INSTITUTUL DE MATEMATICA
AL ACADEMIEI ROMANE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

A PROCEDURE TO COMPUTE AN OPTIMAL ROBUST CONTROLLER IN THE GAP-METRIC

by

Vasile Drăgan, Aristide Halanay, Adrian Stoica

Preprint No. 10/94

BUCURESTI

A PROCEDURE TO COMPUTE AN OPTIMAL ROBUST CONTROLLER IN THE GAP-METRIC

by

V. Drăgan*, A. Halanay** and A. Stoica***

May 1994

*) Institute of Mathematics, Bucharest, Romania

**) Faculty of Mathematics, Bucharest, Romania

***) Department of Aerospace Engineering, Polytechnic University of Bucharest, Romania

A PROCEDURE TO COMPUTE AN OPTIMAL ROBUST CONTROLLER IN THE GAP-METRIC

Vasile Drăgan

Institute of Mathematics,Bucharest,Romania

Aristide Halanay

Faculty of Mathematics,Bucharest,Romania

Adrian Stoica

Department of Aerospace Engineering,Polytechnic University of Bucharest,Romania

Abstract The present paper gives a procedure to determine an optimal robust controller by using balanced realizations in the sense of Jonckheere-Silverman[8] and an optimal solution to the one-block Nehari problem obtained from a suboptimal one via singular perturbations method.

Our procedure is different from the ones analysed in the book of Habets and seems to have some advantages as examples taken from the book of McFarlane and Glover[9] do suggest.

1. Introduction In the last years,many papers have been devoted to the robust stability in the gap-metric(or equivalently with respect to perturbations in the normalized left coprime factorization).We have in mind primarily the books of McFarlane and Glover[9] and Habets[4] and the references therein.

The aim of this present paper is to give a procedure to determine an optimal robust controller by using balanced realizations in the sense of Jonckheere-Silverman[8] and an optimal solution to the one-block Nehari problem obtained from a suboptimal one via singular perturbations method.

Our procedure seems to have some advantages with respect to the ones analysed in the book of Habets.

2. Preliminary known facts

2.1. Jonckheere-Silverman invariants and balancing Consider the minimal system (A,B,C);then the Riccati standard equations:

$$A^TX+XA-XBB^TX+C^TC=0 \tag{1}$$

$$AY+YA^T-YC^TCY+BB^T=0 \tag{2}$$

have stabilizing solutions $X>0$ and $Y>0$.

According to the results of Jonckheere and Silverman[8],the eigenvalues of the matrix XY are positive and invariant with respect to any nonsingular coordinate transformation T in the state space;we shall call these invariants *Jonckheere-Silverman invariants*.

It is said that a realization (A,B,C) is balanced in the sense of Jonckheere-Silverman if the Riccati equations (1) and (2) have the solutions $X=Y=diag(\sigma_1 I_1,..,\sigma_r I_r)$ respectively,with $\sigma_1>\sigma_2>..>\sigma_r>0$ where I_k are $n_k x n_k$ unit matrices,$k=1,..,r$.

If we start with an arbitrary minimal realization (A,B,C) and with the corresponding solutions X,Y to the associated Riccati equations,a balanced realization in the Jonckheere-Silverman sense is obtained by the following algorithm which will represent a first component of our procedure to compute the optimal robust controller:

1^{st} *Step:* Determine the stabilizing positive-definite solutions X and Y respectively,of the Riccati equations (1) and (2);

2^{nd} *Step:* Consider a Cholesky factorization $X=Z^TZ$;

3^{rd} *Step:* Perform the singular value descomposition $ZYZ^T=R\Sigma^2R^T$ with R orthogonal;

4^{th} *Step:* Define $T:=\Sigma^{-\frac{1}{2}}R^TZ$ and compute TAT^{-1}, TB, CT^{-1}.

□

After such transformation we obtain a realization which is balanced in Jonckheere-Silverman sense and we shall work further with such realization denoted again (A,B,C).

2.2. Robustness in the Gap-Metric Given a nominal stable plant G, the robust stabilization problem in the gap metric consists in finding a controller K which stabilizes all the systems G_Δ satisfying $\delta(G,G_\Delta)<r$, where δ denotes the gap between G and G_Δ (see [1]) and $r>0$ is called *robustness radius*.

As it is known this problem is closely related with the robustness with respect to perturbations in the normalized left coprime factorization. Recall that a double coprime factorization of the form $G=\tilde{M}^{-1}\tilde{N}=NM^{-1}$, is normalized if $M^*M+N^*N=I$ and $\tilde{M}\tilde{M}^*+\tilde{N}\tilde{N}^*=I$; such factorization is realised by using stabilizing solutions to the Riccati equations (1) and (2).[4,9].The robust stabilization problem with respect to perturbations in the normalized left coprime factorization consists in finding a stabilizing controller K for all perturbed systems $G_\Delta:=(\tilde{M}+\Delta_{\tilde{M}})^{-1}(\tilde{N}+\Delta_{\tilde{N}})$, where $\Delta_{\tilde{M}}, \Delta_{\tilde{N}}\in RH^\infty$ and $\| \Delta_{\tilde{M}}^T \ \Delta_{\tilde{N}}^T \|_\infty < r$ (see [2,4,9]).

An important result in [9] is the fact that the maximum robustness radius is $r_{max}=\gamma_{min}^{-1}$ where $\gamma_{min}:=\sqrt{1+\sigma_1^2}$ with σ_1^2 beeing the largest Jonckheere-Silverman invariant.

In the following we shall essentialy use the important result in [9,theorem 4.3] which states that a controller K is a solution to the optimal robust stabilization problem with respect to normalized left coprime factorization if and only if, $K=UV^{-1}$ with $U,V\varepsilon RH^\infty$ satisfying:

$$\left\| \begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix} + \begin{bmatrix} U \\ V \end{bmatrix} \right\|_\infty = (1-\gamma_{min}^{-2})^{\frac{1}{2}} \tag{3}$$

$[U^T \; V^T]^T$ is thus a solution to the one-block Nehari problem (3).

3. Optimal solution to the Nehari problem via singular perturbations

It is known[3] that the distance from an antistable minimal sistem $G(s):=(A,B,C,D)$ to the set of stable systems equals the Hankel norm of G which is equal to the largest Moore singular value of (A,B,C).

In this section we start with a minimal plant $G(s):=(A,B,C,D)$ with A antistable and we look for a system $G_s(s):=(A_s,B_s,C_s,D_s)$ with A_s antistable such that $\|G-G_s\|_\infty=\|G\|_H=\mu_1$, where μ_1 is the largest Moore singular value of (A,B,C), that is μ_1^2 is the largest eigenvalue of PQ, where P and Q are the Gramians of (A,B,C) defined by the Lyapunov equations:

$$AP+PA^T-BB^T=0 \tag{4}$$

$$QA+A^TQ-C^TC=0 \tag{5}$$

The system G_s is a solution to the one-block Nehari problem associated to G[3].

Let $\gamma^2>\mu_1^2$ and define $W:=\gamma^{-2}Q(I-\gamma^{-2}PQ)^{-1}$. We shall show in the following that the system:

$$G_s^\gamma(s) := (-(A+BB^TW)^T, WB, -CQ, D) \tag{6}$$

is a solution to the suboptimal Nehari problem $\|G-G_s\|_\infty<\gamma$.

Proposition 1 *The matrix W defined above is a stabilizing pozitive-solution to the game-theoretic Riccati equation:*

$$(A-\gamma^{-2}PC^TC)^TW+W(A-\gamma^{-2}PC^TC)+W(BB^T-\gamma^{-2}PC^TCP)W-\gamma^{-2}C^TC=0 \qquad (7)$$

Proof From (5) we deduce that the following equality is verified:

$$\begin{bmatrix} -A & 0 \\ -C^TC & A^T \end{bmatrix}\begin{bmatrix} I \\ Q \end{bmatrix}=\begin{bmatrix} I \\ Q \end{bmatrix}(-A) \qquad (8)$$

If we define the similarity transformation:

$$T:=\begin{bmatrix} \gamma^2 I & -P \\ 0 & I \end{bmatrix}$$

we obtain from (8):

$$T\begin{bmatrix} -A & 0 \\ -C^TC & A^T \end{bmatrix}T^{-1}T\begin{bmatrix} I \\ Q \end{bmatrix}=T\begin{bmatrix} I \\ Q \end{bmatrix}(-A)$$

and thus:

$$\begin{bmatrix} -A+\gamma^{-2}PC^TC & -AP-PA^T+\gamma^{-2}PC^TCP \\ -\gamma^{-2}C^TC & -\gamma^{-2}C^TCP+A^T \end{bmatrix}\begin{bmatrix} \gamma^2 I-PQ \\ Q \end{bmatrix}=\begin{bmatrix} \gamma^2 I-PQ \\ Q \end{bmatrix}(-A) \qquad (9)$$

Since $\gamma^2 I-PQ$ is nonsingular and $-A$ is stable,we deduce from (9) that $W:=Q(\gamma^2 I-PQ)^{-1}=\gamma^{-2}Q(I-\gamma^{-2}PQ)^{-1}$ is a stabilizing solution to the Riccati equation associated to the hamiltonian first matrix in the left side of equation (8).

