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# FREE PRODUCT OF $C^*$ - ALGEBRAS VIA A FAMILY OF MAPS

by

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# FREE PRODUCT OF $C^*$ - ALGEBRAS VIA A FAMILY OF MAPS

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(preliminary version)

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FREE PRODUCT OF $C^*$ - ALGEBRAS

VIA A FAMILY OF MAPS

by Valentin Ionescu

ABSTRACT. **The existence of the free product for conditional** expectation families (particularly for state families) is established in the category of the unital $C^*$ - algebras.

Let I be an index set and $(\Gamma_i)_{i\in I}$ a family of index sets. For all $n \geq 1$ if $t = (i_1, \ldots, i_n) \in I^n$ denote $\Gamma_t := \Gamma_{i_1} \times \cdots \times \Gamma_{i_n}$. Let also be given a family of maps $\gamma = (\gamma^{(n)})_{n\geq 1}$ such that $\gamma^{(n)}: \coprod_{t\in I^n} \Gamma_t \longrightarrow \bigcup_{t\in I^n} \Gamma_t$, and $\gamma^{(n)}(t, \alpha_t) \in \Gamma_t$, for all $(t, \alpha_t) \in \coprod_{t\in I_n} \Gamma_t$, if $n \geq 1$.

Denote again

$$D_n(I) := \{(i_1, \ldots, i_n) \in I^n;\ i_j \neq i_{j+1},\ 1 \leq j \leq n-1\}$$

for all $n \geq 1$, and

$$\Gamma := \prod_{i\in I} \Gamma_i .$$

( $\coprod$ and $\prod$ are respectively denoting the coproduct and the product in the category of sets.)

1. Let now $\{A_i, \Phi_i\}$ be a couple for all $i \in I$, where $A_i$ is an unital $C^*$ - algebra and $\Phi_i = \{\varphi_i^{(\alpha_i)};\ \alpha_i \in \Gamma_i\}$ is an arbitrary family of states on $A_i$.

In this context one can establish the following theorem:

<u>Theorem 1.1.</u> There exists a couple $\{A,\Phi\}$ , where A is an unital $C^*$ - algebra and $\Phi=\{\varphi^{(\alpha)};\alpha\in\Gamma\}$ is a family of states on A, such that:

(1) There exists an unital $*$ - homomorphism $j_i : A_i \longrightarrow A$ for each $i\in I$, such that A is generated by $\bigcup_{i\in I} j_i(A_i)$ ;

(2) $\varphi^{(\alpha)}\circ j_i=\varphi_i^{(\alpha_i)}$ for each $i\in I$ and $\alpha\in\Gamma$ ;

(3) $\varphi^{(\alpha)}(j_{i_1}(a_1)\cdots j_{i_n}(a_n))=\varphi_{i_1}^{(\alpha_{i_1})}(a_1)\cdots\varphi_{i_n}^{(\alpha_{i_n})}(a_n)$ if $a_j\in \mathrm{Ker}\,\varphi_{i_j}^{(\gamma_j^{(n)}(t,\alpha_t))}$ ( $1\leq j\leq n$), for all $n\geq 1$, $t:=(i_1,\ldots,i_n)\in D_n(I)$, denoting $\alpha_t:=(\alpha_{i_1},\ldots,\alpha_{i_n})$ ; for each $\alpha\in\Gamma$ .

<u>Remark 1.2.</u> i) Let $\Gamma_i=\Lambda$ for all $i\in I$ and $\Delta : I\times\Lambda\longrightarrow\Lambda$, $\Delta(i,\lambda)\equiv\lambda$ for $i\in I$ and $\lambda\in\Lambda$ . Take $\gamma_j^{(n)}(t,\alpha_t)=\Delta(i_j,\alpha_{i_j})$ if $t=(i_1,\ldots,i_n)$ and $\alpha_t=(\alpha_{i_1},\ldots,\alpha_{i_n})$ for $1\leq j\leq n$ $(n\geq 1)$. In this case (3) becomes: $\varphi^{(\lambda)}(j_{i_1}(a_1)\cdots j_{i_n}(a_n))=0$ if $\varphi_{i_j}^{(\lambda_{i_j})}(a_j)=0$ $(1\leq j\leq n)$ for all $n\geq 1$, $(i_1,\ldots,i_n)\in D_n(I)$, $\lambda\in\Lambda^I$. Under this form, (3) recalls D. Voiculescu's notion of <u>free independence</u>, if one takes $|\Lambda|=1$, and $\{A,\Phi\}$ recalls his concept of <u>reduced free product</u> ( [9] ).

