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Extending Factorizations and Minimal Negative Signatures

Tiberiu Constantinescu and Aurelian Gheondea

Abstract. We formulate a problem of extending factorizations of type $X^\sharp X$ in Kreĭn spaces, with control on the negative signatures, and the minimal negative signatures for this problem are computed. As application we determine the minimal negative signatures of an operatorial one-step completion and the minimal negative signatures of defect for a problem of lifting operators in Kreĭn spaces.

1. Introduction

There exists a widely known method in the theory of moment problems which uses a simple framework from operator theory. This method was initiated and developed by M. A. Naĭmark [34] and M. G. Kreĭn [28] and refers to the search of unitary extensions of a given partial isometry, that go beyond the space where the partial isometry acts. This method turned out to be useful for many problems. For instance, the Nevanlinna-Pick problem can be solved in this way (B. Sz. -Nagy and Koranyi [39]), the Hamburger moment problem and the Nehari problem fit well in this approach (see D. Sarason [37]). Also, as shown by R. Arocena [4], the more general abstract problem of lifting of commutants of Sarason, Sz.-Nagy and Foiaş [36] and [38] (see also [27]), can be embedded into this framework and, finally, let us mention that a recent method of M. Cotlar and C. Sadosky [22] for solving moment problems can be viewed as a case of the considered extension problem.

Quite recently, there appeared tentatives for obtaining other variants to solve some other specific completion or moment problems. We mention here three directions. First, in order to solve Nevanlinna-Pick or Nehari problems for meromorphic functions instead of analytic functions, extensions of isometries in spaces with indefinite metrics were considered by V. M. Adamyan, D. Z. Arov and M. G. Kreĭn [1], T. Ya. Azizov [7], J. W. Ball and W. J. Helton [9], [10], M. G. Kreĭn and H. Langer [30], [31], [32], D. Aplay, P. Bruinsma, A. Dijksma and H. S. V.de Snoo [3]. Second, for solving some bidimensional completion problems, a problem of extending pairs of partial isometries was considered by R. Arocena and F. Montans [5] and, third, in order to solve problems as those in the papers of H. Dym and I. Gohberg [26] or J. W. Ball and I. Gohberg [10], a nonstationary variant of the extension of partial isometries was considered in [15] (see also [16]). The interpolation problems or moment problems that we referred to before are mainly concerning Hilbert spaces or Pontryagin spaces.

An important challange in these topics appeared in 1986 when L. de Branges asked the question of adapting the commutant lifting theorem to contractions in Kreĭn spaces. The first answer was given by the authors in [17] in the framework of Pontryagin spaces and, for different situations involving Kreĭn spaces, it was given by M. A. Dritschel [24], M. A. Dritschel and J. Rovnyak [25] (see also the papers [2], [13], [14], and the authors' papers [18], [19] where the more general case of nontrivial negative signatures of defect is considered).

In [18] and [19] we have considered a problem of extending operators in Kreĭn spaces with control of the negative signatures of defect. This problem, denoted here by $E(T_r, T_c; \kappa_1, \kappa_2)$ (see Section 5) is a core of the variants of commutant lifting with control of the negative signatures of defect. In [20], as a consequence of solving a completion problem. denoted in this paper by $\mathbf{C}(K; \kappa_1, \kappa_2)$ (see Section 4), the extension problem is solved completely in the case of finite dimension.

The purpose of this paper is to describe another variant of the method of Naĭmark and Kreĭn in connection with the determination of minimal negative signatures of the extension problem. In order to follow this method we extend factorizations instead of partial isometries, because of the different behaviour of the factorizations of the type $X^\sharp X$ in Kreĭn spaces and, respectively, Hilbert spaces.

The extending factorizations problem, denoted by $\mathbf{EF}(X, Y; \kappa_1, \kappa_2)$ is considered in Section 3. Here the main result is Theorem 3.4 which gives explicit formulae for the minimal negative signature. Using the remark in [15] , we are led to consider the problem $\mathbf{C}(K; \kappa_1, \kappa_2)$ as a problem of extending factorizations (see Proposition 4.6 and Proposition 4.10).

One of the basic tools used in this paper is the Kreĭn space induced by selfadjoint operators. This led also to the investigations of the relations between the Kreĭn spaces induced by a selfadjoint operator and a selfadjoint extension of it. This is done in the first part of Section 4.

The main result in the last section is Theorem 5.5 which gives formulae for the minimal negative signatures of the modified extension problem $E_m(T_r, T_c; \kappa_1, \kappa_2)$. This problem allows Kreĭn spaces with infinite signatures. In case that only Pontryagin spaces are considered, using a slightly different approach, the same formulae can be obtained for the problem $E(T_r, T_c; \kappa_1, \kappa_2)$.

2. Notation and Preliminary Results

The basic properties of Kreĭn spaces and their linear operators that we use in this paper are contained in the monographs [8] and [12]. In this section we fix the notation and recall some results which will be frequently used in this paper.

2.1 Geometry in Kreĭn spaces. If $\mathcal{K}$ is a Kreĭn space then its inner product is usually denoted by $[\cdot,\cdot]$. For a fundamental symmetry (in brief f.s.) of $\mathcal{K}$ we denote by $(\cdot,\cdot)_J$ the corresponding positive definite inner product. Also $\kappa^-[\mathcal{K}]$ and $\kappa^+[\mathcal{K}]$ denote the *negative signature* and, respectively, the *positive signature* of $\mathcal{K}$.

More general, if $\mathcal{L}$ is a subspace of $\mathcal{K}$, denote by $\kappa^-[\mathcal{L}]$, $\kappa^+[\mathcal{L}]$, and $\kappa^0[\mathcal{L}]$ its signatures. $\mathcal{L}^\perp$ stands for the *orthogonal companion* of $\mathcal{L}$ and $\mathcal{L}^0 = \mathcal{L}\cap\mathcal{L}^\perp$ stands for its *isotropic part.* We have $\kappa^0[\mathcal{L}] = \dim[\mathcal{L}^0]$ and $\kappa^\pm[\mathcal{L}^\perp]$ are also called the *cosignatures* of $\mathcal{L}$.

If $\mathcal{H}$ is a Hilbert space then we denote by $[\mathcal{H}\oplus\mathcal{H}]$ the Kreĭn space obtained from the Hilbert space $\mathcal{H}\oplus\mathcal{H}$ with the f.s. J defined by

$$J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{2.1}$$

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Kreĭn spaces. We denote by $\mathcal{K}_1[+]\mathcal{K}_2$ the Kreĭn space direct sum of $\mathcal{K}_1$ and $\mathcal{K}_2$.

A subspace $\mathcal{L}$ of the Kreĭn space $\mathcal{K}$ is called *regular* if $\mathcal{K} = \mathcal{L} + \mathcal{L}^\perp$. In this case we usually write $\mathcal{K} = \mathcal{L}[+]\mathcal{L}^\perp$.

Let $T \in \mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$, where $\mathcal{K}_1$ and $\mathcal{K}_2$ are Kreĭn spaces. Then $T^\sharp$ denotes the *adjoint* of T. If J_1 and J_2 are fixed f.s. of $\mathcal{K}_1$ and $\mathcal{K}_2$ we denote by T^* the adjoint of T with respect to the Hilbert spaces $(\mathcal{K}_1,(\cdot,\cdot)_{J_1})$ and $(\mathcal{K}_2,(\cdot,\cdot)_{J_2})$.

2.2 The Kreĭn space $\mathcal{H}_A$. Let $\mathcal{K}$ be a Kreĭn space and $A \in \mathcal{L}(\mathcal{K})$ be selfadjoint, i.e. $A = A^\sharp$. If J is a f.s. of $\mathcal{K}$ then JA is a selfadjoint operator on the Hilbert space $(\mathcal{K},(\cdot,\cdot)_J)$, hence we can consider its polar decomposition

$$JA = S_{JA}|JA|, \tag{2.2}$$

where $S_{JA} = \operatorname{sgn}(JA)$ is a selfadjoint partial isometry such that $\ker S_{JA} = \ker A$. Then S_{JA} is a symmetry on the Hilbert space $(\overline{\mathcal{R}(JA)},(\cdot,\cdot)_J)$. Denote by $\mathcal{H}_A$ the Kreĭn space $(\overline{\mathcal{R}(JA)},[\cdot,\cdot])$ where the inner product $[\cdot,\cdot]$ is induced by the symmetry S_{JA} as follows:

$$[x,y] = (S_{JA}x,y)_J, \quad x,y\in\mathcal{H}_A. \tag{2.3}$$

Let us remark that the linear manifolds $\mathcal{R}(|JA|)$ and $\mathcal{R}(|JA|^{\frac{1}{2}})$ are dense in $\mathcal{H}_A$ and that the strong topology on the Kreĭn space $\mathcal{H}_A$ is inherited from the strong topology of the original Kreĭn space $\mathcal{K}$. Denote by $\varepsilon_A : \mathcal{K}\to\mathcal{H}_A$ the quotient mapping. Then $\varepsilon_A \in \mathcal{L}(\mathcal{K},\mathcal{H}_A)$ and we have

$$\varepsilon_A^\sharp\varepsilon_A = JS_{JA}. \tag{2.4}$$

The definition of the Kreĭn space $\mathcal{H}_A$ does not depend on the f.s. J in the sense that if a different f.s. is used, the two Kreĭn spaces obtained in this way are unitary equivalent.

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Kreĭn spaces and $T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. Fix f.s. J_1 and J_2 on $\mathcal{K}_1$ and respectively, $\mathcal{K}_2$. Define the operators

$$J_T = \mathrm{sgn}\,(J_1 - T^*J_2T) \quad , \quad J_{T^*} = \mathrm{sgn}\,(J_2 - TJ_1T^*) \tag{2.5}$$

$$D_T = |J_1 - T^*J_2T|^{\frac{1}{2}} \quad , \quad D_{T^*} = |J_2 - TJ_1T^*|^{\frac{1}{2}}, \tag{2.6}$$

and using these elements define the space $\mathcal{D}_T = \overline{\mathcal{R}(D_T)}$ considered as a Kreĭn space with the f.s. J_T and, similarly, define the Kreĭn space $\mathcal{D}_{T^*} = \overline{\mathcal{R}(D_{T^*})}$ with the f.s. J_{T^*}. The Kreĭn spaces $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ are called the *defect spaces* of T and clearly

$$\mathcal{D}_T = \mathcal{H}_{I-T^\sharp T} \, , \quad \mathcal{D}_{T^*} = \mathcal{H}_{I-TT^\sharp}. \tag{2.7}$$

2.3 The Kreĭn space $\mathcal{K}_A$. Let $\mathcal{K}$ be a Kreĭn space and $A \in \mathcal{L}(\mathcal{K})$, $A = A^\sharp$. Define the inner product $[\cdot,\cdot]$ on $\mathcal{K}$,

$$[x,y]_A = [Ax,y]\,, \quad x,y \in \mathcal{K} \tag{2.8}$$

where $[\cdot,\cdot]$ denotes the inner product of the Kreĭn space $\mathcal{K}$.

Notice that $\ker A$ is the isotropic subspace of the inner product space $(\mathcal{K},[\cdot,\cdot]_A)$. Fix J a f.s. of $\mathcal{K}$ and denote $\hat{\mathcal{K}} = J(\ker A)^\perp$. Then consider the Jordan decomposition of the selfadjoint operator JA with respect to the Hilbert space $(\mathcal{K},(\cdot,\cdot)_J)$,

$$JA = (JA)_+ - (JA)_-, \tag{2.9}$$

and denote $\hat{\mathcal{K}}_+ = \overline{(JA)_+\mathcal{K}}$ and $\hat{\mathcal{K}}_- = \overline{(JA)_-\mathcal{K}}$. Then we have

$$\hat{\mathcal{K}} = \hat{\mathcal{K}}_+ + \hat{\mathcal{K}}_-.$$

Notice that $(\hat{\mathcal{K}}_+,[\cdot,\cdot]_A)$ and $(\hat{\mathcal{K}}_-,-[\cdot,\cdot]_A)$ are pre-Hilbert spaces and denote by $\mathcal{K}_A^+$ and $\mathcal{K}_A^-$ their completions to Hilbert spaces. Define

$$\mathcal{K}_A = \mathcal{K}_A^+[+]\mathcal{K}_A^-, \tag{2.10}$$

where the inner product is the extension by continuity of the inner product $[\cdot,\cdot]_A$. Then $(\mathcal{K}_A,[\cdot,\cdot]_A)$ is a Kreĭn space and (2.10) is a fundamental decomposition of $\mathcal{K}_A$. Let π_A denote the quotient mapping $\mathcal{K} \to \hat{\mathcal{K}}$ composed with the embeding of $\hat{\mathcal{K}}$ into $\mathcal{K}_A$. Then $\pi_A \in \mathcal{L}(\mathcal{K},\mathcal{K}_A)$ and

$$\pi_A^\sharp \pi_A = A. \tag{2.11}$$

The next result is a direct consequence of the definitions.

