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(preliminary version)

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FREE INDEPENDENCE WITH RESPECT TO A FAMILY OF STATE SETS

by

Valentin IONESCU

ABSTRACT. A notion of quantum independence is studied. Some of its properties are used to obtain analogues of some classical results.

Let B be a unital ($*$-) algebra over the complex field. Let us consider the category of ($*$-) algebras over B i.e. $A \supseteq B \ni 1$ with morphisms being ($*$-)algebraic homomorphisms, which are the identity on B. The free product with amalgamation of $(A_i)_{i\in I}$ over B, denoted $\ast_{i\in I} {}_B A_i$, is the coproduct (direct sum) in this category.

If B is additionally a C^*-algebra, A_i, $i\in I$, are C^*-algebras over B, and Φ_i, $i\in I$, are sets of norm one projections (or conditional expectations) of A_i onto B, then the existence of a "reduced free product with amalgamation Φ of the family $(\Phi_i)_{i\in I}$, via a family γ of maps" is established in [6] on the "biggest" C^*-algebraic free product with amalgamation over B.

This fact can be detailed, by similar arguments, in the following form.

Let I be an index set, $(\Gamma_i)_{i\in I}$ a family of index sets, and $\Gamma = \prod_{i\in I} \Gamma_i$. For all $n\in \mathbb{N}$, $n\geq 1$, if $t:=(i_1,\ldots,i_n)\in I^n$, denote $\Gamma_t := \prod_{k=1}^{n} \Gamma_{i_k}$. Let also be given a family of maps $\gamma = (\gamma^{(n)})_{n\geq 1}$ such that $\gamma^{(n)}: \coprod_{t\in I^n} \Gamma_t \longrightarrow \bigcup_{t\in I^n} \Gamma_t$, and $\gamma^{(n)}(t,\alpha_t)\in \Gamma_t$, for all $(t,\alpha_t)\in \{t\}\times\Gamma_t$, if $n\geq 1$.

Denote again

$$D_1(I) := I$$

and

$$D_n(I) := \{(i_1, \ldots, i_n) \in I^n \; ; \; i_k \neq i_{k+1} \; , \; 1 \leq k \leq n-1\}$$

for all $n \geq 2$.

($\prod$ and $\coprod$ are respectively denoting the product and the coproduct in the category of sets.)

Let B be a unital C^*-algebra (over $\mathbb{C}$), $\{A_i, \Phi_i\}$ be a couple for all $i \in I$, where A_i is a C^*-algebra over B, and $\Phi_i = \{\varphi_i^{(\alpha_i)} \; ; \alpha_i \in \Gamma_i\}$ is an arbitrary set of norm one projections of A_i onto B.

<u>Theorem</u>. There exists a couple $\{A, \Phi\}$, where A is a C^*-algebra over B and $\Phi = \{\varphi^{(\alpha)} \; ; \alpha \in \Gamma\}$ is a set of conditional expectations of A onto B such that :

(1) There exists a morphism $j_i : A_i \longrightarrow A$ for each $i \in I$, such that A is generated by $\bigcup_{i \in I} j_i(A_i)$;

(2) $\varphi^{(\alpha)} \circ j_i = \varphi_i^{(\alpha_i)}$ for each $i \in I$ and $\alpha \in \Gamma$;

(3) For all $n \geq 1$ and $t := (i_1, \ldots, i_n) \in I^n$:

$$\varphi^{(\alpha)}(j_{i_{\pi(1)}}(a_{\pi(1)}) \cdots j_{i_{\pi(p)}}(a_{\pi(p)})) = \varphi_{i_{\pi(1)}}^{(\alpha_{i_{\pi(1)}})}(a_{\pi(1)}) \cdots \varphi_{i_{\pi(p)}}^{(\alpha_{i_{\pi(p)}})}(a_{\pi(p)}) \; ,$$

if $a_{\pi(k)} \in \operatorname{Ker} \varphi_{i_{\pi(k)}}^{(\gamma_{\pi(k)}^{(n)}(t, \alpha_t))}$ $(1 \leq k \leq p)$, for all $p \geq 1$, $(i_{\pi(1)}, \ldots, i_{\pi(p)}) \in$ $\in D_p(\{i_1, \ldots, i_n\})$, where $\pi(k) \in \{1, \ldots, n\}$, denoting $\alpha_t := (\alpha_{i_1}, \ldots, \alpha_{i_n})$; for each $\alpha \in \Gamma$.

