}} [(\frac{V_0 - V_f}{b} y + V_f)^2 + (\frac{V_0 - V_f}{b})^2 x^2 - V_0^2 - 2g(b-y)] dxdy \leq 0. \tag{4.6}$$

On the other hand, if we take in (2.8) $v = max[u, (V_0 a - \sqrt{V_0^2 + 2gb\,x})^+]$ we obtain, as before:

$$\int_{D\cap\{u<(V_0 a-\sqrt{V_0^2+2gb\,x})^+\}} \{|\nabla[(V_0 a - \sqrt{V_0^2+2gb\,x})^+ - u]|^2 +$$

$$+[V_0^2 + 2g(b-y)][H(u)-1]\}dxdy \leq 0,$$

or:

$$\int_{D\cap\{0<u<(V_0 a-\sqrt{V_0^2+2gb\,x})^+\}} |\nabla(V_0 a - \sqrt{V_0^2+2gb}\,x - u)|^2 dxdy +$$

$$+\int_{D\cap\{0=u<(V_0 a-\sqrt{V_0^2+2gb\,x})^+\}} [V_0^2 + 2gb - V_0^2 - 2g(b-y)]dxdy.$$

Since the two integrals are positive it follows that:

$$u \geq (V_0 a - \sqrt{V_0^2+2gb\,x})^+ \qquad in\ D. \tag{4.7}$$

The inequality (4.7) yields that in $D\cap\{u=0\}$ we have:

$$x \geq \frac{V_0 a}{\sqrt{V_0^2+2gb}}. \tag{4.8}$$

From (4.6) and (4.8) it follows:

$$\int_{D\cap\{0=u<[V_0 a - x(\frac{V_0-V_f}{b}y+V_f)]^+\}} (\frac{V_0-V_f}{b}y+V_f)^2 + (\frac{V_0-V_f}{b})^2\frac{V_0^2 a^2}{V_0^2+2gb} - V_0^2 - 2g(b-y) \leq 0.$$

By using now the hypothesis of b) we obtain:

$$u \geq [V_0 a - x(\frac{V_0-V_f}{b}y+V_f)]^+ \qquad in\ D, \tag{4.9}$$

and, hence:

$$u \geq V_0(a-x)^+ \qquad in\ D. \tag{4.10}$$

From (4.5) and (4.10) it follows that u^{**} is an element of K, where we have denoted by u^{**} the monotone decreasing rearrangement of u in y. Thus the inequality (4.3) holds for u^* replaced by u^{**} and, hence $u = u^{**}$.

Corollary 4.4 *The velocity components satisfy:* $v_1 \geq 0$, $v_2 \leq 0$ *in* D.

Proof. The Corollary is a consequence of Lemma 4.3 and of the definition (2.1).

Corollary 4.5 *The free boundary is given by* $x = l(y)$, *where:*

$$l(y) = sup\{x/u(x,y) > 0\} \quad for\ all\ y \in (0,b). \tag{4.11}$$

Proof. By using the monotonicity of u with respect to x, it follows that if $u(x_0, y_0) = 0$, then $u(x, y_0) = 0$ for all $x \geq x_0$. Hence:

$$\{(x,y) \in D/u(x,y) > 0\} = \{(x,y) \in D/x < l(y)\}$$

and (4.11) holds.

5. Monotonicity of the stream function with respect to the given velocities

Theorem 5.1 *If $V_{0,1}, V_{0,2}$ are two values of the normal velocity of the fluid on MA, with $V_{0,1} < V_{0,2}$ then $u_1 \leq u_2$ in D , where u_i is the solution of problem (2.8) corresponding to $V_{0,i}, i = 1, 2$.*

Proof. We denote by f_i the function defined by (2.5) for $V_0 = V_{0,i}$ and by K_i the convex K for $f = f_i, i = 1, 2$;we obtain two minimum problems for $i = 1, 2$:.

$$\begin{cases} Find\ u_i \in K_i, \\ J_i(u_i) \leq J_i(v_i)\ for\ all\ v_i \in K_i, \end{cases} \tag{5.1}$$

where:

$$J_i(v) = \int_D \{|\nabla v|^2 + [(V_{0,i})^2 + 2g(b-y)]H(v)\}dxdy. \tag{5.2}$$