Lemma 2.1 *If $\mathcal{K}$ is a Kreĭn space, $A \in \mathcal{L}(\mathcal{K})$ is a selfadjoint operator and J is a f.s. of $\mathcal{K}$ used in the definition of the Kreĭn spaces $\mathcal{H}_A$ and $\mathcal{K}_A$, then the linear operator*

$$\mathcal{K}_A \supseteq \hat{\mathcal{K}} \ni x \to |JA|^{\frac{1}{2}} x \in \mathcal{R}(|JA|^{\frac{1}{2}}) \subseteq \mathcal{H}_A \tag{2.12}$$

extends uniquely to a unitary operator $\mathcal{K}_A \to \mathcal{H}_A$. In addition

$$|JA|^{\frac{1}{2}} \pi_A = \varepsilon_A |JA|^{\frac{1}{2}}. \tag{2.13}$$

The definition of the Kreĭn space $\mathcal{K}_A$ is independent on the f.s. J, modulo unitary equivalence (see [21]).

2.4 Operator signatures. Let $\mathcal{K}$ be a Kreĭn space and $A \in \mathcal{L}(\mathcal{K})$ be a selfadjoint operator. The *signatures* of A are, by definition, the cardinal numbers

$$\kappa^{\pm}(A) = \kappa^{\pm}[\mathcal{K}_A], \quad \kappa^0(A) = \dim \ker(A). \tag{2.14}$$

As a consequence of Lemma 2.1 it follows

$$\kappa^{\pm}(A) = \kappa^{\pm}(\mathcal{H}_A). \tag{2.15}$$

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Kreĭn spaces and consider $T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. The cardinal numbers $\kappa^{\pm}(I - T^{\sharp}T)$, $\kappa^{\pm}(I - TT^{\sharp})$, $\kappa^0(I - T^{\sharp}T)$ and $\kappa^0(I - TT^{\sharp})$ are called the *signatures of defect* of T. These signatures verify the following equalities (see [21])

$$\kappa^{\pm}(I - T^{\sharp}T) + \kappa^{\pm}[\mathcal{K}_2] = \kappa^{\pm}(I - TT^{\sharp}) + \kappa^{\pm}[\mathcal{K}_1], \tag{2.16}$$

$$\kappa^0(I - T^{\sharp}T) = \kappa^0(I - TT^{\sharp}). \tag{2.17}$$

2.5 Some spectral properties. Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{L}(\mathcal{H}), A = A^*$. A real number t is *isolated on the left (on the right)* with respect to the spectrum of A, here denoted by $\sigma(A)$, if there exists $\varepsilon > 0$ such that $(t - \varepsilon, t) \cap \sigma(A) = \emptyset$ (respectively, $(t, t + \varepsilon) \cap \sigma(A) = \emptyset$).

Let now $\mathcal{K}$ be a Kreĭn space and $A \in \mathcal{L}(\mathcal{K}), A = A^{\sharp}$. If for some f.s. J of $\mathcal{K}$, 0 is isolated on the left (on the right) with respect to $\sigma(JA)$ then the same is true for any other f.s. of $\mathcal{K}$.

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Kreĭn spaces and $T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. With respect to fixed f.s. J_1 and J_2 on $\mathcal{K}_1$ and, respectively, on $\mathcal{K}_2$, we introduce the spectral properties:

$(\alpha)_+$ *0 is isolated on the right with respect to $\sigma(J_1 - T^* J_2 T)$,*

$(\alpha)_-$ *0 is isolated on the left with respect to $\sigma(J_1 - T^* J_2 T)$.*

The properties $(\alpha)_+$ and $(\alpha)_-$ do not depend on the f.s. J_1 and J_2. Moreover, if T has the property $(\alpha)_+$ or $(\alpha)_-$ then $T^{\sharp}$ shares the same property (see [21]).

2.6 Indefinite factorizations. Let $A \in \mathcal{L}(\mathcal{K}_1), A = A^\sharp$ and $C \in \mathcal{L}(\mathcal{K}_2), C = C^\sharp$ be given. We are interested in factorizations of the type

$$A = B^\sharp C B, \tag{2.18}$$

where $B \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$, Under certain conditions, this kind of factorizations produce unitary operators acting between the Kreĭn spaces induced by A and C. Here we recall two criteria of different type. The first one is a consequence of a well-known extension lemma [29], [33] , [35], [23].

Lemma 2.2 *Let $B \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ be surjective and satisfy (2.18). Then:*

(i) *B induces a unitary operator in $\mathcal{L}(\mathcal{K}_A, \mathcal{K}_C)$.*

(ii) *If J_1 and J_2 are f.s. with respect to which $\mathcal{H}_A$ and $\mathcal{H}_C$ are defined, there exists a uniquely determined unitary operator $V \in \mathcal{L}(\mathcal{H}_A, \mathcal{H}_C)$ such that*

$$V|J_1A|^{\frac{1}{2}} = |J_2C|^{\frac{1}{2}}B.$$

For the proof of the second criterion see [21].

Lemma 2.3 *Let $B \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ have dense range and satisfy the equality*

$$A = B^\sharp B.$$

Moreover, assume that for some (equivalently, for all) f.s. J_1 of $\mathcal{K}_1$, 0 is isolated either on the left or on the right with respect to $\sigma(J_1A)$. Then:

(i) *B induces a unitary operator on $\mathcal{L}(\mathcal{K}_A, \mathcal{K}_2)$.*

(ii) *If J_1 is a f.s. of $\mathcal{K}_1$ used in the definition of $\mathcal{H}_A$ then there exists a uniquely determined operator $V \in \mathcal{L}(\mathcal{H}_A, \mathcal{K}_2)$ such that*

$$V|J_1A|^{\frac{1}{2}} = B.$$

2.7 Elementary rotations. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Kreĭn spaces and $T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. An *elementary rotation* of T is a triplet $(U; \mathcal{K}_1', \mathcal{K}_2')$, where $\mathcal{K}_1'$ and $\mathcal{K}_2'$ are Kreĭn spaces, the operator $U \in \mathcal{L}(\mathcal{K}_1[+]\mathcal{K}_1', \mathcal{K}_2[+]\mathcal{K}_2')$ is unitary and extends T, i.e.

$$P_{\mathcal{K}_2}U|\mathcal{K}_1 = T,$$

and one of the following equivalent minimality conditions holds

$$\mathcal{K}_2 \vee U\mathcal{K}_1 = \mathcal{K}_2[+]\mathcal{K}_2', \quad \mathcal{K}_1 \vee U^\sharp\mathcal{K}_2 = \mathcal{K}_1[+]\mathcal{K}_1'.$$

For any operator $T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ there exists an elementary rotation (cf. [6]). Here we refer to a certain elementary rotation denoted by $(R(T); \mathcal{D}_{T^*}, \mathcal{D}_T)$, where

$$R(T) = \begin{bmatrix} T & D_{T^*} \\ D_T & -L_{T^*}J_{T^*} \end{bmatrix}, \tag{2.19}$$

$L_{T^*} \in \mathcal{L}(\mathcal{D}_{T^*}, \mathcal{D}_T)$ being a uniquely determined operator (see [6], [21]).

2.8 A unitary extension. Let $\mathcal{H}$ be a Hilbert space and consider the Kreĭn space $[\mathcal{H} \oplus \mathcal{H}]$ defined as in (2.1). Also, let $\mathcal{K}$ be a Kreĭn space and $T \in \mathcal{L}(\mathcal{H}, \mathcal{K}), \mathcal{L} = \mathcal{R}(T)$. In this paper we will use the following result (e.g. see [7]).

Lemma 2.4 *In order to exist a unitary extension $U \in \mathcal{L}([\mathcal{H} \oplus \mathcal{H}], \mathcal{K})$ of T it is necessary and sufficient that T be injective and $\mathcal{L}^{\perp} = \mathcal{L}$.*

3. The Problem of Extending Factorizations

The problem we are concerning with has the following statement:

EF$(X, Y; \kappa_1, \kappa_2)$:

There are given Kreĭn spaces $\mathcal{H}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ and operators $X \in \mathcal{L}(\mathcal{H}, \mathcal{G}_1), Y \in \mathcal{L}(\mathcal{H}, \mathcal{G}_2)$ such that
$$X^{\sharp}X = Y^{\sharp}Y = Z \in \mathcal{L}(\mathcal{H}).$$
Given cardinal numbers κ_1 and κ_2, it is required to determine a quintuple $(\hat{X}, \hat{Y}; \mathcal{G}_1', \mathcal{G}_2'; W)$ such that:

(*i*) $\mathcal{G}_1'$ and $\mathcal{G}_2'$ are Kreĭn spaces and
$$\kappa^-[\mathcal{G}_1'] = \kappa_1, \quad \kappa^-[\mathcal{G}_2'] = \kappa_2.$$

(*ii*) $\hat{X} \in \mathcal{L}(\mathcal{H}, \mathcal{G}[+]\mathcal{G}_1'$ is an extension of X and $\hat{Y} \in \mathcal{L}(\mathcal{H}, \mathcal{G}_2[+]\mathcal{G}_2'$ is an extension of Y, such that
$$\hat{X}^{\sharp}\hat{X} = \hat{Y}^{\sharp}\hat{Y} = Z.$$

(*iii*) $W \in \mathcal{L}(\mathcal{G}_1[+]\mathcal{G}_1', \mathcal{G}_2[+]\mathcal{G}_2')$ is unitary such that
$$W\hat{X} = \hat{Y}.$$

(*iv*) $\mathcal{G}_1 \vee W^{\sharp}\mathcal{G}_2 = \mathcal{G}_1[+]\mathcal{G}_1'$.

Before considering this extending factorization problem we need to recall a known result (e.g. see [18]).

Lemma 3.1 *Let $\mathcal{H}$ and $\mathcal{G}$ be Kreĭn spaces, $X \in \mathcal{L}(\mathcal{H}, \mathcal{G})$ and $Z \in \mathcal{L}(\mathcal{H}), Z = Z^{\sharp}$ be such that*

$$X^{\sharp}X = Z.$$

If J denotes a fixed f.s. on $\mathcal{H}$ then X is uniquely represented by

$$X = [V|JZ|^{\frac{1}{2}} \quad X_0] \tag{3.1}$$

where $V : \mathcal{R}(|JZ|^{\frac{1}{2}}) (\subseteq \mathcal{H}_Z) \to \mathcal{G}$ *is isometric such that* $V|JZ|^{\frac{1}{2}}$ *is bounded,* $X_0 \in \mathcal{L}(\ker Z, \mathcal{G})$ *is such that* $\mathcal{R}(X_0)$ *is neutral and included in* $\mathcal{R}(V)^{\perp}$. *In particular* $\mathcal{R}(X_0) = \mathcal{R}(X)^0$ *(the isotropic part of* $\mathcal{R}(X)$*).*

Let X be an operator as in Lemma 3.1. In the following it will be needed to consider a technical condition that we denote by (γ):

$$(\gamma)\left\{\begin{array}{l}\text{The operator } V \text{ from the representation (3.1) extends} \\ \text{(uniquely) to an isometry in } \mathcal{L}(\mathcal{H}_Z, \mathcal{G}).\end{array}\right.$$

Lemma 3.2 *Let* X *be as in Lemma 3.1. Then* X *has the property* (γ) *if and only if* $X|J(\ker Z)^{\perp}$ *extends (uniquely) to an isometry in* $\mathcal{L}(\mathcal{K}_Z, \mathcal{G})$. *In addition, if* X *has the property* (γ) *then* $\overline{\mathcal{R}(X)}$ *is a pseudo-regular subspace of* $\mathcal{G}$ *and* $\overline{\mathcal{R}(X)}^{0} = \overline{\mathcal{R}(X_0)}$.