According to [9] , Φ can be called a <u>reduced free product with amalgamation</u> of $(\Phi_i)_{i \in I}$ <u>over</u> B <u>via</u> γ , and (3) of Theorem <u>the free independence property with amalgamation on</u> A , <u>via</u> γ .

One can denote

$$\Phi = \mathop{\ast}_{i \in I}^{(\gamma)}{}_B \, \Phi_i \quad , \quad (A, \Phi) = \mathop{\ast}_{i \in I}^{(\gamma)}{}_B (A_i, \Phi_i) \; .$$

1. Let us delimit the frame of this paper in the preceding context. Take $\Gamma_i = \Lambda$ for all $i \in I$ and $\chi^{(n)} : I^n \longrightarrow \Lambda^n$, $n \geq 1$. Take $\gamma^{(n)} : I^n \times \Lambda^n \longrightarrow \Lambda^n$ by $\gamma^{(n)}(t, \bar{\lambda}) := \chi^{(n)}(t)$ if $t \in I^n$, and $\bar{\lambda} \in \Lambda^n$, for each $n \geq 1$.

Let us call <u>states</u> of A all linear maps $\varphi : A \longrightarrow B$ which are projections ($\varphi|_B = id_B$) and B-B-bimodule maps ($\varphi(b_1 a b_2) = b_1 \varphi(a) b_2$ if $b_1, b_2 \in B$ and $a \in A$), in the case of a unital algebra over B. At the $*$-algebraic level require additionally φ be positive.

Consider $\Phi_i = \{\varphi_i^{(\lambda)} ; \lambda \in \Lambda\}$ i.e. a state set of A_i, and fix $\lambda_0 \in \Lambda$.

Denote

$$\varphi_i := \varphi_i^{(\lambda_0)} \quad , \quad \Psi_i := \{\varphi_i^{(\lambda)} ; \lambda \in \Lambda \setminus \{\lambda_0\}\} .$$

$(i \in I)$

In view of previous Theorem one can derive the following assertion.

<u>Theorem 1.1.</u> If A is the $*$-algebraic free product with amalgamation of $(A_i)_{i \in I}$ over B, then there exists a state φ of A such that relative to the canonical morphisms $j_i : A_i \longrightarrow A$, $i \in I$:

(1) $\varphi \circ j_i = \varphi_i$ for each $i \in I$;

(2) For all $n \geq 1$ and $t := (i_1, \ldots, i_n) \in I^n$:

$$\varphi(j_{i_{\pi(1)}}(a_{\pi(1)}) \cdots j_{i_{\pi(p)}}(a_{\pi(p)})) = \varphi_{i_{\pi(1)}}(a_{\pi(1)}) \cdots \varphi_{i_{\pi(p)}}(a_{\pi(p)}) ,$$

if $a_{\pi(k)} \in Ker\, \varphi_{i_{\pi(k)}}^{(\chi^{(n)}_{\pi(k)}(t))}$, $\pi(k) \in \{1, \ldots, n\}$, $(1 \leq k \leq p)$, for all $p \geq 1$ and $(i_{\pi(1)}, \ldots, i_{\pi(p)}) \in D_p(\{i_1, \ldots, i_n\})$.

Thus, φ can be called a <u>reduced free product with amalgamation</u> of $(\varphi_i)_{i \in I}$ <u>over</u> B, <u>with respect to</u> $(\Psi_i)_{i \in I}$, <u>via</u> χ, and (2) of Theorem 1.1 <u>the free</u> Ψ <u>-independence property with amalgamation</u> on A via χ.