To prove that W is positive-definite we consider a Cholesky factorization $Q=R^TR$ and we obtain:

$$W=\gamma^{-2}Q(I-\gamma^{-2}PQ)^{-1}=\gamma^{-2}R^TR(I-\gamma^{-2}PR^TR)^{-1}=\gamma^{-2}R^T(I-\gamma^{-2}RPR^T)^{-1}R$$

Since $\gamma^2>RPR^T$(since $\gamma^2>PQ$),from the last equality above it follows that $W>0$.

□

Corollary *The matrix* $-(A+BB^TX)^T$ *is stable.*

Proof Write equation (7) in the equivalent form:

$$-(A+BB^TW)^TW-W(A+BB^TW)+WBB^TW+\gamma^{-2}(I+WP)C^TC(I+PW)=0 \qquad (10)$$

Since the last two terms in equation (10) are positive semidefinite and the pair $(B^TW,-(A+BB^TW))$ is detectable ($-(A+BB^TW)+BB^TW=-A$ is stable, since A is assumed antistable), taking into account that W is positive definite according to Proposition 1, we conclude that the matrix $-(A+BB^TX)$ is stable.

□

The Corollary shows that G_s^γ defined by (6) is stable.

Proposition 2 *We have* $\|G-G_s^\gamma\|_\infty<\gamma$.

Proof Consider the difference-system $G_d:=G-G_s^\gamma$ obtained when connecting in parallel G and $-G_s^\gamma$; then using (7), a realization of G_d is (A_d,B_d,C_d,D_d), where:

$$A_d=\begin{bmatrix} A & 0 \\ 0 & -(A+BB^TW)^T \end{bmatrix};\ B_d=\begin{bmatrix} B \\ WB \end{bmatrix};\ C_d=[C \quad CP] \tag{11}$$

We shall prove now that the Riccati equation:

$$A_d\Pi+\Pi A_d^T+\gamma^{-2}\Pi C_d^TC_d\Pi+B_dB_d^T=0 \tag{12}$$

is verified by:

$$\Pi:=\begin{bmatrix} -P & I \\ I & W \end{bmatrix} \tag{13}$$

Indeed, when substituting (11) and (13) in (12) one obtains the following equations corresponding to the partitioned form of (12):

- The block (1,1) of (12): $-AP-PA^T+BB^T=0$;
- The block (1,2) of (12): $A-A-BB^TW+BB^TW=0$;
- The block (2,2) of (12):

$$-(A+BB^TW)^TW-W(A+BB^TW)+WBB^TW+\gamma^{-2}(I+WP)C^TC(I+PW)=0$$

The first equation above is verified because of (4), the second is obvious and the third is just equation (10) which is true according to Proposition 1. Therefore we conclude that Π is a solution to the Riccati equation (12).

In order to prove that $\|G_d\|_\infty \le \gamma$, consider the adjoint system of G_d:

$$\dot{x} = -A_d^T x - C_d^T u$$
$$y = B_d^T x$$

A direct calculation using the equations above and (12) shows that if u is an L^2-input then we have:

$$\int_{-\infty}^{\infty} y^T y\,dt = \int_{-\infty}^{\infty} x^T B_d B_d^T x\,dt = \int_{-\infty}^{\infty} x^T(-A_d\Pi - \Pi A_d^T - \gamma^{-2}\Pi C_d^T C_d \Pi)\,x\,dt =$$

$$\int_{-\infty}^{\infty} [(-\dot{x}^T + u^T C_d)\Pi x + x^T \Pi(-\dot{x} + C_d^T u)\,x - \gamma^{-2} x^T \Pi C_d^T C_d \Pi x]\,dt -$$

$$\int_{-\infty}^{\infty} \left[-\frac{d}{dt}(x^T \Pi x) + \gamma^2 u^T u - (\gamma u + \gamma^{-1} C_d \Pi x)^T (\gamma u + \gamma^{-1} C_d \Pi x)\right] dt$$

Since A_d is dichotomic the term:

$$\int_{-\infty}^{\infty} \frac{d}{dt}(x^T(t)\Pi x(t))\,dt$$

vanishes and hence we deduce that:

$$\int_{-\infty}^{\infty} y^T y\,dt \le \gamma^2 \int_{-\infty}^{\infty} u^T u\,dt$$

Therefore the L^∞-norm of $(-A_d^T, -C_d^T, B_d^T)$ is less or equal than γ; since the norm of an operator equals the norm of its adjoint, we conclude that $\|G_d\|_\infty \le \gamma$.

□

Remark For the two-blocks Nehari problem the corresponding formulae to (6) and arguments are in [7].

□

In the following we are interested in studying the behaviour of G_s^γ as $\gamma \to \mu_1$. To this end we shall assume that (A,B,C,D) is a balanced realization in the sense that $P=Q=diag(\mu_1 I_1, .., \mu_p I_p)$, where $\mu_1 > .. > \mu_p > 0$ and I_k are $n_k x n_k$ unit matrices, $k=1,..,p$.

Denote $M_{22} := diag(\mu_2 I_2, .., \mu_p I_p)$ hence $P=Q=diag(\mu_1 I_1, M_{22})$.

Take $\gamma^2 := \mu_1^2 + \varepsilon$ and compute:

$$W := \gamma^{-2}Q(I-\gamma^{-2}PQ)^{-1} = \begin{bmatrix} \frac{\mu_1}{\varepsilon} I_1 & 0 \\ 0 & W_{22}(\varepsilon) \end{bmatrix} \qquad (14)$$

where:

$$W_{22}(\varepsilon) := diag\left\{ \frac{\mu_1}{\mu_1^2 - \mu_i^2 + \varepsilon} \right\}_{i=2,\dots,p} \qquad (15)$$

Consider the partitions af A, B and C conformally with the partition of W as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix};\ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix};\ C = [\, C_1 \quad C_2 \,] \qquad (16)$$

Then,we obtain from (6):

$$A_s^\gamma := -(A+BB^TW)^T = \begin{bmatrix} -\left(A_{11}^T + \frac{1}{\varepsilon}\mu_1 B_1 B_1^T\right) & -\left(A_{21}^T + \frac{\mu_1}{\varepsilon} B_1 B_2^T\right) \\ -(A_{12}^T + W_{22}(\varepsilon) B_2 B_1^T) & -(A_{22}^T + W_{22}(\varepsilon) B_2 B_2^T) \end{bmatrix}$$

$$B_s^\gamma := WB = \begin{bmatrix} \frac{\mu_1}{\varepsilon} B_1 \\ W_{22}(\varepsilon) B_2 \end{bmatrix};\ C_s^\gamma := -CP := -[\mu_1 C_1 \ - C_2 M_{22}]\ ;\ D_s^\gamma = D \qquad (17)$$

The realization above corresponds to a singularly perturbed (two-time-scale) system;the corresponding equations may be written as:

$$\begin{cases} \varepsilon \dot{x}_1 = -(\varepsilon A_{11}^T + \mu_1 B_1 B_1^T) x_1 - (\varepsilon A_{21}^T + \mu_1 B_1 B_2^T) x_2 + \mu_1 B_1 u \\ \dot{x}_2 = -(A_{12}^T + W_{22}(\varepsilon) B_2 B_1^T) x_1 - (A_{22}^T + W_{22}(\varepsilon) B_2 B_2^T) x_2 + W_{22}(\varepsilon) B_2 u \\ y = -\mu_1 C_1 x_1 - C_2 M_{22} x_2 + Du \end{cases} \qquad (18)$$

According to the theory of singular perturbations this system can be reduced if $-B_1B_1^T$ is stable,that is if $B_1B_1^T$ is nonsingular.

