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by

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On the universal completion of an

Archimedean Riesz space

by Ileana Bucur

1. Introduction. If $(E, \leq)$ is a Riesz space then E is called Dedekind complete if any subset of E (or only of E_+ , the set of all positive elements of E) which is bounded from above has a supremum in E. As well-known, a Dedekind complete Riesz space is called universally complete (or inextensable Riesz space), if it is isomorphic with the space $\mathcal{C}_\infty(K)$ of all continuous functions $f : K \to [-\infty, \infty]$ such that the set $\{x \in K : |f(x)| < \infty\}$ is dense in K, where K is an extremal, Hausdorff, compact space. As it was shown by A.G. Pinsker in 1949 a Dedekind complete Riesz space E is universally complete iff every disjoint system in E_+ has a supremum in E. (This is so called "Pinsker criterium for universality" and moreover it is taken now as definition for "universally complete"). A subset F of a Dedekind complete Riesz space $(E, \leq)$ is called a foundation of E if F is an ideal of E (i.e. F is a linear subspace of E such that for any $x \in F$ the order interval $[-|x|, |x|]$ is included in F) and any $x \in E_+$ is the supremum of the set $\{y \in F_+; y \leq x\}$. It is known that for any Dedekind Riesz space $(E, \leq)$ there exists a universally complete Riesz space $\tilde{E}$ such that E is a foundation of E. This space $(\tilde{E})$ is uniquely determined up to an isomorphysme. The well-known proof of this result makes use of Kakutany representation theorem which asserts that any Dedekind complete Riesz space $(E, \leq)$ with strong unit is isomorphic with the linear ordered space of all real continuous functions on an extremal, Hausdorff, compact space.

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Starting with Pinsker criterium of universality in [7] A.C. Zaanen gives a proof for the above result that does not employ any representation theory.

In this note we present an other criterium for universality and then, starting with it, we give also a construction for the universal completion of a Dedekind complete Riesz space which is more intuitive.

The notion of Archimedean subset that we introduce here was inspired by reding the paper [1].

The proof for our criterium of universality that I gave in [2] makes also use of Kakutany representation theorem.

2. Archimedean subsets

Let $(E, <)$ be a Dedekind complete Riesz space. For any two non-empty subsets A, B of E we put

$$A + B := \{a + b \mid a \in A,\ b \in B\}$$

and for any real number r we denote

$$rA := \{ra \mid a \in A\}$$

As usually, for any non-empty subset M of E we shall denote by $\bigvee M$ (resp. $\bigwedge M$) the supremum (resp. infimum) of M if does it exist.

Definition. A subset M of E_+ is termed:

sup-stable if for any a', $a'' \in M$ we have $a' \vee a'' \in M$,

solid if for any $m \in M$ and any $p \in E_+$ with $p \leq m$ we have $p \in M$,

closed in order from below (c.o.f.b.) if for any non-empty subset M' of M which is bounded from above we have $\bigvee M' \in M$.

Remark 2.1. For any family $(M_i)_{i \in I}$ of subsets from E_+ which are closed in order from below the set $\bigcap_{i \in I} M_i$ is also closed in order from below. Hence for any subset M of E_+ there exists a smallest subset of E_+, denoted by $\overline{M}$, which is closed in order from below such that $M \subset \overline{M}$.

Remark 2.2. For any subset M of E_+ we have

$$\overline{M} = \{x \in E_+ \mid (\exists)\, M_x \subset M;\ \ \bigvee M_x = x\}.$$

Indeed, we have obviously

$$M \subset \{x \in E_+ \mid (\exists)\, M_x \subset M : \bigvee M_x = x\} \subset \overline{M}.$$

On the other hand if $(x_i)_{i \in I}$ is a family from the set $\{x \in E_+ \mid (\exists)\, M_x \subset M; \bigvee M_x = x\}$ which is bounded from above then we have

$$\bigvee_{i \in I} x_i = \bigvee_{i \in I} \bigvee M_{x_i} = \bigvee \{\bigcup_{i \in I} M_{x_i}\} \in \{x \in E_+ \mid (\exists)\, M_x \subset M;\ x = \bigvee M_x\}.$$

