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Dualities associated to binary operations on $\overline{R}$

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Summary. Continuing our papers [14]-[16] and [6]-[9], where we have given an axiomatic approach to generalized conjugation theory, we introduce and study dualities $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ associated to a binary operation $*$ on $\overline{R}$, where X and W are two arbitrary sets and $\overline{R} = [-\infty, +\infty]$, which encompass, as particular cases, conjugations, $\vee$-dualities and $\perp$-dualities in the sense of [14] and [7]. We show that this class of dualities can be extended so as to encompass also the $*$-dualities $\Delta : \overline{A}^X \longrightarrow \overline{A}^W$ in the sense of [8], where $\overline{A}$ is the canonical enlargement of a complete totally ordered group.

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0. Introduction

Since various concepts of conjugation have important applications to duality in optimization theory, an axiomatic approach to generalized conjugation theory was started in [14] and continued in [15], [16] and [6]-[9]. Let us recall that if X and W are two sets (which we shall assume non-empty throughout the sequel), a mapping $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ (where $\overline{R}^X$ denotes the family of all functions $f : X \longrightarrow \overline{R} = [-\infty, +\infty]$) is called

a) a *duality* ([16], [6]), if for any index set I we have

$$(\inf_{i\in I} f_i)^\Delta = \sup_{i\in I} f_i^\Delta \qquad (\{f_i\}_{i\in I} \subseteq \overline{R}^X), \qquad (0.1)$$

where $\inf_{i\in I} f_i \in \overline{R}^X$ and $\sup_{i\in I} f_i^\Delta \in \overline{R}^W$ are defined pointwise on X and W respectively (i.e., $(\inf_{i\in I} f_i)(x) = \inf_{i\in I} f_i(x)$ for all $x \in X$), with the usual conventions

$$\inf \emptyset = +\infty, \qquad \sup \emptyset = -\infty; \tag{0.2}$$

b) a *conjugation* [14], if we have (0.1) and

$$(f \dot{+} d)^\Delta = f^\Delta \underset{\bullet}{+} -d \qquad (f \in \overline{R}^X, d \in \overline{R}), \tag{0.3}$$

where we identify each $d \in \overline{R}$ with the constant function taking everywhere the value d, the operations $\dot{+}$ and $\underset{\bullet}{+}$ on $\overline{R}^X$ (respectively, $\overline{R}^W$) are defined pointwise, and the binary operations $\dot{+}$ and $\underset{\bullet}{+}$ on $\overline{R}$ are the "upper addition" and "lower addition" defined ([10], [11]) by

$$a \dot{+} b = a \underset{\bullet}{+} b = a + b \text{ if } R \cap \{a, b\} \neq \emptyset \text{ or } a = b = \pm\infty, \tag{0.4}$$

$$a \dot{+} b = +\infty,\ a \underset{\bullet}{+} b = -\infty \text{ if } a = -b = \pm\infty; \tag{0.5}$$

c) a $\vee$-*duality* [7], if we have (0.1) and

$$(f \vee d)^\Delta = f^\Delta \wedge -d \qquad (f \in \overline{R}^X, d \in \overline{R}), \tag{0.6}$$

where $\vee$ and $\wedge$ stand for (pointwise) sup and inf, in $\overline{R}^X$ and $\overline{R}^W$ respectively;

d) a $\perp$-*duality* [7], if we have (0.1) and

$$(f \perp d)^\Delta = f^\Delta \top - d \qquad (f \in \overline{R}^X, d \in \overline{R}), \tag{0.7}$$

where $\perp$ and $\top$ are the binary operations defined [6] on $\overline{R}$ by

$$a \perp b = \begin{cases} a & \text{if } a < b \\ +\infty & \text{if } a \geq b \end{cases}, \tag{0.8}$$

$$a \top b = \begin{cases} a & \text{if } a > b \\ -\infty & \text{if } a \leq b \end{cases}, \tag{0.9}$$

and extended pointwise to $\overline{R}^X$ and $\overline{R}^W$.

Let us also recall (see e.g. [16], [6]) that if $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ is a duality, then so is the dual mapping $\Delta' : \overline{R}^W \longrightarrow \overline{R}^X$ defined by

$$g^{\Delta'} = \inf_{\substack{h \in \overline{R}^X \\ h^\Delta \leq g}} h \qquad (g \in \overline{R}^W), \tag{0.10}$$

and for any $f \in \overline{R}^X$ and $g \in \overline{R}^W$ we have the equivalence

$$f^{\Delta} \leq g \Longleftrightarrow g^{\Delta'} \leq f. \tag{0.11}$$

In the above mentioned papers, among other results, various "representation theorems" have been given for dualities (with the aid of functions $G : X \times W \times \overline{R} \longrightarrow \overline{R}$), conjugations, $\vee$-dualities and $\perp$-dualities (with the aid of "coupling functions" from $X \times W$ into $\overline{R}$) $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ and for their duals $\Delta' : \overline{R}^W \longrightarrow \overline{R}^X$, as well as for the "second dual" $f^{\Delta\Delta'} = (f^{\Delta})^{\Delta'} \in \overline{R}^X$ of a function $f \in \overline{R}^X$. Also, it has been shown there that the dual $\Delta' : \overline{R}^W \longrightarrow \overline{R}^X$ of a conjugation $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ is a conjugation, but the dual of a $\vee$-duality is a $\perp$-duality (actually, this was the main motivation for introducing in [6] the operations $\perp$ and $\top$ and the concept of $\perp$-dualities) and the dual of a $\perp$-duality is a $\vee$-duality.

Furthermore, in the recent paper [8], we have generalized the theory of conjugations (0.1), (0.3) to certain mappings $\Delta : \overline{A}^X \longrightarrow \overline{A}^W$, called (in [8]) *$*$-dualities*, where $\overline{A} = (\overline{A}, \leq, *, *)$ is the "canonical enlargement" of a complete totally ordered group $A = (A, \leq, *)$, which contain, as particular cases, the conjugations $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ and various kinds of known polarities (e.g., polarities $\Delta : \overline{R}_+^X \longrightarrow \overline{R}_+^W$ in the sense of Moreau [10], p. 92, formula (14.4), Rockafellar [13], p. 136, Elster and Wolf [4], with applications to fractional programming duality, etc.).

In the present paper, continuing to develop these axiomatic approaches to generalized conjugation theory, we shall introduce and study a kind of dualities $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$, namely, *dualities associated to a binary operation $*$ on $\overline{R}$* satisfying "condition (α)" (i.e., condition (1.1) below), called, briefly, *$*$-dualities*, which encompass, as particular cases, the conjugations, $\vee$-dualities and $\perp$-dualities mentioned above. Also, we shall show how this theory can be extended to the case when $\overline{R}$ is replaced by the canonical enlargement $\overline{A}$ of a complete totally ordered group $A = (A, \leq, *)$, so as to encompass also the "$*$-dualities" in the sense of [8], as particular cases.

In Section 1 we shall introduce the class of binary operations $*$ on $\overline{R}$ satisfying *condition* (α) (defined by (1.1)), which contain, as particular cases, the binary operations $+$, $\vee$ and $\perp$ mentioned above. Also, we shall introduce and study "*the (left) epi-hypo-inverse*" $*_l$ and "*the (left) conjugate*" $\bar{*}$ of such a binary operation $*$, which will be needed in the sequel.

In Section 2 we shall introduce the concept of a duality $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ with respect to a binary operation $*$ on $\overline{R}$ satisfying condition (α), called, briefly, a *$*$-duality*, with the aid of a suitable "second condition" (besides (0.1)), namely, condition (2.1), encompassing, among other particular cases, (0.3), (0.6) and (0.7). Also, we shall determine the dual of a $*$-duality, from which one recovers, in particular, the above mentioned results of [14] and [7] on the duals of conjugations, $\vee$-dualities and $\perp$-dualities.

In Section 3 we shall obtain some results on the representations of $*$-dualities $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ and of their duals $\Delta' : \overline{R}^W \longrightarrow \overline{R}^X$, with the aid of coupling functions $\psi : X \times W \longrightarrow \overline{R}$, which contain, as particular cases, the results of [14] and [7] on the representation of conjugations, $\vee$-dualities and $\perp$-dualities, mentioned above.

Finally, in Section 4 (Appendix) we shall show that the concept of $*$-duality of the present paper can be extended to a more general notion of "$(*, s)$-duality", which encompasses, as particular cases, also the "$*$-dualities" $M : \overline{A}^X \longrightarrow \overline{A}^W$, in the sense of [8], where $\overline{A} = (\overline{A}, \leq, *, *)$ is the "canonical enlargement" of a complete totally ordered group $A = (A, \leq, *)$. Thus, we shall obtain a unifying framework for the results of the present paper and those of [8].