Assume that $B_1B_1^T$ is nonsingular and perform the usual reduction,by taking:

$$x_1 = -(B_1 B_1^T)^{-1}(B_1 B_2^T x_2 - B_1 u) \tag{19}$$

Hence we obtain the reduced order system G_{sr} of G_s^γ:

$$\begin{cases} \dot{x}_2 = A_{sr} x_2 + B_{sr} u \\ y = C_{sr} x_2 + D_{sr} u \end{cases}$$

where:

$$\begin{aligned}
A_{sr} &:= [A_{12}^T + W_{22}(0) B_2 B_1^T](B_1 B_1^T)^{-1} B_1 B_2^T - A_{22}^T - W_{22}(0) B_2 B_2^T \\
B_{sr} &:= -[A_{12}^T + W_{22}(0) B_2 B_1^T](B_1 B_1^T)^{-1} + W_{22}(0) B_2 \\
C_{sr} &:= \mu_1 C_1 (B_1 B_1^T)^{-1} B_1 B_2^T - C_2 M_{22} \\
D_{sr} &:= -\mu_1 C_1 (B_1 B_1^T)^{-1} B_1 + D
\end{aligned} \tag{20}$$

A useful result concerning G_{sr} is stated in the following theorem:

Theorem *The system G_{sr} defined above is a solution to the optimal Nehari problem for G.*

Proof When writing equation (10) in a partitioned form according to (14) and (17), we obtain:

- The block (1,1) of (10):

$$-\left(\frac{\mu_1}{\varepsilon} A_{11}^T + \frac{\mu_1^2}{\varepsilon^2} B_1 B_1^T\right) - \left(\frac{\mu_1}{\varepsilon} A_{11} + \frac{\mu_1^2}{\varepsilon^2} B_1 B_1^T\right) + \frac{\mu_1^2}{\varepsilon^2} B_1 B_1^T + \frac{1}{\mu_1^2 + \varepsilon}\Bigg(C_1^T C_1 +$$
$$\frac{\mu_1^2}{\varepsilon} C_1^T C_1 + \frac{\mu_1^2}{\varepsilon} C_1 C_1^T + \frac{\mu_1^4}{\varepsilon^2} C_1^T C_1\Bigg) = 0$$

and hence:

$$\mu_1^2 (B_1 B_1^T - C_1^T C_1) + \varepsilon(\mu_1 A_{11}^T + \mu_1 A_{11} - C_1^T C_1) = 0$$

When writing the Lyapunov equation (5) in the partitioned form with $Q = diag(\mu_1 I_1, M_{22})$, one obtains $\mu_1 A_{11}^T + \mu_1 A_{11} - C_1^T C_1 = 0$, therefore the equation above gives $B_1 B_1^T = C_1^T C_1$. On the other hand it is known[8] that for a balanced realization $B_1B_1^T=C_1^TC_1$; we conclude that the relation above gives no new information.

- The block (1,2) of (10):

$$-\left(A_{21}^T+\frac{\mu_1}{\varepsilon}B_1B_2^T\right)W_{22}(\varepsilon)-\frac{\mu_1}{\varepsilon}[A_{12}+B_1B_2^TW_{22}(\varepsilon)]+\frac{\mu_1}{\varepsilon}B_1B_2^TW_{22}(\varepsilon)+$$
$$\frac{1}{\mu_1^2+\varepsilon}\left[C_1^TC_2+\frac{\mu_1^2}{\varepsilon}C_1^TC_2+C_1^TC_2M_{22}W_{22}(\varepsilon)+\frac{\mu_1^2}{\varepsilon}C_1^TC_2M_{22}W_{22}(\varepsilon)\right]=0$$

From the expression (14) of W it results that $W_2(\varepsilon)=M_{22}(\gamma^2I-M_{22}^2)^{-1}$, therefore:

$$I+M_{22}W_{22}(\varepsilon)=I+M_{22}^2(\gamma^2I-M_{22}^2)^{-1}=(\gamma^2I-M_{22}^2)^{-1} \tag{21}$$

Using (21), after some algebraic manipulations in the equation corresponding to the block (1,2), we obtain:

$$\mu_1[A_{12}+B_1B_2^TW_{22}(\varepsilon)]+\varepsilon A_{21}^TW_{22}(\varepsilon)-C_1^TC_2[(\mu_1^2+\varepsilon)I-M_{22}^2]^{-1}=0$$

and hence, for $\varepsilon=0$ it follows that:

$$A_{12}+B_1B_2^TW_{22}(0)=\mu_1^{-1}C_1^TC_2(\mu_1^2I-M_{22}^2)^{-1} \tag{22}$$

- The block (2,2) of (10):

$$-[A_{22}+B_2B_2^TW_{22}(\varepsilon)]^TW_{22}(\varepsilon)-W_{22}(\varepsilon)[A_{22}+B_2B_2^TW_{22}(\varepsilon)]+W_{22}(\varepsilon)B_2B_2^TW_{22}(\varepsilon)+$$
$$\frac{1}{\mu_1^2+\varepsilon}[I+M_{22}W_{22}(\varepsilon)]^TC_2^TC_2[I+M_{22}W_{22}(\varepsilon)]=0$$

Using (21), from equation above we obtain:

$$A_{22}^TW_{22}(\varepsilon)+W_{22}(\varepsilon)A_{22}=\frac{1}{\mu_1^2+\varepsilon}[(\mu_1^2+\varepsilon)I-M_{22}^2]^{-1}C_2^TC_2[(\mu_1^2+\varepsilon)I-M_{22}^2]^{-1}-$$
$$W_{22}(\varepsilon)B_2B_2^TW_{22}(\varepsilon)$$

therefore, for $\varepsilon=0$ it follows that:

$$A_{22}^T W_{22}(0) + W_{22}(0) A_{22} = \mu_1^{-2} (\mu_1^2 I - M_{22}^2)^{-1} C_2^T C_2 (\mu_1^2 I - M_{22}^2)^{-1} - W_{22}(\varepsilon)(0) B_2 B_2^T W_{22}(0) \tag{23}$$

We shall prove now that A_{sr} is stable;using (20) and (23) it follows that:

$$\begin{aligned} A_{sr} W_{22}(0) + W_{22}(0) A_{sr}^T &= [(A_{12}^T + W_{22}(0) B_2 B_1^T)(B_1 B_1^T)^{-1} B_1 B_2^T - A_{22}^T - \\ & W_{22}(0) B_2 B_2^T] W_{22}(0) + W_{22}(0) [B_2 B_1^T (B_1 B_1^T)^{-1} (A_{12} + B_1 B_2^T W_{22}(0)) - A_{22} - \\ & B_2 B_2^T W_{22}(0)] = [A_{12}^T + W_{22}(0) B_2 B_1^T] (B_1 B_1)^{-1} B_1 B_2^T W_{22}(0) + \\ & W_{22}(0) B_2 B_1^T (B_1 B_1^T)^{-1} [A_{12} + B_1 B_2^T W_{22}(0)] - W_{22}(0) B_2 B_2^T W_{22}(0) - \\ & \mu_1^{-2} (\mu_1^2 I - M_{22}^2)^{-1} C_2^T C_2 (\mu_1^2 I - M_{22}^2)^{-1} \end{aligned} \tag{24}$$

We have also from (22):

$$[A_{12} + B_1 B_2^T W_{22}(0)]^T (B_1 B_1^T)^{-1} [A_{12} + B_1 B_2^T W_{22}(0)] =$$

$$\mu_1^{-2} (\mu_1^2 I - M_{22}^2)^{-1} C_2^T C_1 (B_1 B_1)^{-1} C_1^T C_2 (\mu_1^2 I - M_{22}^2)^{-1}$$

Since $B_1 B_1^T = C_1^T C_1$,from equation above and from the inequality $C_1 (C_1^T C_1)^{-1} C_{11}^T \leq I$ we obtain:

$$[A_{12} + B_1 B_2^T W_{22}(0)]^T (B_1 B_1^T)^{-1} [A_{12} + B_1 B_2^T W_{22}(0)] \leq$$

$$\mu_1^{-2} (\mu_1^2 I - M_{22}^2)^{-1} C_2^T C_2 (\mu_1^2 I - M_{22}^2)^{-1}$$

therefore,taking into account the expression (20) for B_{sr},from (24) it follows that:

$$A_{sr} W_{22}(0) + W_{22}(0) A_{sr}^T \leq -B_{sr} B_{sr}^T \tag{25}$$

From (20) we deduce that $A_{sr} + B_{sr} B_2^T = -A_{22}^T$;since (A,B,C) is a balanced realization with A antistable it follows that A_{22} is antistable too (see[3,theorem 4.2]).From the equality above we conclude that the pair (A_{sr}, B_{sr}) is stabilizable.Since $W_{22}(0)>0$ from the inequality (25) we conclude that A_{sr} is stable.