Proof. The first part of the statement is a direct consequence of Lemma 2.1. For the second one, assuming that X has the property (γ), let V denote also the isometric extension to the whole Kreĭn space $\mathcal{H}_Z$ and $\mathcal{L} = V\mathcal{H}_Z$. Then $\mathcal{L}$ is regular subspace of $\mathcal{G}$ and, from Lemma 3.1, it follows that X has the representation

$$X = \begin{bmatrix} V|JZ|^{\frac{1}{2}} & 0 \\ 0 & X_0 \end{bmatrix} : \begin{array}{c} \ker Z \\ \oplus \\ J(\ker Z)^{\perp} \end{array} \to \begin{array}{c} \mathcal{L} \\ [+] \\ \mathcal{L}^{\perp} \end{array}$$

From here it follows immediately that

$$\overline{\mathcal{R}(X)} = \mathcal{L}[+]\overline{\mathcal{R}(X_0)} \tag{3.2}$$

then, this $\overline{\mathcal{R}(X_0)}$ is also a neutral subspace and it follows that $\overline{\mathcal{R}(X)}$ is pseudo-regular and its isotropic part is $\overline{\mathcal{R}(X_0)}$. ∎

We can now consider the problem $\mathbf{EF}(X, Y; \kappa_1, \kappa_2)$.

Lemma 3.3 *Assume that both of* X *and* Y *have the property* (γ). *If* $\mathbf{EF}(X, Y; \kappa_1, \kappa_2)$ *has solutions then the following identity holds:*

$$\kappa_1 + \operatorname{rank}(X|\ker Z) + \kappa^{-}[\mathcal{R}(X)^{\perp}] = \kappa_2 + \operatorname{rank}(Y|\ker Z) + \kappa^{-}[\mathcal{R}(Y)^{\perp}]. \tag{3.3}$$

Proof. Let $(\hat{X}, \hat{Y}; \mathcal{G}_1', \mathcal{G}_2'; W)$ be a solution of the problem $\mathbf{EF}(X, Y; \kappa_1, \kappa_2)$. Then $\hat{X}$ has the representation

$$\hat{X} = [X \quad X_1]^t : \mathcal{H} \to \mathcal{G}_1[+]\mathcal{G}_1' \tag{3.4}$$

such that

$$\hat{X}^{\sharp}\hat{X} = X^{\sharp}X + X_1^{\sharp}X_1 = Z = X^{\sharp}X,$$

hence $X_1^{\sharp}X_1 = 0$, i.e. $\mathcal{R}(X_1)$ is a neutral submanifold of $\mathcal{G}_1'$.

Consider now the representation (3.1) of X and, since X has the property (γ), we have the decomposition (3.2), where $\mathcal{L}$ is regular. Denoting

$$\hat{X}_0 = [X_0 \quad X_1]^t : \ker Z \to \mathcal{G}_1[+]\mathcal{G}_1' , \tag{3.5}$$

it follows that

$$\overline{\mathcal{R}(\hat{X})} = \mathcal{L}[+]\overline{\mathcal{R}(\hat{X}_0)}, \tag{3.6}$$

in particular the subspace $\overline{\mathcal{R}(\hat{X})}$ is pseudo-regular and $\overline{\mathcal{R}(\hat{X}_0)}$ is the isometric part of $\overline{\mathcal{R}(\hat{X})}$.

Similarly, $\hat{Y}$ has the representation

$$\hat{Y} = [Y \quad Y_1]^t : \mathcal{H} \to \mathcal{G}_2[+]\mathcal{G}_2' \tag{3.7}$$

such that Y_1 has neutral range and, taking into account that Y has the property (γ) and denoting

$$\hat{Y}_0 = [Y_0 \quad Y_1]^t : \ker Z \to \mathcal{G}_2[+]\mathcal{G}_2', \tag{3.8}$$

where $Y_0 = Y|\ker Z$, it follows that

$$\overline{\mathcal{R}(\hat{Y})} = \mathcal{S}[+]\overline{\mathcal{R}(\hat{Y}_0)} \tag{3.9}$$

in particular this means that $\overline{\mathcal{R}(\hat{Y})}$ is a pseudo-regular subspace and $\overline{\mathcal{R}(\hat{Y}_0)}$ is the isotropic part of $\overline{\mathcal{R}(\hat{Y})}$.

Taking into account the factorization relation

$$W\hat{X} = \hat{Y},$$

since W is unitary we obtain $W\overline{\mathcal{R}(\hat{X})} = \overline{\mathcal{R}(\hat{Y})}$ and then, using (3.6) and (3.9) it follows

$$\overline{\mathcal{R}(\hat{Y})} = W\mathcal{L}[+]\overline{\mathcal{R}(\hat{Y}_0)}. \tag{3.10}$$

The identity (3.3) is now a consequence of the identity

$$\kappa^-[\mathcal{L}^\perp] = \kappa^-[(W\mathcal{L})^\perp],$$

where the orthogonal complements are computed with respect to the Kreĭn spaces $\mathcal{G}_1[+]\mathcal{G}_1'$ and, respectively, $\mathcal{G}_2[+]\mathcal{G}_2'$.

Indeed, from (3.6) we obtain

$$\kappa^-[\mathcal{L}^\perp] = \kappa^-[\mathcal{G}_1'] + \operatorname{rank}(X|\ker Z) + \kappa^-[\mathcal{R}(X)^\perp]$$

(this time we consider $\mathcal{R}(X)$ as a subspace of $\mathcal{G}_1$), and similarly, from (3.10) we obtain

$$\kappa^-[(W\mathcal{L})^\perp] = \kappa^-[\mathcal{G}_2'] + \operatorname{rank}(Y|\ker Z) + \kappa^-[\mathcal{R}(Y)^\perp]$$

(viewing $\mathcal{R}(Y)$ as a subspace of $\mathcal{G}_2$). ∎

The main result concerning the problem $\mathbf{EF}(X, Y; \kappa_1, \kappa_2)$ is the computation of the minimal negative signatures κ_1 and κ_2.

Theorem 3.4 *Assume that X and Y satisfy the property* (γ) *and, in addition,*

$$\operatorname{rank}(Y|\ker Z) < \infty\ , \quad \operatorname{rank}(X|\ker Z) < \infty. \tag{3.11}$$

Then, the set of pairs (κ_1, κ_2) *for which the problem* $\mathbf{EF}(X, Y; \kappa_1, \kappa_2)$ *has solutions, has a minimum which is simultaneously attained and given by*

$$\kappa_1^{min} = \operatorname{rank}(Q(I-P)) + \max\{0, \kappa^-[\mathcal{R}(Y)^\perp] - \kappa^-[\mathcal{R}(X)^\perp]\}, \tag{3.12}$$

$$\kappa_2^{min} = \operatorname{rank}(P(I-Q)) + \max\{0, \kappa^-[\mathcal{R}(X)^\perp] - \kappa^-[\mathcal{R}(Y)^\perp]\}, \tag{3.13}$$

where, with respect to a fixed f.s. J on $\mathcal{H}$*, we denote* $P = P^{\mathcal{H}}_{\mathcal{R}(X_0^*)}, Q = P^{\mathcal{H}}_{\mathcal{R}(Y_0^*)}$ *and* $X_0 = X|\ker Z, Y_0 = Y|\ker Z$.

Proof. Let κ_1 and κ_2 be cardinal numbers for which the problem $\mathbf{EF}(X, Y; \kappa_1, \kappa_2)$ has solutions and let $(\hat{X}, \hat{Y}; \mathcal{G}_1', \mathcal{G}_2'; W)$ be a solution. Then we consider the representations (3.4) and (3.7) of $\hat{X}$ and $\hat{Y}$ and consider the operators $\hat{X}_0$ and $\hat{Y}_0$ with neutral ranges, introduced in (3.5) and (3.8). Restricting the operator identity $W\hat{X} = \hat{Y}$ to the subspace $\ker Z$ it follows

$$W\hat{X}_0 = \hat{Y}_0. \tag{3.14}$$

We fix now f.s. J, J_1, J_2, J_1', J_2' on $\mathcal{H}, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1'$ and, respectively, $\mathcal{G}_2'$ and consider the corresponding Hilbert spaces.

With respect to these Hilbert spaces, we obtain from (3.14) that

$$\mathcal{R}(\hat{X}_0^*) = \mathcal{R}(\hat{Y}_0^*). \tag{3.15}$$

Consider now the decomposition

$$\mathcal{R}(X_0^*) = \mathcal{R}(X_0^*) \cap \mathcal{R}(Y_0^*) \oplus \mathcal{R}(P(I-Q)), \tag{3.16}$$

and

$$\mathcal{R}(Y_0^*) = \mathcal{R}(X_0^*) \cap \mathcal{R}(Y_0^*) \oplus \mathcal{R}(Q(I-P)), \tag{3.17}$$

with the remark that the assumptions (3.11) imply that both of P and Q have finite ranks. Taking into account that

$$\mathcal{R}(P(I-Q)) \cap \mathcal{R}(Q(I-P)) = 0,$$

from (3.15) and (3.17) we obtain

$$\kappa_1 = \kappa^-[\mathcal{G}_1'] \geq \operatorname{rank}(X_1) \geq \operatorname{rank}(Q(I-P)) \tag{3.18}$$

and similarly, from (3.15) and (3.16) we obtain

$$\kappa_2 = \kappa^-[\mathcal{G}_2'] \geq \operatorname{rank}(Y_1) \geq \operatorname{rank}(P(I-Q)). \tag{3.19}$$

Let us remark now that from (3.16) and (3.17) it follows

$$\operatorname{rank}(X_0) = \operatorname{rank}(X_0^*) = \operatorname{rank}(P \wedge Q) + \operatorname{rank}(P(I-Q)) \tag{3.20}$$

and

$$\operatorname{rank}(Y_0) = \operatorname{rank}(Y_0^*) = \operatorname{rank}(P \wedge Q) + \operatorname{rank}(Q(I-P)), \tag{3.21}$$

where $P \wedge Q$ denotes the orthogonal projection onto $\mathcal{R}(X_0^*) \cap \mathcal{R}(Y_0^*)$. Using (3.20) and (3.21), from Lemma 3.3 we obtain

$$\kappa_1 + \operatorname{rank}(P(I-Q)) + \kappa^-[\mathcal{R}(X)^\perp] = \kappa_2 + \operatorname{rank}(Q(I-P)) + \kappa^-[\mathcal{R}(Y)^\perp] \tag{3.22}$$

and finally, from here, (3.18) and (3.19) we obtain $\kappa_1 \geq \kappa_1^{min}$ and $\kappa_2 \geq \kappa_2^{min}$, where κ_1^{min} and κ_2^{min} are introduced at (3.12) and (3.13).

Conversely, let κ_1^{min} and κ_2^{min} be given by the formulae in (3.12) and (3.13). We will construct in the following a solution of the problem $\mathbf{EF}(X, Y; \kappa_1^{min}, \kappa_2^{min})$.