1. Inverses and conjugates of binary operations on $\overline{R}$

Definition 1.1. We shall say that a binary operation $*$ on $\overline{R}$ satisfies *condition* (α), if for any index set I we have

$$(\inf_{i \in I} b_i) * c = \inf_{i \in I} (b_i * c) \qquad (\{b_i\}_{i \in I} \subseteq \overline{R}, c \in \overline{R}). \tag{1.1}$$

Remark 1.1. a) For $I = \emptyset$, condition (1.1) yields

$$+\infty * c = +\infty \qquad (c \in \overline{R}). \tag{1.2}$$

b) For each $c \in \overline{R}$, define $k_c : \overline{R} \longrightarrow \overline{R}$ by

$$k_c(b) = b * c \qquad (b \in \overline{R}). \tag{1.3}$$

Then, condition (α) means that for any index set I we have

$$k_c(\inf_{i \in I} b_i) = \inf_{i \in I} k_c(b_i) \qquad (\{b_i\}_{i \in I} \subseteq \overline{R}, c \in \overline{R}), \tag{1.4}$$

or, *equivalently* (see [6], lemma 2.1), that for each $c \in \overline{R}$ the function k_c *is non-decreasing and upper semi-continuous.*

Let us give now some examples of binary operations $*$ on $\overline{R}$ satisfying condition (α).

Example 1.1. Let $* = \dot{+}$. Then, by [11], formula (4.7), $*$ satisfies condition (α).

Example 1.2. Let $* = \vee$. Then it is well-known (and immediate) that $*$ satisfies condition (α).

Example 1.3. Let $* = \perp$ (of (0.8)). Then, by [7], formula (1.21), $*$ satisfies condition (α).

We shall denote by min (respectively, max), an inf (respectively, sup) which is attained.

Proposition 1.1. *Let $*$ be a binary operation on $\overline{R}$, satisfying condition (α). Then there exists a unique binary operation $*_l$ on $\overline{R}$ such that for any $a, b, c \in \overline{R}$ we have the equivalence*

$$a \leq b * c \Longleftrightarrow a *_l c \leq b, \tag{1.5}$$

namely,

$$a *_l c = \min \{b' \in \overline{R} \mid a \leq b' * c\} \qquad (a, c \in \overline{R}). \tag{1.6}$$

Proof. For any $a, c \in \overline{R}$ we have $\inf \{b' \in \overline{R} \mid a \leq b' * c\} = \inf \{b' \in \overline{R} \mid a \leq k_c(b')\}$ (with k_c of (1.3)), which is attained, since the set $\{b' \in \overline{R} \mid a \leq k_c(b')\}$ is closed (because k_c is upper semi-continuous, by remark 1.1 b)). Thus, the min in (1.6) exists.

Let us show now that the binary operation $*_l$ on $\overline{R}$ defined by (1.6) satisfies (1.5).

If $a \leq b * c$, then $b \in \{b' \in \overline{R} \mid a \leq b' * c\}$, whence, by (1.6), $a *_l c \leq b$. Conversely, assume now that $a *_l c = \min \{b' \in \overline{R} \mid a \leq b' * c\} \leq b$, so there exists $b' \in \overline{R}$ such that $a \leq b' * c = k_c(b')$, $b' \leq b$. Then, since k_c is non-decreasing (by remark 1.1 b)), we obtain $a \leq k_c(b') \leq k_c(b) = b * c$.

Finally, if we have (1.5), then for any $a, c \in \overline{R}$ we have

$$a *_l c = \min \{b' \in \overline{R} \mid a *_l c \leq b'\} = \min \{b' \in \overline{R} \mid a \leq b' * c\}. \qquad \sharp$$

Definition 1.2. Let $*$ be a binary operation on $\overline{R}$, satisfying condition (α). Then the unique binary operation $*_l$ on $\overline{R}$, of proposition 1.1, will be called *the (left) epi-hypo-inverse of $*$.*

Remark 1.2. a) The theory of inversion of functions $k : R \longrightarrow \overline{R}$ (see e.g. [2], pp. 208-211, and [12]) can be easily extended to functions $k : \overline{R} \longrightarrow \overline{R}$, which is, in fact, its natural framework. Then, for any $c \in \overline{R}$, since k_c of (1.3) is non-decreasing and upper semi-continuous, it admits (see e.g. [12], proposition 2.6, extended to this framework), a unique "epi-hypo-inverse", i.e., a unique function $j_c : \overline{R} \longrightarrow \overline{R}$ such that for any $a, b \in \overline{R}$ we have the equivalence

$$a \leq k_c(b) \Longleftrightarrow j_c(a) \leq b, \tag{1.7}$$

namely,

$$j_c(a) = \min \{b' \in \overline{R} \mid a \leq k_c(b')\} \qquad (a \in \overline{R}). \qquad (1.8)$$

Then, by proposition 1.1, we have

$$j_c(a) = a *_l c \qquad (a \in \overline{R}), \qquad (1.9)$$

which motivates the terminology of definition 1.2.

b) If $*$ is commutative and satisfies condition (α), then, by (1.5),

$$a *_l c \leq b \Longleftrightarrow a *_l b \leq c. \qquad (1.10)$$

c) For a binary operation $*$ on $\overline{R}$, we shall also consider the binary operation $*-$ on $\overline{R}$, defined by

$$a *- c = a * (-c) \qquad (a, c \in \overline{R}). \qquad (1.11)$$

Then, by (1.6) and (1.11), we have $(*-)_l = *_l-$, since

$$a(*-)_l c = \min \{b' \in \overline{R} \mid a \leq b' *- c = b' * (-c)\} = a *_l (-c) \qquad (a, c \in \overline{R}). \qquad (1.12)$$

d) If $*$ satisfies condition (α), then so does $*-$. Since $(*-)- = *$, the converse is also true.

If a binary operation $*$ on $\overline{R}$ satisfies condition (α), then $*_l$ need not satisfy it, but we shall show that $*_l$ has a "dual" property.

Definition 1.3. We shall say that a binary operation $*$ on $\overline{R}$ satisfies *condition* (β), if for any index set I we have

$$(\sup_{i \in I} a_i) * c = \sup_{i \in I} (a_i * c) \qquad (\{a_i\}_{i \in I} \subseteq \overline{R}, c \in \overline{R}). \qquad (1.13)$$

Remark 1.3. If $*$ satisfies condition (β), then, by (1.13) for $I = \emptyset$, we have

$$-\infty * c = -\infty \qquad (c \in \overline{R}). \qquad (1.14)$$

Proposition 1.2 *If a binary operation $*$ on $\overline{R}$ satisfies condition (α), then $*_l$ satisfies condition (β) .*

Proof. By (α), the minima in (1.6) are attained, and hence

$$(\sup_{i \in I} a_i) *_l c = \min \{b' \in \overline{R} \mid \sup_{i \in I} a_i \leq b' * c\} = \min \bigcap_{i \in I} \{b' \in \overline{R} \mid a_i \leq b' * c\} =$$

$$= \sup_{i \in I} \min \{b' \in \overline{R} \mid a_i \leq b' * c\} = \sup_{i \in I} (a_i *_l c) \qquad (\{a_i\}_{i \in I} \subseteq \overline{R}, c \in \overline{R}). \quad \sharp$$

Remark 1.4. a) One can also give the following alternative proof of proposition 1.2: If $a, a', c \in \overline{R}$, $a \leq a'$, then $\{b' \in \overline{R} \mid a' \leq k_c(b')\} \subseteq \{b' \in \overline{R} \mid a \leq k_c(b')\}$, whence, by (1.8), $j_c(a) \leq j_c(a')$, so j_c is *non-decreasing.* Also, by (1.7), for each $b \in \overline{R}$ we have $\{a \in \overline{R} \mid j_c(a) \leq b\} = \{a \in \overline{R} \mid a \leq k_c(b)\}$, which is a closed set, so j_c is *lower semi-continuous.* But, by [6], lemma 2.1, we have these two properties if and only if

$$j_c(\sup_{i \in I} a_i) = \sup_{i \in I} j_c(a_i) \qquad (\{a_i\}_{i \in I} \subseteq \overline{R}, c \in \overline{R}), \qquad (1.15)$$

i.e. (by (1.9)), if and only if $*_l$ satisfies condition (β). ♯

b) By the above, k_c of (1.3) is the "*hypo-epi-inverse*" of j_c (of (1.9)), in the inversion theory (of [2],pp. 208-211, and [12]) extended to functions $k : \overline{R} \longrightarrow \overline{R}$, and therefore one can say that the binary operation $*$ is the *(left) hypo-epi-inverse of* $*_l$, in symbols, $* = (*_l)_u = *_{lu}$. Indeed, from (1.5) we obtain

$$b * c = \max\{a' \in \overline{R} \mid a' \leq b * c\} = \max\{a' \in \overline{R} \mid a' *_l c \leq b\} \qquad (b, c \in \overline{R}). \quad (1.16)$$

More generally, if we start with any $*$ satisfying condition (β), then one can define, in the obvious way, the *(left) hypo-epi-inverse* $*_u$ of $*$, and we have $* = (*_u)_l = *_{ul}$. Also, dually to proposition 1.2, we have that *if a binary operation* $*$ *on* $\overline{R}$ *satisfies condition* (β), *then* $*_u$ *satisfies condition* (α).

c) Concerning the above notations, let us mention that l (and u) stand to indicate the lower (and, respectively, the upper) semi-continuity of $*_l$ (respectively, $*_u$) in the first component. The word "left" in the above terminology is used to indicate that we are dealing with the first component of $*_l$ (respectively, $*_u$); in the sequel, we shall omit the word "left", since this will lead to no confusion. Note also that a binary operation $*$ on $\overline{R}$ satisfying both (α) and (β) (with $*_l$ replaced by $*$) has both inverses $*_l$ and $*_u$, but they need not coincide.