One can denote $\varphi = \underset{i\in I}{\overset{(\chi)}{*}}{}_B^{\Psi_i} \varphi_i$, $(A,\varphi) = \underset{i\in I}{\overset{(\chi)}{*}}{}_B^{\Psi_i} (A_i,\varphi_i)$.

Let be in the following $B = \mathbb{C}\cdot 1$.

Consider the (non-commutative) probability space (A, φ) given by Theorem 1.1. The elements of A can be called (non-commutative) random variables.

According to [8],[3], for a random variable $w = j_{i_1}(a_1)\cdots j_{i_n}(a_n)$, with $t:=(i_1,\ldots,i_n)\in I^n$ and $a_k\in A_{i_k}$ $(1\leq k\leq n)$, the expectations of the form

$$\varphi_{i_\ell}(a_{\pi(1)}\cdots a_{\pi(r)}) \quad \text{or} \quad \varphi_{i_\ell}^{(\chi_\ell^{(n)}(t))}(a_{\pi(1)}\cdots a_{\pi(r)}) \quad ,$$

where $\pi(1),\ldots,\pi(r)\in\{1,\ldots,n\}$ and $i_{\pi(1)}=\ldots=i_{\pi(r)}=i_\ell$ (i.e. $a_{\pi(1)},\ldots,a_{\pi(r)}\in A_{i_\ell}$), can be called <u>elementary moments</u> of w .

<u>Remark 1.2.</u> Let $w = j_{i_1}(a_1)\ldots j_{i_n}(a_n)$ be a random variable with $t:=(i_1,\ldots,i_n)\in I^n$, $a_k\in A_{i_k}$ $(1\leq k\leq n)$. Let us decompose $a_k = \varphi_{i_k}^{(\chi_k^{(n)}(t))}(a_k)\cdot 1 + a_k^0$,where $a_k^0\in \operatorname{Ker}\varphi_{i_k}^{(\chi_k^{(n)}(t))}$ depends of χ and t . Denote $w^0 = j_{i_1}(a_1^0)\ldots j_{i_n}(a_n^0)$. Then it is easy to observe that each elementary moment of the random variable w^0 can be expressed as a sum of products of elementary moments of w , and conversely. One can also decompose

$$\varphi(j_{i_1}(a_1)\cdots j_{i_n}(a_n)) = \sum_{\pi,\sigma} \varphi_{i_{\pi(1)}}^{(\chi_{\pi(1)}^{(n)}(t))}(a_{\pi(1)})\cdots\varphi_{i_{\pi(r)}}^{(\chi_{\pi(r)}^{(n)}(t))}(a_{\pi(r)})\,\varphi(a^0_{\sigma(1)}\cdots a^0_{\sigma(n-r)})$$

where the sum runs over all partitions of $\{1,\ldots,n\}$ into two ordered sets $\pi = (\pi(1),\ldots,\pi(r))$ and $\sigma = (\sigma(1),\ldots,\sigma(n-r))$, $r\geq 1$, considering $\pi=\phi \iff r=0$.

Thus it is easy to observe, similarly to [8],[3], that the free Ψ -independence property on A allows the calculation of all moments of w relative to φ , if all elementary moments of w are given. The calculation follows the same recursive procedure as in [8],[3] .

The following properties can now be established like in [8] and [3] by similar arguments.

Proposition 1.3. Let $n \geqslant 1$, $(i_1,\dots,i_n) \in I^n$ such that $i_k \neq i_\ell$ for all $k \neq \ell$ and $a_k \in A_{i_k}$ $(1 \leq k \leq n)$.