Let us first notice that the following decomposition holds,

$$\mathcal{G}_1 = \mathcal{L}[+](\mathcal{R}(X_0) \oplus J_1\mathcal{R}(X_0))[+]\mathcal{L}',$$

where $\mathcal{L} = V\mathcal{H}_Z$ is a regular subspace of $\mathcal{G}_1$, and

$$\mathcal{R}(X)^\perp = \mathcal{R}(X_0)[+]\mathcal{L}'$$

and, similarly,

$$\mathcal{G}_2 = \mathcal{S}[+](\mathcal{R}(Y_0) \oplus J_2\mathcal{R}(Y_0))[+]\mathcal{S}',$$

where $\mathcal{S} = U\mathcal{H}_Z$ is a regular subspace of $\mathcal{G}_2$ (U corresponds to V in the decomposition (3.1) of Y) and

$$\mathcal{R}(Y)^\perp = \mathcal{R}(Y_0)[+]\mathcal{S}'.$$

We consider now $\mathcal{H}_1$ a copy of the Hilbert space $\mathcal{R}(Q(I-P))$ and take X_1 the natural embedding onto the first component

$$X_1 : \mathcal{R}(Q(I-P)) \to [\mathcal{H}_1 \oplus \mathcal{H}_1]$$

Then extend X_1 trivially onto the whole $\ker Z$ and define

$$\hat{X}_0 = [X_0 \quad X_1]^t : \ker Z \to \mathcal{G}_1[+][\mathcal{H}_1 \oplus \mathcal{H}_1].$$

Similarly, consider $\mathcal{H}_2$ a copy of the Hilbert space $\mathcal{R}(P(I-Q))$ and take Y_1 the natural embedding onto the first component

$$Y_1 : \mathcal{R}(P(I-Q)) \to [\mathcal{H}_1 \oplus \mathcal{H}_2],$$

then extend Y_1 trivially onto the whole ker Z and define

$$\hat{Y}_0 = [Y_0 \quad Y_1]^t : \ker Z \to \mathcal{G}_2[+][\mathcal{H}_2 \oplus \mathcal{H}_2].$$

With these definitions it is now easy to check that

$$\mathcal{R}(\hat{X}_0^*) = \mathcal{R}(\hat{Y}_0^*),$$

hence there exists a (uniquely determined) invertible operator $T : \mathcal{R}(\hat{X}_0) \to \mathcal{R}(\hat{Y}_0)$ such that

$$T\hat{X}_0 = \hat{Y}_0.$$

Using Lemma 2.4 we extend T to a unitary operator

$$W : (\mathcal{R}(X_0) \oplus J_1\mathcal{R}(X_0))[+][\mathcal{H}_1 \oplus \mathcal{H}_1] \to (\mathcal{R}(Y_0) \oplus J_2\mathcal{R}(Y_0))[+][\mathcal{H}_2 \oplus \mathcal{H}_2],$$

in particular we also have

$$W\hat{X}_0 = \hat{Y}_0.$$

We define $\hat{X} \in \mathcal{L}(\mathcal{H}, \mathcal{G}_1[+][\mathcal{H}_1 \oplus \mathcal{H}_1])$ and $\hat{Y} \in \mathcal{L}(\mathcal{H}, \mathcal{G}_2[+][\mathcal{H}_2 \oplus \mathcal{H}_2])$ by

$$\hat{X} = [V|JZ|^{\frac{1}{2}} \quad \hat{X}_0]$$

and

$$\hat{Y} = [U|JZ|^{\frac{1}{2}} \quad \hat{Y}_0].$$

Extending the unitary operator W such that

$$W|\mathcal{L} = UV^{-1} : \mathcal{L} \to \mathcal{S}$$

we have

$$W\hat{X} = \hat{Y}. \tag{3.23}$$

Finally, let $S \in \mathcal{L}(\mathcal{L}', \mathcal{S}')$ be such that

$$\kappa^-[I - SS^\sharp] = \max\{0, \kappa^-[\mathcal{R}(Y)^\perp] - \kappa^-[\mathcal{R}(X)^\perp]\}$$

$$\kappa^-[I - S^\sharp S] = \max\{0, \kappa^-[\mathcal{R}(X)^\perp] - \kappa^-[\mathcal{R}(Y)^\perp]\},$$

extend W with an elementary rotation of S, say it $(R(S), \mathcal{D}_{S^*}, \mathcal{D}_S)$ (see (2.19)) and notice that (3.23) still holds. Denoting

$$\mathcal{G}_1' = [\mathcal{H}_1 \oplus \mathcal{H}_1][+]\mathcal{D}_{S^*}$$

$$\mathcal{G}_2' = [\mathcal{H}_2 \oplus \mathcal{H}_2][+]\mathcal{D}_S,$$

it follows that

$$\kappa^-[\mathcal{G}_1'] = \kappa_1^{min} \quad , \kappa^-[\mathcal{G}_2' = \kappa_2^{min}.$$

Then $(\hat{X}, \hat{Y}; \mathcal{G}_1', \mathcal{G}_2'; W)$ is a solution of the problem $\mathbf{EF}(X, Y; \kappa_1^{min}, \kappa_2^{min})$. ∎

An important particular case of the problem $\mathbf{EF}(X, Y; \kappa_1, \kappa_2)$ is for $\kappa_1 = \kappa_2 = 0$; in this situation we write simply $\mathbf{EF}(X, Y)$. Before specializing Theorem 3.4 to this case it is worth noticing that we can state $\mathbf{EF}(X, Y)$ in the following equivalent form:

$\mathbf{EF}(X, Y)$ {

There are given Kreĭn spaces $\mathcal{H}, \mathcal{G}_1$, and $\mathcal{G}_2$ and operators $X \in \mathcal{L}(\mathcal{H}, \mathcal{G}_1)$, $Y \in \mathcal{L}(\mathcal{H}, \mathcal{G}_2)$ such that:

$$X^\sharp X = Y^\sharp Y = Z \in \mathcal{L}(\mathcal{H}).$$

It is required to determine a triplet $(\mathcal{G}_1', \mathcal{G}_2'; W)$ such that :

(i) $\mathcal{G}_1'$ and $\mathcal{G}_2'$ are Hilbert spaces.

(ii) $W : \mathcal{G}_1[+]\mathcal{G}_1' \to \mathcal{G}_2[+]\mathcal{G}_2'$ is a unitary operator such that

$$WX = Y.$$

(iii) $\mathcal{G} \vee W^\sharp \mathcal{G}_2 = \mathcal{G}_1[+]\mathcal{G}_1'$.

Indeed, this follows from the remark that, since $\mathcal{G}_1'$ and $\mathcal{G}_2'$ are Hilbert spaces, in this case the extended operators $\hat{X}$ and $\hat{Y}$ are trivial extensions of X and, respectively, Y.

Corollary 3.5 *Assume that X and Y satisfy the property (γ) and also that the conditions in (3.11) hold. Then, the problem* $\mathbf{EF}(X, Y)$ *has solutions if and only if $\mathcal{R}(X_0^*) = \mathcal{R}(Y_0^*)$ and $\kappa^-[\mathcal{R}(X)^\perp] = \kappa^-[\mathcal{R}(Y)^\perp]$, where $X_0 = X|\ker Z$, $Y_0 = Y|\ker Z$ and the $*$ operation has to be understood with respect to a fixed f.s. J on $\mathcal{H}$.*

Proof. Indeed, using Theorem 3.4 it follows that $\mathbf{EF}(X, Y)$ has solutions if and only if $\kappa_1^{min} = \kappa_2^{min} = 0$, and using formulae (3.12) and (3.13) this means $Q(I - P) = P(I - Q) = 0$ and $\kappa^-[\mathcal{R}(X)^\perp] = \kappa^-[\mathcal{R}(Y)^\perp]$. It remains to notice that $Q(I - P) = P(I - Q) = 0$ if and only if $\mathcal{R}(X_0^*) = \mathcal{R}(Y_0^*)$. ∎

4. The Completion Problem

In this section we will consider a problem of selfadjoint completing of a certain type partial block-matrix. As a first task, we investigate the relations between the induced Kreĭn spaces of a selfadjoint operator and that of a compression to a regular subspace. Without restricting the generality, we can consider the spaces onto which this block-matrix acts to be Hilbert spaces, instead of Kreĭn spaces, since otherwise we can use fundamental symmetries.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and selfadjoint operator $H \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ given by the following block-matrix:

$$H = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}. \tag{4.1}$$

Consider a linear operator $\chi_{H,C} \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_H)$ defined by

$$\chi_{H,C} h = |H|^{\frac{1}{2}} h\ , \quad h \in \mathcal{H}_2. \tag{4.2}$$

Then, it is easy to show that

$$\chi^{\sharp}_{H,C}\chi_{H,C} = C \tag{4.3}$$

hence, using Lemma 3.1 with respect to the decomposition $\mathcal{H}_2 = (\mathcal{H}_2 \ominus \ker C) \oplus \ker C$ we have the representation

$$\chi_{H,C} = [V|C|^{\frac{1}{2}} \quad \chi_{H,C}|\ker C], \tag{4.4}$$

where $V : \mathcal{R}(|C|^{\frac{1}{2}})(\subseteq \mathcal{H}_H) \to \mathcal{H}_H$ is isometric.

By definition, we say that the Kreĭn space $\mathcal{H}_C$ is *canonically embedded* into the Kreĭn space $\mathcal{H}_H$ if the isometry V is bounded and, in this case $\mathcal{H}_C$ has to be identified with the regular subspace $V\mathcal{H}_C$ of $\mathcal{H}_H$.

Consider now the operator $\rho_{H,C} \in \mathcal{L}(\mathcal{H}_2, \mathcal{K}_H)$ defined by

$$\rho_{H,C} h = \pi_H h\ , \quad h \in \mathcal{H}_2. \tag{4.5}$$

Then we also have

$$\rho^{\sharp}_{H,C}\rho_{H,C} = C \tag{4.6}$$

and from here it follows that the linear operator $\rho_{H,C}|\mathcal{H}_2 \ominus \ker C : \mathcal{H}_2 \ominus \ker C (\subseteq \mathcal{K}_C) \to \mathcal{K}_H$ is isometric.

By definition, the Kreĭn space $\mathcal{K}_C$ is *canonically embedded* in $\mathcal{K}_H$ if the isometry $\rho_{H,C}|\mathcal{H}_2 \ominus \ker C$ is bounded, in this case the Kreĭn space $\mathcal{K}_C$ being identified with the regular subspace $\rho_{H,C}\mathcal{K}_C$ of $\mathcal{K}_H$.

Lemma 4.1 *$\mathcal{H}_C$ is canonically embedded in $\mathcal{H}_H$ if and only if $\mathcal{K}_C$ is canonically embedded in $\mathcal{K}_H$.*

Proof. From the definitions of $\rho_{H,C}$ and $\chi_{H,C}$ we have

$$V|C|^{\frac{1}{2}} h = |H|^{\frac{1}{2}} \rho_{H,C} h\ , \quad h \in \mathcal{H}_2 \ominus \ker C.$$

Taking into account that $|C|^{\frac{1}{2}} \in \mathcal{L}(\mathcal{K}_C, \mathcal{H}_C)$ and $|H|^{\frac{1}{2}} \in \mathcal{L}(\mathcal{K}_H, \mathcal{H}_H)$ are unitary operators, it follows that the isometric operators V and $\rho_{H,C}|\mathcal{H}_2 \ominus \ker C$ are simultaneously bounded. ∎

Also as a consequence of Lemma 3.1, $\mathcal{R}(\chi_{H,C}|\ker C)$ is the isotropic part of $\mathcal{R}(\chi_{H,C})$ and similarly, $\mathcal{R}(\rho_{H,C}|\ker C)$ is the isotropic part of $\mathcal{R}(\rho_{H,C})$. The dimensions of these isotropic parts can be computed in terms of the data of the selfadjoint block-matrix H.

Lemma 4.2 $\operatorname{rank}(\chi_{H,C}|\ker C) = \operatorname{rank}(B|\ker C) = \operatorname{rank}(\rho_{H,C}|\ker C)$.