Let us consider now some examples.

Example 1.1 (continued). If $* = \dot{+}$, then $*_l = \underset{\bullet}{+}-$ (since these $*$ and $*_l$ satisfy (1.5), by [11], formula (3.3)).

Example 1.2 (continued). If $* = \vee$, then $*_l = \top$ (since these $*$ and $*_l$ satisfy (1.5), by [7], formula (1.5)).

Example 1.3 (continued). If $* = \perp$, then $*_l = \wedge$ (since these $*$ and $*_l$ satisfy (1.5), by [7], formula (1.6)).

In the sequel, for simplicity, for any binary operation $*$ on $\overline{R}$ we shall write $-a * c$ instead of $(-a) * c$, which will lead to no confusion (with $-(a * c)$).

Definition 1.4. Let $*$ be a binary operation $\overline{R}$. Then the binary operation $\bar{*}$ on $\overline{R}$, defined by

$$a \bar{*} c = -(-a * c) \qquad (a, c \in \overline{R}), \qquad (1.17)$$

will be called *the (left) conjugate of* $*$.

Example 1.1 (continued). If $* = \dot{+}$, then $\bar{*} = \underset{\cdot}{+}-$ (since $-(-a \dot{+} c) = a \underset{\cdot}{+} -c$, by [11], formula (2.2)).

Example 1.2 (continued). If $* = \vee$, then $\bar{*} = \wedge-$ (since $-(-a \vee c) = a \wedge -c$).

Example 1.3 (continued). If $* = \perp$, then $\bar{*} = \top-$ (since $-(-a \perp c) = a \top - c$, by [7], formula (1.12)).

Remark 1.5. a) By (1.17), we have

$$-(a\bar{*}c) = -a * c \qquad (a, c \in \overline{R}), \qquad (1.18)$$

$$a * c = -(-a\bar{*}c) \qquad (a, c \in \overline{R}). \qquad (1.19)$$

b) By (1.17)(applied to $\bar{*}$ instead of $*$) and (1.19), for the "biconjugate" $\bar{\bar{*}} = \overline{(\bar{*})}$ of a binary operation $*$ on $\overline{R}$ we have $\bar{\bar{*}} = *$, since

$$a\bar{\bar{*}}c = -(-a\bar{*}c) = a * c \qquad (a, c \in \overline{R}). \qquad (1.20)$$

c) For any binary operation $*$ on $\overline{R}$ we have

$$\overline{*-} = \bar{*} - . \qquad (1.21)$$

Indeed, by (1.17) and (1.11),

$$a\overline{*-}c = -(-a *- c)) = -(-a * (-c)) = a\bar{*}(-c) = a\bar{*} - c \qquad (a, c \in \overline{R}). \qquad (1.22)$$

Proposition 1.3. *a) A binary operation* $*$ *on* $\overline{R}$ *satisfies* (α) *(respectively,* (β)*) if and only if* $\bar{*}$ *satisfies* (β) *(respectively,* (α)*).*

b) If $*$ *satisfies* (α), *then for any* $a, b, c \in \overline{R}$ *we have the equivalence*

$$a\bar{*}c \leq b \Longleftrightarrow -b *_l c \leq -a. \qquad (1.23)$$

If $*$ *is also commutative, then we also have the equivalence*

$$a\bar{*}c \leq b \Longleftrightarrow -b *_l -a \leq c. \qquad (1.24)$$

c) $*$ *is commutative if and only if* $\bar{*}$ *is "anti-commutative", i.e.,*

$$a\bar{*}c = -c\bar{*} - a \qquad (a, c \in \overline{R}). \qquad (1.25)$$

Proof. a) If $*$ satisfies condition (α), then

$$(\sup_{i \in I} a_i)\bar{*}c = -\left[(-\sup_{i \in I} a_i) * c\right] = -\left[(\inf_{i \in I} (-a_i)) * c\right] =$$

$$= -\inf_{i\in I}(-a_i * c) = \sup_{i\in I}[-(-a_i * c)] = \sup_{i\in I}(a_i \bar{*} c) \qquad (\{a_i\}_{i\in I} \subseteq \overline{R}, c \in \overline{R}),$$

so $\bar{*}$ satisfies condition (β). Dually, interchanging sup and inf, we obtain that if $*$ satisfies (β), then $\bar{*}$ satisfies (α). Hence, if $\bar{*}$ satisfies (β) (respectively, (α)) then $* = \bar{\bar{*}}$ satisfies (α) (respectively, (β)).

b) By (1.17) and (1.5) we have

$$a\bar{*}c \leq b \Longleftrightarrow -(-a * c) \leq b \Longleftrightarrow -a * c \geq -b \Longleftrightarrow -b *_l c \leq -a.$$

If $*$ is also commutative, then, by (1.23) and (1.10), we have (1.24).

c) If $*$ is commutative, then

$$a\bar{*}c = -(-a * c) = -(c * -a) = -c\bar{*} - a \qquad (a, c \in \overline{R}).$$

Dually, *if $*$ is anti-commutative, then $\bar{*}$ is commutative.* Hence, if $\bar{*}$ is anti-commutative, then $* = \bar{\bar{*}}$ is commutative. ♯

Since $\bar{*}$ is defined for any binary operation $*$ (not necessarily satisfying condition (α)), we may consider the conjugate of the epi-hypo-inverse of a binary operation $*$ satisfying condition (α), i.e., the binary operation

$$a\overline{*_l}c = -(-a *_l c) = -\min\{b' \in \overline{R} \mid -a \leq b' * c\} =$$

$$= \max\{-b' \in \overline{R} \mid a \geq -(b' * c)\} = \max\{b \in \overline{R} \mid a \geq -(-b * c)\} =$$

$$= \max\{b \in \overline{R} \mid a \geq b\bar{*}c\} \qquad (a, c \in \overline{R}). \tag{1.26}$$

Remark 1.6. By (1.12) and (1.21) (applied to $*_l$ instead of $*$), for any binary operation $*$ on $\overline{R}$ we have

$$\overline{(*-)_l} = \overline{*_l -} = (\overline{*_l}) - . \tag{1.27}$$

Theorem 1.1. *If a binary operation $*$ on $\overline{R}$ satisfies condition (α), then so does the binary operation $\overline{*_l}$, and we have*

$$(\overline{*_l})_l = \bar{*}, \tag{1.28}$$

$$\overline{(\overline{*_l})_l} = *. \tag{1.29}$$

Proof. By proposition 1.2, $*_l$ satisfies condition (β). Hence, by proposition 1.3, $\overline{*_l}$ satisfies condition (α), and therefore $(\overline{*_l})_l$ is well defined. Then, by (1.6) (applied to $\overline{*_l}$), (1.5) and (1.17), we obtain

$$a(\overline{*_l})_l c = \min\{b' \in \overline{R} \mid a \leq b'\overline{*_l}c\} = \min\{b' \in \overline{R} \mid a \leq -(-b' *_l c)\} =$$

$$= -\max\{b \in \overline{R} \mid -a \geq b *_l c\} = -\max\{b \in \overline{R} \mid b \leq -a * c\} =$$

$$= -(-a * c) = a \bar{*} c \qquad (a, c \in \overline{R}),$$

which proves (1.28). Finally, (1.28) and (1.29) are equivalent (by (1.20)). ♯

Remark 1.7. By the first part of theorem 1.1, applied to $\overline{*_l}$, and by (1.29), *if* $\overline{*_l}$ *satisfies condition* (α), *then so does* $*$.

Example 1.1 (continued). If $* = \dot{+}$, so $*_l = \underset{\cdot}{+}-$, then $\overline{*_l} = \dot{+}$ (since $-(-a \underset{\cdot}{+} -c) = a \dot{+} c$). Hence, by theorem 1.1, $(\overline{*_l})_l = \bar{*} = \underset{\cdot}{+}-$, $\overline{(\overline{*_l})_l} = * = \dot{+}$.

Example 1.2 (continued). If $* = \vee$, so $*_l = \top$, then $\overline{*_l} = \bot-$ (since $-(-a \top c) = a \bot - c$). Hence, by theorem 1.1, $(\overline{*_l})_l = \bar{*} = \wedge-$, $\overline{(\overline{*_l})_l} = * = \vee$.

Example 1.3 (continued). If $* = \bot$, so $*_l = \wedge$, then $\overline{*_l} = \vee-$ (since $-(-a \wedge c) = a \vee -c$). Hence, by theorem 1.1, $(\overline{*_l})_l = \bar{*} = \top-$, $\overline{(\overline{*_l})_l} = * = \bot$.

Starting with $\overline{*_l}$ instead of $*$, from example 1.2 and theorem 1.1 above we obtain

Example 1.4. If $* = \bot- = \overline{\vee_l}$, then $*_l = \wedge-$, $\overline{*_l} = \vee$. Hence, $(\overline{*_l})_l = \top$, $\overline{(\overline{*_l})_l} = \bot-$.