ii) Let $|\Lambda|=2$; for example $\Lambda=\{1,2\}$ , $\Phi_i=\{\varphi_i^{(1)},\varphi_i^{(2)}\}$. Take $\Delta: I\times\Lambda\longrightarrow\Lambda$ given by $\Delta(i,1)\equiv 2$, $\Delta(i,2)\equiv 1$ ( $i\in I$) and again $\gamma_j^{(n)}(t,\alpha_t)=\Delta(i_j,\alpha_{i_j})$ for $t=(i_1,\ldots,i_n)$ , $\alpha_t=(\alpha_{i_1},\ldots,\alpha_{i_n})$ , n... In this case, (3) contains, renoting $\varphi^{(1,\ldots,1,\ldots)}$ by $\varphi^{(1)}$ and $\varphi^{(2,\ldots,2,\ldots)}$ by $\varphi^{(2)}$:

$$\varphi^{(1)}(j_{i_1}(a_1)\cdots j_{i_n}(a_n))=\varphi_{i_1}^{(1)}(a_1)\cdots\varphi_{i_n}^{(1)}(a_n)\quad \text{if}\quad \varphi_{i_k}^{(2)}(a_k)=0\ (1\leq k\leq n)$$

and

$$\varphi^{(2)}(j_{i_1}(a_1)\cdots j_{i_n}(a_n))=\varphi_{i_1}^{(2)}(a_1)\cdots\varphi_{i_n}^{(2)}(a_n)\quad \text{if}\quad \varphi_{i_k}^{(1)}(a_k)=0\ (1\leq k\leq n)$$

for all $n\geq 1$, $(i_1,\ldots,i_n)\in D_n(I)$.

Thus, (3) recalls the notions of "<u>$\psi$ - independence</u>" and "<u>$\varphi$ - independence</u>"introduced by M. Bozejko and R. Speicher ([3] ), and $\{A,\Phi\}$ recalls their <u>symmetrized product</u>.

Proof of the Theorem 1.1.

A simple recourse to the existence of the coproduct in the category of unital algebras ([2]), supplies an object A having an unital algebra structure ( the free product of the unital algebras $A_i$, $i \in I$) and the canonical homomorphisms $j_i : A_i \longrightarrow A$ , for $i \in I$, which are unital, such that A is algebrically generated by $\bigcup_{i \in I} j_i(A_i)$ .

But, of course, one can easily make $j_i$ hermitian and organize A like a $*$ - algebra, by appealing to the existence of the involution on each $A_i$.

Thus, the condition (2) and (3) remain to be explored.

Let $i \in I$ be arbitrarily fixed. Let then $(\pi_i^{(\alpha_i)}, H_i, \xi_i)$, $\alpha_i \in \Gamma_i$ be the unital $*$ - representations given by Gelfand-Neumark--Segal construction which is respectively applied to the state $\varphi_i^{(\alpha_i)}$, $\alpha_i \in \Gamma_i$ , of the family $\Phi_i$ , taking the same Hilbert space $H_i$ and the same vector $\xi_i \in H_i$ , $\|\xi_i\| = 1$ , for each $\alpha_i \in \Gamma_i$ One can proceed in this manner by a recourse to the infinite tensorial product ( [6] , [4] ), but one may loose the cyclicity of $\xi_i$ .

One can now consider the free product $\{H, \xi\}$ of the family $\{H_i, \xi_i\}_{i \in I}$ .

Thus ([1],[9]), one has $\|\xi\| = 1$, and,denoting $H_i^\circ := H_i \ominus \mathbb{C}\xi_i$ , $i \in I$, one can write:

$$H = \mathbb{C}\xi \oplus \bigoplus_{n \geq 1} \bigoplus_{(i_1, \dots, i_n) \in D_n(I)} H_{i_1}^\circ \otimes \dots \otimes H_{i_n}^\circ \qquad (1.1.$$

As it is known, if $H_{(i)} := \mathbb{C}\xi \oplus \bigoplus_{n \geq 1} \bigoplus_{\substack{(i_1, \dots, i_n) \in D_n(I) \\ i_1 \neq i}} H_{i_1}^\circ \otimes \dots \otimes H_{i_n}^\circ$ then $H \simeq H_i \otimes H_{(i)}$ and this isomorphism is given by the following unitary operato operator $V_i$ :

$$V_i \xi = \xi_i \otimes \xi \quad ,$$

. / .

$$V_i(h_1^\circ\otimes\cdots\otimes h_n^\circ)=\begin{cases} h_1^\circ\otimes(h_2^\circ\otimes\cdots\otimes h_n^\circ) & \text{if } i_1=i \text{ and } n\geq 2\\ h_1^\circ\otimes\xi & \text{if } i_1=i \text{ and } n=1\\ \xi_i\otimes(h_1^\circ\otimes\cdots\otimes h_n^\circ) & \text{if } i_1\neq i\end{cases}$$

for all $n\geq 1$, $(i_1,\ldots,i_n)\in D_n(I)$, $h_j^\circ\in H_{i_j}^\circ$ $(1\leq j\leq n)$.
One can therefore give, with respect to the previous decomposition of H , the following unitary $*$ - representation $\lambda_i$ of $B(H_i)$ into $B(H)$ :

$$\lambda_i(T):=V_i^{-1}(T\otimes id)V_i \quad , \text{ for all } T\in B(H_i) .$$

Moreover, if one identifies $\xi\equiv\xi_i$ in (1.1) one can consider the following orthogonal decomposition of H:

$$H=H_i\oplus H_i^\perp .$$

Then, one can also observe that the subspaces $H_i$ and $H_i^\perp$ of H are invariant under the action of $\lambda_i(T)$ for all $T\in B(H_i)$.

Let $\alpha^\circ=(\alpha_i^\circ)_{i\in I}\in\Gamma$ be fixed: $\alpha_i^\circ\in\Gamma_i$ for each $i\in I$.