Proof. Let us denote $B_2 = B|\ker C$. Then for $h \in \ker C, h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$ we have

$$[\chi_{H,C}h, |H|^{\frac{1}{2}}(h_1 \oplus h_2)]_{S_H} = (Hh, h_1 \oplus h_2) = (B_2 h, h_1 \oplus h_2) = (HP_{\overline{\mathcal{R}(B_2^*)}}h, h_1 \oplus h_2)$$

$$= [\chi_{H,C}P_{\overline{\mathcal{R}(B_2^*)}}h, |H|^{\frac{1}{2}}(h_1 \oplus h_2)]_{S_H}.$$

This implies the first equality. The second one follows in a similar way. ∎

We present now two criteria, of different character, which insure the existence of the canonical embedings. For this purpose it is convenient to introduce the notation

$$B_1 = B|\mathcal{H}_2 \ominus \ker C\,, \quad B_2 = B|\ker C. \tag{4.7}$$

Then, the selfadjoint operator H in (4.1) is represented by

$$H = \begin{bmatrix} A & B_1 & B_2 \\ B_1^* & C & 0 \\ B_2^* & 0 & 0 \end{bmatrix}. \tag{4.8}$$

Lemma 4.3 *Assume that $B_1 = \Gamma C$ for a certain $\Gamma \in \mathcal{L}(\mathcal{H}_2 \ominus \ker C, \mathcal{H}_1)$. Then $\mathcal{K}_C$ is canonically embedded in $\mathcal{K}_H$ and, in addition,*

$$\kappa^{\pm}[\mathcal{R}(\rho_{H,C})^{\perp}] = \kappa^{\pm}(P_{\ker B_2^*}(A - \Gamma C\Gamma^*)|\ker B_2^*).$$

Proof. If $B_1 = \Gamma C$ then using the representation (4.8) we obtain the factorization of H as

$$\begin{bmatrix} I & \Gamma & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A - \Gamma C\Gamma^* & 0 & B_2 \\ 0 & C & 0 \\ B_2^* & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ \Gamma^* & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Defining $R \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ by

$$R = \begin{bmatrix} I & 0 & 0 \\ \Gamma^* & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and using Lemma 2.2, it follows that R^{-1} induces a unitary operator

$$\mathcal{K}_C[+]\mathcal{K}_{\begin{bmatrix} A - \Gamma C\Gamma^* & B_2 \\ B_2^* & 0 \end{bmatrix}} \longrightarrow \mathcal{K}_H.$$

Since this unitary operator is an extension of $\rho_{H,C}|\mathcal{H}_2 \ominus \ker C$, it follows that $\mathcal{K}_C$ is canonically embedded in $\mathcal{K}_H$. Moreover,

$$\kappa^{\pm}[\mathcal{R}(\rho_{H,C})^{\perp}] = \kappa^{\pm}(\begin{bmatrix} A - \Gamma C \Gamma^* & B_2 \\ B_2^* & 0 \end{bmatrix})$$

$$= \kappa^{\pm}(P_{\ker B_2}(A - \Gamma C \Gamma^*)|\ker B_2),$$

and the proof is finished. ∎

Lemma 4.4 *Assume that* $B_1 = \Delta|C|^{\frac{1}{2}}$ *for a certain* $\Delta \in \mathcal{L}(\mathcal{H}_2 \ominus \ker C\mathcal{H}_1)$, *and, in addition, assume that 0 is isolated either on the left or on the right with respect to* $\sigma(H)$. *Then* $\mathcal{H}_C$ *is canonically embedded in* $\mathcal{H}_H$ *and*

$$\kappa^{\pm}[(\mathcal{R}_{H,C})^{\perp}] = \kappa^{\pm}(P_{\ker B_2^*}(A - \Delta S_C \Delta^*)|\ker B_2^*).$$

Proof. We consider the decompositions

$$\mathcal{H}_1 = \ker B_2^* \oplus (\mathcal{H}_1 \ominus \ker B_2^*),$$

$$\ker C = (\ker C \ominus \ker B_2) \oplus \ker B_2,$$

and, with respect to the first decomposition let Δ be represented by $\Delta = [\Delta_1 \quad \Delta_2]^t$. Then (4.8) becomes

$$H = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & \Delta_1|C|^{\frac{1}{2}} \\ A_{21} & A_{22} & B_2 & 0 & \Delta_2|C|^{\frac{1}{2}} \\ 0 & B_2^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ |C|^{\frac{1}{2}}\Delta_1^* & |C|^{\frac{1}{2}}\Delta_2^* & 0 & 0 & C \end{bmatrix}. \tag{4.9}$$

We consider now the Kreĭn space $\mathcal{H}_{A_{11}-\Delta_1 S_C \Delta_1^*}[+][\overline{\mathcal{R}(B_2)} \oplus \overline{\mathcal{R}(B_2)}][+]\mathcal{H}_C = \mathcal{K}$ with the specified f.s.

$$J = \begin{bmatrix} \operatorname{sgn}(A_{11} - \Delta_1 S_C \Delta_1^*) & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & S_C \end{bmatrix}. \tag{4.10}$$

Define the operator $T \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{K})$ by

$$T = \begin{bmatrix} |A_{11} - \Delta_1 S_C \Delta_1^*|^{\frac{1}{2}} & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ A_{21} - \Delta_2 S_C \Delta_1^* & \frac{1}{2}(A_{22} - \Delta_2 S_C \Delta_2^*) & B_2 & 0 & 0 \\ S_C \Delta_1^* & S_C \Delta_2^* & 0 & 0 & |C|^{\frac{1}{2}} \end{bmatrix}. \tag{4.11}$$

Using (4.9)–(4.11) it is easy to verify that the following factorization holds

$$H = T^* J T.$$

Since T has dense range and 0 is isolated either on the left or on the right with respect to $\sigma(H)$, using Lemma 2.3 it follows that there exists a uniquely determined unitary operator $U \in \mathcal{L}(\mathcal{H}_H, \mathcal{K})$ such that

$$U|H|^{\frac{1}{2}} = T.$$

Taking into account the defnition of T it follows that

$$U^{-1}|C|^{\frac{1}{2}}h = \chi_{H,C}h, \quad h \in \mathcal{H}_2 \ominus \ker C,$$

hence $\mathcal{H}_C$ is canonically embedded in $\mathcal{H}_H$. Also, using the unitary operator U it follows

$$\kappa^{\pm}[\mathcal{R}(\chi_{H,C})^{\perp}] = \kappa^{\pm}(A_{11} - \Delta_1 S_C \Delta_1^*)$$

The proof is finished. ∎

Remark. Just from the definitions of $\rho_{H,C}$ and $\chi_{H,C}$ it follows

$$|H|^{\frac{1}{2}} \rho_{H,C} = \chi_{H,C}.$$

Considering $|H|^{\frac{1}{2}}$ as a unitary operator in $\mathcal{L}(\mathcal{K}_H, \mathcal{H}_H)$ (see Proposition 2.1) it follows that $\mathcal{R}(\rho_{H,C})^{\perp}$ and $\mathcal{R}(\chi_{H,C})^{\perp}$ and unitary equivalent, hence their signatures are the same. ∎

We are now in a position to formulate the completion problem.

This is

$$\mathbf{C}(K;\kappa_1,\kappa_2)\left\{\begin{array}{l}\text{Let } \mathcal{H}_1, \mathcal{H}_2 \text{ and } \mathcal{H}_3 \text{ be Kreĭn spaces and,}\\ \text{with respect to the Kreĭn space } \mathcal{H}_1[+]\mathcal{H}_2[+]\mathcal{H}_3,\\ \text{let be given the selfadjoint partial block-matrix}\\ \qquad K = \begin{bmatrix} A & B & \\ B^\sharp & C & D \\ & D^\sharp & E \end{bmatrix}.\\ \text{Denote}\\ \qquad H = \begin{bmatrix} A & B \\ B^\sharp & C \end{bmatrix} \text{ and } G = \begin{bmatrix} C & D \\ D^\sharp & E \end{bmatrix}.\\ \text{Given cardinal numbers } \kappa_1 \text{ and } \kappa_2, \text{ it is required}\\ \text{to determine a triplet } (F;\mathcal{G}_1',\mathcal{G}_2'), \text{ where}\\ F \in \mathcal{L}(\mathcal{H}_3,\mathcal{H}_1), \text{ and } \mathcal{G}_1', \mathcal{G}_2' \text{ are Kreĭn spaces,}\\ \text{such that:}\\ (i)\ \kappa^-[\mathcal{G}_1'] = \kappa_1,\quad \kappa^-[\mathcal{G}_2'] = \kappa_2.\\ (ii) \text{ considering the selfadjoint completion of } K,\\ \qquad K(F) = \begin{bmatrix} A & B & F \\ B^\sharp & C & D \\ F^\sharp & D^\sharp & E \end{bmatrix}\\ \quad \text{we have, modulo canonical embedings, that}\\ \quad \mathcal{K}_{K(F)} = \mathcal{K}_H[+]\mathcal{G}_1' \text{ and } \mathcal{K}_{K(F)} = \mathcal{K}_G[+]\mathcal{G}_2'.\end{array}\right.$$

We will first show that the problem $\mathbf{C}(K;\kappa_1,\kappa_2)$ can be restated into the framework of a problem of type $\mathbf{EF}(X,Y;\kappa_1,\kappa_2)$.

To this end, let $(F;\mathcal{G}_1',\mathcal{G}_2')$ be a solution of the problem $\mathbf{C}(K;\kappa_1,\kappa_2)$. By definition, there exist unitary operators $\omega_H : \mathcal{K}_{K(F)} \to \mathcal{K}_H[+]\mathcal{G}_1'$ and $\omega_G : \mathcal{K}_{K(F)} \to \mathcal{K}_G[+]\mathcal{G}_2'$ such that ω_H^{-1} is an extension of the canonical embeding of $\mathcal{K}_H$ into $\mathcal{K}_{K(F)}$ and similarly, ω_G^{-1} is an extension of the canonical embeding of $\mathcal{K}_G$ into $\mathcal{K}_{K(F)}$. Define the unitary operator $W \in \mathcal{L}(\mathcal{K}_H[+]\mathcal{G}_1',\mathcal{K}_G[+]\mathcal{G}_2')$ by

$$W = \omega_G\omega_H^{-1}, \tag{4.12}$$

and then define the operator $\hat{X} \in \mathcal{L}(\mathcal{H}_2,\mathcal{K}_H[+]\mathcal{G}_1')$ by

$$\hat{X} = \omega_H\pi_{K(F)}|\mathcal{H}_2, \tag{4.13}$$

and, similarly, let $\hat{Y} \in \mathcal{L}(\mathcal{H}_2,\mathcal{K}_G[+]\mathcal{G}_2')$ be defined by

$$\hat{Y} = \omega_G\pi_{K(F)}|\mathcal{H}_2. \tag{4.14}$$

Proposition 4.5 *Let $(F;\mathcal{G}_1',\mathcal{G}_2')$ be a solution of the problem $\mathbf{C}(K;\kappa_1,\kappa_2)$ and, let $\hat{X},\hat{Y}$ and W be defined as in (4.12)-(4.14). Then $(\hat{X},\hat{Y},\mathcal{G}_1',\mathcal{G}_2';W)$ is a solution of the problem $\mathbf{EF}(\rho_{H,C},\rho_{G,C};\kappa_1,\kappa_2)$.*

Proof. Let us first remark that from (4.6) it follows

$$X^{\sharp}X = Y^{\sharp}Y = C$$

where $X = \rho_{H,C}$ and $Y = \rho_{G,C}$, hence the problem $\mathbf{EF}(X,Y;\kappa_1,\kappa_2)$ makes sense. It remains to prove that the quintuple $(\hat{X},\hat{Y};\mathcal{G}_1',\mathcal{G}_2';W)$ constructed as indicated in (4.12)-(4.14) is a solution of the problem $\mathbf{EF}(X,Y;\kappa_1,\kappa_2)$.

Condition (i) in $\mathbf{EF}(X,Y;\kappa_1,\kappa_2)$ is clearly satisfied since it coincides with the condition (i) in $\mathbf{C}(K;\kappa_1,\kappa_2)$.

In order to prove that the condition (ii) holds, let us first note that $\hat{X} = \omega_H \rho_{K(F),C}$ hence

$$\hat{X}^{\sharp}\hat{X} = \rho^{\sharp}_{K(F),C}\rho_{K(F),C} = C.$$

Similarly we obtain

$$\hat{Y}^{\sharp}\hat{Y} = C.$$

Let $h \in \mathcal{H}_2, h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$. Then

$$[\hat{X}h, h_1 \oplus h_2]_H = [\omega_H \pi_{K(F)} h, h_1 \oplus h_2]_H = [\pi_{K(F)} h, \omega_H^{\sharp}(h_1 \oplus h_2)]_{K(F)}$$

$$= [\omega_H^{\sharp}\pi_H h, \omega_H^{\sharp}(h_1 \oplus h_2)]_{K(F)} = [\pi_H h, h_1 \oplus h_2]_H = [Xh, h_1 \oplus h_2]_H,$$

which proves that $\hat{X}$ is an extension of X. Similarly one proves that $\hat{Y}$ is an extension of Y and thus the condition (ii) also holds.