Proposition 2.3. a) For any subset M of E_+ which is solid (resp. sup-stable) the set $\overline{M}$ is solid (resp. sup-stable).

b) For any two non-empty subsets M', M'' of E_+ which are solid and sup-stable the set M' + M'' is solid and sup-stable.

c) For any two non-empty subsets M', M'' of E_+ which are solid and c.o.f.b. the set M' + M'' is also solid and c.o.f.b.

Proof. a) Let M be a non-empty solid (resp. sup-stable) subset of E_+ and let x, y be two elements of $\overline{M}$. We consider M_x , M_y subsets of M such that

$$x = \bigvee M_x\ ,\ y = \bigvee M_y$$

If M is solid and $z \in E_+$ is such that $z \leq x$ we have

$$z = x \wedge z = z \wedge (\bigvee M_x) = \bigvee \{z \wedge m \mid m \in M_x\} \in \overline{M}$$

If M is sup-stable then we have

$$x \vee y = (\bigvee M_x) \vee (\bigvee M_y) = \bigvee \{M_x \cup M_y\} \in \overline{M}.$$

b) If M', M'' are non-empty, solid and sup-stable we consider $m' \in M'$, $m'' \in M''$ arbitrary and $x \in E_+$ such that

$$x \leq m' + m''$$

Using the Riesz decomposition property we can find x', x'' in E_+ such that

$$x = x' + x'',\quad x' \leq m',\quad x'' \leq m''$$

From the fact that M', M'' are solid we deduce $x' \in M'$, $x'' \in M''$ and therefore $x \in M' + M''$ i.e. M' + M'' is a solid subset of E_+.

If moreover we suppose M', M'' solid and sup-stable then taking x, y two elements from M' + M'' we may consider $u', v' \in M'$; $u'', v'' \in M''$ such that

$$x = u' + u'', \quad y = v' + v''$$

The elements $w' := u' \vee v'$, $w'' := u'' \vee v''$ belong to M', respectively M'' and $x \vee y \leq w' + w''$. Since $M' + M''$ is solid we get $x \vee y \in M' + M''$.

c) Let M', M'' two non-empty subsets of E_+ which are solid and c.o.f.b. From the proof of the preceding point b) we deduce that $M' + M''$ is a solid part of E_+.

Let now $x \in E_+$ and let $(x_i' + x_i'')_{i \in I}$ be a family in $M' + M''$ such that $x_i' \in M'$, $x_i'' \in M''$ for any $i \in I$ and such that

$$x = \bigvee_{i \in I} (x_i' + x_i'')$$

Since the families $(x_i')_{i \in I}$, $(x_i'')_{i \in I}$ are dominated by x and since M' and M'' are c.o.f.b we get

$$\bigvee_{i \in I} x_i' \in M', \quad \bigvee_{i \in I} x_i'' \in M'', \quad x \leq (\bigvee_{i \in I} x_i') + (\bigvee_{i \in I} x_i'')$$

Using the fact that $M' + M''$ is solid we get $x \in M' + M''$.

For any $x \in E_+$ we shall denote by $[0, x]$ the order interval

$$[0, x] = \{ y \in E_+ \mid y \leq x \}.$$

Proposition 2.4. Let A be a non-empty, solid, sup-stable subset of E_+ which is c.o.f.b. The following assertions are equivalent:

a) For any $x, y \in E_+$ we have

$$A + [0, x] \subset A + [0, y] \Rightarrow x \leq y$$

b) There is no element $p \in E_+$, $p \neq 0$ such that $np \in A$ for any $n \in N$

c) For any $p \in E_+$ we have

$$\bigwedge_{n \in N^*} (\bigvee \{ p \wedge (\tfrac{1}{n} a) \mid a \in A \}) = 0$$

d) For any two subsets B and C at E_+ which are non-empty, solid, sup-stable and c.o.f.b. we have

$$A + B \subset A + C \Rightarrow B \subset C$$

Proof. b) $\Rightarrow$ c). Let $p \in E_+$ and let q be the element of E_+ defined by

$$q := \bigwedge_{n \in N^*} (\bigvee \{ p \wedge (\tfrac{1}{n} a) \mid a \in A \})$$