Remark 1.8. a) Example 1.4 differs from example 1.3 only by the minus signs, but we shall use it in Section 3. Similarly, starting with $* = \vee- = \overline{\bot_l}$, one obtains an example which differs from example 1.2 only by the minus sign (however, we shall not use it in the sequel).

b) The following table summarizes examples 1.1-1.4 above:

	Example 1.1	Example 1.2	Example 1.3	Example 1.4
$*$	$\dot{+}$	$\vee$	$\bot$	$\bot-$
$*_l$	$\underset{\cdot}{+}-$	$\top$	$\wedge$	$\wedge-$
$\overline{*_l}$	$\dot{+}$	$\bot-$	$\vee-$	$\vee$
$(\overline{*_l})_l = \bar{*}$	$\underset{\cdot}{+}-$	$\wedge-$	$\top-$	$\top$

Each binary operation $*$ on $\overline{R}$ can be extended to $\overline{R}^X$, where X is any set, as follows.

Definition 1.4. For any $f, h \in \overline{R}^X$, let

$$(f * h)(x) = f(x) * h(x) \qquad (x \in X). \qquad (1.30)$$

2. $*$-dualities and their duals

Definition 2.1. Let X and W be two sets and let $*$ be a binary operation on $\overline{R}$. A duality $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ (see (0.1)) will be called a *$*$-duality*, if

$$(f * d)^\Delta = f^\Delta \bar{*} d \qquad (f \in \overline{R}^X, d \in \overline{R}), \qquad (2.1)$$

where we identify each $d \in \overline{R}$ with the constant function $h_d \in \overline{R}^X$ defined by $h_d(x) = d \ \ (x \in X)$.

Example 2.1. Let $* = \dot{+}$. Then $\bar{*} = \underset{\cdot}{+}-$ (see example 1.1), and thus condition (2.1) means that we have (0.3), i.e., that $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ is a conjugation.

Example 2.2. Let $* = \vee$. Then $\bar{*} = \wedge-$ (see example 1.2), and thus condition (2.1) means that we have (0.6), i.e., that $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ is a $\vee$-duality.

Example 2.3. Let $* = \perp$. Then $\bar{*} = \top\dot{-}$ (see example 1.3), and thus condition (2.1) means that we have (0.7), i.e., that $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ is a $\perp$-duality.

Remark 2.1. If Δ is a $*$-duality, then it is also a $(*-)$-duality. Indeed, by (1.11), (2.1) and (1.21), we have

$$(f *- d)^\Delta = (f * (-d))^\Delta = f^\Delta \bar{*} - d = f \overline{*-} d \qquad (f \in \overline{R}^X, d \in \overline{R}).$$

Proposition 2.1. *Let X and W be two sets and let $*$ be a commutative binary operation on $\overline{R}$, satisfying condition (α). Then, a duality $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ is a $*$-duality if (and only if) we have (2.1) for all $d \in R$.*

Proof. Assume that $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ is a duality, satisfying (2.1) for all $d \in R$. Then, by the commutativity of $*$, (1.2) and (0.1) for $I = \emptyset$ (with the conventions (0.2)), we have

$$(f * +\infty)^\Delta = (+\infty * f)^\Delta = +\infty^\Delta = -\infty \qquad (f \in \overline{R}^X). \ (2.2)$$

On the other hand, since $\bar{*}$ satisfies condition (β) (by proposition 1.3 a)) we have, by (1.25) and (1.14) (applied to $\bar{*}$),

$$f^\Delta \bar{*} + \infty = -\infty \bar{*} - f^\Delta = -\infty \qquad (f \in \overline{R}^X). \qquad (2.3)$$

Thus, by (2.2) and (2.3), we have (2.1) for $d = +\infty$.

Finally, by the commutativity of $*$, (1.1), (0.1), (2.1) for all $d \in R$, (1.25) and (1.13) for $\bar{*}$ (by proposition 1.3 a)), we obtain

$$(f * -\infty)^{\Delta} = (-\infty * f)^{\Delta} = ((\inf_{d\in R} d) * f)^{\Delta} = (\inf_{d\in R} (d * f))^{\Delta} =$$

$$= (\inf_{d\in R} (f * d))^{\Delta} = \sup_{d\in R} (f * d)^{\Delta} = \sup_{d\in \overline{R}} (f^{\Delta} \bar{*} d) = \sup_{d\in \overline{R}} (-d \bar{*} - f^{\Delta}) =$$

$$= (\sup_{d\in R} (-d)) \bar{*} - f^{\Delta} = +\infty \bar{*} - f^{\Delta} = f^{\Delta} \bar{*} - \infty \qquad (f \in \overline{R}^X),$$

so Δ satisfies (2.1) also for $d = -\infty$. ♯

Theorem 2.1. *Let X and W be two sets and let $*$ be a binary operation on $\overline{R}$, satisfying condition (α).*

a) If $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ is a $*$-duality, then its dual $\Delta' : \overline{R}^W \longrightarrow \overline{R}^X$ is a $\overline{*_l}$-duality.

b) If $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ is a $\overline{*_l}$-duality, then its dual $\Delta' : \overline{R}^W \longrightarrow \overline{R}^X$ is a $*$-duality.

Proof. a) If Δ is a $*$-duality, then so is Δ' (see Section 0), and, by (0.10), (1.23), (2.1), (0.11) and (1.5), we have

$$(g \overline{*_l} d)^{\Delta'} = \inf_{h^{\Delta} \le g \overline{*_l} d} h = \inf_{h^{\Delta} \le -(-g *_l d)} h = \inf_{-h^{\Delta} \ge -g *_l d} h =$$

$$= \inf_{g \ge h^{\Delta} \bar{*} d} h = \inf_{g \ge (h * d)^{\Delta}} h = \inf_{g^{\Delta'} \le h * d} h =$$

$$= \inf_{g^{\Delta'} *_l d \le h} h = g^{\Delta'} *_l d = g^{\Delta'} \bar{\bar{*}}_l d \qquad (g \in \overline{R}^W, d \in \overline{R}).$$

b) If Δ is a $\overline{*_l}$-duality, then, by part a) (applied to $\overline{*_l}$ instead of $*$) and (1.29), Δ' is a $*$-duality. ♯

Corollary 2.1. *Let X and W be two sets and let $*$ be a binary operation on $\overline{R}$, satisfying condition (α). Then*

a) Every $*$-duality is the dual of a $\overline{*_l}$-duality.

b) Every $\overline{*_l}$-duality is the dual of a $*$-duality.

Proof. a) If Δ is a $*$-duality, then $\Delta = (\Delta')'$ (see Section 0), where Δ' is a $\overline{*_l}$-duality (by theorem 2.1 a)).

b) If Δ is a $\overline{*_l}$-duality, then $\Delta = (\Delta')'$, where Δ' is a $*$-duality (by theorem 2.1 b)). ♯

Corollary 2.2 *Let X, W and $*$ be as above. Then*

a) A mapping $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ is a $*$-duality if and only if Δ' is a $\overline{*_l}$-duality.

b) A mapping $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ is a $\overline{*_l}$-duality if and only if Δ' is a $*$-duality.

Remark 2.2. a) For $* = \dot{+}$, $\vee$, or $\perp$, theorem 2.1 yields again that the dual of a conjugation, or $\vee$-duality, or $\perp$-duality, is a conjugation, or a $\perp$-duality, or a

$\vee$-duality, respectively (see Section 0). Similar remarks can be made for corollaries 2.1 and 2.2.

b) One can generalize definition 2.1 as follows. Let K be a family of non-decreasing upper semi-continuous functions $k : \overline{R} \longrightarrow \overline{R}$. A duality $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ is a K-*duality*, if

$$(k \circ f)^\Delta = -k \circ (-f^\Delta) \qquad (k \in K, f \in \overline{R}^X). \tag{2.4}$$

Then, in particular, Δ *is a* $*$*-duality if and only if it is a* K_0*-duality, where*

$$K_0 = \{k_c \mid c \in \overline{R}\}, \tag{2.5}$$

with k_c *of* (1.3). Indeed, by (1.3) and (1.30), we have

$$(k_c \circ f)(x) = k_c(f(x)) = f(x) * c = (f * c)(x) \qquad (f \in \overline{R}^X, c \in \overline{R}, x \in X),$$

whence

$$(k_c \circ f)^\Delta = (f * c)^\Delta \qquad (f \in \overline{R}^X, c \in \overline{R}), \tag{2.6}$$

and, on the other hand, by (1.3), (1.17) and (1.30), we have

$$(-k_c \circ (-f^\Delta))(w) = -k_c(-f^\Delta(w)) = -(-f^\Delta(w) * c) = f^\Delta(w) \bar{*} c =$$

$$= (f^\Delta \bar{*} c)(w) \qquad (f \in \overline{R}^X, c \in R, w \in W); \tag{2.7}$$

thus, (2.4) (for K_0 of (2.5)) is equivalent to (2.1). ♯

3. Representations of $*$-dualities and their duals, with the aid of coupling functions

Definition 3.1. Let $*$ be a binary operation on $\overline{R}$. An element $e \in \overline{R}$ is called

a) a *left neutral element for* $*$, if

$$e * c = c \qquad (c \in \overline{R}); \tag{3.1}$$

b) a *right neutral element for* $*$, if

$$c * e = c \qquad (c \in \overline{R}); \tag{3.2}$$

c) a *neutral element for* $*$, if it is both a left and a right neutral element for $*$.

Note that a neutral element is necessarily unique.

Example 3.1. Let $* = \dot{+}$. Then $e = 0$ is the neutral element for $*$.