Taking into account the definitions of W, $\hat{X}$ and $\hat{Y}$ we have

$$W\hat{X} = W\omega_H \pi_{K(F)} | \mathcal{H}_2 = \omega_G \pi_{K(F)} | \mathcal{H}_2 = \hat{Y}.$$

Finally,

$$\mathcal{K}_H \vee W^{\sharp}\mathcal{K}_G = \mathcal{K}_H \vee \omega_H \omega_G^{\sharp}\mathcal{K}_G = \omega_H(\omega_H^{\sharp}\mathcal{K}_H \vee \omega_G^{\sharp}\mathcal{K}_G)$$

$$= \omega_H(\pi_{K(F)}(\mathcal{H}_1 \oplus \mathcal{H}_2) \vee \pi_{K(F)}(\mathcal{H}_2 \oplus \mathcal{H}_3))$$

$$= \omega_H \pi_{K(F)}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3) = \omega_H \mathcal{K}_{K(F)} = \mathcal{K}_H[+]\mathcal{G}_1'.$$

We proved that $(\hat{X},\hat{Y};\mathcal{G}_1',\mathcal{G}_2';W)$ is a solution of the problem $\mathbf{EF}(X,Y;\kappa_1,\kappa_2)$. ∎

Lemma 4.6 *Assume that $\mathcal{K}_C$ is canonically embedded in both of $\mathcal{K}_H$ and $\mathcal{K}_G$. If the problem $\mathbf{C}(K;\kappa_1,\kappa_2)$ is solvable then the following equality holds*

$$\kappa_1 + \operatorname{rank}(B|\ker C) + \kappa^-[\mathcal{R}(\rho_{H,C})^{\perp}]$$

$$= \kappa_2 + \operatorname{rank}(D^{\sharp}|\ker C) + \kappa^-[\mathcal{R}(\rho_{G,C})^{\perp}].$$

Proof. As before, we take $X = \rho_{H,C}$ and $Y = \rho_{G,C}$. Just from the definitions it follows that X has the property (γ) if and only if $\mathcal{K}_C$ is canonically embedded into $\mathcal{K}_H$. Thus, assuming that $\mathcal{K}_C$ is canonically embedded into $\mathcal{K}_H$ and $\mathcal{K}_G$ it follows that X and Y have the property (γ) hence we can apply Lemma 3.3, via Proposition 4.6, and get the required equality. ∎

For the choice $X = \rho_{H,C}$ and $Y = \rho_{G,C}$, we have now to show how the solutions of the problem $\mathbf{EF}(X, Y; \kappa_1, \kappa_2)$ produce solutions of the problem $\mathbf{C}(K; \kappa_1, \kappa_2)$. We need first two preliminary results.

Lemma 4.7 *Let $(\hat{X}, \hat{Y}; \mathcal{G}_1', \mathcal{G}_2'; W)$ be a solution of the problem $\mathbf{EF}(X, Y; \kappa_1, \kappa_2)$, where $X = \rho_{H,C}$ and $Y = \rho_{G,C}$. Define $F \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_1)$ by*

$$F = P_{\mathcal{H}_1} \pi_H^\sharp P_{\mathcal{K}_H} W^\sharp \pi_G | \mathcal{H}_3, \tag{4.15}$$

and $T_H \in \mathcal{L}(\mathcal{H}_1[+]\mathcal{H}_2[+]\mathcal{H}_3, \mathcal{K}_H[+]\mathcal{G}_1')$ by

$$T_H = [\pi_H | \mathcal{H}_1 \quad \hat{X} \quad W^\sharp \pi_G | \mathcal{H}_3]. \tag{4.16}$$

Then T_H has dense range and satisfies the following equality

$$\begin{bmatrix} A & B & F \\ B^\sharp & C & D \\ F^\sharp & D^\sharp & E \end{bmatrix} = T_H^\sharp T_H. \tag{4.17}$$

Proof. From the definition of the operator T_H we have

$$\mathcal{R}(T_H) = \mathcal{R}(\pi_H | \mathcal{H}_1) + \mathcal{R}(\hat{X}) + \mathcal{R}(W^\sharp \pi_G | \mathcal{H}_3).$$

Since $\hat{X}$ is an extension of $X = \pi_H | \mathcal{H}_2$ it follows that

$$\mathcal{R}(T_H) \supseteq \mathcal{R}(\pi_H).$$

Since $\hat{Y}$ is an extension of $Y = \pi_G | \mathcal{H}_2$ and $W^\sharp \hat{Y} = \hat{X}$ it follows that

$$\mathcal{R}(T_H) \supseteq (W^\sharp \pi_G).$$

Taking into account the minimality condition (iv) in the problem $\mathbf{EF}(X, Y; \kappa_1, \kappa_2)$, we obtain thus that $\mathcal{R}(T_H)$ is dense in $\mathcal{K}_H[+]\mathcal{G}_1'$.

In order to prove the equality (4.17) we compute the entries of $T_H^\sharp T_H$ regarded as a block-matrix with respect to the decomposition

$$\mathcal{H}_1[+]\mathcal{H}_2[+]\mathcal{H}_3.$$

To this end let $h_i, k_i \in \mathcal{H}_i, i = 1, 2, 3$. Then

$$[T_H^\sharp T_H h_1, k_1] = [\pi_H h_1, \pi_H k_1]_H = [H h_1, k_1] = [A h_1, k_1].$$

Since $\hat{X}$ is an extension of $X = \pi_H|\mathcal{H}_2$ we have

$$[T_H^\sharp T_H h_2, k_1] = [\hat{X}h_2, \pi_H k_1]_H = [\pi_H h_2, \pi_H k_1]_H = [Hh_2, k_1] = [Bh_2, k_1].$$

Also, since $\hat{X}^\sharp \hat{X} = C$ it follows

$$[T_H^\sharp T_H h_2, k_2] = [\hat{X}h_2, \hat{X}k_2]_H = [\hat{X}^\sharp \hat{X}h_2, k_2] = [Ch_2, k_2].$$

Using the definition of F in (4.12) we have

$$[T_H^\sharp T_H h_3, k_1] = [W^\sharp \pi_G h_3, \pi_H k_1] = [Fh_3, k_1].$$

Taking into account that $\hat{X} = W^\sharp \hat{Y}$ and $\hat{Y}$ is an extension of $\pi_G|\mathcal{H}_2$, it follows

$$[T_H^\sharp T_H h_3, k_2] = [W^\sharp \pi_G, \hat{X}k_2]_H$$

$$= [\pi_G h_3, W\hat{X}k_2]_G = [\pi_G h_3, \hat{Y}k_2]_G = [\pi_G h_3, \pi_G k_2]_G = [Dh_3, k_2]$$

and finally

$$[T_h^\sharp T_H h_3, k_3] = [W^\sharp \pi_G h_3, W^\sharp \pi_G k_3]_H = [\pi_G h_3, \pi_G k_3]_G = [Eh_3, k_3].$$

We proved thus the equality (4.17). ∎

Lemma 4.8 *Let* $(\hat{X}, \hat{Y}; \mathcal{G}_1', \mathcal{G}_2'; W)$ *be a solution of the problem* $\mathbf{EF}(X, Y; \kappa_1, \kappa_2)$, *where* $X = \rho_{H,C}$ *and* $Y = \rho_{G,C}$. *Define* F *by (4.15) and then define*

$$T_G \in \mathcal{L}(\mathcal{H}_1[+]\mathcal{H}_2[+]\mathcal{H}_3, \mathcal{K}_G[+]\mathcal{G}_2')$$

by

$$T_G = [W\pi_H|\mathcal{H}_1 \quad \hat{Y} \quad \pi_G|\mathcal{H}_3]. \tag{4.18}$$

Then T_G *has dense range and satisfy the following equality*

$$\begin{bmatrix} A & B & F \\ B^\sharp & C & D \\ F^\sharp & D^\sharp & E \end{bmatrix} = T_G^\sharp T_G. \tag{4.19}$$

The proof of this lemma is similar with that of Lemma 4.8 and will be omitted.

Proposition 4.9 *If* $F \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_1)$ *is defined as in (4.15), where* $(\hat{X}, \hat{Y}; \mathcal{G}_1', \mathcal{G}_2'; W)$ *is a solution of the problem* $\mathbf{EF}(X, Y; \kappa_1, \kappa_2)$ *with* $X = \rho_{H,C}$ *and* $Y = \rho_{G,C}$ *and assuming that the operator* T_H *in (4.16) (equivalently the operator* T_G *in (4.18)) has the property* (γ), *then* $(F; \mathcal{G}_1', \mathcal{G}_2')$ *is a solution of the problem* $\mathbf{C}(K; \kappa_1, \kappa_2)$.

Proof. Let us first remark that from (4.16) and (4.18), it follows

$$WT_H = T_G,$$

hence T_H and T_G have simultaneously the property (γ). Assuming that these hold, since T_H and T_G have dense ranges, it follows using Lemma 2.3 and (4.17) that T_H uniquely induces a unitary operator $\omega_H : \mathcal{K}_{K(F)} \to \mathcal{K}_H[+]\mathcal{G}_1'$ and, similarly, from (4.19), T_G uniquely induces a unitary operator $\omega_G : \mathcal{K}_{K(F)} \to \mathcal{K}_G[+]\mathcal{G}_2'$.

On the other hand, from (4.16) it follows that $T_H|\mathcal{H}_1[+]\mathcal{H}_2$ is an extension of π_H hence $\omega_H^\sharp$ is an extension of the canonical embedding of $\mathcal{K}_H$ into $\mathcal{K}_{K(F)}$. Similarly one proves that $\omega_G^\sharp$ is an extension of the canonical embedding of $\mathcal{K}_G$ into $\mathcal{K}_{K(F)}$. Thus we proved that $(F;\mathcal{G}_1',\mathcal{G}_2')$ is a solution of the problem $\mathrm{C}(K;\kappa_1,\kappa_2)$. ∎

Remark. Assume that $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ are *finite dimensional* Hilbert spaces. Notice that in this case, with respect to the decomposition $\mathcal{H}_2 = \ker C \oplus (\mathcal{H}_2 \ominus \ker C)$, we have the representations

$$B = [B_2 \quad \Delta_1 |C|^{\frac{1}{2}}],$$

and

$$D = [D_2 \quad |C|^{\frac{1}{2}} \Delta_2]^t,$$

and denote

$$A_{22} = P_{\ker B_2^*}(A - \Delta_1 S_C \Delta_1^*)|\ker B_2^*,$$

$$E_{22} = P_{\ker D_2}(E - \Delta_2^* S_C \Delta_2)|\ker D_2.$$

Also, let P and Q be the orthogonal projections of $\ker C$ onto $\mathcal{R}(B_2^*)$ and, respectively, $\mathcal{R}(D_2)$. Using Proposition 4.6, Proposition 4.10 and Theorem 3.4 we get

$$\min\{\kappa^-(K(F)) | F \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_1\}$$

$$= \kappa^-(C) + \mathrm{rank}(P) + \mathrm{rank}(Q(I-P)) + \max\{\kappa^-(A_{22}), \kappa^-(E_{22})\}.$$

(compare with [20], Theorem 4.1).