Let $n\geq 1$, $t=(i_1,\ldots,i_n)\in I^n$. Denote $\alpha_t^\circ:=(\alpha_{i_1}^\circ,\ldots,\alpha_{i_n}^\circ)\in\Gamma_t$

With the above considerations, one can now construct the unita $*$ - representation $\sigma^{(i_k,\alpha_{i_k}^\circ)}$ of $A_{i_k}$ into $B(H)$ given by

$$\sigma^{(i_k,\alpha_{i_k}^\circ)}(a)=\begin{bmatrix}\lambda_{i_k}\pi_{i_k}^{(\alpha_{i_k}^\circ)}(a) & 0\\ & \\ 0 & \lambda_{i_k}\pi_{i_k}^{(\gamma_k^{(n)}(t,\alpha_t^\circ))}(a)\end{bmatrix} \tag{1.2}$$

for $a\in A_{i_k}$, with respect to the precedent orthogonal decomposition of H into $H_{i_k}\oplus H_{i_k}^\perp$, $(1\leq k\leq n)$.

Remark that the $*$ - representations $\sigma^{(i_k,\alpha_{i_k}^\circ)}: A_{i_k}\longrightarrow B(H)$, $1\leq k\leq n$, have the following properties:

1° $\sigma^{(i_k,\alpha_{i_k}^\circ)}(a)\xi\equiv\varphi_{i_k}^{(\alpha_{i_k}^\circ)}(a)\xi$ (mod $H_{i_k}^\circ$) for $a\in A_{i_k}$

2° $\sigma^{(i_1,\alpha_{i_1}^\circ)}(a_1)\ldots\sigma^{(i_{n-1},\alpha_{i_{n-1}}^\circ)}(a_{n-1})H_{i_n}^\circ\subset H_{i_1}^\circ\otimes\cdots\otimes H_{i_{n-1}}^\circ\otimes H_{i_n}^\circ$
if $a_k\in \operatorname{Ker}\varphi_{i_k}^{(\gamma_k^{(n)}(t,\alpha_t^\circ))}$ $(1\leq k\leq n-1)$; for all $n\geq 2$,
$t=(i_1,\ldots,i_n)\in D_n(I)$ .

In fact,

$$\begin{aligned}\sigma^{(i_k,\alpha^\circ_{i_k})}(a)\xi &= \lambda_{i_k}(\pi_{i_k}^{(\alpha^\circ_{i_k})}(a))\xi \oplus 0\\ &= V_{i_k}^{-1}(\pi_{i_k}^{(\alpha^\circ_{i_k})}(a)\otimes id)V_{i_k}\xi\\ &= V_{i_k}^{-1}(\pi_{i_k}^{(\alpha^\circ_{i_k})}(a)\otimes id)(\xi_{i_k}\otimes\xi)\\ &= V_{i_k}^{-1}(\pi_{i_k}^{(\alpha^\circ_{i_k})}(a)\xi_{i_k}\otimes\xi)\\ &= \pi_{i_k}^{(\alpha^\circ_{i_k})}(a)\xi_{i_k}\ ,\end{aligned}$$

and then

$$\begin{aligned}\langle\xi,\sigma^{(i_k,\alpha^\circ_{i_k})}(a)\xi-\varphi_{i_k}^{(\alpha^\circ_{i_k})}(a)\xi\rangle &= \langle\xi,\sigma^{(i_k,\alpha^\circ_{i_k})}(a)\xi\rangle-\varphi_{i_k}^{(\alpha^\circ_{i_k})}(a)\\ &= \langle\xi,\pi_{i_k}^{(\alpha^\circ_{i_k})}(a)\xi_{i_k}\rangle-\varphi_{i_k}^{(\alpha^\circ_{i_k})}(a)\\ &\equiv \langle\xi_{i_k},\pi_{i_k}^{(\alpha^\circ_{i_k})}(a)\xi_{i_k}\rangle-\varphi_{i_k}^{(\alpha^\circ_{i_k})}(a)\\ &= 0\ ,\end{aligned}$$

therefore ($1^\circ$) is simply a transcription of the fact that $\pi_{i_k}^{(\alpha^\circ_{i_k})}$ is the GNS representation (with the vector $\xi_{i_k}$ ) of the state $\varphi_{i_k}^{(\alpha^\circ_{i_k})}$ given on $A_{i_k}$.

Also, if $a\in \operatorname{Ker}\varphi_{i_j}^{(\gamma_j^{(m)}(t,\alpha^\circ_t))}$ for $i_j\in I_t:=\{i_1,\dots,i_n\}$, it results $\pi_{i_j}^{(\gamma_j^{(m)}(t,\alpha^\circ_t))}(a)\xi_{i_j}\in H^\circ_{i_j}$ and then, for $(i_j,i_l)\in D_2(I_t)$ and $(i_1,\dots,i_k)\in D_k(I_t)$, because $H^\circ_{i_1}\otimes\dots\otimes H^\circ_{i_k}\subset H^\perp_{i_j}$, one has