We define for $i = 1, 2$: $U_i = \frac{u_i}{V_{0,i}}, F_i = \frac{f_i}{V_{0,i}}$, $\mathcal{K}_i$ the convex K corresponding to F_i and:

$$\mathcal{J}_i(v_i) = \int_D \{|\nabla v_i|^2 + [1 + \frac{2g(b-y)}{V_{0,i}^2}]H(v_i)\}dxdy \quad v_i \in \mathcal{K}_i. \tag{5.3}$$

From (5.1) we obtain for $i = 1, 2$:

$$\begin{cases} Find\ U_i \in \mathcal{K}_i, \\ \mathcal{J}_i(U_i) \leq \mathcal{J}_i(v_i)\ for\ all\ v_i \in \mathcal{K}_i. \end{cases} \tag{5.4}$$

By adding the two inequalities for $v_1 = min(U_1, U_2)$ *and* $v_2 = max(U_1, U_2)$,since $V_{0,1} < V_{0,2}$ it follows:

$$\int_{D\cap\{U_1>U_2\}} \{2g(b-y)[H(U_1) - H(U_2)]\}dxdy \leq 0 \tag{5.5}$$

and hence:

(i) $mes(D \cap \{U_1 > U_2\}) = 0$ or

(ii) $H(U_1) = H(U_2) = 1$ in $D \cap \{U_1 > U_2\}$.

In the second case, we obtain as in the proof of Theorem 4.1 $U_1 \leq U_2$ in D. Hence $u_1 \leq u_2$ in D.

We shall prove next a similar result for $V_{f,1} < V_{f,2}$.

Theorem 5.2 *If two given velocities on the porous wall satisfy $V_{f,1} < V_{f,2}$,the corresponding stream function decreases,i.e. $u_1 \geq u_2$ in D.*

Proof. Denoting now f_i the function f for $V_f = V_{f,i}$ and K_i being the set defined in (2.6) for $f = f_i, i = 1,2$ we get the variational problems:

$$\begin{cases} Find\ u_i \in K_i, \\ J(u_i) \leq J(v_i)\ for\ all\ v_i \in K_i. \end{cases} \tag{5.6}$$

Since $u_1 \geq u_2$ on Γ we can take $v_1 = max(u_1, u_2), v_2 = min(u_1, u_2)$ and we obtain:

$$J(u_1) + J(u_2) = J(max(u_1, u_2)) + J(min(u_1, u_2)) \tag{5.7}$$

Hence, from (5.6) and (5.7) it follows:

$$J(u_1) = J(max(u_1, u_2)); J(u_2) = J(min(u_1, u_2)). \tag{5.8}$$

By using now the uniqueness of the solution of problem (5.6) we get $u_1 = max(u_1, u_2)$ and $u_2 = min(u_1, u_2)$ which ends the proof.

The last result of this paper shows the variation of the stream function with respect to $\frac{V_0}{V_f}$.

Theorem 5.3 *With the notations of Theorem 5.1 and Theorem 5.2 for the velocities V_0 and V_f , if $V_{0,1} < V_{0,2}$ and $V_{f,1} < V_{f,2}$, but $\frac{V_{0,1}}{V_{f,1}} = \frac{V_{0,2}}{V_{f,2}}$ we obtain for the corresponding stream functions $u_1 \leq u_2$ in D.*

Proof. We consider for $i = 1,2$ the functions U_i defined in Theorem 5.1 .Since $U_i \in \mathcal{K}_i$ for $i = 1,2$ and $U_1 = U_2$ on Γ , we can take $v_1 = min(U_1, U_2)$ and $v_2 = max(U_1, U_2)$ in (5.4) and, with the same technique as in Theorem 5.1 we obtain the conclusion of this Theorem.

6. Conclusions

The flow of a fluid in a gravity field, impacting on a porous wall was studied by using variational methods. These methods are useful, since they permit us to establish existence and uniqueness theorems and to obtain properties of the flow region, of the stream function and of the free boundary, properties which cannot be used as hypotheses of the problem because the flow region and the free boundary are unknown. Moreover a computational approach may be performed (as in Stavre(1991)) by employing a finite element method.

These methods can also solve the two-dimensional asymmetric and the axially symmetric problems but not the three-dimensional asymmetric case.

We hope to extend the methods presented here to the coupled problem of a jet penetrating a porous medium.

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