5. The Extension Problem

In [19] it was considered the following extension problem with prescribed negative signatures of defect.

$$\mathbf{E}(T_r, T_c; \kappa_1', \kappa_2') \begin{cases} \text{There are given Kreĭn spaces } \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_1', \mathcal{K}_2' \\ \text{and two linear operators } T_r \in \mathcal{L}(\mathcal{K}_1[+]\mathcal{K}_1', \mathcal{K}_2) \\ \text{and } T_c \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2[+]\mathcal{K}_2') \text{ such that} \\ \quad T_r|\mathcal{K}_1 = P_{\mathcal{K}_2} T_c. \\ \text{Given cardinal numbers } \kappa_1' \text{ and } \kappa_2' \text{ it is required} \\ \text{to determine an operator} \\ \quad \widetilde{T} \in \mathcal{L}(\mathcal{K}_1[+]\mathcal{K}_1', \mathcal{K}_2[+]\mathcal{K}_2') \\ \text{such that:} \\ (i)\ \widetilde{T}|\mathcal{K}_1 = T_c \text{ and } P_{\mathcal{K}_2}\widetilde{T} = T_r. \\ (ii)\ \kappa^-(I - \widetilde{T}^\sharp \widetilde{T}) = \kappa^- \text{ and} \\ \quad \kappa^-(I - \widetilde{T}\widetilde{T}^\sharp) = \kappa_2'. \end{cases}$$

Motivated by the approach adopted in this paper, we formulate a modified extension problem as follows

$$\mathbf{E}_m(T_r, T_c; \kappa_1, \kappa_2) \begin{cases} \text{Assume that } \mathcal{K}_1, \mathcal{K}_1', \mathcal{K}_2, \mathcal{K}_2' \text{ are Kreĭn spaces and} \\ T_r \in \mathcal{L}(\mathcal{K}_1[+]\mathcal{K}_1', \mathcal{K}_2) \text{ and } T_c \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2[+]\mathcal{K}_2') \\ \text{are operators such that} \\ \qquad T_r|\mathcal{K}_1 = P_{\mathcal{K}_2} T_c. \\ \text{Given cardinal numbers } \kappa_1 \text{ and } \kappa_2, \text{ it is required} \\ \text{to determine a triple } (\widetilde{T}; \mathcal{G}_1', \mathcal{G}_2') \text{ such that} \\ \widetilde{T} \in \mathcal{L}(\mathcal{K}_1[+]\mathcal{K}_1', \mathcal{K}_2[+]\mathcal{K}_2') \text{ satisfies} \\ (i)\ \widetilde{T}|\mathcal{K}_1 = T_c \text{ and } P_{\mathcal{K}_2}\widetilde{T} = T_r, \\ (ii)\ \kappa^-[\mathcal{G}_1'] = \kappa_1 \text{ and } \kappa^-[\mathcal{G}_2'] = \kappa_2, \\ (iii) \text{ modulo canonical embedings, we have} \\ \mathcal{R}_{I-\tilde{T}^\sharp\tilde{T}} = \mathcal{K}_{I-T_c^\sharp T_c}[+]\mathcal{G}_1,\ \mathcal{K}_{I-\tilde{T}\tilde{T}^\sharp} = \mathcal{K}_{I-T_r T_r^\sharp}[+]\mathcal{G}_2'. \end{cases}$$

It is clear that if $(\widetilde{T}; \mathcal{G}_1', \mathcal{G}_2')$ is a solution of the problem $\mathbf{E}_m(T_r, T_c; \kappa_1, \kappa_2)$ then $\widetilde{T}$ is a solution of the problem $\mathbf{E}(T_r, T_c; \kappa_1', \kappa_2')$ where

$$\kappa_1' = \kappa_1 + \kappa^-(I - T_c^\sharp T_c)\ , \quad \kappa_2' = \kappa_2 + \kappa^-(I - T_r T_r^\sharp). \tag{5.1}$$

Conversely, if (5.1) holds and $\widetilde{T}$ is a solution of the problem $\mathbf{E}(T_r, T_c; \kappa_1', \kappa_2')$ then $\widetilde{T}$ produces a solution of the problem $\mathbf{E}_m(T_r, T_c; \kappa_1, \kappa_2)$ if and only if $\mathcal{K}_{I-T_c^\sharp T_c}$ and $\mathcal{K}_{I-T_r T_r^\sharp}$ are canonically embedded in $\mathcal{K}_{I-\tilde{T}^\sharp\tilde{T}}$ and, respectively, $\mathcal{K}_{I-\tilde{T}\tilde{T}^\sharp}$ and κ_1 and κ_2 coincide with the negative cosignatures of $\mathcal{K}_{I-T_c^\sharp T_c}$ and, respectively, $\mathcal{K}_{I-T_r T_r^\sharp}$ with respect to these canonical embedings.

The following result illustrates a situation when the two extension problems do coincide.

Lemma 5.1 *Assume that*

$$\kappa_1' = \kappa^-(I - T_c^\sharp T_c) < \infty \ , \ \kappa_2' = \kappa^-(I - T_r T_r^\sharp) < \infty. \tag{5.2}$$

Then any solution $\widetilde{T}$ *of the problem* $\mathbf{E}(T_r, T_c; \kappa_1', \kappa_2')$ *produces a solution of the problem* $\mathbf{E}_m(T_r, T_c; 0, 0)$.

Proof. Let $\widetilde{T}$ be a solution of the problem $\mathbf{E}(T_r, T_c; \kappa_1', \kappa_2')$. Then $\widetilde{T}$ is a row extension on T_c, hence

$$\widetilde{T} = [T_c \quad *]$$

and this produces a representation

$$I - \widetilde{T}^\sharp \widetilde{T} = \begin{bmatrix} I - T_c^\sharp T_c & * \\ * & * \end{bmatrix}$$

where we have denoted by " $*$ " the operators entries which are of no importance here. Taking into account the first condition in (5.2) it follows that the mapping $\rho_{I-\widetilde{T}^\sharp\widetilde{T}, I-T_c^\sharp T_c}$ (see (4.5) for the definition) induces a densely defined isometric operator acting between Pontryagin spaces of the same negative signature hence, using a Pontryagin Lemma type argument (e.g. see [18], Corollary 1.3) it follows that this isometric operator is bounded, i.e. $\mathcal{K}_{I-T_c^\sharp T_c}$ is canonically embedded into $\mathcal{K}_{I-\widetilde{T}^\sharp\widetilde{T}}$. Also, there exists a Hilbert space $\mathcal{G}_1'$ such that, modulo the canonical embeding,

$$\mathcal{K}_{I-\widetilde{T}^\sharp\widetilde{T}} = \mathcal{K}_{I-T_c^\sharp T_c}[+]\mathcal{G}_1'.$$

Similarly, using the second condition in (5.2) we prove that there exists a Hilbert space $\mathcal{G}_2'$ such that, modulo the canonical embeding,

$$\mathcal{K}_{I-\widetilde{T}\widetilde{T}^\sharp} = \mathcal{K}_{I-T_c T_c^\sharp}[+]\mathcal{G}_2'.$$

Thus $(\widetilde{T}; \mathcal{G}_1', \mathcal{G}_2')$ is a solution of the problem $\mathbf{E}_m(T_r, T_c, 0, 0)$. ∎

We will now embed the problem $\mathbf{E}_m(T_r, T_c; \kappa_1, \kappa_2)$ into a completion problem as the one studied in Section 4. For this reason let us denote

$$T_r|\mathcal{K}_1 = P_{\mathcal{K}_2} T_c = T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2).$$

Then there exist uniquely determined operators $B \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2')$, $D \in \mathcal{L}(\mathcal{K}_1', \mathcal{K}_2)$ such that

$$T_c = [T \quad B]^t, \quad T_r = [T \quad D]. \tag{5.3}$$

Using these objects we consider the selfadjoint partial block-matrix K acting on the Kreĭn space $\mathcal{K}_2'[+]\mathcal{K}_2[+]\mathcal{K}_1[+]\mathcal{K}_1'$ and defined by

$$K = \begin{bmatrix} I & 0 & B & \\ 0 & I & T & D \\ B^\sharp & T^\sharp & I & 0 \\ & D^\sharp & 0 & I \end{bmatrix}. \tag{5.4}$$

Also, let us remark that $\widetilde{T} \in \mathcal{L}(\mathcal{K}_1[+]\mathcal{K}_1', \mathcal{K}_2[+]\mathcal{K}_2')$ satisfies

$$\widetilde{T}|\mathcal{K}_1 = T_c, \quad P_{\mathcal{K}_2}\widetilde{T} = T_r,$$

if and only if for some $F \in \mathcal{L}(\mathcal{K}_1', \mathcal{K}_2')$ we have

$$\widetilde{T} = \begin{bmatrix} T & D \\ B & F \end{bmatrix}. \tag{5.5}$$

Proposition 5.2 *The formula (5.5) establishes a bijective correspondence between the set of solutions $(\widetilde{T}; \mathcal{G}_1', \mathcal{G}_2')$ of the problem $\mathbf{E}_m(T_r, T_c; \kappa_1, \kappa_2)$ and the set of solutions $(F; \mathcal{G}_1', \mathcal{G}_2')$ of the problem $\mathbf{C}(K; \kappa_1, \kappa_2)$ where K is defined as in (5.4).*

Proof. Let $F \in \mathcal{L}(\mathcal{K}_1', \mathcal{K}_2')$ be arbitrary, consider $\widetilde{T}$ as in (5.5) and define the selfadjoint operator

$$K(F) = \begin{bmatrix} I & 0 & B & F \\ 0 & I & T & D \\ B^\sharp & T^\sharp & I & 0 \\ F^\sharp & D^\sharp & 0 & I \end{bmatrix} \tag{5.6}$$

Then,

$$K(F) = \begin{bmatrix} I & \widetilde{T} \\ \widetilde{T}^\sharp & I \end{bmatrix} \tag{5.7}$$

and the following factorization holds

$$K(F) = \begin{bmatrix} I & 0 \\ \widetilde{T}^\sharp & I \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & I - \widetilde{T}^\sharp\widetilde{T} \end{bmatrix} \cdot \begin{bmatrix} I & \widetilde{T} \\ 0 & I \end{bmatrix}. \tag{5.8}$$

Then, using Lemma 2.2, from (5.7) we obtain a unitary operator

$$\mathcal{K}_{K(F)} \to \mathcal{K}_1'[+]\mathcal{K}_2[+]\mathcal{K}_{I-\widetilde{T}^\sharp\widetilde{T}}. \tag{5.9}$$

In accordance with the notation in Section 4 we denote

$$H = \begin{bmatrix} I & 0 & B \\ 0 & I & T \\ B^\sharp & T^\sharp & I \end{bmatrix}, \quad G = \begin{bmatrix} I & T & D \\ T^\sharp & I & 0 \\ D^\sharp & 0 & I \end{bmatrix} \tag{5.10}$$

We notice that the following representation holds

$$H = \begin{bmatrix} I & T_c \\ T_c^\sharp & I \end{bmatrix},$$

and using a factorization of H similar with that in (5.7) we obtain a unitary operator

$$\mathcal{K}_H \to \mathcal{K}_2'[+]\mathcal{K}_2[+]\mathcal{K}_{I-T_c^\sharp T_c}. \tag{5.11}$$

Let now $(F;\mathcal{G}_1',\mathcal{G}_2')$ be a solution of the problem $\mathbf{C}(K;\kappa_1,\kappa_2)$ and define $\widetilde{T}$ as in (5.5). By definition, we have a unitary operator

$$\mathcal{K}_H[+]\mathcal{G}_1' \to \mathcal{K}_{K(F)}, \tag{5.12}$$

which extends the canonical embeding of $\mathcal{K}_H$ into $\mathcal{K}_{K(F)}$. Using this operator and the unitary operators in (5.8) and (5.10) we obtain a unitary operator

$$\mathcal{K}_2'[+]\mathcal{K}_2[+]\mathcal{K}_{I-T_c^\sharp T_c}[+]\mathcal{G}_1' \to \mathcal{K}_2'[+]\mathcal{K}_2[+]\mathcal{K}_{I-\widetilde{T}^\sharp\widetilde{T}}.$$

This unitary operator maps $\mathcal{K}_2'[+]\mathcal{K}_2$ onto itself and extends the canonical embeding of $\mathcal{K}_{I-T_c^\sharp T_c}$ into $\mathcal{K}_{I-\widetilde{T}^\sharp\widetilde{T}}$, hence, modulo this canonical embeding, we have

$$\mathcal{K}_{I-\widetilde{T}^\sharp\widetilde{T}} = \mathcal{K}_{I-T_c^\sharp T_c}[+]\mathcal{G}_1'.$$

Similarly we prove that $\mathcal{K}_{I-T_rT_r^\sharp}$ is canonically embedded into $\mathcal{K}_{I-\widetilde{T}\widetilde{T}^\sharp}$ and modulo this embeding we have

$$\mathcal{K}_{I-\widetilde{T}\widetilde{T}^\sharp} = \mathcal{K}_{I-T_rT_r^\sharp}[+]\mathcal{G}_2'.$$

Thus $(\widetilde{T};\mathcal{G}_1',\mathcal{G}_2)$ is a solution of the problem $\mathbf{E}_m(T_r,T_c;\kappa_1,\kappa_2)$.