$$\begin{aligned}\sigma^{(i_j,\alpha^\circ_{i_j})}(a)(H^\circ_{i_1}\otimes\dots\otimes H^\circ_{i_k}) &= 0\oplus\lambda_{i_j}(\pi_{i_j}^{(\gamma_j^{(m)}(t,\alpha^\circ_t))}(a))(H^\circ_{i_1}\otimes\dots\otimes H^\circ_{i_k})\\ &\equiv V_{i_j}^{-1}(\pi_{i_j}^{(\gamma_j^{(m)}(t,\alpha^\circ_t))}(a)\otimes id)V_{i_j}(H^\circ_{i_1}\otimes\dots\otimes H^\circ_{i_k})\\ &= V_{i_j}^{-1}(\pi_{i_j}^{(\gamma_j^{(m)}(t,\alpha^\circ_t))}(a)\otimes id)(\xi_{i_j}\otimes(H^\circ_{i_1}\otimes\dots\otimes H^\circ_{i_k}))\\ &= V_{i_j}^{-1}(\pi_{i_j}^{(\gamma_j^{(m)}(t,\alpha^\circ_t))}(a)\xi_{i_j}\otimes(H^\circ_{i_1}\otimes\dots\otimes H^\circ_{i_k}))\\ &= \pi_{i_j}^{(\gamma_j^{(m)}(t,\alpha^\circ_t))}(a)\xi_{i_j}\otimes(H^\circ_{i_1}\otimes\dots\otimes H^\circ_{i_k})\\ &\subset H^\circ_{i_j}\otimes H^\circ_{i_1}\otimes\dots\otimes H^\circ_{i_k}\ .\end{aligned}$$

Therefore, it results $2^\circ$ by a simple induction.

Observe that another transcription of $1^\circ$ is $\langle\xi,\sigma^{(i_k,\alpha^\circ_{i_k})}(a)\xi\rangle=\varphi_{i_k}^{(\alpha^\circ_{i_k})}(a)$

From $1^o$ and $2^o$ it results the following property for the $*$ - representations $\sigma^{(i_k,\alpha^o_{i_k})}: A_{i_k} \longrightarrow B(H)$ $(1 \leq k \leq n)$ :

$$3^o \quad \langle \xi, \sigma^{(i_1,\alpha^o_{i_1})}(a_1) \cdots \sigma^{(i_n,\alpha^o_{i_n})}(a_n)\xi\rangle = \varphi_{i_1}^{(\alpha^o_{i_1})}(a_1) \cdots \varphi_{i_n}^{(\alpha^o_{i_n})}(a_n)$$

if $a_k \in \operatorname{Ker} \varphi_{i_k}^{(\gamma_k^{(n)}(t,\alpha^o_t))}$ $(1 \leq k \leq n)$

for all $n \geq 1$, $t=(i_1, \ldots, i_n) \in D_n(I)$ .

In fact, under these considerations, one has, for $h^o_{i_n} \in H^o_{i_n}$, by $1^o$ :

$$\langle \xi, \sigma^{(i_1,\alpha^o_{i_1})}(a_1) \cdots \sigma^{(i_{n-1},\alpha^o_{i_{n-1}})}(a_{n-1})\sigma^{(i_n,\alpha^o_{i_n})}(a_n)\xi\rangle =$$
$$= \langle \xi, \sigma^{(i_1,\alpha^o_{i_1})}(a_1) \cdots \sigma^{(i_{n-1},\alpha^o_{i_{n-1}})}(a_{n-1}) h^o_{i_n}\rangle +$$
$$+ \langle \xi, \sigma^{(i_1,\alpha^o_{i_1})}(a_1) \cdots \sigma^{(i_{n-1},\alpha^o_{i_{n-1}})}(a_{n-1}) \varphi_{i_n}^{(\alpha^o_{i_n})}(a_n)\xi\rangle$$
$$= \langle \xi, \sigma^{(i_1,\alpha^o_{i_1})}(a_1) \cdots \sigma^{(i_{n-1},\alpha^o_{i_{n-1}})}(a_{n-1})\xi\rangle \varphi_{i_n}^{(\alpha^o_{i_n})}(a_n) ,$$

because one has $\sigma^{(i_1,\alpha^o_{i_1})}(a_1) \cdots \sigma^{(i_{n-1},\alpha^o_{i_{n-1}})}(a_{n-1}) h^o_{i_n} \perp \xi$ , by $2^o$, in H. Therefore a trivial induction furnishes the assertion $3^o$

One has for each $n \geq 1$ and $(i_1, \ldots, i_n) \in I^n$ the $*$ - homomorphisms $\sigma^{(i_k,\alpha^o_{i_k})}$ , $1 \leq k \leq n$ , from $A_{i_k}$ into B(H).

Then, using, a procedure of inductive limit, for example in the category of unital algebras, one can obtain the existence of an unital homomorphism $\sigma^{(\alpha^o)}: A \longrightarrow B(H)$ such that:

$$\sigma^{(\alpha^o)}(j_{i_1}(a_1) \cdots j_{i_n}(a_n)) = \sigma^{(i_1,\alpha^o_{i_1})}(a_1) \cdots \sigma^{(i_n,\alpha^o_{i_n})}(a_n)$$

for all $n \geq 1$ and $(i_1, \ldots, i_n) \in D_n(I)$ , if $a_k \in A_{i_k}$ $(1 \leq k \leq n)$.
In our context it results that $\sigma^{(\alpha^o)}$ is hermitian, therefore one obtaines, in this manner, an unital $*$ - representation of the unital $*$ - algebra A into B(H).