Example 3.2. Let $* = \vee$. Then $e = -\infty$ is the neutral element for $*$.

Example 3.3. Let $* = \perp$. Then $e = +\infty$ is the (unique) right neutral element for $*$ (by [7], formula (1.15)), but there exists no left neutral element for $*$.

Definition 3.2. Let X be a set and let $*$ be a binary operation on $\overline{R}$, which admits a left (or right) neutral element e. Then, for any subset S of X, the *generalized indicator function of* S *(with respect to* e *)* is the function $\chi_S : X \longrightarrow \{e, +\infty\}$ defined by

$$\chi_S(y) = \begin{cases} e & \text{if } y \in S \\ +\infty & \text{if } y \in X \setminus S. \end{cases} \tag{3.3}$$

Example 3.1 (continued). If $* = \dot{+}$, so $e = 0$, then χ_S is the usual indicator function of S.

Example 3.2 (continued). If $* = \vee$, so $e = -\infty$, then χ_S is the "representation function" of S, introduced by Flachs and Pollatschek [5].

Lemma 3.1 *Let* X *be a set and let* $*$ *be a binary operation on* $\overline{R}$*, satisfying (1.2) and admitting a left neutral element* e*. Then, for any function* $f \in \overline{R}^X$ *we have*

$$f = \inf_{x \in X} \{\chi_{\{x\}} * f(x)\}, \tag{3.4}$$

where $\chi_{\{x\}}$ *is the generalized indicator function of the singleton* $\{x\}$.

Proof. By (3.3), (3.1) and (1.2) we have, for any $x, y \in X$,

$$\chi_{\{x\}}(y) * f(x) = \begin{cases} e * f(x) = f(x) = f(y) & \text{if } x = y \\ +\infty * f(x) = +\infty & \text{if } x \neq y, \end{cases}$$

whence

$$\inf_{x \in X} \{\chi_{\{x\}}(y) * f(x)\} = \inf \{f(y), +\infty\} = f(y) \qquad (y \in X). \quad \sharp$$

We recall (see [11]) that if X and W are two sets, then every function $\psi : X \times W \longrightarrow \overline{R}$ is called a *coupling function.*

Theorem 3.1. *Let* X *and* W *be two sets and let* $*$ *be a binary operation on* $\overline{R}$*, satisfying (1.2) and admitting a left neutral element* e*. Then for each* $*$*-duality* $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ *there exists a coupling function* $\psi : X \times W \longrightarrow \overline{R}$*, for example,*

$$\psi(x, w) = (\chi_{\{x\}})^{\Delta}(w) \qquad (x \in X, w \in W), \tag{3.5}$$

such that we have

$$f^{\Delta}(w) = \sup_{x \in X} \{\psi(x, w) \bar{*} f(x)\} \qquad (f \in \overline{R}^X, w \in W). \tag{3.6}$$

Moreover, if $*$ *is also commutative, then* ψ *of (3.5) is unique (i.e., the unique coupling function for which we have (3.6)).*

Proof. By lemma 3.1 and definition 2.1, for any $*$-duality $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ we have

$$f^\Delta = (\inf_{x\in X} \{\chi_{\{x\}} * f(x)\})^\Delta = \sup_{x\in X}\{(\chi_{\{x\}})^\Delta \bar{*} f(x)\} \qquad (f \in \overline{R}^X),$$

i.e., (3.6), with ψ of (3.5).

Moreover, if e is a neutral element for $*$ and $x \in X$, then, applying (3.6) to $f = \chi_{\{x\}}$ and using (1.17), (3.3), (3.2) and (1.2), we obtain

$$(\chi_{\{x\}})^\Delta(w) = \sup_{x'\in X} \{\psi(x', w)\bar{*}\chi_{\{x\}}(x')\} = \sup_{x'\in X} \{-(-\psi(x', w) * \chi_{\{x\}}(x'))\} =$$

$$= -(-\psi(x, w)) = \psi(x, w) \qquad (w \in W). \quad \sharp$$

Remark 3.1. a) If $*$ is a binary operation on $\overline{R}$, satisfying (1.2) and admitting a left neutral element e, then, by (3.3), (3.1) and (1.2), for any $x, y \in X$ and any $d \in \overline{R}$ we have

$$\chi_{\{x\}}(y) * d = \begin{cases} e * d = d & \text{if } x = y \\ +\infty * d & \text{if } x \neq y. \end{cases} \qquad (3.7)$$

Now, by part of [6], theorem 3.1, for any duality $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ we have

$$f^\Delta(w) = \sup_{x\in X} G_\Delta(x, w, f(x)) \qquad (f \in \overline{R}^X, w \in W), \qquad (3.8)$$

where $G_\Delta : X \times W \times \overline{R} \longrightarrow \overline{R}$ is the function defined by

$$G_\Delta(x, w, d) = (\varphi_{x,d})^\Delta(w) \qquad (x \in X, w \in W, d \in \overline{R}), \qquad (3.9)$$

with $\varphi_{x,d} : X \longrightarrow \overline{R}$ defined by

$$\varphi_{x,d}(y) = \begin{cases} d & \text{if } x = y \\ +\infty & \text{if } x \neq y. \end{cases}$$

But, by (3.10) and (3.7), we have

$$\varphi_{x,d} = \chi_{\{x\}} * d \qquad (x \in X, d \in \overline{R}), \qquad (3.11)$$

and hence, if Δ is a $*$-duality, then, by (3.9), (3.11) and (2.1),

$$G_\Delta(x, w, d) = (\chi_{\{x\}} * d)^\Delta(w) = (\chi_{\{x\}})^\Delta(w)\bar{*}d \quad (x \in X, w \in W, d \in \overline{R}), \qquad (3.12)$$

which, together with (3.8), yields again (3.6), with ψ of (3.5).

b) One can also prove that if $*$ and Δ are as in a) above, then $\psi : X \times W \longrightarrow \overline{R}$ of (3.5) is the unique coupling function satisfying, for any index set I,

$$-\psi(x,w) * \inf_{i\in I} d_i = \inf_{i\in I} \{-\psi(x,w) * d_i\} \qquad (x \in X, w \in W, \{d_i\}_{i\in I} \subseteq \overline{R}). \quad (3.13)$$

In the converse direction to theorem 3.1, we have

Theorem 3.2. *Let X and W be two sets, $*$ a binary operation on $\overline{R}$, $\psi : X\times W \longrightarrow \overline{R}$ a coupling function and $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ the mapping defined by (3.6).*

a) If $*$ is commutative and satisfies condition (α), then Δ is a duality.

b) If $*$ is associative, then Δ satisfies (2.1).

Hence, if $*$ is commutative, associative and satisfies condition (α), then Δ is a $*$-duality.

Proof. a) For any $\{f_i\}_{i\in I} \subseteq \overline{R}^X$ and $w \in W$ we have, by (3.6), (1.17), the commutativity of $*$, and (1.1),

$$(\inf_{i\in I} f_i)^\Delta(w) = \sup_{x\in X} \{\psi(x,w)\bar{*}\inf_{i\in I} f_i(x)\} = \sup_{x\in X} \{-\{-\psi(x,w) * \inf_{i\in I} f_i(x)\}\} =$$

$$= \sup_{x\in X} \{-\{\inf_{i\in I} f_i(x) * -\psi(x,w)\}\} = -\inf_{x\in X} \{\inf_{i\in I} f_i(x) * -\psi(x,w)\} =$$

$$= -\inf_{x\in X} \{\inf_{i\in I} \{f_i(x) * -\psi(x,w)\}\} = -\inf_{i\in I} \{\inf_{x\in X} \{f_i(x) * -\psi(x,w)\}\} =$$

$$= \sup_{i\in I} \{-\inf_{x\in X} \{f_i(x) * -\psi(x,w)\}\} = \sup_{i\in I} \sup_{x\in X} \{-(f_i(x) * -\psi(x,w))\} =$$

$$= \sup_{i\in I} \sup_{x\in X} \{-(-\psi(x,w) * f_i(x))\} = \sup_{i\in I} \{\psi(x,w)\bar{*}f_i(x)\} = \sup_{i\in I} f_i^\Delta(w).$$

b) For any $f \in \overline{R}^X$, $d \in \overline{R}$ and $w \in W$ we have, by (3.6), (1.30), (1.17) and the associativity of $*$,

$$(f * d)^\Delta(w) = \sup_{x\in X} \{\psi(x,w)\bar{*}(f(x) * d)\} = \sup_{x\in X} \{-\{-\psi(x,w) * (f(x) * d)\}\} =$$

$$= \sup_{x\in X} \{-((-\psi(x,w) * f(x)) * d)\} = \sup_{x\in X} \{(\psi(x,w)\bar{*}f(x)) * d\} =$$

$$= f^\Delta(w)\bar{*}d = (f^\Delta\bar{*}d)(w). \quad \|$$

Remark 3.2. One can also prove that Δ of (3.6) is a duality whenever ψ satisfies (3.13) (even if $*$ is not commutative or does not satisfy condition (α)).