Obviously we have $q \leq p$ and therefore

$$q = q \wedge q = \bigwedge_{n \in N^*} (\bigvee \{q \wedge p \wedge (\tfrac{1}{n} a) \mid a \in A\}) = \bigwedge_{n \in N^*} (\bigvee q \wedge (\tfrac{1}{n} a) \mid a \in A\})$$

Hence for any $n \in N^*$ we have

$$q = \bigvee \{q \wedge (\tfrac{1}{n} a) \mid a \in A\} ; \quad nq = \bigvee \{(nq) \wedge a \mid a \in A\}$$

and therefore A being solid and c.o.f.b. we get

$$nq \in A$$

Using the assertion b) it follows $q = 0$.

c) $\Rightarrow$ b). If $p \in E_+$ is such that $np \in A$ for any $n \in N$ then we have $p = \bigvee \{p \wedge (\tfrac{1}{n} a) \mid a \in A\}$ for any $n \in N^*$ and therefore

$$p = \bigwedge_{n \in N^*} (\bigvee \{p \wedge (\tfrac{1}{n} a) \mid a \in A\})$$

Using c) we get $p = 0$.

a) $\Rightarrow$ b). Let $p \in E_+$ be such that $np \in A$ for any $n \in N$ and let a be an element of A. From the relations

$$p - \frac{1}{n} a \leq p \wedge n(p - \frac{1}{n} a)^+ = p \wedge (np - a)^+ \leq p \wedge [(np - a) \vee 0] \leq p$$

for any $n \in N^*$ we deduce

$$a + p < (a + p) \wedge (np \vee a) + \frac{1}{n} a \leq a + p + \frac{1}{n} a \qquad (\forall)\ n \in N^*,$$

$$a + p = \bigvee_{n \in N} [(a + p) \wedge (np \vee a)]$$

The set A being solid, sup-stable and c.o.f.b. we get

$$a + p \in A$$

The element $a \in A$ being arbitrary it follows that

$$A + p \in A, \quad A + [0, p] \subset A = A + [0, 0]$$

and therefore, using the assertion a), $p = 0$.

c) $\Rightarrow$ d). If B and C are non-empty subsets of E_+ which are solid, sup-stable an c.o.f.b. such that $A + B \subset A + C$ then by induction we deduce $A + nB \subset A + nC$ for any $n \in N^*$ and therefore we have

$$B \subset \frac{1}{n} A + B \subset \frac{1}{n} A + C$$

Let $b \in B$ be arbitrary and let $a_n \in A$ and $c_n \in C$ be such that $b = \frac{1}{n} a_n + c_n$.

We bave

$$b = b \wedge (\tfrac{1}{n} a) + (b \wedge c_n) \leq \bigvee \{ b \wedge (\tfrac{1}{n} a) \mid a \in A \ + \bigvee \{ b \wedge c \mid c \in C \}.$$

for any $n \in N^*$ and therefore

$$b \leq \bigwedge_{n \in N^*} (\vee \{ b \wedge (\tfrac{1}{n} a) \mid a \in A \}) + \bigvee \{ b \wedge c \mid c \in C \} = \bigvee \{ b \wedge c \mid c \in C \}.$$

The set C being solid and c.o.f.b. we get $b \in C$. The assertion d) $\Rightarrow$ a) is obvious.

Definition. A non-empty subset A of E_+ which satisfies the condition c) from the above proposition will be called archimedean.