We shall prove now that G_{sr} satisfies the inequality:

$$\|G-G_{sr}\|_\infty \le \gamma$$

for all $\gamma>\mu_1$.

Let define the difference system $G_{dr}:=G-G_{sr}$ which has the realization $(A_{dr},B_{dr},C_{dr},D_{dr})$, where:

$$A_{dr}=\begin{bmatrix} A & 0 \\ 0 & A_{sr} \end{bmatrix};\ B_{dr}=\begin{bmatrix} B \\ B_{sr} \end{bmatrix};\ C_{dr}=[\,C \quad -C_{sr}\,]\ ;\ D_{dr}=D-D_{sr} \tag{26}$$

Consider now the Riccati inequation:

$$\begin{aligned}&A_{dr}\Pi_r+\Pi_r A_{dr}^T+(\Pi_r C_{dr}^T+B_{dr}D_{dr}^T)(\gamma^2 I-D_{dr}D_{dr}^T)^{-1}(C_{dr}\Pi_r+D_{dr}B_{dr}^T)+\\ &B_{dr}B_{dr}^T\le 0\end{aligned} \tag{27}$$

We shall show now that this inequation is satisfied by:

$$\Pi_r:=\left[\begin{array}{c|c} -P & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ \hline [0 \quad I] & W_{22}(0) \end{array}\right] \tag{28}$$

where the dimension of I equals the number of columns of A_{12} and $P=diag(\mu_1 I, M_{22})$.

When substituting (28) into (27) we obtain, using (20) and (26):

- The block (1,1) of (27):

$$-AP-PA^T+BB^T=0$$

- The block (1,2) of (27):

$$\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}+\begin{bmatrix} 0 \\ A_{sr}^T \end{bmatrix}+B\cdot B_{sr}^T=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- The block (2,2) of (27):

$$A_{sr}W_{22}(0)+W_{22}(0)A_{sr}^T+[C_2^T-W_{22}(0)C_{sr}^T-\mu_1 B_{sr}^T B_1(B_1B_1^T)^{-1}C_1^T]\cdot$$
$$[\gamma^2 I-\mu_1^2 C_1(B_1B_1^T)^{-1}C_1^T]^{-1}[C_2-C_{sr}W_{22}(0)-\mu_1 C_1(B_1B_1^T)^{-1}B_1^T B_{sr}]+ \tag{29}$$
$$B_{sr}B_{sr}^T\leq 0$$

In order to prove (29),we compute first,using (20),(22) and (23):

$$A_{sr}W_{22}(0)+W_{22}(0)A_{sr}^T=\mu_1^{-1}(\mu_1^2 I-M_{22}^2)^{-1}C_2^T C_1(B_1B_1^T)^{-1}B_1B_2^T W_{22}(0)+$$
$$\mu_1^{-1}W_{22}(0)B_2B_1^T(B_1B_1^T)^{-1}C_1^T C_2(\mu_1^2 I-M_{22}^2)^{-1}-\mu_1^{-2}(\mu_1^2 I-M_{22}^2)^{-1}C_2^T C_2\cdot \tag{30}$$
$$(\mu_1^2 I-M_{22}^2)^{-1}-W_{22}(0)B_2B_2^T W_{22}(0)$$

A direct calculation using (20) shows that:

$$C_2^T-W_{22}(0)C_{sr}^T-\mu_1 B_{sr}^T B_1(B_1B_1^T)^{-1}=(\mu_1^2 I-M_{22}^2)^{-1}C_2^T[I-C_1(B_1B_1)^T C_1^T]$$

and hence,since $B_1B_1^T=C_1^T C_1$ and $\gamma^2>\mu_1^2$,we have:

$$[C_2^T-W_{22}(0)C_{sr}^T-\mu_1 B_{sr}^T B_1(B_1B_1^T)^{-1}C_1^T]\ [\gamma^2 I-\mu_1^2 C_1(B_1B_1^T)^{-1}C_1^T]^{-1}\cdot$$
$$[C_2-C_{sr}W_{22}(0)-\mu_1 C_1(B_1B_1^T)^{-1}B_1^T B_{sr}]=(\mu_1^2 I-M_{22}^2)^{-1}C_2^T\cdot$$
$$[I-C_1(C_1^T C_1)^{-1}C_1^T]\,[\gamma^2 I-\mu_1^2 C_1(C_1^T C_1)^{-1}C_1^T]^{-1}[I-C_1(C_1^T C_1)^{-1}C_1^T]\cdot \tag{31}$$
$$C_2(\mu_1^2 I-M_{22}^2)^{-1}\leq\mu_1^{-2}(\mu_1^2 I-M_{22}^2)^{-1}C_2^T[I-C_1(C_1^T C_1)^{-1}C_1^T]\,C_2(\mu_1^2 I-M_{22}^2)^{-1}$$

For the last two terms in the left side of (29),we obtain with (20) and (22):

$$B_{sr}B_{sr}^T=-\mu_1^{-2}(\mu_1^2 I-M_{22}^2)^{-1}C_2^T C_1(B_1B_1^T)^{-1}C_1^T C_2(\mu_1^2 I-M_{22}^2)^{-1}-$$
$$\mu_1^{-1}(\mu_1^2 I-M_{22}^2)^{-1}C_2^T C_1(B_1B_1^T)^{-1}B_1B_2^T W_{22}(0)-\mu_1^{-1}W_{22}(0)\cdot \tag{32}$$
$$B_2B_1^T(B_1B_1^T)^{-1}C_1^T C_2(\mu_1^2 I-M_{22}^2)^{-1}+W_{22}(0)B_2B_2^T W_{22}(0)$$

When replacing (30),(31) and (32) in (29) one can directly verify that this inequality is true.

Consider now the adjoint of G_{dr},i.e.:

$$\dot{x} = -A_{dr}^T x - C_{dr}^T u$$
$$y = B_{dr}^T x + D_{dr}^T u \qquad (33)$$

A direct calculation using (27) gives for an arbitrary $u \varepsilon L^2(-\infty,\infty)$:

$$\int_{-\infty}^{\infty} y^T y dt = \int_{-\infty}^{\infty} (x^T B_{dr} + u^T D_{dr})(B_{dr}^T x + D_{dr}^T u)\, dt \le -\int_{-\infty}^{\infty} \{x^T [A_{dr}\Pi_r +$$
$$\Pi_r A_{dr}^T + (\Pi_r C_{dr}^T + B_{dr} D_{dr}^T)(\gamma^2 I - D_{dr} D_{dr}^T)^{-1}(C_{dr}\Pi_r + D_{dr} B_{dr}^T)]\, x +$$
$$u^T D_{dr} B_{dr}^T x + x^T B_{dr} D_{dr}^T u + u^T D_{dr} D_{dr}^T u\}\, dt = \int_{-\infty}^{\infty} [-\dot{x}^T \Pi_r x + u^T C_{dr} \Pi_r x - \qquad (34)$$
$$x^T \Pi_r \dot{x} + x^T \Pi_r C_{dr}^T - x^T (\Pi_r C_{dr}^T + B_{dr} D_{dr}^T)(\gamma^2 I - D_{dr} D_{dr}^T)^{-1} \cdot$$
$$(C_{dr}\Pi_r + D_{dr} B_{dr}^T)\, x + u^T D_{dr} B_{dr}^T x + x^T B_{dr} D_{dr}^T + u^T D_{dr} D_{dr}^T u]\, dt$$

Since A_{dr} is dichotomic, the term:

$$\int_{-\infty}^{\infty} (\dot{x}^T \Pi_r x + x^T \Pi_r \dot{x})\, dt = \int_{-\infty}^{\infty} \frac{d}{dt}(x^T \Pi_r x)\, dt$$

vanishes, then from (34) it follows that:

$$\int_{-\infty}^{\infty} y^T y dt \le -\int_{-\infty}^{\infty} [x^T(-\Pi_r C_{dr}^T + B_{dr} D_{dr}^T)(\gamma^2 I - D_{dr} D_{dr}^T) - u](\gamma^2 I - D_{dr} D_{dr}^T) \cdot$$
$$[(\gamma^2 I - D_{dr} D_{dr}^T)(-C_{dr}\Pi_r + D_{dr} B_{dr}^T)\, x - u^T]\, dt + \gamma^2 \int_{-\infty}^{\infty} u^T u dt \qquad (35)$$

From (35) we deduce that:

$$\int_{-\infty}^{\infty} y^T y dt \le \gamma^2 \int_{-\infty}^{\infty} u^T u dt$$

for all $u \varepsilon L^2(-\infty,\infty)$, therefore the L^∞-norm of $(-A_d^T, -C_d^T, B_d^T, D_d^T)$ is less or equal than γ. Since the L^∞-norm of a system equals the L^∞-norm of its adjoint, it follows that:

$$\|G_{dr}\|_\infty \le \gamma$$

The inequality above holds for all $\gamma>\mu_1$;according to Glover's theorem[3],$\|G_{dr}\|_\infty=\|G-G_{sr}\|_\infty\ge\mu_1$,then we obtain $\|G-G_{sr}\|_\infty=\mu_1$,therefore G_{sr} is a solution to the optimal Nehari problem.