Conversely, if $(\widetilde{T};\mathcal{G}_1',\mathcal{G}_2')$ is a solution of the problem $\mathbf{E}_m(T_r,T_c;\kappa_1,\kappa_2)$ then (5.5) uniquely determines $F\in\mathcal{L}(\mathcal{K}_1',\mathcal{K}_2')$. We use again the unitary operators in (5.8) and (5.10) to produce a unitary extension of the canonical embeding of $\mathcal{K}_H$ in $\mathcal{K}_{K(F)}$ as in (5.11). Similarly we get a unitary extension of the canonical embeding of $\mathcal{K}_G$ into $\mathcal{K}_{K(F)}$, with the complementary Kreĭn space $\mathcal{G}_2'$. Thus $(F;\mathcal{G}_1',\mathcal{G}_2)$ is a solution of the problem $\mathbf{C}(K;\kappa_1,\kappa_2)$. ∎

As it was pointed out during Section 4, the problem $\mathbf{C}(K;\kappa_1,\kappa_2)$ is stated into the framework of a problem $\mathbf{EF}(X,Y;\kappa_1,\kappa_2)$. In accordance with the notation in previous sections, we consider the selfadjoint operators H and G as in (5.9) and, in addition

$$C = \begin{bmatrix} I & T \\ T^\sharp & I \end{bmatrix}. \tag{5.13}$$

Lemma 5.3 *Assume that either $\kappa_1+\kappa^-(I-T_c^\sharp T_c)$ or $\kappa_2+\kappa^-(I-T_rT_r^\sharp)$ are finite. If $(\hat{X},\hat{Y};\mathcal{G}_1',\mathcal{G}_2';W)$ is a solution of the problem $\mathbf{EF}(X,Y;\kappa_1,\kappa_2)$, with $X=\rho_{H,C}$ and $Y=\rho_{G,C}$, then, letting F be defined by (4.12) and $\widetilde{T}$ as in (5.5), $(\widetilde{T};\mathcal{G}_1',\mathcal{G}_2')$ is a solution of the problem $\mathbf{E}_m(T_r,T_c;\kappa_1,\kappa_2)$.*

Proof. Assume that $\kappa_1+\kappa^-(I-T_c^\sharp T_c)$ is finite. We first prove that if $\widetilde{T}$ is obtained from a solution $(\hat{X},\hat{Y};\mathcal{G}_1',\mathcal{G}_2';W)$ as indicated in the statement of the lemma, then $\kappa^-(I-\widetilde{T}^\sharp\widetilde{T})$ is finite.

To this end, let us consider the unitary operator in (5.8) and denote by $\mathcal{S}$ the regular subspace of $\mathcal{K}_{K(F)}$ which is mapped by this unitary operator onto $\mathcal{K}_2'[+]\mathcal{K}_2$, in particular we have

$$\kappa^-(I-\widetilde{T}^\sharp\widetilde{T})=\kappa^-[\mathcal{S}^\perp]. \tag{5.14}$$

From (5.7) it follows that $\mathcal{S}=\pi_{K(F)}(\mathcal{K}_2'[+]\mathcal{K}_2)\subseteq\mathcal{R}(\pi_{K(F)})$.

We consider now the operator $T_H\in\mathcal{L}(\mathcal{K}_2'[+]\mathcal{K}_2[+]\mathcal{K}_1[+]\mathcal{K}_1',\mathcal{K}_H[+]\mathcal{G}_1')$ defined by

$$T_H=\begin{bmatrix}\pi_H|\mathcal{K}_2' & \hat{X} & W^\sharp\pi_G|\mathcal{K}_1'\end{bmatrix} \tag{5.15}$$

and using Lemma 4.8 we obtain that T_H induces an isometric operator ω_H with domain $\mathcal{R}(\pi_{K(F)})$ and dense range in $\mathcal{K}_H[+]\mathcal{G}_1'$. An argument of Pontryagin Lemma type shows that

$$\kappa^-[\mathcal{S}^\perp]\leq\kappa^-[(\omega_H\mathcal{S})^\perp]. \tag{5.16}$$

Let us consider now the representation (3.4) of $\hat{X}$. Remark that $\mathcal{R}(\hat{X}_1)$ is a neutral subspace of $\mathcal{G}_1'$, hence it has finite dimension. From (5.14) it follows

$$\omega_H\mathcal{S}+\mathcal{R}(\hat{X}_0)=\pi_H(\mathcal{K}_2'[+]\mathcal{K}_2)+\mathcal{R}(\hat{X}_1)\supseteq\pi_H(\mathcal{K}_2'[+]\mathcal{K}_2).$$

Making use of the unitary operator in (5.10), this yields

$$\kappa^-[(\omega_H\mathcal{S}+\mathcal{R}(\hat{X}_1))^\perp]\leq\kappa_1+\kappa^-(I-T_c^\sharp T_c)$$

and finally, from here, (5.13), (5.15) and the fact that $\mathcal{R}(\hat{X}_1)$ is finite dimensional, we conclude that $\kappa^-(I-\widetilde{T}^\sharp\widetilde{T})$ is finite.

Since $\kappa^-(I-\widetilde{T}^\sharp\widetilde{T})$ is finite it follows that $\widetilde{T}$ has either the property $(\alpha)_+$ or the property $(\alpha)_-$. Taking into account the factorization (5.7), it follows that the hypothesis of Proposition 4.10 are fulfilled, hence $(F;\mathcal{G}_1',\mathcal{G}_2')$ is a solution of $\mathrm{C}(K;\kappa_1,\kappa_2)$ and using Proposition 5.2 we conclude that $(\widetilde{T};\mathcal{G}_1',\mathcal{G}_2')$ is a solution of $\mathrm{E}_m(T_r,T_c;\kappa_1,\kappa_2)$. ∎

We focus now on determining the minimal negative signatures for which the extension problem is solvable. For this purpose we need to fix f.s. J_1,J_1',J_2 and J_2' on $\mathcal{K}_1,\mathcal{K}_1',\mathcal{K}_2$ and respectively $\mathcal{K}_2'$. With respect to these f.s. we consider the defect operators D_T and D_{T^*} (see (2.6)).

There exist uniquely determined operators $\Lambda_2=J_2'B|\ker D_T$ and $\Gamma_2:\mathcal{R}(D_T)\to\mathcal{K}_1'$ such that

$$J_2'B=[\Lambda_2\quad\Gamma_2D_T], \tag{5.17}$$

and similarly, there exist uniquely determined operators $\Lambda_1=P_{\ker D_{T^*}}J_2D:\mathcal{K}_1'\to\ker D_{T^*}$ and $\Gamma_1':\mathcal{R}(D_{T^*})\to\mathcal{K}_1'$ such that

$$D^*J_2=[\Lambda_1^*\quad\Gamma_1'D_{T^*}]. \tag{5.18}$$

Lemma 5.4 *Assume that the operators Γ_2 and Γ_1', defined in (5.16) and (5.17) are bounded, and denote $\Gamma_1 = \Gamma_1'^* \in \mathcal{L}(\mathcal{K}_1', \mathcal{D}_{T^*})$. Then, in order for the problem $\mathbf{E}_m(T_r, T_c; \kappa_1, \kappa_2)$ to be solvable, the following equality must hold*

$$\kappa_1 + \operatorname{rank}(\Lambda_2) + \kappa^-(P_{\ker \Lambda_2^*}(J_2' - \Gamma_2 J_T \Gamma_2^*)|\ker \Lambda_2^*)$$

$$= \kappa_2 + \operatorname{rank}(\Lambda_1) + \kappa_1(P_{\ker \Lambda_1}(J_1' - \Gamma_1^* J_{T^*} \Gamma_1)|\ker \Lambda_1).$$

Proof. Using Lemma 4.3, since Γ_1 and Γ_2 are bounded, we obtain that $\mathcal{K}_C$ is canonically embedded into both of $\mathcal{K}_H$ and $\mathcal{K}_G$, equivalently, $\rho_{H,C}$ and $\rho_{G,C}$ have the property (γ) (see Section 3). In adition, also from Lemma 4.3, we obtain

$$\kappa^-[\mathcal{R}(\rho_{G,C})^\perp] = \kappa^-(P_{\ker \Lambda_1}(J_1' - \Gamma_1^* J_{T^*} \Gamma_1)|\ker \Lambda_1), \tag{5.19}$$

and

$$\kappa^-[\mathcal{R}(\rho_{H,C})^\perp] = \kappa^-(P_{\ker \Lambda_2^*}(J_2' - \Gamma_2 J_T \Gamma_2^*)|\ker \Lambda_2^*). \tag{5.20}$$

We apply now Proposition 5.2 and Lemma 4.7, taking into account the formulae (5.18), (5.19) and the definitions of Λ_1 and Λ_2 and get the required formula. ∎

Theorem 5.5 *Assume that the hypothesis of Lemma 5.4 are fulfilled and, in addition, that Λ_1 and Λ_2 have finite ranks and also that $\kappa^-(I - T_c^\sharp T_c)$ and $\kappa^-(I - T_r T_r^\sharp)$ are finite. Then, the set of pairs (κ_1, κ_2) for which the problem $\mathbf{E}_m(T_r, T_c; \kappa_1, \kappa_2)$ is solvable has a minimum, simultaneously attained, which is given by the following formulae*

$$\begin{aligned}\kappa_1^{min} = \operatorname{rank}(Q(I-P)) + \max\{0, \kappa^-(P_{\ker \Lambda_1}(J_1' - \Gamma_1^* J_{T^*} \Gamma_1)|\ker \Lambda_1) \\ -\kappa^-(P_{\ker \Lambda_2^*}(J_2' - \Gamma_2 J_T \Gamma_2^*)|\ker \Lambda_2^*)\}\end{aligned} \tag{5.21}$$

and

$$\begin{aligned}\kappa_2^{min} = \operatorname{rank}(P(I-Q)) + \max\{0, \kappa^-(P_{\ker \Lambda_2^*}(J_2' - \Gamma_2 J_T \Gamma_2^*)|\ker \Lambda_2^*) \\ -\kappa^-(P_{\ker \Lambda_1}(J_1' - \Gamma_1^* J_{T^*} \Gamma_1)|\ker \Lambda_1)\end{aligned} \tag{5.22}$$

where we have denoted $P = P_{\mathcal{R}(\Lambda_1)}$ and $Q = P_{J_2 T \mathcal{R}(\Lambda_2^*)}$.

Proof. First recall the considerations during the proof of Lemma 5.4. Then notice that since Λ_1 and Λ_2 have finite ranks and $\kappa^-(I - T_c^\sharp T_c)$ and $\kappa^-(I - T_r T_r^\sharp)$ are finite, in order to determine the minimal signatures of the problem $\mathbf{E}_m(T_r, T_c; \kappa_1, \kappa_2)$, taking into account Lemma 5.4, it follows that we are interested only in those pairs (κ_1, κ_2) such that either κ_1 or κ_2 is finite. Thus, Lemma 5.3 works in this case and we can apply Theorem 3.4. It remais only to notice that the orthogonal projections P and

Q can be considered as in the statement of the theorem, since the partial matrix K in (5.4) gives the same bound for κ_1 and κ_2 as the partial matrix K

$$K' = \begin{bmatrix} I & 0 & B & \\ 0 & I & 0 & D \\ B^\sharp & 0 & I - T^\sharp T & -T^\sharp D \\ & D & -D^\sharp T & I \end{bmatrix}. \tag{5.23}$$

The proof is finished. ∎

Remark. In formulae (5.20) and (5.21) we optionally can take $P = P_{J_1 T^* \mathcal{R}(A_1)}$ and $Q = P_{\mathcal{R}(A_2^*)}$. This follows by considering instead of K' in (5.22) the partial block-matrix

$$K'' = \begin{bmatrix} I & -BT^\sharp & B & \\ -TB^\sharp & I - TT^\sharp & 0 & D \\ B^\sharp & 0 & I & 0 \\ & D^\sharp & 0 & I \end{bmatrix}. \tag{5.24}$$

Corollary 5.6 *Assume that the hypothesis of Theorem 5.5 are fulfilled and let κ_1' and κ_2' be defined as in (5.2). Then the problem $\mathbf{E}(T_r, T_c; \kappa_1', \kappa_2')$ is solvable if and only if*

$$\mathcal{R}(A_1) = J_2 T \mathcal{R}(A_2^*) \tag{5.25}$$

and

$$\begin{aligned} &\kappa^-(P_{\ker A_1}(J_1' - \Gamma_1^* J_{T^*} \Gamma_1)|\ker A_1) = \\ &\qquad \kappa^-(P_{\ker A_2^*}(J_2' - \Gamma_2 J_T \Gamma_2^*)|\ker A_2^*). \end{aligned} \tag{5.26}$$

Proof. This is a consequence of Lemma 5.1 and Theorem 5.5 ∎

We conclude by noticing that Corollary 5.7 is a generalization of [18, Theorem 5.1] (see also Remark 5.6 in [19] and Remark 5.7 in [20]).

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