Remark 2.5. From the definition it follows that

a) If A is an archimedean subset of E_+ then any non-empty subset B of A is also arhimedean

b) If A is archimedean then the smallest subset B of E_+ which is sup-stable, solid such that $A \subset B$ then B is also archimedean.

Proposition 2.6. a) If A is arhimedean then $\bar{A}$ is also arhimedean.

b) If A and B are two arhimedean subsets of E_+ then A + B and rA are also arhimedean for any real number r, $r \geq 0$.

Proof. a) Let $p \in E_+$ be arbitrary and let x be an element of $\bar{A}$. Then there exists a subset Ax of A such that $x = \bigvee Ax$. For any $n \in N^*$ we have

$$\frac{1}{n} x = \bigvee \{ \frac{1}{n} a \mid a \in Ax \}, \quad p \wedge (\tfrac{1}{n} x) = \bigvee \{ p \wedge (\tfrac{1}{n} a) \mid a \in Ax \},$$

$$p \wedge (\tfrac{1}{n} x) \leq \bigvee \{ p \wedge (\tfrac{1}{n} a) \mid a \in A \}$$

Hence

$$\bigvee \{ p \wedge (\tfrac{1}{n} x) \mid x \in \bar{A} \} \leq \bigvee \{ p \wedge (\tfrac{1}{n} a) \mid a \in A \} \qquad (\forall)\ n \in N^*,$$

$$\bigwedge_{n\in N^*} \left(\bigvee\{ p\wedge(\tfrac{1}{n} x) \mid x\in \overline{A}\}\right) \leq \bigwedge_{n\in N^*} \left(\bigvee\{p\wedge(\tfrac{1}{n} a) \mid a\in A\}\right) = 0$$

i.e. $\overline{A}$ is arhimedean.

b) If A and B are archimedean and $p\in E_+$ we have

$$p\wedge(\tfrac{1}{n} a + \tfrac{1}{n} b) \leq p\wedge(\tfrac{1}{n} a) + p\wedge(\tfrac{1}{n} b) \qquad (\forall)\ a\in A,\ b\in B,\ n\in N^*$$

Hence

$$\bigvee\{ p\wedge(\tfrac{1}{n}(a+b)) \mid a\in A,\ b\in B\} \leq \bigvee\{p\wedge(\tfrac{1}{n} a) \mid a\in A\} + \bigvee\{p\wedge(\tfrac{1}{n} b) \mid b\in B\}$$

for any $n\in N^*$ and therefore

$$\bigwedge_{n\in N^*} \left(\bigvee\{ p\wedge\tfrac{1}{n}(a+b) \mid a\in A,\ b\in B\}\right) \leq \bigwedge_{n\in N^*} \left(\bigvee\{p\wedge\tfrac{1}{n} a \mid a\in A\} + \right.$$

$$\left. + \bigvee\{p\wedge\tfrac{1}{n} b \mid b\in B\}\right) = \bigwedge_{n\in N^*} \left(\bigvee\{p\wedge\tfrac{1}{n} a \mid a\in A\}\right) + \bigwedge_{n\in N^*} \left(\bigvee\{p\wedge\tfrac{1}{n} b \mid b\in B\}\right) = 0.$$

i.e. A + B is arhimedean.

Analogously one can prove that for any arhimedean subset A of E_+ and any $r\in R_+$ the set rA is also arhimedean.

3. Universal completenes criterium

We remember that a Dedekind complete Riesz space $(E, \leq)$ is called universally complete if every disjoint system in the positive cone E_+ has a supremum (or equivalently, is bounded).

In this section we show the following assertion:

A Dedekind complete Riesz space $(E, \leq)$ is universally oomplete if and only if any archimedean subset of E_+ is bounded.

To this porpose, first we prove

Proposition 3.1. Every disjoint system in the positive cone E_+ is an archimedean subset of E_+.