□

We summarize the steps leading to the optimal solution to the Nehari problem:

1st Step: Compute the Gramians P and Q associated to (A,B,C);

2nd Step: Perform the balancing transformation T in the sense of Moore;this transformation can be obtained using the algorithm described in Section 2.1 in which X and Y must be replaced by Q and P respectively.

Replace $A\leftarrow TAT^{-1};B\leftarrow TB;C\leftarrow CT^{-1}$;

3rd Step: Perform the partitions (16) conformally with W given by (14);

4th Step: Compute the realization $(A_{sr},B_{sr},C_{sr},D_{sr})$ of the optimal solution G_{sr} to the Nehari one-block problem,using (20).

□

Remark The above procedure is described for the case when $B_1B_1^T$ is invertible.

From (13) it follows that if $B_1=0$,then there are no singular perturbations and letting $\varepsilon=0$ we get directly the optimal Nehari aproximation.

If $B_1B_1^T$ is singular,by an orthogonal transformation it takes the form:

$$\begin{bmatrix} \overline{B}_1\overline{B}_1^T & 0 \\ 0 & 0 \end{bmatrix}$$

where $\overline{B}_1\overline{B}_1^T$ is nonsingular.After the corresponding transformation we obtain a singularly perturbed system for which the fast component has the dimension equal to the rank of $B_1B_1^T$ and we may procede as in the generic situation when $B_1B_1^T$ is invertible.

We deduce that the dimension of the optimal Nehari approximation obtained by our procedure equals $n\text{-}rank(B_1B_1^T)$.

4. The optimally robust controller

We shall apply the procedure in the previous section to determine the solution to the optimal robust stabilization problem with respect to normalized left coprime factorization.

Recall that the normalized left coprime factorization is obtained by using the stabilizing solution to the Riccati equation (2), and if the realization (A,B,C) is balanced in the Jonckheere - Silverman sense, then we shall have:

$$\begin{bmatrix} -\tilde{N}^*(s) \\ \tilde{M}^*(s) \end{bmatrix} := \left(-(A-\Sigma C^TC)^T, -C^T, \begin{bmatrix} -B^T \\ -C\Sigma \end{bmatrix}, I \right) \tag{36}$$

Taking into account the structure of the system (36), the optimal solution to the Nehari problem (3) has the form:

$$\begin{bmatrix} U(s) \\ V(s) \end{bmatrix} := \left(A_o, B_o, \begin{bmatrix} C_{o1} \\ C_{o2} \end{bmatrix}, \begin{bmatrix} D_{o1} \\ D_{o2} \end{bmatrix} \right) \tag{37}$$

Since a realization of V^{-1} is $(A_o-B_oD_{o2}^{-1}C_{o2}, B_oD_{o2}^{-1}, -D_{o1}^{-1}C_{o2}, D_{o2}^{-1})$, the optimal controller $K=UV^{-1}$ has the following state space representation:

$$\begin{aligned} \dot{x}_1 &= (A_o-B_oD_{o2}^{-1}C_{o2})x_1+B_oD_{o2}^{-1}u \\ \dot{x}_2 &= A_ox_2+B_o(-D_{o2}^{-1}C_{o2}x_1+D_{o2}^{-1}u) \\ y &= C_{o1}x_2+D_{o1}(-D_{o2}^{-1}C_{o2}x_1+D_{o2}^{-1}u) \end{aligned} \tag{38}$$

When substracting the first two equations (38) one obtains:

$$\dot{x}_1-\dot{x}_2 = A_o(x_1-x_2)$$

Since A_o is stable (because (37) is a stable solution to the optimal Nehari problem) we conclude that (38) has an uncontrollable stable part; after removal of this part we obtain the following equivalent realization of (38):

$$K(s) := (A_o - B_o D_{o2}^{-1} C_{o2},\ B_o D_{o2}^{-1},\ C_{o1} - D_{o1} D_{o2}^{-1} C_{o2},\ D_{o1} D_{o2}^{-1}) \tag{39}$$

A direct calculation based on the fact that Σ verifies (1) and (2) shows that the controllability and observability Gramians associated to the system (36) are $\Sigma(I+\Sigma^2)^{-1}$ and Σ respectively.

A further transformation defined by $T = (I+\Sigma^2)^{\frac{1}{4}}$ leads to a balanced realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ of (36) with the Gramians $\tilde{P} = \tilde{Q} = \Sigma(I+\Sigma^2)^{-\frac{1}{2}}$.

When performing the partition $\Sigma = diag(\sigma_1 I_1, \Sigma_{22})$, we obtain:

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix};\ \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix};\ \tilde{C} = [\tilde{C}_1 \quad \tilde{C}_2] \tag{40}$$

where:

$$\begin{aligned}
&\tilde{A}_{11} = -A_{11}^T + \sigma_1 C_1^T C_1\ ;\ \tilde{A}_{12} = (1+\sigma_1^2)^{\frac{1}{4}} (-A_{21}^T + C_1^T C_2 \Sigma_{22}) (I+\Sigma_{22}^2)^{-\frac{1}{4}}\ ; \\
&\tilde{A}_{21} = (1+\sigma_1^2)^{-\frac{1}{4}} (I+\Sigma_{22}^2)^{\frac{1}{4}} (-A_{12}^T + \sigma_1 C_2^T C_1)\ ; \\
&\tilde{A}_{22} = (I+\Sigma_{22}^2)^{\frac{1}{4}} (-A_{22}^T + C_2^T \overbrace{C_2)}^{\Sigma_{22}} (I+\Sigma_{22}^2)^{-\frac{1}{4}}\ ; \\
&\tilde{B}_1 = -\sigma_1 C_1^T\ ;\ \tilde{B}_2 = -\Sigma_{22} C_2^T\ ; \\
&\tilde{C}_1 = \begin{bmatrix} -(1+\sigma_1^2)^{-\frac{1}{4}} B_1^T \\ -\sigma_1 (I+\Sigma_{22}^2)^{-\frac{1}{4}} C_1 \end{bmatrix};\ \tilde{C}_2 = \begin{bmatrix} -(1+\sigma_1^2)^{-\frac{1}{4}} B_2^T \\ -(I+\Sigma_{22}^2)^{-\frac{1}{4}} C_2 \Sigma_{22} \end{bmatrix};
\end{aligned} \tag{41}$$