Then

$$\varphi(j_{i_1}(a_1)\cdots j_{i_n}(a_n)) = \varphi_{i_1}(a_1)\cdots\varphi_{i_n}(a_n) \;.\square$$

Example : Identify $j_i(A_i) = A_i$. Consider $w = ab$ with $a \in A_{i_1}$, $b \in A_{i_2}$, $i_1 \neq i_2$. Put $t := (i_1, i_2)$.

Decomposing

$$b = \varphi_{i_2}^{(\chi_2^{(2)}(t))}(b)\cdot 1 + b^\circ \quad \text{with} \quad b^\circ \in \operatorname{Ker}\varphi_{i_2}^{(\chi_2^{(2)}(t))} \;,$$

one can write :

$$\begin{aligned}\varphi(ab) &= \varphi_{i_2}^{(\chi_2^{(2)}(t))}(b)\,\varphi(a) + \varphi(ab^\circ)\\ &= \varphi_{i_2}^{(\chi_2^{(2)}(t))}(b)\,\varphi_{i_1}(a) + \varphi(ab^\circ)\end{aligned}$$

Then, decomposing

$$a = \varphi_{i_1}^{(\chi_1^{(2)}(t))}(a)\cdot 1 + a^\circ \;, \quad \text{with} \quad a^\circ \in \operatorname{Ker}\varphi_{i_1}^{(\chi_1^{(2)}(t))} \;,$$

one can continue

$$\begin{aligned}\varphi(ab^\circ) &= \varphi_{i_1}^{(\chi_1^{(2)}(t))}(a)\,\varphi(b^\circ) + \varphi(a^\circ b^\circ)\\ &= \varphi_{i_1}^{(\chi_1^{(2)}(t))}(a)\,\varphi_{i_2}(b^\circ) + \varphi(a^\circ b^\circ)\;.\end{aligned}$$

But $\varphi(a^\circ b^\circ) = \varphi_{i_1}(a^\circ)\,\varphi_{i_2}(b^\circ)$, by the free Ψ -independence property.

And

$$\varphi_{i_1}(a^\circ) = \varphi_{i_1}(a) - \varphi_{i_1}^{(\chi_1^{(2)}(t))}(a) \;, \quad \varphi_{i_2}(b^\circ) = \varphi_{i_2}(b) - \varphi_{i_2}^{(\chi_2^{(2)}(t))}(b)\;.$$

Thus, one obtains

$$\varphi(ab) = \varphi(a)\,\varphi(b) \;.\square$$

By induction one can also obtain the following two properties.

<u>Proposition 1.4.</u> Let be $n \geq 1$, and $(i_1,\dots,i_n) \in D_n(I)$, denoting $p := \#\{i_1,\dots,i_n\}$.
Consider $a_k \in A_{i_k}$ $(1 \leq k \leq n)$ and $w = j_{i_1}(a_1)\dots j_{i_n}(a_n)$.

Then the expectation $\varphi(w)$ can be expressed as a sum of products of elementary moments of w, each term containing at least p factors.∎

<u>Proposition 1.5.</u> Let be $n \geq 1$, $t := (i_1,\dots,i_n) \in D_n(I)$ such that there exists $j \in \{1,\dots,n\}$ with $i_j \neq i_k$ for $k \in \{1,\dots,n\} \setminus \{j\}$, and there exist $\ell, \ell' \in \{1,\dots,n\}$ with $\ell < j < \ell'$ but $i_\ell = i_{\ell'}$. Let also be $p := \#\{i_1,\dots,i_n\}$.

Consider $w = j_{i_1}(a_1)\dots j_{i_n}(a_n)$, $a_k \in A_{i_k}$ $(1 \leq k \leq n)$.

Then the expectation $\varphi(w)$ can be expressed in the following form:

$$\varphi(w) = \varphi_{i_j}^{(\chi_j^{(n)}(t))}(a_j)\,\varphi(a_1\cdots a_{j-1}\,a_{j+1}\cdots a_n) + \sum$$

where $\sum$ is a sum of products of elementary moments of w, each term containing at least $p+1$ factors.∎

Example : Identify $j_i(A_i) = A_i$. Consider $w = aba$ with $a \in A_{i_1}$, $b \in A_{i_2}$, $i_1 \neq i_2$, $t := (i_1, i_2, i_1)$; $p = 2$.