Proof. Let $(p_i)_{i\in I}$ be a system of elements in E_+ such that $p_i\wedge p_j = 0$ for any $i, j\in I$, $i\neq j$ and let q be an arbitrary element of E_+. We denote by q_0

the element of E_+ defined by

$$q_0 = \bigwedge_{n \in N^*} \left(\bigvee \{ q \wedge (\tfrac{1}{n} p_i) \mid i \in I \} \right)$$

Since for any $n \in N^*$ we have

$$q_0 \leqslant \bigvee \{ q \wedge (\tfrac{1}{n} p_i) \mid i \in I \}$$

we deduce the relations:

$$q_0 \leqslant q; \ q_0 = q_0 \wedge (\bigvee \{ q \wedge (\tfrac{1}{n} p_i) \mid i \in I \}) = \bigvee \{ q_0 \wedge (\tfrac{1}{n} p_i) \mid i \in I \},$$

$$q_0 \wedge p_j = \bigvee \{ q_0 \wedge p_j \wedge (\tfrac{1}{n} p_i) \mid i \in I \} = q_0 \wedge (\tfrac{1}{n} p_j) \qquad (\forall) \ j \in I$$

Hence $q_0 \wedge p_j < \frac{1}{n} p_j$ for any $j \in I$ and any $n \in N^*$ and therefore the linear ordered space $(E, \leqslant)$ being archimedean it follows that $q_0 \wedge p_j = 0$ for any $j \in I$. Finally we get

$$q_0 = \bigvee \{ q_0 \wedge p_i \mid i \in I \} = 0$$

i.e. the set $\{ p_i \mid i \in I \}$ is an archimedean subset of E_+.

Corollary 3.2. If any archimedean subset of E_+ is bounded then $(E, \leqslant)$ is universally complete.

Now we shall prove the converse assertion

Proposition 3.3. If $(E, \leqslant)$ is universally complete then any arhimedean subset of E_+ is bounded.

Proof. If $(E, <)$ is universally complete then $(E, \leqslant)$ may be identified with the set $\mathcal{C}_\infty(K)$ of all functions $f : K \to [-\infty, \infty]$ which are continuous and densely finite where K is a Hausdorff, compact, extremal space i.e. for any open subset G of K its closure is also open. Let M be an arhimedean subset of $\mathcal{C}_\infty^+(K)$ and let $\varphi : K \to [0, \infty]$ be the function on K given by

$$\varphi(x) = \sup \{ m(x) \mid m \in M \}$$

It is known that the upper semi-continuous regularization $\check{\varphi}$ of φ is a continuous function on K and the set $\{ x \in K \mid \check{\varphi}(x) > \varphi(x) \}$ is nowhere dense in K.

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It remains to show that $\check{\varphi} \in \mathcal{C}_\infty(K)$ or equivalently that the interior of the set $\{x \in K \mid \varphi(x) = +\infty\}$ is empty.

If we suppose the contrary then there exists an open-closed, non-empty subset G of K such that $\varphi(x) = \infty$ for any $x \in G$. If $g : K \to [0, 1]$ is the characteristic function of G we have, for any $n \in N^*$,

$$ng \leq \varphi \ , \quad \bigvee \{ (ng) \wedge m \mid m \in M \} = ng$$

and therefore

$$g = \bigwedge_{n \in N^*} \left(\bigvee \{ g \wedge (\tfrac{m}{n}) \mid m \in M \}\right)$$

The last relation contradicts the fact that M is archimedean.

Theorem 3.4. A Dedekind complete Riesz space $(E, \leqslant)$ is universally complete iff any archimedean subset of E_+ is bounded or iff any non-empty, solid, sup-stable, c.o.f.b. and arhimedean subset of E_+ is bounded.

4. The universal completion

The porpose of this section is to associate to a given Dedekind complete Riesz space $(E, \leqslant)$ an universally complete space $(\tilde{E}, \leqslant)$ such that E_+ is isomorphic with a convex subcone S of $\tilde{E}_+$ which is solid and increasingly dense in $\tilde{E}_+$ i.e every element of $\tilde{E}_+$ is the supremum of its minorants from S. The space $(\tilde{E}, \leqslant)$ is uniquely determined up to an isomorphism of Riesz spaces and will be called the universal completion of $(E, \leqslant)$.