Since $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is balanced we can apply the procedure described in the previous section to get the solution to the optimal Nehari problem associated to (36). Let us remark the fact that the partition (40) generated by the partition of Σ is the same to that induced by the partition considered for the Nehari problem; indeed, since $\tilde{P} = \tilde{Q} = \Sigma(I+\Sigma^2)^{-\frac{1}{2}}$, we obtain $\tilde{\mu}_i^2 = \sigma_i^2/(1+\sigma_i^2)$, $i=1,..,r$, therefore the degrees of multiplicity of $\tilde{\mu}_1$ and σ_1 respectively, are equal. Hence, the partition (40) can be applied in formulae (20) which give:

$$
\begin{aligned}
&\tilde{A}_{sr} = -A_{22}^T + \tilde{B}_{sr} C_2 \Sigma_{22} \,; \\
&\tilde{B}_{sr} = \sigma_1^{-1} (1+\sigma_1)^{\frac{1}{4}} (I+\Sigma_{22}^2)^{-\frac{1}{4}} (-A_{21} + \Sigma_{22} C_2^T C_1) (C_1^T C_1)^{-1} C_1^T + \\
&\qquad \tilde{W}_{22}(0) \Sigma_{22} C_2^T [C_1 (C_1^T C_1)^{-1} C_1^T - I] \,; \\
&\tilde{C}_{sr} = \begin{bmatrix} (1+\sigma_1^2)^{-\frac{1}{4}} [-\tilde{\mu}_1 \sigma_1^{-1} B_1^T (C_1^T C_1)^{-1} C_1^T C_2 \Sigma_{22} + B_2^T \tilde{M}_{22}] \\ (I+\Sigma_{22}^2)^{-\frac{1}{4}} [-\tilde{\mu}_1 C_1 (C_1^T C_1)^{-1} C_1^T + I] \, C_2 \Sigma_{22} \end{bmatrix} ; \\
&\tilde{D}_{sr} = \begin{bmatrix} -\tilde{\mu}_1 (1+\sigma_1^2)^{-\frac{1}{4}} \sigma_1^{-1} B_1^T (C_1^T C_1)^{-1} C_1^T \\ -\tilde{\mu}_1 (I+\Sigma_{22}^2)^{-\frac{1}{4}} C_1 (C_1^T C_1)^{-1} C_1^T + I \end{bmatrix} ;
\end{aligned}
\tag{42}
$$

where:

$$
\begin{aligned}
&\tilde{\mu}_1 = \sigma_1 (1+\sigma_1^2)^{-\frac{1}{2}} \\
&\tilde{W}_{22}(0) = (1+\sigma_1^2) \, diag \left\{ \frac{\sigma_i \sqrt{1+\sigma_i^2}}{\sigma_1^2 - \sigma_i^2} \right\}_{i=2,\dots,r} \\
&\tilde{M}_{22} = diag \left\{ \sigma_i (1+\sigma_i^2)^{-\frac{1}{2}} \right\}_{i=2,\dots,r}
\end{aligned}
\tag{43}
$$

Since:

$$
A_o = \tilde{A}_{sr} \,;\; B_o = \tilde{B}_{sr} \,;\; C_o = -\tilde{C}_{sr} \,;\; D_o = -\tilde{D}_{sr} \tag{44}
$$

using the expressions above and (39) we obtain the realization of the optimal controller K.

We present now the global algorithm to get an optimal robust controller for a minimal plant (A,B,C) in the gap metric(or equivalently with respect to normalized left-coprime factorization):

1^{st} *Step*: Determine a balanced realization of (A,B,C) in the sense of Jonckheere-Silverman,applying the procedure described in Section 2.1;

2^{nd} *Step*: Compute an optimal solution (A_o,B_o,C_o,D_o) to the Nehari problem (3) using the formulae (42),(43),(44);

3^{rd} Step: Determine an optimal robust controller K with the realization (39).

We suppliment this algorithm with a testing procedure;we know that a controller is optimal if when it is coupled to a fictious plant[9,eq.(4.31)],one obtains a system whose H^∞-norm equals the optimal robust radius given by formula $r_{max}=(1+\sigma_1^2)^{-1/2}$.Our testing procedure consists in performing the H^∞-norm of the system obtained by coupling the controller obtained by using the above algorithm to the fictious plant.We use in the algorithm for computing the H^∞-norm a procedure described in [5].

5.Examples

We shall apply the algorithm described in the previous section to compute robust controllers for some examples detailed by McFarlane and Glover in[9,chapter 7].The examples given in [9] are loop-shaping problems which are solved by determining a shaped plant $G_s=WG$,where W is a shaping function,adequately choosen and G is the nominal given plant;the the suboptimal controller denoted K_∞ for G_s is determined with respect to left coprime factorization and the solution to the loop-shaping problem is obtained when taking $K=WK_\infty$.

Although we have used a testing procedure which consists in computing the H^∞-norm of the resulting system obtained when coupling the optimal robust controller to the fictious plant,it is desirable to make other tests as are:

• the calculation of the eigenvalues of the resulting system obtained when coupling the optimal controller to the given plant;

• the calculation of the stability radius structured or unstructured in the sense of Hinrichsen and Prichard according to the formula $r=(\|E(sI-A_R)^{-1}D\|_\infty)^{-1}$ [6],where A_R is the matrix corresponding to the closed-loop system $GK(I-GK)^{-1}$ and the matrices

D and E are chosen in order to introduce the affine perturbations of the form $A_R+D\Delta E$ with Δ unknown.

In the following we present the results obtained when applying the method of optimal controllers described in Section 4,for examples 2 and 3 from [9,chapter 7].

Example 1 It is a loop-shaping design example for the attitude control of a flexible space platform.The nominal plant G is given in Appendix and it has 10 states,4 inputs and 5 outputs[9,p.193]; the shaping function considered is $W=diag\{4200W_c;4200W_c;4200I_3\}$ where $W_c:=(s+0.1)/s$ [9,p.145];thus a shaped system G_s with 12 states has been obtained.

When applying for G_s the algorithm described in Section 4 we obtain the optimal robust controller K_∞.We use for this controller the test described at the end of the previous section; the optimal robustness radius equals for this example 0.26320135805525,while the H^∞-norm of the system obtained when coupling the controller to the corresponding fictious plant equals 0.26320135805398.

We remark that our optimal robust controller K_∞ has 11 states, while the one obtained in [9] after reductions has only 10 states, but we mention the fact that the matrices describing our final controller $K=WK_\infty$ has the largest coefficient $2.7703\cdot10^4$ while for the suboptimal controller given in [9,p.195] these value is $6.3086\cdot10^5$.A realization of the final optimal controller K and the eigenvalues of the closed-loop system obtained when coupling this controller to G,are given in Appendix.

The unstructured stability radius in the sense of Hinrichsen-Prichard,computed for $D=E=I_{23x23}$ is $0.2999\cdot10^{-4}$.

Figures 1a and 1b show $1/(\overline{\sigma}(GK(I-GK)^{-1}))$ and $1/(\overline{\sigma}(K(I-GK)^{-1}))$ respectively,determined with the optimal robust controller.

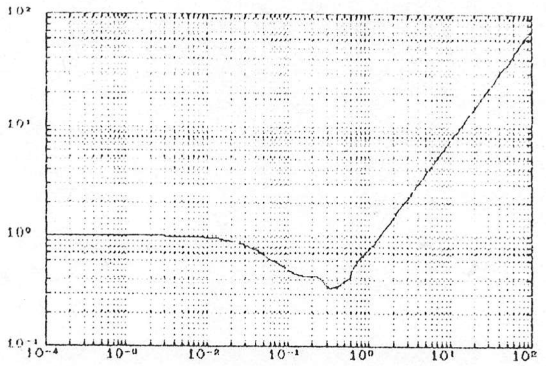

Figure 1a: $1/(\overline{\sigma}(GK(I-GK)^{-1})$

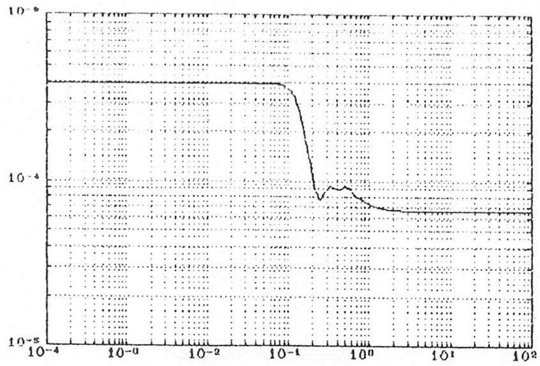

Figure 1b: $1/\overline{\sigma}(K(I-GK)^{-1})$

Example 2 This example is a loop-shaping problem for an aircraft having a nominal model given in Appendix(see also [9, p.198]),for which a shaping function $W=diag\{24W_c,12W_c,24W_c\}$ with $W_c=(s+0.4)/s$ has been chosen ([9,p.163]),obtaining thus a shaped system $G_s=WG$ with 8 states,3 inputs and 3 outputs.The optimal stability radius for this system is 0.37860962986518.

Using for G_s the algorithm described in Section 4,it results a robust optimal controller K_*;after coupling it to the fictious corresponding plant one obtains a resulting system which H^∞-norm equals 0.37860962986501.

The final optimal controller $K=WK_*$ has 10 states,while the one obtained in [9,p.199] after reduction has 9 states.The largest element in the realization of our optimal controller K is $3.1745\cdot10^3$,while for the suboptimal controller given in [9,p.199] this value is $9.1670\cdot10^3$.