Proposition 3.1. *Under the assumptions of theorem 3.2 a), we have*

$$f^{\Delta}(w) = \min_{\substack{d\in\overline{R} \\ -d*_l-\psi(\cdot,w)\leq f}} d \qquad (f \in \overline{R}^X, w \in W). \tag{3.14}$$

Proof. By (3.6) and (1.24), for any $f \in \overline{R}^X$ and $w \in W$ we have

$$f^{\Delta}(w) = \min_{\substack{d\in\overline{R} \\ f^{\Delta}(w)\leq d}} d = \min_{\substack{d\in\overline{R} \\ \psi(\cdot,w)\dot{*}f\leq d}} d = \min_{\substack{d\in\overline{R} \\ -d*_l-\psi(\cdot,w)\leq f}} d. \quad \sharp$$

Remark 3.3. For $* = \dot{+}$ and, respectively, $* = \vee$, proposition 3.1 yields [14], proposition 3.1 and, respectively, [7], corollary 2.2.

Definition 3.3. We shall say that a binary operation $*$ on $\overline{R}$ satisfies *condition* (r), if $*$ is commutative, associative and admits a neutral element e.

From theorems 3.1 and 3.2, we obtain

Theorem 3.3. *Let X and W be two sets and let $*$ be a binary operation on $\overline{R}$, satisfying conditions (α) and (r). For a mapping $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$, the following statements are equivalent:*

1°. *Δ is a $*$-duality.*

2°. *There exists a coupling function $\psi : X \times W \longrightarrow \overline{R}$, such that we have (3.6).*

Moreover, in this case ψ of 2° is unique, namely, it is the function (3.5).

Remark 3.4. a) One can prove that the equivalence 1° $\Longleftrightarrow$ 2° also holds for an associative binary operation $*$ on $\overline{R}$ satisfying (1.2) and having a neutral element (instead of satisfying conditions (α) and (r)). Under these assumptions, if Δ is a $*$-duality, then ψ of (3.5) is the unique coupling function satisfying (3.13) and such that we have (3.6).

b) By theorem 3.3 and a) above, for $*$ satisfying conditions (α) and (r) (or, alternatively, being associative, satisfying (1.2) and having a neutral element), we have a one-to-one correspondence between $*$-dualities $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ and coupling functions $\psi : X \times W \longrightarrow \overline{R}$. We shall call $\Delta = \Delta(*, \psi)$ of (3.6) (respectively, $\psi = \psi_{\Delta,*}$ of (3.5)) *the $*$-duality associated to the coupling function ψ* (respectively, *the coupling function associated to the $*$-duality Δ*).

c) In particular, for $* = \dot{+}$ and, respectively, $* = \vee$ (which satisfy conditions (α) and (r)), from theorem 3.3 we obtain again the results of [14] and [7] on the relations between conjugations, respectively, $\vee$-dualities, and coupling functions ([14], example 2.1 and theorem 3.1 and, respectively, [7], example 2.1 and theorem 2.1).

d) By (3.7), one can replace (3.4) of lemma 3.1 by

$$f = \inf_{(x,d)\in \mathrm{Epi}\, f} \{\chi_{\{x\}} * d\}, \tag{3.15}$$

where Epi $f = \{(x,d) \in X \times R \mid f(x) \leq d\}$, the epigraph of f. Then, by the above arguments, using (3.15) and (0.1) with $I =$ Epi f (which is $\emptyset$ for $f = +\infty$), we obtain, for any $*$-duality Δ and any $f \in \overline{R}^X$,

$$f^{\Delta} = \sup_{(x,d)\in \mathrm{Epi}\, f} \{(\chi_{\{x\}})^{\Delta} \bar{*} d\} = \sup_{(x,d)\in \mathrm{Epi}\, f} \{\psi_{\Delta}(x,w) \bar{*} d\}. \tag{3.16}$$

Let us consider now the dual mappings Δ' (defined by (0.10)).

Theorem 3.4. *Let X and W be two sets, $*$ a commutative binary operation on $\overline{R}$, satisfying condition (α), $\psi : X \times W \longrightarrow \overline{R}$ a coupling function, and $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ the mapping defined by (3.6). Then*

$$g^{\Delta'}(x) = \sup_{w\in W} \{-g(w) *_l -\psi(x,w)\} \qquad (g \in \overline{R}^W, x \in X). \tag{3.17}$$

Proof. By (0.10), (3.6) and (1.24), we have

$$g^{\Delta'} = \inf_{h^{\Delta}\leq g} h = \inf_{\psi(\cdot,\cdot)\bar{*}h\leq g} h = \inf_{-g*_l-\psi(\cdot,\cdot)\leq h} h \qquad (g \in \overline{R}^W), \tag{3.18}$$

whence

$$g^{\Delta'} \geq \sup_{w\in W} \{-g *_l -\psi(\cdot,w)\} \qquad (g \in \overline{R}^W). \tag{3.19}$$

On the other hand, for any $g \in \overline{R}^W$, the function h_g defined by

$$h_g(x) = \sup_{w\in W} \{-g(x) *_l -\psi(x,w)\} \qquad (x \in X),$$

belongs to the set $\{h \in \overline{R}^X \mid -g *_l - \psi(\cdot,\cdot) \leq h\}$, whence, by (3.18), we obtain $g^{\Delta'} \leq h_g$, which, together with (3.19), yields (3.17). ∥

Under the assumptions of theorem 3.4, Δ of (3.6) is a duality (by theorem 3.2), and hence so is Δ' of (3.17); however, we do not know whether Δ is a $*$-duality. In the next result we obtain the same conclusion (3.17), with different assumptions on $*$ and Δ.

Theorem 3.5. *Let X and W be two sets, $*$ a binary operation on $\overline{R}$ satisfying condition (α) and admitting a left neutral element e, $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ a $*$-duality and $\psi : X \times W \longrightarrow \overline{R}$ the coupling function (3.5). Then we have (3.17) (and, by theorem 3.1, we have also (3.6)).*

Proof. By part of [6], theorem 3.5, for any duality $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ we have

$$g^{\Delta'}(x) = \sup_{w \in W} G_{\Delta'}(w, x, g(w)) \qquad (g \in \overline{R}^W, x \in X), \tag{3.20}$$

where

$$G_{\Delta'}(w, x, b) \doteq \min_{\substack{a \in \overline{R} \\ G_\Delta(x,w,a) \leq b}} a \qquad (w \in W, x \in X, b \in \overline{R}), \tag{3.21}$$

with G_Δ of (3.9), (3.10). But, since now $*$ satisfies (1.2) and admits a left neutral element e, and since Δ is a $*$-duality, we have (3.12) (see remark 3.1). Thus, by (3.21), (3.12), (1.24) and (3.5), we obtain

$$G_{\Delta'}(w, x, b) = -b *_l -\psi(x, w) \qquad (w \in W, x \in X, b \in \overline{R}), \tag{3.22}$$

which, together with (3.20), yields (3.17). ♯

Theorem 3.6. *Let X and W be two sets, $*$ a commutative binary operation on $\overline{R}$, and $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ a duality for which there exist a unique coupling function $\psi = \psi_{\Delta,*} : X \times W \longrightarrow \overline{R}$ such that*

$$f^\Delta(w) = \sup_{x \in X} \{\psi_{\Delta,*}(x, w) \bar{*} f(x)\} \qquad (f \in \overline{R}^X, w \in W), \tag{3.23}$$

and a coupling function $\psi : X \times W \longrightarrow \overline{R}$ such that $\Delta' : \overline{R}^W \longrightarrow \overline{R}^X$ satisfies (3.17). Then ψ of (3.17) is unique, namely, we have

$$\psi = \psi_{\Delta,*}. \tag{3.24}$$

Proof. By $\Delta = (\Delta')'$, (0.10) (applied to Δ' instead of Δ), (3.17) and (1.24), we have

$$f^\Delta(w) = \inf_{\substack{g \in \overline{R}^W \\ g^{\Delta'} \leq f}} g(w) = \inf_{\substack{g \in \overline{R}^W \\ \sup_{x \in X}\{\psi(x,w) \bar{*} f(x)\} \leq g}} g(w) =$$
$$= \sup_{x \in X} \{\psi(x, w) \bar{*} f(x)\} \qquad (f \in \overline{R}^X, x \in W),$$

which by our assumption of uniqueness of $\psi_{\Delta,*}$ in (3.23), implies (3.24). ♯

From theorems 3.1, 3.4 and 3.6 we obtain

Theorem 3.7 *Let X and W be two sets and let $*$ be a commutative binary operation on $\overline{R}$, satisfying condition (α) and admitting a neutral element e. Then, for each $*$-duality $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ there exists a unique coupling function $\psi : X \times W \longrightarrow \overline{R}$*

such that we have (3.17). Namely, ψ coincides with the unique coupling function for which we have (3.6), i.e., ψ is the function (3.5).

Remark 3.5. In particular, for $* = \dot{+}$ and $* = \vee$, from theorem 3.7 we obtain again the results of [14] and [7] on the representation of conjugations, $\vee$-dualities and their duals, with the aid of coupling functions.

Let us consider now, for a $*$-duality $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$, the "second dual" (called also the $\Delta'\Delta$-hull) $f^{\Delta\Delta'} = (f^{\Delta})^{\Delta'} \in \overline{R}^X$ of a function $f \in \overline{R}^X$.