First we recall some definition which will be used in the sequel

A non-empty set C endowed with: an addition operation $C \times C \ni (x, y) \to x+y \in C$ and a multiplication with positive real numbers $\mathbb{R}_+ \times C \ni (r, c) \to rc \in C$ such that

a_1) $(x + y) + z = x + (y + z)$, $x + y = y + z$ $\quad (\forall)\ x, y, z \in C$

a_2) there exists a neutral element denoted 0_C or simply 0 i.e.

$$0 + x = x \qquad (\forall)\ x \in C$$

a_3) If $x, y \in C$ and $x + y = 0$ then $x = y = 0$

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a_4) If $x, y, z \in C$ and $x + z = y + z$ then $x = y$

m_1) $r(x + y) = rx + ry$, $(r + r')x = rx + r'x$ $(\forall)\ x, y \in C;\ r, r' \in \mathbb{R}_+$

m_2) $(rr')x = r(r'x)$, $1x = x$ $(\forall)\ x \in C;\ r, r' \in \mathbb{R}_+$

will be called an abstract convex cone.

Giving an abstract convex cone C we may introduce an order relation on C which will be called the specific order on C in the following way

$$x, y \in C,\ x \leq y \overset{\text{def}}{\Longleftrightarrow} (\exists)\ z \in C;\ x + z = y.$$

Obviously the element 0 becomes the smallest element of C and if any subset of C has an infimum then any subset of C which is bounded from above has a supremum (in C). Moreover in this cas there exists a Dedekind complete Riesz space $(E, \leq)$ such that the cone C may be identified with the convex cone E_+ of all positive elements of E([3]).

Let $(E, \leq)$ be a Dedekind complete Riesz space and let E_+ be its convex cone of positive elements. We shall denote by C the set of all non-empty, solid, sup-stable, c.o.f.b. and archimedean subsets of E_+ and we shall introduce in C an addition operation and a multiplication with positive, real numbers as follows

$$A, B \in C, \quad A + B = \{a + b \mid a \in A,\ b \in B\}$$

$$A \in C,\ r \in R_+, \quad rA = \{ra \mid a \in A\}.$$

From Propositions 2.3., 2.6. it follows that for any $A, B \in C$, $r \in R_+$ we have $A + B \in C$, $rA \in C$ and from Proposition 2.4. we deduce that the property a_4) from above is fulfilled.

We remark that for any $x \in E_+$ the order interval $[0, x] := \{y \in E_+ \mid y \leq x\}$ is an element of C. Particularly the interval $[0, 0]$ which will be denoted simply 0 is the neutral element of C with respect to the addition operator. The properties a_1), a_2), a_3), m_1), m_2) may be easily verified. Hence the set C endowed with the above operations is an abstract convex cone.

Proposition 4.1. The map $\varphi : E_+ \longrightarrow C$ given by $\varphi(x) = [0, x]$ has the following properties

a) $x, y \in E_+$, $\varphi(x) = \varphi(y) \Longleftrightarrow x = y$

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b) $x, y \in E_+$, $r \in R_+ \Rightarrow \varphi(x+y) = \varphi(x) + \varphi(y)$; $\varphi(rx) = r\ \varphi(x)$

c) $x, y \in E_+$, $x \leq y \Leftrightarrow \varphi(x) \subset \varphi(y)$

d) If $A \in C$ and $x \in E_+$ are such that $A \subset \varphi(x)$ then $A \in \varphi(E_+)$.

Proof. The properties a) and c) are obvious. If $r \in R_+$ and $x, y \in E_+$ we have $\varphi(x) + \varphi(y) \subset \varphi(x+y)$. For the converse inclusion we consider $z \in \varphi(x+y)$ i.e. $z \in E_+$, $z \leq x+y$ and let (from Riesz decomposition property) $x', y' \in E_+$ be such that $x' \leq x$, $y' \leq y$, $z = x' + y'$. Hence $z \in \varphi(x) + \varphi(y)$ and therefore the property b) is shown.