Then

$$\varphi(aba) = \varphi_{i_2}^{(\chi_2^{(3)}(t))}(b)\,\varphi(aa) + \varphi(ab^\circ a)$$

by the decomposition $b = \varphi_{i_2}^{(\chi_2^{(3)}(t))}(b)\cdot 1 + b^\circ$ with $b^\circ \in \operatorname{Ker} \varphi_{i_2}^{(\chi_2^{(3)}(t))}$.

By using similar decompositions for a, one can finally obtain

$$\varphi(ab^\circ a) = \varphi(a)\varphi(b)\varphi(a) - \varphi_{i_2}^{(\chi_2^{(3)}(t))}(b)\,\varphi(a)\varphi(a).$$

Therefore

$$\varphi(aba) = \varphi_{i_2}^{(\chi_2^{(3)}(t))}(b)\,\varphi_{i_1}(a^2) + \sum$$

where

$$\sum = \varphi_{i_1}(a)\varphi_{i_2}(b)\varphi_{i_1}(a) - \varphi_{i_2}^{(\chi_2^{(3)}(t))}(b)\,\varphi_{i_1}(a)\varphi_{i_1}(a).$$

Observe that only the term $\varphi_{i_2}^{(\chi_2^{(3)}(t))}(b)\,\varphi_{i_1}(a^2)$ has exactly $p=2$ factors.

Let now us introduce some facts from [3], [7] and [8].

Let be $n \geq 1$. Denote by $p(\{1,\dots,n\})$ the set of all partitions $\pi = (\pi_1,\dots,\pi_p)$ of $\{1,\dots,n\}$, consisting of ordered and mutually disjoint sets.

One can denote $\pi_\ell = (\pi_\ell(1),\dots,\pi_\ell(n_\ell))$, $\ell = \overline{1,p}$.

There exists a natural correspondence between I^n and $p(\{1,\dots,n\})$:

$$t \longmapsto \pi^t .$$

If $t = (i_1,\dots,i_n)$, $p := \#\{i_1,\dots,i_n\}$ and $\{i(1),\dots,i(p)\} = \{i_1,\dots,i_n\}$ is an enumeration, then $\pi^t = (\pi_1^t,\dots,\pi_p^t)$ where $\pi_\ell^t = (k = \overline{1,n};\ i_k = i(\ell))$, $\ell = \overline{1,p}$.

This correspondence is onto, but it is not one-to-one.

Therefore it induces an one-to-one correspondence on the set of the equivalence classes relative to the following relation "$\sim$" on I^n given by

$$(i_1,\dots,i_n) \sim (i_1',\dots,i_n') \iff (i_k = i_\ell \iff i_k' = i_\ell').$$

Definition 1.6. A partition $\pi = (\pi_1,\dots,\pi_p)$ of $P(\{1,..,n\})$ is called crossing if there exists two sets π_{ℓ_1} and π_{ℓ_2} in π such that there exist $k_1, k_1' \in \pi_{\ell_1}$ and $k_2, k_2' \in \pi_{\ell_2}$ with $k_1 < k_2 < k_1' < k_2'$. Otherwise π is called non-crossing.

Denote the set of all non-crossing partitions of $\{1,\dots,n\}$ by $P_{nc}(\{1,\dots,n\})$.

Definition 1.7. A set π_ℓ of a partition $\pi \in P_{nc}(\{1,\dots,n\})$ is called inner if there exists a set $\pi_{\ell'} = (\pi_{\ell'}(1),\dots,\pi_{\ell'}(n_{\ell'}))$ of π such that

$$\pi_{\ell'}(1) < \pi_\ell(k) < \pi_{\ell'}(n_{\ell'}) \quad \text{for all} \quad \pi_\ell(k) \in \pi_\ell .$$

Otherwise, π_ℓ is called outer.