Theorem 3.8. *Under the assumptions of theorem 3.7, for any $*$-duality Δ : $\overline{R}^X \longrightarrow \overline{R}^W$ we have*

$$f^{\Delta\Delta'}(x) = \sup_{w\in W} \{-f^{\Delta}(w) *_l -\psi(x,w)\} =$$

$$= \sup_{w\in W} \min_{\substack{b\in\overline{R} \\ \psi(x,w)\bar{*}b\leq f^{\Delta}(w)}} b \qquad (f \in \overline{R}^X, x \in X), \tag{3.25}$$

with $\psi : X \times W \longrightarrow \overline{R}$ of (3.5).

Proof. The first equality follows from (3.17) applied to $g = f^{\Delta}$. Furthermore, by (1.6), the commutativity of $*$ and (1.17), for any $f \in \overline{R}^X$, $x \in X$ and $w \in W$ we have

$$-f^{\Delta}(w) *_l -\psi(x,w) = \min \{b \in \overline{R} \mid -f^{\Delta}(w) \leq b * -\psi(x,w) = -\psi(x,w) * b\} =$$

$$= \min \{b \in \overline{R} \mid f^{\Delta}(w) \geq -(-\psi(x,w) * b) = \psi(x,w)\bar{*}b\},$$

which yields the second equality in (3.25). ♯

Theorem 3.9. *Under the assumptions of theorem 3.5, for any $*$-duality Δ : $\overline{R}^X \longrightarrow \overline{R}^W$ we have*

$$f^{\Delta\Delta'}(x) = \sup_{\substack{w\in W, b\in\overline{R} \\ b*_l-\psi(\cdot,w)\leq f}} \{b *_l -\psi(x,w)\} \qquad (f \in \overline{R}^X, x \in X), \tag{3.26}$$

with $\psi : X \times W \longrightarrow \overline{R}$ of (3.5).

Proof. By [6], theorem 3.6, for any duality $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ we have

$$f^{\Delta\Delta'}(x) = \sup_{\substack{w\in W, b\in\overline{R} \\ G_{\Delta'}(w,\cdot,b)\leq f}} G_{\Delta}(w,x,b) \qquad (f \in \overline{R}^X, x \in X), \tag{3.27}$$

with $G_{\Delta'}$ of (3.21), where G_{Δ} is that of (3.9), (3.10). But, by the above proof of theorem 3.5, we have now (3.22), which, together with (3.27), yields (3.26). ♯

Remark 3.6. a) Theorem 3.9 shows that, under the assumptions of theorem 3.7, for any $*$-duality $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ the $\Delta'\Delta$-hull of f coincides with the "Φ -convex hull" of f, in the sense of [3], where

$$\Phi = \{b *_l -\psi(\cdot, w) \mid w \in W, b \in \overline{R}\}, \tag{3.28}$$

or, in other words, that for any $*$-duality $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$, the "elementary functions", in a sense similar to that of [11], are the functions $\gamma_{w,b} = b *_l -\psi(\cdot w) \in \overline{R}^X \quad (w \in W, b \in \overline{R})$.

b) In particular, for $* = \dot{+}$ and $* = \vee$, from theorems 3.8 and 3.9 we obtain again the main results of [14] and [7] on the representation of second conjugates and second $\vee$-duals of f, with the aid of coupling functions.

Let us observe now that *the above results can be "dualized" as follows*: Let X and W be two sets, $*$ a binary operation on $\overline{R}$, satisfying condition (α), and $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$. Then $\overline{*_l}$ satisfies condition (α) and $\Delta' : \overline{R}^W \longrightarrow \overline{R}^X$ is a $\overline{*_l}$-duality (by theorems 1.1 and 2.1 a)). Hence, replacing the assumptions of the above results by the same assumptions on $\overline{*_l}$, and using (1.20), (1.28), we obtain representations of Δ' and $\Delta = (\Delta')'$ with the aid of the coupling function $\psi' : W \times X \longrightarrow \overline{R}$ defined by

$$\psi'(w, x) = (\chi_{\{w\}})^{\Delta'}(x) \qquad (w \in W, x \in X), \tag{3.29}$$

or, equivalently, with the aid of the coupling function $\psi : X \times W \longrightarrow \overline{R}$ defined by

$$\psi(x, w) = \psi'(w, x) = (\chi_{\{w\}})^{\Delta'}(x) \qquad (x \in X, w \in W). \tag{3.30}$$

For example, dualizing in this way theorem 3.1, we arrive at

Theorem 3.10. *Let X and W be two sets and let $*$ be a binary operation on $\overline{R}$, such that $\overline{*_l}$ admits a left neutral element e, $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ a $*$-duality, and $\psi : X \times W \longrightarrow \overline{R}$ the coupling function (3.30). Then we have*

$$g^{\Delta'}(x) = \sup_{w \in W} \{\psi(x, w) *_l g(w)\} \qquad (g \in \overline{R}^W, x \in X). \tag{3.31}$$

Moreover, if $\overline{*_l}$ is also commutative, then ψ of (3.30) is the only coupling function for which we have (3.31).

Similarly, dualizing theorem 3.7, we arrive at

Theorem 3.11. *Let X and W be two sets and let $*$ be a binary operation on $\overline{R}$, satisfying condition (α) and such that $\overline{*_l}$ is commutative, satisfies condition (α) and admits a neutral element e. Then for each $*$-duality $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ there exists a* *unique coupling function* $\psi : X \times W \longrightarrow \overline{R}$, *namely,* ψ *of (3.30), such that*

$$f^{\Delta}(w) = \sup_{x \in X} \{-f(x) \bar{*} - \psi(x,w)\} \qquad (f \in \overline{R}^X, w \in W). \qquad (3.32)$$

Moreover, the same ψ *is the unique coupling function for which we have (3.31).*

Remark 3.7. In particular, let $* = \perp -$. Then $*$ satisfies condition (α) (by remark 1.2 d)) and $\overline{*_l} = \vee$ (see example 1.4), so $\overline{*_l}$ satisfies the assumptions of theorem 3.11. Also, $\bar{*} = \top$ and $*_l = \wedge -$ (see example 1.4). Hence, for $* = \perp -$, from theorem 3.11, combined with remark 2.1, we obtain again the results of [7] on the representation of $\perp$-dualities and their duals, with the aid of coupling functions. However, note that in [7] we have also obtained another expression for the coupling function ψ of (3.30) (see [7], formula (3.9)), by exploiting the *special* properties of $\perp$ and $\top$, and this has also implied another expression for the coupling function ψ occurring in theorem 3.1, i.e., for ψ of (3.5) (see [7], formula (4.10)).

Proposition 3.2. *Under the assumptions of theorem 3.11, we have (3.14).*

Proof. By (3.32) and (1.23), for any $f \in \overline{R}^X$ and $w \in W$ we have

$$f^{\Delta}(w) = \min_{\substack{d \in \overline{R} \\ f^{\Delta}(w) \leq d}} d = \min_{\substack{d \in \overline{R} \\ -f\bar{*}-\psi(\cdot,w) \leq d}} d = \min_{\substack{d \in \overline{R} \\ -d *_l - \psi(\cdot,w) \leq f}} d. \quad \sharp$$

Remark 3.8. For $* = \perp -$ we have $*_l = \wedge -$ (see example 1.4), so proposition 3.2, combined with remark 2.1, yields again [7], corollary 3.1.

Finally, let us consider the second duals $f^{\Delta\Delta'}$.

Theorem 3.12. *Under the assumptions of theorem 3.11, for any* $*$*-duality* $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ *we have*

$$f^{\Delta\Delta'}(x) = \sup_{w \in W} \{\psi(x,w) *_l f^{\Delta}(w)\} = \sup_{w \in W} \min_{\substack{b \in \overline{R} \\ -b\bar{*}-\psi(x,w) \leq f^{\Delta}(w)}} b \quad (f \in \overline{R}^X, x \in X), \qquad (3.33)$$

with $\psi : X \times W \longrightarrow \overline{R}$ *of (3.30).*

Proof. The first equality follows from (3.31) applied to $g = f^{\Delta}$. Furthermore, by proposition 1.3 c) (applied to $\overline{*_l}$), $*_l = \overline{*_l}$ is anti-commutative. Hence, by (1.6) and (1.19),

$$\psi(x,w) *_l f^{\Delta}(w) = -f^{\Delta}(w) *_l -\psi(x,w) = \min\{b \in \overline{R} \mid -f^{\Delta}(w) \leq b * -\psi(x,w)\} =$$

$$= \min\{b \in \overline{R} \mid -f^{\Delta}(w) \leq -(-b\bar{*}-\psi(x,w))\} = \min\{b \in \overline{R} \mid f^{\Delta}(w) \geq -b\bar{*}-\psi(x,w)\},$$

which yields the second equality in (3.33). $\sharp$

Theorem 3.13. *Under the assumptions of theorem 3.11, for any $*$-duality $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ we have*

$$f^{\Delta\Delta'}(x) = \sup_{\substack{w\in W, b\in\overline{R} \\ \psi(\cdot,w)*_l b\leq f}} \{\psi(x,w) *_l b\} \qquad (f \in \overline{R}^X, w \in W), \tag{3.34}$$

with $\psi : X \times W \longrightarrow \overline{R}$ of (3.30).