Denoting by $\omega_\xi$ the vector state on B(H) corresponding to $\xi$ , one has

$$(\omega_\xi \circ \sigma^{(\alpha^o)}) \circ j_i = \varphi_i^{(\alpha^o_i)}$$

for all $i \in I$

and, using $3^o$, $(\omega_\xi \circ \sigma^{(\alpha^o)})(j_{i_1}(a_1) \cdots j_{i_n}(a_n)) = \varphi_{i_1}^{(\alpha^o_{i_1})}(a_1) \cdots \varphi_{i_n}^{(\alpha^o_{i_n})}(a_n)$
if $a_k \in \operatorname{Ker} \varphi_{i_k}^{(\gamma_k^{(n)}(t,\alpha^o_t))}$ , $(1 \leq k \leq n)$, $(i_1, \ldots, i_n) \in D_n(I)$, for all $n \geq 1$.

Therefore, it remains to consider the state $\varphi^{(\alpha^o)} := \omega_\xi \circ \sigma^{(\alpha^o)}$ corresponding to the representation $\sigma^{(\alpha^o)}$ and the properties (2) and (3) are also satisfied.

The $*$ - representation of the free product A of the algebras $A_i$, $i \in I$, are in one - to - one correspondence with families of $*$ - representations of $A_i$, $i \in I$, which act on the same Hilbert space.

By analogy with the tensor product, one can introduce the following pre - $C^*$ - norm on A :

$$\|a\| = \sup \{ \|\pi(a)\| \ ; \pi = * \text{-representation of } A \}$$

that extends the $C^*$ - norm on $A_i$, $i \in I$. Then, taking for A its completion in this norm, one can easily see that this construction defines a coproduct in the category of $C^*$-algebras.

This change of $\sigma^{(\alpha)}, \alpha \in \Gamma$ doesn't distort $\varphi^{(\alpha)}, \alpha \in \Gamma$

<u>Remark 1.3.</u> One can observe there also exists another family $\Phi^{\#} = \{\varphi^{\#(\alpha)} ; \alpha \in \Gamma\}$ of states on A such that

(2') $\varphi^{\#(\alpha)} \circ j_i = \varphi_i^{(\gamma^{(1)}(i,\alpha_i))}$ for all $i \in I$ and $\alpha \in \Gamma$ ;

(3') $\varphi^{\#(\alpha)}(j_{i_1}(a_1) \cdots j_{i_n}(a_n)) = \varphi_{i_1}^{(\gamma_1^{(n)}(t,\alpha_t))}(a_1) \cdots \varphi_{i_n}^{(\gamma_n^{(n)}(t,\alpha_t))}(a_n)$ ,

if $a_j \in \operatorname{Ker} \varphi_{i_j}^{(\alpha_{i_j})}$ , ( $1 \leq j \leq n$ ), for all $n \geq 1$,

$t := (i_1, \ldots, i_n) \in D_n(I)$, $\alpha_t := (\alpha_{i_1}, \ldots, \alpha_{i_n})$ ;

for each $\alpha \in \Gamma$ .

In fact, for all $\alpha \in \Gamma$ one can also construct the unital $*$ - representations $\sigma^{\#(i_k, \alpha_{i_k})}$ of $A_{i_k}$ into B(H), $1 \leq k \leq n$, for each $n \geq 1$ and $t := (i_1, \ldots, i_n) \in I^n$ (denoting $(\alpha_{i_1}, \ldots, \alpha_{i_n}) =: \alpha_t$), given by

(1.3)

$$\sigma^{\#(i_k,\alpha_{i_k})}(a) = \begin{bmatrix} \lambda_{i_k}\pi_{i_k}^{(\gamma_k^{(m)}(t,\alpha_t))}(a) & 0 \\ 0 & \lambda_{i_k}\pi_{i_k}^{(\alpha_{i_k})}(a) \end{bmatrix} \tag{1.3.}$$

for $a \in A_{i_k}$, with respect to the same orthogonal decompositions of H into $H_{i_k} \oplus H_{i_k}^{\perp}$, $1 \leq k \leq n$.

The * - representations $\sigma^{\#(i_k,\alpha_{i_k})}$, $1 \leq k \leq n$, have the following properties:

$$\langle \xi, \sigma^{\#(i_k,\alpha_{i_k})}(a)\xi \rangle = \varphi_{i_k}^{(\gamma_k^{(m)}(t,\alpha_t))}(a) \quad \text{for } A_{i_k} \ni a \quad ;$$

$$\langle \xi, \sigma^{\#(i_1,\alpha_{i_1})}(a_1)\cdots\sigma^{\#(i_n,\alpha_{i_n})}(a_n)\xi \rangle = \varphi_{i_1}^{(\gamma_1^{(m)}(t,\alpha_t))}(a_1)\cdots\varphi_{i_n}^{(\gamma_n^{(m)}(t,\alpha_t))}(a_n)$$

if $a_k \in \operatorname{Ker}\varphi_{i_k}^{(\alpha_{i_k})}$ $(1 \leq k \leq n)$, for all $n \geq 1$, $t := (i_1,\ldots,i_n) \in D_n(I)$, $\alpha_t := (\alpha_{i_1},\ldots,\alpha_{i_n})$.

Therefore, the existence of $\Phi^{\#}$ having the properties (2') and (3') becomes clear.

A couple $\{A,\Phi\}$ given by the Theorem 1.1 can be called a <u>reduced free product</u> of $\{A_i,\Phi_i\}_{i\in I}$ <u>via</u> $\gamma$, and (3) the <u>free independence property on A via $\gamma$</u>.