<u>Definition 1.8.</u> Let be $n \geqslant 1$. An element $t=(i_1,\dots,i_n) \in I^n$ is called <u>non-crossing</u> if the partition $\pi^t \in P(\{1,\dots,n\})$ corresponding to t is non-crossing.

In the opposite case, t is called <u>crossing</u>.

Similarly to [8] and [3] one can deduce the following consequences of the preceding propositions.

<u>Corollary 1.9.</u> Let be $n \geqslant 1$, $t=(i_1,\dots,i_n) \in I^n$ such that t is crossing and $p := \#\{i_1,\dots,i_n\}$.

Consider $a_k \in A_{i_k}$ $(1 \leq k \leq n)$ and $w = j_{i_1}(a_1)\dots j_{i_n}(a_n)$.

Then the expectation $\varphi(w)$ can be expressed as a sum of products of elementary moments of w, each term containing at least $p+1$ factors.

Example: Identify $j_i(A_i) = A_i$. Consider $w = abab$ with $a \in A_{i_1}$, $b \in A_{i_2}$, $i_1 \neq i_2$, $t := (i_1,i_2,i_1,i_2)$, $p=2$.

Decomposing $b = \varphi_{i_2}^{(\chi_4^{(4)}(t))}(b)\cdot 1 + b^\circ$, with $b^\circ \in \operatorname{Ker} \varphi_{i_2}^{(\chi_4^{(4)}(t))}$, one can write

$$\varphi(abab) = \varphi_{i_2}^{(\chi_4^{(4)}(t))}(b)\,\varphi(aba) + \varphi(abab^\circ).$$

But

$$\varphi(aba) = \varphi_{i_2}^{(\chi_2^{(4)}(t))}(b)\,\varphi_{i_1}(a^2) + \sum, \text{like in the precedent}$$

Example, where

$$\sum = \varphi_{i_1}(a)\,\varphi_{i_2}(b)\,\varphi_{i_1}(a) - \varphi_{i_2}^{(\chi_2^{(4)}(t))}(b)\,\varphi_{i_1}(a)\,\varphi_{i_1}(a).$$

So, the summand $\varphi_{i_2}^{(\chi_4^{(4)}(t))}(b)\,\varphi(aba)$ contains products with at least $p+1=3$ factors.

Decomposing $a = \varphi_{i_1}^{(\chi_3^{(4)}(t))}(a)\cdot 1 + a^\circ$ with $a^\circ \in \operatorname{Ker} \varphi_{i_1}^{(\chi_3^{(4)}(t))}$,

one can continue :

$$\varphi(abab^\circ) = \varphi_{i_1}^{(\chi_3^{(4)}(t))}(a)\,\varphi(abb^\circ) + \varphi(aba^\circ b^\circ).$$

And so on, decomposing successively $b = \varphi_{i_2}^{(\chi_2^{(4)}(t))}(b)\cdot 1 + b^{\circ\prime}$ with $b^{\circ\prime} \in \operatorname{Ker} \varphi_{i_2}^{(\chi_2^{(4)}(t))}$, and $a = \varphi_{i_1}^{(\chi_1^{(4)}(t))}(a)\cdot 1 + a^{\circ\prime}$ with $a^{\circ\prime} \in \operatorname{Ker} \varphi_{i_1}^{(\chi_1^{(4)}(t))}$

one can write :

$$\varphi(aba^0b^0) = \varphi_{i_2}^{(\chi_2^{(4)}(t))}(b)\,\varphi(aa^0b^0) + \varphi(ab^{0\prime}a^0b^0)$$

and

$$\varphi(ab^{0\prime}a^0b^0) = \varphi_{i_1}^{(\chi_1^{(4)}(t))}(a)\,\varphi(b^{0\prime}a^0b^0) + \varphi(a^{0\prime}b^{0\prime}a^0b^0).$$

But $\varphi(b^{0\prime}a^0b^0) = \varphi_{i_2}(b^{0\prime})\varphi_{i_1}(a^0)\varphi_{i_2}(b^0)$ and $\varphi(a^{0\prime}b^{0\prime}a^0b^0) = \varphi_{i_1}(a^{0\prime})\varphi_{i_2}(b^{0\prime})\varphi_{i_1}(a^0)\varphi_{i_2}(b^0)$, by the free Ψ-independence property.