As for the last property, if $A \in C$ and $x \in E_+$ are such that $A \subset \varphi(x)$ then the element x is a majorant of A and therefore A has a supremum a in E_+ (or equaly in E) and $a \leq x$. Since A is c.o.f.b we have $a \in A$ and therefore $A \subset [0, a]$. The set A being solid we get $A = [0, a] = \varphi(a)$.

Proposition 4.2. a) If $A, B \in C$ and $A \subset B$ then there exists $A' \in C$ such that $A + A' = B$. i.e the order relation on C given by the inclusion relation coincides with the specific order on C.

b) For any family $(A_i)_{i \in I}$ from C the subset $\bigcap_{i \in I} A_i$ of E_+ is an element of C and it is just the infimum of the family $(A_i)_{i \in I}$ in C with respect to the specific order.

c) If $(A_i)_{i \in I}$ is an upper directed family in C with respect to the inclusion (or equaly w.r. to the specific order of C) and if this family is dominated in C then the subset $\overline{\bigcup A_i}$ of E_+ is an element of C and it is the supremum of the family $(A_i)_{i \in I}$ in C w.r. to the specific order.

Proof. For any $A \in C$ we consider the "projection" operator on A defined on E_+, $x \to x_A$ given by

$$x_A = \bigvee\{a \mid a \in A, a \leq x\} = \bigvee\{x \wedge a \mid a \in A\}.$$

The following properties of this operator are almost trivial

1) $x \in E_+$, $x \in A \Leftrightarrow x_A = x$

2) $x, y \in E_+$, $x \leq y \Rightarrow x_A \leq y_A$ and $x_A = x \wedge y_A$

3) $x, y \in E_+ \Rightarrow (x \wedge y)_A = x_A \wedge y_A$; $(x \vee y)_A = x_A \vee y_A$

4) $x, y \in E_+ , x \leq y \Rightarrow x - x_A \leq y - y_A$

5) $x, y \in E_+ \Rightarrow (x - x_A)\vee(y - y_A) = x \vee y - (x\vee y)_A; (x - x_A)\wedge (y - y_A) = x\wedge y - (x \wedge y)_A$

6) $t, x \in E_+, t \leq x - x_A \Rightarrow (\exists) y \in E_+ , y \leq x; t = y - y_A.$

For example the assertion 4) may be derived as follows:

$$x \leq y \Rightarrow x + y_A = x \wedge y_A + x \vee y_A = x_A + x \vee y_A \leq x_A + y; x - x_A \leq y - y_A.$$

The inequality $(x - x_A) \vee (y - y_A) \leq (x\vee y) - (x \vee y)_A$ in 5) follows from the assertion 4). On the other hand we have $x_A \leq (x\vee y)_A$, $y_A \leq (x\vee y)_A$ and therefore

$$x - x_A \geq x - (x \vee y)_A , y - y_A \geq y - (x\vee y)_A , (x - x_A)\vee(y - y_A) > (x\vee y) - (x\vee y)_A$$

Hence the relation 5) is verified.

As for the assertion 6) let $t, x \in E_+$ be such that $t \leq x - x_A$. We consider the element y of E_+ , $y = t + x_A$. We have $x_A \leq y \leq x$ and therefore, using the assertion 2) we get

$$y_A = x_A \quad y = x_A , t = y - x_A = y - y_A.$$

We prove now the assertion a). Let $A, B \in C$ be such that $A \subset B$ and let D be the subset of E_+ given by

$$D = \{ b - b_A \mid b \in B \}$$

From the above properties 5), 6) we deduce that D is a non-empty solid, sup-stable subset of E_+. It is also archimedean since $D \subset B$. The relation $A + D = B$ may be shown as follows: for any $b \in B$ we have $b = b_A + (b - b_A) \in A + D$ i.e. $B \subset A + D$. For any $a \in A$ and $b \in B$ the element $a\vee b$ belongs to B and we have, using the above properties 3), 1)

$$a + (b - b_A) \leq a + (a\vee b - (a\vee b)_A) = a + a\vee b - a = a\vee b; a + (b - b_A) \in B$$