Moreover, one can interpret a triple $\{A,\Phi,\Phi^{\#}\}$ like a <u>symmetrized free product</u> of $\{A_i,\Phi_i\}_{i\in I}$ <u>via</u> $\gamma$, and the assertions (3) and (3') like the <u>symmetrical free independence property on A via $\gamma$</u>.

<u>OBSERVATION 1.4.</u>

Let $\Gamma_i = \Lambda$ for each $i \in I$, $\Delta : I \times \Lambda \longrightarrow \Lambda$ given by $\Delta(i,\lambda) \equiv \sigma(\lambda)$ with a bijective map $\sigma : \Lambda \longrightarrow \Lambda$, and again $\gamma_j^{(m)}(t,\alpha_t) = \Delta(i_j,\alpha_{i_j})$ if $t = (i_1,\ldots,i_n)$, $\alpha_t = (\alpha_{i_1},\ldots,\alpha_{i_n})$.

Take $\Lambda = \{1,2,\ldots,m\}$ $(m \geq 1)$ and $\sigma \in \mathfrak{S}_m$ (the symmetric group of order m).

Then $\{\varphi_i^{(\Delta(i,k))}, k \in \Lambda\} = \Phi_i$ for all $i \in I$.

./.

If $\sigma \neq \begin{pmatrix} 1 & 2 & \dots & m \\ m & m-1 & \dots & 1 \end{pmatrix}$, it results $\Phi^{\#} \neq \Phi$. Therefore, there exist, in this case, a reduced free product $\{A, \Phi\}$ and a symmetrized free product $\{A, \Phi, \Phi^{\#}\}$ of $\{A_i, \Phi_i\}_{i\in I}$, via $\gamma$, which are different.

If $\sigma = \begin{pmatrix} 1 & 2 & \dots & m \\ m & m-1 & \dots & 1 \end{pmatrix}$, it results $\Phi^{\#} = \Phi$. Therefore, for this choice of $\gamma$, the objects furnished by the Theorem 1. and the Remark 1.3. are identical.

2. Let B be an unital $C^*$ - algebra.

Consider now $\{A_i, \Phi_i\}$ for all $i \in I$, where $A_i$ is an unital $C^*$ - algebra such that $A_i \supset B \ni 1_{A_i}$ and $\Phi_i = \{\varphi_i^{(\alpha_i)}; \alpha_i \in \Gamma_i\}$ is an arbitrary family of norm one projections of the $C^*$ - algebra $A_i$ onto its $C^*$- subalgebra B.

Theorem 1.1 can be generalized in this context.

<u>Theorem 2.1</u>. There exists a couple $\{A, \Phi\}$, where A is an unital $C^*$ -algebra and $\Phi = \{\varphi^{(\alpha)}; \alpha \in \Gamma\}$ is a family of conditional expectations of A onto B such that:

(1) There exists an unital * - homomorphism $j_i : A_i \longrightarrow A$ for each $i \in I$, such that $j_i(B) = B$, i.e. B identifies with a subalgebra of A so that $A \supset B \ni 1_A$, and A is generated by $\bigcup_{i \in I} j_i(A_i)$;

(2) $\varphi^{(\alpha)} \circ j_i = \varphi_i^{(\alpha_i)}$ for each $i \in I$ and $\alpha \in \Gamma$;

(3) $\varphi^{(\alpha)}(j_{i_1}(a_1) \cdots j_{i_n}(a_n)) = \varphi_{i_1}^{(\alpha_{i_1})}(a_1) \cdots \varphi_{i_n}^{(\alpha_{i_n})}(a_n)$ if $a_j \in \operatorname{Ker} \varphi_{i_j}^{(\gamma_j^{(n)}(t, \alpha_t))}$ $(1 \le j \le n)$, for all $n \ge 1$, $t = (i_1, \ldots, i_n) \in D_n(I)$, denoting $\alpha_t := (\alpha_{i_1}, \ldots, \alpha_{i_n})$, for each $\alpha \in \Gamma$.

Some preliminaries are necessary.

The projections of norm one and the conditional expectations are respectively considered in the sense of [11] and [1o] - [12].

The terminology for Hilbert $C^*$-modules is that of [8] (see also [5], [7]). But the principal reference for 2. is [9].

2.2. Let X and Y be (right) B-modules, equipped with B - valued inner semi-products (i.e. non-negative hermitian sesquilinear maps, which are assumed to be antilinear in the first argument) and vector space structures over the complex field, respecting the module actions. Assume moreover that is given an unital $*$ - homomorphism $\chi : B \longrightarrow L(Y)$ .

Then one can canonically define a B - valued inner semi-product on the algebraic tensor product of X with Y over B (by analogy to a inner semi-product inducing on a $C^*$- algebra via a state).

Let $X \otimes_B Y$ be denoting the pre-Hilbert (right) B-module obtained by separation with respect to the semi-norm defined by the previous B-valued inner semi-product.

If X and Y are Hilbert (right) B-modules and $T \in L(X)$ , one can easily observe that $T \otimes id \in L ( X \otimes_B Y)$.