The realization for the final optimal controller K determined by applying the algorithm from Section 4,and the eigenvalues of the resulting system obtained when coupling K to G are given in Appendix.

The unstructured stability radius in the sense of Hinrichsen-Prichard,computed for $D=E=I_{15x15}$ is 0.1304.

Figures 2a and 2b show $1/(\overline{\sigma}(GK(I-GK)^{-1}))$ and $1/(\overline{\sigma}(K(I-GK)^{-1}))$ respectively,determined with the optimal robust controller.

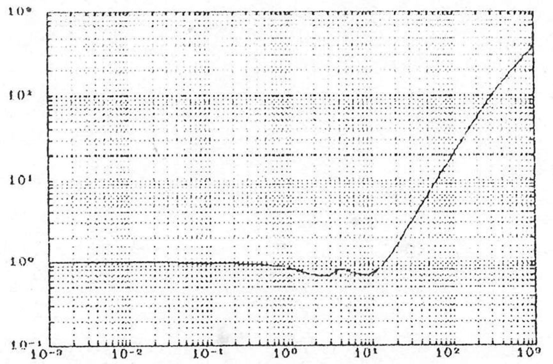

Figure 2a:$1/(\bar{\sigma}(GK(I-GK)^{-1})$

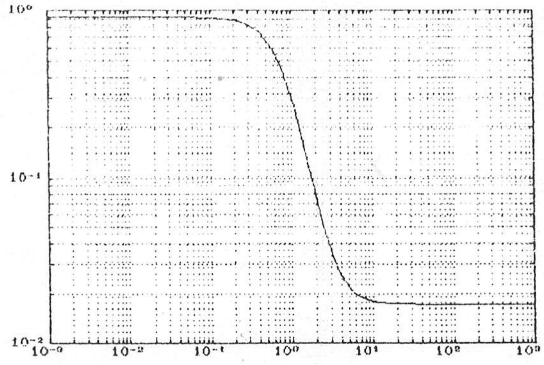

Figure 2b:$1/\bar{\sigma}(K(I-GK)^{-1})$

6.References

[1] A.K. El Sakkary-The gap metric:Robustness of stabilization of feedback systems,*IEEE Trans.Aut.Control*-30(1985),240-247;

[2] T.T.Georgiou,M.C.Smith-Optimal Robustness in the Gap Metric, *Preprint*,1989;

[3] K.Glover-All optimal Hankel-norm approximations of linear multivariable systems and their L^{∞}-error bounds,*Int.J.Control*, 1984,vol.39,8,1115-1193;

[4] L.Habets-Robust Stabilization in the Gap topology,*Springer-Verlag*,1991;

[5] A.Halanay,A.Stoica-An iterative procedure to compute H^{∞}-norms, *submitted for publication*;

[6] D.Hinrichsen,A.J.Prichard-Real and Complex Stability Radii;A Survey,*Control of Uncertain Systems;Preceedings of an International Workshop,Bremen,West Germany,June 1989*;Birkhäuser, 1990,119-162;

[7] E.A.Jonckheere,J.C.Huang,L.M.Silverman-Spectral Theory of the Linear-Quadratic and H^{∞}-Problems,*Linear Algebra and Its Applications*,122/124(1989),273-300;

[8] E.A.Jonckheere,L.M.Silverman-A New Set of Invariants for Linear Systems,*IEEE Trans.Aut.Control*,AC-28,10(1983),953-962;

[9] D.C.McFarlane,K.Glover-Robust Control Design Using Normalized Coprime Factor Plant Descriptions,*Springer-Verlag*,1990.

Appendix

Example 1

G(s):=[A,B,C,D] where

A=

Columns 1 through 5

0.0000e+000	1.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	1.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	-1.1870e-001
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000

Columns 6 through 10

0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
1.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
-6.9000e-003	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	1.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	-2.8190e-001	-1.0600e-002	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	1.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	-5.8050e-001	-1.5200e-002

B=

0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
1.7000e-003	-1.0000e-003	-1.3100e-002	2.0000e-004
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
1.8000e-003	1.1000e-003	6.4000e-003	6.7000e-003
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
1.3000e-003	5.0000e-004	-1.1400e-002	8.8000e-003
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
2.8000e-003	3.0000e-004	5.8000e-003	3.8000e-003
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
3.8000e-003	-1.5000e-003	2.5000e-003	-6.3000e-003

```
C=

Columns 1 through 5

  1.7000e-003  0.0000e+000  1.8000e-003  0.0000e+000  1.3000e-003
 -1.0000e-003  0.0000e+000  1.1000e-003  0.0000e+000  5.0000e-004
  0.0000e+000  1.7000e-003  0.0000e+000  1.8000e-003  0.0000e+000
  0.0000e+000 -1.0000e-003  0.0000e+000  1.1000e-003  0.0000e+000
  0.0000e+000  1.7000e-003  0.0000e+000  1.8000e-003  0.0000e+000

Columns 6 through 10

  0.0000e+000  2.8000e-003  0.0000e+000  3.8000e-003  0.0000e+000
  0.0000e+000  3.0000e-004  0.0000e+000 -1.5000e-003  0.0000e+000
  1.3000e-003  0.0000e+000  2.8000e-003  0.0000e+000  3.8000e-003
  5.0000e-004  0.0000e+000  3.0000e-004  0.0000e+000 -1.5000e-003
  6.4000e-003  0.0000e+000 -6.0000e-004  0.0000e+000 -5.5000e-003

D=

  0.0000e+000  0.0000e+000  0.0000e+000  0.0000e+000
  0.0000e+000  0.0000e+000  0.0000e+000  0.0000e+000
  0.0000e+000  0.0000e+000  0.0000e+000  0.0000e+000
  0.0000e+000  0.0000e+000  0.0000e+000  0.0000e+000
  0.0000e+000  0.0000e+000  0.0000e+000  0.0000e+000
```

K(s):=(AK,BK,CK,CK) where

AK=
Columns 1 through 5

0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
-4.4423e+002	-7.2311e+002	-5.9339e-002	8.7054e-003	-1.4304e-002
-8.2733e+001	1.8843e+002	1.9489e-002	-7.4569e-002	4.0956e-004
8.9876e+002	4.6327e+002	4.7633e-002	3.8062e-002	-2.4314e-002
-5.5275e+002	4.6051e+002	3.0429e-002	8.0291e-002	2.6429e-001
-1.2899e+003	-7.6209e+002	3.5626e-002	8.1879e-002	6.0754e-002
1.6463e+002	-3.5450e+001	-6.4700e-002	-1.4877e-002	-1.0510e-001
-2.0593e+003	-2.0885e+003	7.9983e-002	6.2393e-003	4.2556e-002
1.2136e+003	1.3760e+003	7.6166e-017	1.5624e-018	-1.2854e-016
-2.7703e+003	-1.7457e+003	-2.9758e-016	-8.3374e-017	3.0887e-016
-5.5711e+002	-1.9642e+003	-1.5075e-016	1.6571e-017	2.2189e-016
7.1383e+002	-6.5693e+002	-1.0325e-017	-2.8992e-017	1.1645e-018

Columns 6 through 10

0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
-6.8791e-002	-5.0108e-002	4.1755e-002	-6.7776e-001	-1.5690e-001
-1.0430e-001	-1.1510e-001	2.9941e-002	5.8912e-002	5.1666e-003
-2.4760e-001	1.3240e-002	1.0844e-001	7.6827e-001	1.2539e-001
-3.8758e-002	-1.7637e-001	-2.5821e-001	-2.5028e-002	1.4094e-001
9.3811e-002	-9.2973e-002	-1.0440e-001	-1.0522e+000	-3.8228e-001
3.0854e-001	9.3368e-002	-4.3697e-004	7.4968e-002	3.7269e-001
-1.3899e-001	-2.7186e-001	-2.9045e-002	-2.3960e+000	-3.9675e-001
-9.7531e-002	2.4729e-001	-3.3793e-001	1.2887e+000	2.9903e-001
-6.5077e-002	1.7765e-001	1.1029e-001	-2.7160e+000	-4.5947e-001
5.4588e-002	-1.0373e-001	-5.5718e-001	-1.3137e+000	-3.1364e-001
1.7608e-001	2.3479e-001	2.6317e-002	1.1729e-001	-3.2197e-002