Hence $A + D \subset B$ i.e. $A + D = B$. The assertion a) follows now from the relation $B = \overline{B} = \overline{A + D} = \overline{A} + \overline{D} = A + \overline{D}$ and from the fact that $\overline{D} \in C$.

The assertions b) and c) may be derived directly from the assertion a) and using Propositions 2.3., 2.6.

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Proposition 4.3. Any archimedean subset M of the cone C is bounded w.r. to the specific order of C.

Proof. Let M be an archimedean subset of C with respect to the specific order i.e. for any $A \in C$ we have

$$\bigwedge_{n \in N^*} \left(\bigvee \{ A \wedge (\tfrac{1}{n} B) \mid B \in M \}\right) = [0,0]$$

where for any subset H of C we have denoted $\bigwedge H$ (resp. $\bigvee A$) the infimum (resp. the supremum) of H in C with respect to the specific order.

To show that M is bounded we may suppose, using Proposition 2.5. that M is a non-empty solid, sup-stable, c.o.f.b. and archimedean subset of C. Since M is sup-stable and solid in C we deduce that the subset M_0 of E_+ defined by

$$M_0 = \bigcup_{A \in M} A$$

is also solid and sup-stable. We want to show that M_0 is an archimedean subset of E_+. Let a be an element of E_+. From the relation

$$\bigwedge_{n \in N^*} \left(\bigvee \{ [0, a] \cap (\tfrac{1}{n} B) \mid B \in M \}\right) = [0, 0]$$

and from Proposition 4.1, d) we deduce that for any $n \in N^*$ and any $B \in M$ there exists an element b(n, B) in the set $\frac{1}{n} B$ such that

$$[0, a] \cap (\tfrac{1}{n} B) = [0, b(n, B)], \qquad b(n, B) \leq a.$$

The set M being sup-stable it follows that for any $n \in N^*$ the family $\{ b(n, B) \mid B \in M \}$ is upper directed and dominated by a.

Using Proposition 4.2., c) we get

$$\bigvee \{ [0, a] \cap (\tfrac{1}{n} B) \mid B \in M \} = \overline{\bigcup_{B \in M} [0, b(n, B)]} = [0, b_n]$$

where $b_n = \bigvee \{ b(n, B) \mid B \in M \}$. One can easily see that

$$b_n = \bigvee \{ a \wedge (\tfrac{1}{n} b) \mid b \in M_0 \}$$

and since $\bigcap_{n \in N^*} [0, b_n] = [0, 0]$ we get $\bigwedge_{n \in N^*} b_n = 0$ i.e. M_0 is an archimedean subset of E_+. From the above considerations $\overline{M}_0 \in C$ and $B \subset M_0 \subset \overline{M}_0$ for any $B \in M$. Hence M is bounded.

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Theorem 4.4. For any Dedekind complete Riesz space $(E, \leq)$ there exists a universally complete space $(\breve{E}, \leq)$ such that $(E, \leq)$ may be identified with a linear subspace of $\tilde{E}$ and E_+ is a solid convex subcone of $\breve{E}_+$ such that any element of $\tilde{E}_+$ is the supremum in $(\breve{E}, \leq)$ of the set of its minorants from E_+.

Proof. With the above notations we consider the Dedekind complete Riesz space $(\tilde{E}, \leq)$ such that $\tilde{E}_+$ may be identified with the abstract convex cone C constructed in this section. It remains only to identify E_+ with $\varphi(E_+)$ where φ is the map defined in Proposition 4.1. and to apply Theorem 3.4.

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