2.3. Let $H_i$ be a (pre-) Hilbert (right) B-module, $\xi_i \in H_i$ , such that $\langle \xi_i , \xi_i \rangle = 1_B$ , and $\chi_i : B \longrightarrow L(H_i)$ be an unital $*$ - homomorphism, for all $i \in I$.
Then, one can construct ( $H: = \bigotimes_{i \in I}{}_B^{\xi} H_i \; ; \chi$ ) (the analogue of the infinite tensor product of [6]; see also [4] ) by an inductive limit procedure. It is endowed with a ( pre-)Hilbert (right) B-module structure and an unital $*$ - homomorphism $\chi : B \longrightarrow L($

This infinite tensor product is also associative and distributive with respect to the direct sums of (pre-)Hilbert (right) B-modules.

NOTE

Here, $\langle \, , \rangle$ and L(E) are respectively denoting the inner semi-product and the bounded operators on the B-module E equipped with $\langle \, , \rangle$ .

B is endowed with the inner product $\langle b_1 , b_2 \rangle = b_1^* b_2$ . For a B-module E, equipped with a B-valued inner semi-product, one naturally define the orthogonal complement of each B-submodule of E.

Sketch of Theorem 2.1's proof

One can proceed like in 1. In the following we review especially the differences. The recourse to the existence of the coproduct in the category of unital algebras furnishes the unital albegra A and the unital $*$ - homomorphisms $j_i : A_i \longrightarrow A$, $i \in I$, such that A is generated by $\bigcup_{i\in I} j_i(A_i)$. One can identify B with $j_{i}(B)$ by a simple algebrical exercise.

Let $i \in I$ be arbitrarily fixed. Let then $(\pi_i^{(\alpha_i)}, H_i, \xi_i, \chi_i, \chi_i^\circ)$, $\alpha_i \in \Gamma_i$, be the unital $*$ - representations supplied by the analogue of the Gelfand - Neumark - Segal construction performed in [9], which is respectively applied to the projections of norm one $\varphi_i^{(\alpha_i)}$, $\alpha_i \in \Gamma_i$, of the family $\Phi_i$, taking the same Hilbert (right) B-module $H_i = B \oplus H_i^\circ$ and the same vector $\xi_i = 1_B \oplus 0 \in B \oplus H_i^\circ$ for each $\alpha_i \in \Gamma_i$.

One can proceed in this manner by recourse to 2.2 -2.3. Thus, one obtaines the unital $*$ - homomorphisms $\chi_i : B \longrightarrow L(H_i)$, $\chi_i^\circ : B \longrightarrow L(H_i^\circ)$ so that $\chi_i(b')(b \oplus h) = b'b \oplus \chi_i^\circ(b')h$. One can observe that $\pi_i^{(\alpha_i)}(b) \equiv b$ for $b \in B$.

The Hilbert B-module free product $\{H, \xi\}$ of the family $\{H_i, \xi_i\}_{i\in I}$ is defined by

$$H = B \oplus \bigoplus_{n\geq 1} \bigoplus_{(i_1, \dots, i_n)\in D_n(I)} H_{i_1}^\circ \otimes_B \cdots \otimes_B H_{i_n}^\circ = B \oplus H^\circ$$

$$\xi = 1_B \oplus 0 \in B \oplus H^\circ$$

Considering $H_{(i)} := B \oplus \bigoplus_{n\geq 1} \bigoplus_{\substack{(i_1, \dots, i_n)\in D_n(I) \\ i_1 \neq i}} H_{i_1}^\circ \otimes_B \cdots \otimes_B H_{i_n}^\circ \subset H$, then the isomorphism $H \simeq H_i \otimes_B H_{(i)}$ is given by $V_i : H \longrightarrow H_i \otimes_B H_{(i)}$ such that

$$V_i \xi = \xi_i \otimes \xi ,$$

$$V_i(h_1^\circ \otimes \dots \otimes h_n^\circ) = \begin{cases} h_1^\circ \otimes (h_2^\circ \otimes \dots \otimes h_n^\circ) & \text{if } i_1 = i \text{ and } n \geq 2 \\ h_1^\circ \otimes \xi & \text{if } i_1 = i \text{ and } n = 1 \\ \xi_i \otimes (h_1^\circ \otimes \dots h_n^\circ) & \text{if } i_1 \neq i \end{cases}$$

for all $n \geq 1$, $(i_1, \dots, i_n) \in D_n(I)$, $h_j^\circ \in H_{i_j}^\circ$ $(1 \leq j \leq n)$.

Thus one defines the unital $*$-homomorphism $\lambda_i : L(H_i) \longrightarrow L(H)$ by $\lambda_i(T) := V_i^{-1}(T \otimes id)V_i$ .

One can also consider the following orthogonal decomposition of H

$$H = H_i \oplus H_i^\perp$$

and one can also observe that the submodules $H_i$ and $H_i^\perp$ of H are invariant under the action of $\lambda_i(T)$ for all $T \in L(H_i)$.

Let $\alpha \in \Gamma$ be fixed.