Columns 11 through 13

0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000
2.4341e-001	5.0341e-002	-1.7475e-001
2.2427e-002	4.1548e-002	2.1069e-001
-2.5811e-001	1.7033e-002	-2.6494e-001
8.8794e-002	2.4726e-001	4.2356e-001
5.3335e-002	1.8972e-001	1.3203e-001
-1.4321e-001	3.8285e-001	-1.8339e-001
9.6883e-001	-6.1123e-003	-2.9552e-003
-4.7508e-001	5.8928e-002	8.6894e-002
4.3500e-001	2.3561e-001	7.0744e-001
4.9252e-001	-9.5093e-001	-9.8025e-001
-2.4277e-001	3.4038e-002	-1.0045e+000

BK=

1.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	1.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
-4.4423e+003	-7.2311e+003	-1.7725e+003	-1.5054e+003	-1.5980e+003
-8.2733e+002	1.8843e+003	-3.3685e+002	5.9055e+002	-1.4267e+002
8.9876e+003	4.6327e+003	3.2704e+003	1.2341e+003	9.6722e+002
-5.5275e+003	4.6051e+003	-1.8249e+003	1.1422e+003	1.2623e+003
-1.2899e+004	-7.6209e+003	-5.1733e+003	-1.5072e+003	-3.4121e+003
1.6463e+003	-3.5450e+002	-4.2136e+002	-9.3721e+001	1.4991e+002
-2.0593e+004	-2.0885e+004	-7.4044e+003	-4.4034e+003	-3.7160e+003
1.2136e+004	1.3760e+004	3.7456e+003	2.9929e+003	2.1757e+003
-2.7703e+004	-1.7457e+004	-1.2703e+004	-3.6122e+003	-3.1964e+003
-5.5711e+003	-1.9642e+004	-1.9256e+003	-5.5549e+003	-7.4891e+003
7.1383e+003	-6.5693e+003	2.2709e+003	-1.4744e+003	-1.1848e+003

CK=

Columns 1 through 5

-4.1999e+002	5.0018e+002	-1.0345e-016	-5.6453e-017	3.3653e-017
1.1883e+002	-1.4152e+002	-8.1845e-019	-4.9640e-018	-4.2589e-018
8.0881e+002	-9.6324e+002	-1.5302e-016	2.1010e-017	2.5160e-017
-2.0888e+001	2.4877e+001	8.1637e-018	-3.4491e-017	-1.2560e-017

Columns 6 through 10

-7.0704e-017	5.7432e-017	3.9681e-017	2.0051e-017	-6.3205e-004
-7.4593e-018	-3.2266e-017	1.2065e-018	-2.1292e-017	-3.4213e-003
-3.4196e-018	-2.4608e-017	1.4207e-017	7.1293e-017	1.8709e-004
-1.2702e-017	-2.5984e-017	2.8386e-018	-9.5545e-018	5.2554e-004

Columns 11 through 13

2.2621e-001	-9.6133e-002	5.1731e-001
-2.4585e-002	7.6900e-002	-1.5260e-001
1.1143e-001	-4.9132e-002	-1.0961e+000
7.2327e-002	4.0249e-001	1.8911e-002

DK=

-4.1999e+003	5.0018e+003	-2.0122e+003	1.5312e+003	-6.4373e+002
1.1883e+003	-1.4152e+003	5.6934e+002	-4.3324e+002	1.8214e+002
8.0881e+003	-9.6324e+003	3.8752e+003	-2.9488e+003	1.2397e+003
-2.0888e+002	2.4877e+002	-1.0008e+002	7.6157e+001	-3.2017e+001

The eigenvalues of the closed-loop system G coupled with K:

-0.1230+0.7365i;-0.1230-0.7365i;-0.1485+0.7098i;-0.1485-0.7098i;
-0.0750+0.5303i;-0.0750-0.5303i;-0.0947+0.5198i;-0.0947-0.5198i;
-0.3383+0.2155i;-0.3383-0.2155i;-0.0612+0.2240i;-0.0612-0.2240i;
-0.0824+0.2444i;-0.0824-0.2444i;-0.1933+0.2238i;-0.1933-0.2238i;
-0.6967;-0.5231;-0.3089;-0.1064;-0.1006;-0.999;-0.0997.

Example 2

G(s):=[A,B,C,D] where

A=

0.0000e+000	0.0000e+000	1.1320e+000	0.0000e+000	-1.0000e+000
0.0000e+000	-5.3800e-002	-1.7120e-001	0.0000e+000	7.0500e-002
0.0000e+000	0.0000e+000	0.0000e+000	1.0000e+000	0.0000e+000
0.0000e+000	4.8500e-002	0.0000e+000	-8.5560e-001	-1.0130e+000
0.0000e+000	-2.9090e-001	0.0000e+000	1.0532e+000	-6.8590e-001

B=

0.0000e+000	0.0000e+000	0.0000e+000
-1.2000e-001	1.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000
4.4190e+000	0.0000e+000	-1.6650e+000
1.5750e+000	0.0000e+000	-7.3200e-002

C=

1.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	1.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	1.0000e+000	0.0000e+000	0.0000e+000

D=

0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000

K(s):=(AK,BK,CK,DK) where

AK=

Columns 1 through 5

0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
1.2952e+001	4.6424e-001	-2.0074e+001	-3.9202e-001	-6.4827e-004
-1.4299e+001	-4.5280e-001	1.5746e+001	-1.9510e-002	-3.7092e-001
-3.5898e-001	-5.7122e-003	-3.6848e+000	-4.2943e-005	1.1136e-002
-5.1044e+002	-1.8016e+001	7.5694e+002	-2.7812e-001	2.3896e-001
8.5445e+002	2.7923e+001	-1.2698e+003	1.9060e-015	8.6184e-003
6.3553e+001	4.3998e+001	-1.1478e+002	-1.6373e-015	-2.8901e-002
6.6080e+001	2.6076e+000	-7.3006e+000	2.8272e-016	-2.6479e-001

Columns 6 through 10

0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000	0.0000e+000
-9.5670e-005	5.2292e+000	1.4016e+000	-5.3309e-002	1.9048e+000
2.4541e-002	-4.8436e+000	-1.4814e+000	-1.9970e-001	-1.1678e+000
-3.7152e-001	6.9480e-001	2.5686e-001	-4.3824e-001	6.3866e-001
3.0918e-002	-1.9657e+002	-5.0518e+001	5.3737e+000	-6.3289e+001
-1.1010e-001	3.2280e+002	7.0366e+001	-8.8226e+000	1.0216e+002
1.3572e+000	2.7031e+001	7.4885e+000	-3.0426e+001	9.1730e+000
-1.5172e-001	7.5562e+000	-1.9017e-001	-1.4963e+000	-7.3467e+000

BK=

1.0000e+000	0.0000e+000	0.0000e+000
0.0000e+000	1.0000e+000	0.0000e+000
0.0000e+000	0.0000e+000	1.0000e+000
3.2379e+001	1.1606e+000	-5.0186e+001
-3.5747e+001	-1.1320e+000	3.9366e+001
-8.9746e-001	-1.4280e-002	-9.2119e+000
-1.2761e+003	-4.5039e+001	1.8923e+003
2.1361e+003	6.9808e+001	-3.1745e+003
1.5888e+002	1.0999e+002	-2.8694e+002
1.6520e+002	6.5191e+000	-1.8251e+001

```
CK=

Columns 1 through 5

  1.4535e+000  3.0591e-002  9.5448e-001 -3.5807e-016 -2.5312e-016
 -6.4976e-001 -1.3675e-002 -4.2669e-001  7.9251e-017 -5.9574e-017
  1.9540e+001  4.1125e-001  1.2832e+001  1.4518e-016  2.3765e-016

Columns 6 through 10

  1.1164e-016  1.9207e-016  1.5088e+000  2.9914e-001  2.8630e-002
 -1.4730e-016 -3.7730e-017  2.1918e-001 -2.0573e+000  4.9882e-002
  8.4142e-016  3.9031e-018  1.5265e-002 -2.6525e-002 -3.5460e+000

DK=

  3.6337e+000  7.6477e-002  2.3862e+000
 -1.6244e+000 -3.4188e-002 -1.0667e+000
  4.8850e+001  1.0281e+000  3.2079e+001
```

The eigenvalues of the closed-loop system G coupled with K:

-7.6840+7.7107i;-7.6840-7.7107i;-2.3730+2.3477i;-2.3730-2.3477i;
-0.3995+0.0023i;-0.3995-0.0023i;-99.7302;-17.2173;-11.8992;-12.0501;
-3.3435;-0.3962;-0.4000;-0.3994;-0.3999;