Proof. Since Δ' is a $\overline{*_l}$-duality (by theorem 2.1 a)), we have, by (3.12) (applied to Δ' and $\bar{*}_l$), (1.20) (for $*_l$) and (3.30),

$$G_{\Delta'}(w, x, b) = (\chi_{\{w\}})^{\Delta'}(x) *_l b = \psi(x, w) *_l b \qquad (w \in W, x \in X, b \in \overline{R}), \tag{3.35}$$

which, together with (3.27), yields (3.34). ♯

Remark 3.9. a) One can make an observation similar to remark 3.5, with Φ of (3.28) replaced by

$$\Phi = \{\psi(\cdot, w) *_l b \mid w \in W, b \in \overline{R}\}, \tag{3.36}$$

and with the "elementary functions" $\gamma_{w,b} = \psi(\cdot, w) *_l b \quad (w \in W, b \in \overline{R})$.

b) In particular, for $* = \perp-$ we have $*_l = \wedge-$ (see example 1.4), and thus theorems 3.12 and 3.13, combined with remark 2.1, yield again the main results of [7] on the representation of second $\perp$-duals of f, with the aid of coupling functions.

4. Appendix: A Unifying framework for the above results and those of [8]

Let us first recall some concepts from [8].

Let $A = (A, \leq, *)$ be a *complete totally ordered group*, i.e. (see e.g. [1], Ch. 14) a set endowed with a total order $\leq$ such that $(A, \leq)$ is a conditionally complete lattice (that is, every non-empty order-bounded subset of A admits a supremum and an infimum in A) and with a binary operation $*$ for which $(A, *)$ is a group, such that all group translations are isotone; then, by a result of Iwasawa (see e.g. [1], Ch. 14, theorem 20), $*$ is commutative. In the paper [8], assuming that A is not a singleton, we have adjoined to it a greatest element $+\infty$ and a least element $-\infty$, i.e., we have considered the set

$$\overline{A} = A \cup \{+\infty\} \cup \{-\infty\}, \tag{4.1}$$

with the order $\leq$ extended to $\overline{A}$ by

$$-\infty \leq a \leq +\infty \qquad (a \in \overline{A}), \tag{4.2}$$

and we have extended the binary operation $*$ on A to two different binary operations $\dot{*}$ and $\underset{\bullet}{*}$ on $\overline{A}$ (called *upper* and *lower composition*, respectively), by the rules

$$a \dot{*} b = a \underset{\bullet}{*} b = a * b \qquad (a, b \in A), \tag{4.3}$$

$$+\infty \dot{*} a = a \dot{*} +\infty = +\infty \qquad (a \in \overline{A}), \tag{4.4}$$

$$-\infty \dot{*} a = a \dot{*} -\infty = -\infty \qquad (a \in A \cup \{-\infty\}), \tag{4.5}$$

$$+\infty \underset{\bullet}{*} a = a \underset{\bullet}{*} +\infty = +\infty \qquad (a \in A \cup \{+\infty\}), \tag{4.6}$$

$$-\infty \underset{\bullet}{*} a = a \underset{\bullet}{*} -\infty = -\infty \qquad (a \in \overline{A}). \tag{4.7}$$

Then, $\overline{A} = (\overline{A}, \leq, \dot{*}, \underset{\bullet}{*})$ has been called (in [8]) *the canonical enlargement of* $(A, \leq, *)$. Furthermore, a mapping $M : \overline{A}^X \longrightarrow \overline{A}^W$ has been called ([8], definition 2.3) a $*$-duality, if for any index set I we have

$$(\inf_{i \in I} f_i)^M = \sup_{i \in I} f_i^M \qquad (\{f_i\}_{i \in I} \subseteq \overline{A}^X), \tag{4.8}$$

$$(f \dot{*} a)^M = f^M \underset{\bullet}{*} a^{-1} \qquad (f \in \overline{A}^X, a \in \overline{A}), \tag{4.9}$$

where inf, $\dot{*}$ (in $\overline{A}^X$) and sup, $\underset{\bullet}{*}$ (in $\overline{A}^W$) are defined pointwise on $\overline{A}$, each $a \in \overline{A}$ is identified with the constant function $f_a(x) = a$ $(x \in X)$, and if $a \in A$, then a^{-1} denotes the inverse of a in the Abelian group $(A, *)$, while the "inverses" of $a \in \overline{A} \backslash A$ are defined by $(+\infty)^{-1} = -\infty$, $(-\infty)^{-1} = +\infty$. In particular, clearly, for $A = R$, with the usual total order $\leq$ on R and with $* = +$, the usual addition on R, $\dot{*}$ and $\underset{\bullet}{*}$ are nothing else than the upper and lower additions (0.4), (0.5) on $\overline{R}$ and the $*$-dualities are the conjugations (0.1), (0.3).

Now we can give the following unifying framework for the results of the present paper and those of [8].

Definition 4.1. Let $(\overline{A}, \leq)$ be a complete chain (i.e., a complete lattice, where $\leq$ is a total order on $\overline{A}$), and let $s : (\overline{A}, \leq) \longrightarrow (\overline{A}, \leq)$ be a bijective duality (i.e., a bijective mapping $s : \overline{A} \longrightarrow \overline{A}$ such that $s(\inf_{i \in I} a_i) = \sup_{i \in I} s(a_i)$ for every index set I and every family $\{a_i\}_{i \in I} \subseteq A$). Given a binary operation $*$ on $\overline{A}$, we define a new binary operation $*^s$ on $\overline{A}$, called *the s-conjugate of* $*$, by

$$a *^s c = s(s(a) * c) \qquad (a, c \in \overline{A}). \tag{4.10}$$

Remark 4.1. a) If $\overline{A} = \overline{R}$, endowed with the usual total order $\leq$ and if $s : (\overline{R}, \leq) \longrightarrow (\overline{R}, \leq)$ is the mapping defined by

$$s(a) = -a \qquad (a \in \overline{R}), \tag{4.11}$$

then s is a bijective duality and, by (4.10) and (1.17), for any binary operation $*$ on $\overline{R}$ we have

$$a *^s c = -(-a * c) = a \bar{*} c \qquad (a, c \in \overline{R}). \qquad (4.12)$$

b) If $\overline{A} = (\overline{A}, \leq, \dot{*}, \underset{\cdot}{*})$ is the canonical enlargement of a complete totally ordered group $A = (A, \leq, *)$ and if $s : (\overline{A}, \leq) \longrightarrow (\overline{A}, \leq)$ is the mapping defined by

$$s(a) = a^{-1} \qquad (A \in \overline{A}), \qquad (4.13)$$

then s is a bijective duality (by [8], lemma 1.1) and, by (4.10) with $*$ being now the binary operation $\dot{*}$ of $(\overline{A}, \leq, \dot{*}, \underset{\cdot}{*})$ and [8], lemma 1.3, we have

$$a \dot{*}^s c = (a^{-1} \dot{*} c)^{-1} = a \underset{\cdot}{*} c^{-1} \qquad (a, c \in \overline{A}). \qquad (4.14)$$

Definition 4.2. Let $(\overline{A}, \leq)$, s and $*$ be as in definition 4.1. A mapping $\Delta : \overline{A}^X \longrightarrow \overline{A}^W$ is called a $(*, s)$-*duality*, if it is a duality (in the sense (0.1), with $\overline{R}$ replaced by $\overline{A}$) and if

$$(f * a)^\Delta = f^\Delta *^s a \qquad (f \in \overline{A}^X, a \in \overline{A}), \qquad (4.15)$$

where each $a \in \overline{A}$ is identified with the constant function $f_a(x) = a \ \ (x \in X)$ and where $*$ (in $\overline{A}^X$) and $*^s$ (in $\overline{A}^W$) are defined pointwise on $\overline{A}$.

Remark 4.2. a) If $(\overline{A}, \leq)$, s and $*$ are as in remark 4.1 a), then, by (4.6) and (4.3), $\Delta : \overline{R}^X \longrightarrow \overline{R}^W$ is a $(*, s)$-duality if and only if it is a $*$-duality in the sense of definition 2.1.

b) If $(\overline{A}, \leq)$, s and $*$ are as in remark 4.1 b), then, by (4.15) and (4.14), $\Delta : \overline{A}^X \longrightarrow \overline{A}^W$ is a $(*, s)$-duality if and only if it is a $*$-duality in the sense of [8] (i.e., in the sense of (4.8), (4.9) above, with $M = \Delta$).

By remarks 4.1 a) and 4.2 a) and by our assumptions on $(\overline{A}, \leq)$ and s, *the results of the present paper can be extended to results on* $(*, s)$-*dualities*, which, by remarks 4.1 b) and 4.2 b), encompass, as particular cases, also the results of [8] on $*$-dualities (in the sense of [8]); indeed, note that if $\overline{A} = (\overline{A}, \leq, \dot{*}, \underset{\cdot}{*})$ is the canonical enlargement of a complete totally ordered group $A = (A, \leq, *)$, then, by [8], lemma 1.4, the binary operation $\dot{*}$ on $\overline{A}$ satisfies condition (α) (i.e., (1.1) with $*$ replaced by $\dot{*}$ and with inf taken in $\overline{A}$), so the "extended" definition 1.2 (of $*_l$) can be applied to $(\overline{A}, \leq)$, s and $*$ of remark 4.1 b), and one obtains (by (1.6) and [8], lemma 1.5)

$$a \dot{*}_l c = a \underset{\cdot}{*} c^{-1} \qquad (a, c \in \overline{A}). \qquad (4.16)$$

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