For all $n \geq 1$ and $t:=(i_1, \dots, i_n) \in D_n(I)$ one can also construct the unital $*$ - homomorphisms $\sigma^{(i_k,\alpha_{i_k})} : A_{i_k} \longrightarrow L(H)$, $1 \leq k \leq n$, given by

$$\sigma^{(i_k,\alpha_{i_k})}(a) = \begin{bmatrix} \lambda_{i_k} \pi_{i_k}^{(\alpha_{i_k})}(a) & 0 \\ 0 & \lambda_{i_k} \pi_{i_k}^{(\gamma_k^{(n)}(t,\alpha_t))}(a) \end{bmatrix}.$$

The $*$-homomorphisms $\sigma^{(i_k,\alpha_{i_k})} : A_{i_k} \longrightarrow L(H)$ , $1 \leq k \leq n$, have the following properties:

$1^\circ$ $\sigma^{(i_k,\alpha_{i_k})}(a)\xi \equiv \xi \cdot \varphi_{i_k}^{(\alpha_{i_k})}(a) \pmod{H_{i_k}^\circ}$ for $a \in A_{i_k}$ ;

$2^\circ$ $\sigma^{(i_1,\alpha_{i_1})}(a_1) \dots \sigma^{(i_{n-1},\alpha_{i_{n-1}})}(a_{n-1}) H_{i_n}^\circ \subset H_{i_1}^\circ \otimes_B \dots \otimes_B H_{i_{n-1}}^\circ \otimes_B H_{i_n}^\circ$ ,
if $a_k \in \operatorname{Ker} \varphi_{i_k}^{(\gamma_k^{(n)}(t,\alpha_t))}$ , $(1 \leq k \leq n-1)$, for all $n \geq 2$, $t = (i_1, \dots, i_n) \in D_n(I)$, where $\alpha_t := (\alpha_{i_1}, \dots, \alpha_{i_n})$ ;

$3^\circ$ $\langle \xi, \sigma^{(i_1,\alpha_{i_1})}(a_1) \dots \sigma^{(i_n,\alpha_{i_n})}(a_n)\xi \rangle = \varphi_{i_1}^{(\alpha_{i_1})}(a_1) \dots \varphi_{i_n}^{(\alpha_{i_n})}(a_n)$ ,
if $a_k \in \operatorname{Ker} \varphi_{i_k}^{(\gamma_k^{(n)}(t,\alpha_t))}$ $(1 \leq k \leq n)$, for all $n \geq 1$, $t = (i_1, \dots, i_n) \in D_n(I)$, where $\alpha_t := (\alpha_{i_1}, \dots, \alpha_{i_n})$ ;

$4^\circ$ $\sigma^{(i_k,\alpha_{i_k})}(b) \equiv b$ for $b \in B$.

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Therefore, one can obtain an unital $*$ - homomorphism family $\sigma^{(\alpha)}: A \longrightarrow L(H)$, $\alpha \in \Gamma$ and so on... ; moreover $\sigma^{(\alpha)}(b) = b$ for alll $b \in B$.

Denoting by $\omega_\xi : L(H) \longrightarrow B$ $\omega_\xi(T) := \langle \xi, T\xi \rangle$ , one can observe that $\omega_\xi$ is a positivity preserving projection of the $C^*$- algebra L (H) onto its $C^*$ -subalgebra B.

Thus, $\varphi^{(\alpha)} := \omega_\xi \circ \sigma^{(\alpha)}$ is also a positivity preserving projection of A onto B and moreover $\varphi^{(\alpha)}(ab) = \varphi^{(\alpha)}(a) \cdot b$ , for all $a \in A$, $b \in B$ . So that $\varphi^{(\alpha)}, \alpha \in \Gamma$ , are conditional expectations and satisfy the conditions (2) and (3).

The $*$ -representations of the free product A of the $*$ - algebras $A_i$, $i \in I$, are also in one-to-one correspondence with families of $*$ - representations of $A_i$ , $i \in I$, which act on the same Hilbert B-module by bounded operators.

A couple $\{A, \Phi\}$ given by the Theorem 2.1 can be called a <u>reduced free product with amalgamation</u> of $\{A_i, \Phi_i\}_{i \in I}$ via $\gamma$ and (3) of Theorem 2.1 the <u>free independence property with amalgamation on A via $\gamma$</u>.

<u>Remark 2.4.</u> One can also obtain the existence of another family $\Phi^{\#} = \{\varphi^{\#(\alpha)} ; \alpha \in \Gamma\}$ of conditional expectations of A onto B such that

(2') $\varphi^{\#(\alpha)} \circ j_i = \varphi_i^{(\gamma^{(1)}(i,\alpha_i))}$ for each $i \in I$ and $\alpha \in \Gamma$ ;

(3') $\varphi^{\#(\alpha)}(j_{i_1}(a_1) \cdots j_{i_n}(a_n)) = \varphi_{i_1}^{(\gamma_1^{(n)}(t,\alpha_t))}(a_1) \cdots \varphi_{i_n}^{(\gamma_n^{(n)}(t,\alpha_t))}(a_n)$ ,

if $a_j \in \operatorname{Ker} \varphi_{i_j}^{(\alpha_{i_j})}$ , $1 \leq j \leq n$, for all $n \geq 1$, $t := (i_1, \dots i_n) \in D_n(I)$, $\alpha_t := (\alpha_{i_1}, \dots, \alpha_{i_n})$ ; for each $\alpha \in \Gamma$ .

One can interpret such a triple $\{A, \Phi, \Phi^{\#}\}$ like a <u>symmetrized free product with amalgamation</u> of $\{A_i, \Phi_i\}_{i \in I}$ via $\gamma$, and the last assertions (3) and (3') like the <u>symmetrical free independence property with amalgamation on $A$ via $\gamma$</u>.

<u>Comment</u>. This study will be continued elsewhere. For example, as a fist step, one can establish a limit theorem, one can define the white noises and so on, generalizing the results of[3] .

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