One has also $\varphi(abb^0) = \varphi_{i_1}(a)\,\varphi_{i_2}(bb^0)$ and $\varphi(aa^0b^0) = \varphi_{i_1}(aa^0)\,\varphi_{i_2}(b^0)$ like in the first Example. Because $\varphi_{i_2}(bb^0) = \varphi_{i_2}(b^2) - \varphi_{i_2}^{(\chi_4^{(4)}(t))}(b)\,\varphi_{i_2}(b)$ and $\varphi_{i_1}(aa^0) = \varphi_{i_1}(a^2) - \varphi_{i_1}^{(\chi_3^{(4)}(t))}(a)\,\varphi_{i_1}(a)$ it becomes clear that the summand $\varphi(ab\,a\,b^0)$ also contains products with at least $p+1=3$ factors.

<u>Corollary 1.10.</u> Let be $n \geq 1$, $t = (i_1,\ldots,i_n) \in I^n$ such that t is non-crossing and $p := \#\{i_1,\ldots,i_n\}$.

Consider $a_k \in A_{i_k}$ $(1 \leq k \leq n)$ and $w = j_{i_1}(a_1)\ldots j_{i_n}(a_n)$.

Then the expectation $\varphi(w)$ can be expressed in the following form:

$$\varphi(w) = \prod_{\ell=1}^{p} \varphi_{i(\ell)}^{(\widetilde{\chi}_{\pi_\ell}^{(m)}(t))}(w_{\pi_\ell}) + \sum$$

where

$$\pi_\ell = (\pi_\ell(1),\ldots,\pi_\ell(m_\ell)) = (k = \overline{1,n}\,;\ i_k = i(\ell))$$

$$w_{\pi_\ell} := a_{\pi_\ell(1)}\cdots a_{\pi_\ell(m_\ell)} \quad (\in A_{i(\ell)}),$$

$$\widetilde{\chi}_{\pi_\ell}^{(m)}(t) := \begin{cases} \chi_{\pi_\ell(m_\ell)}^{(m)}(t) & \text{if } \pi_\ell \text{ is inner} \\ \lambda_0 & \text{otherwise} \end{cases}$$

$\ell = \overline{1,p}$, $\pi^t = (\pi_1,\ldots,\pi_p)$ being the non-crossing partition corresponding to t, and $\sum$ is a sum of products of elementary moments of w, in which each term contains at least $p+1$ factors.

<u>Note 1.11</u> In the definition of $\chi_{\pi_\ell}^{(m)}$ one can choose any element of π_ℓ.

2. Let now us specialize the frame of the precedent section.

Take $\Lambda = N$.

Let $\{A, \Phi\}$ be a couple, where A is a C^*-algebra over B and $\Phi = \{\varphi^{(n)};\ n \in N\}$ is a specified state set of A.

Consider $\{A_i, \varphi_i, \Psi_i\}_{i\in I}$, where $A_i := A$, $\varphi_i := \varphi^{(o)}$, and $\Psi_i := \Psi := \{\varphi^{(n)};\ n \geq 1\}$, for each $i \in I$.

If $\hat{A} = \underset{i\in I}{*}{}_B A_i$ (the $*$-algebraic free product with amalgamation over B), then there exists a state $\hat{\varphi}$ of $\hat{A}$, so that relative to the canonical morphisms $j_i : A_i = A \longrightarrow \hat{A}$ one has

(1) $\hat{\varphi} \circ j_i = \varphi^{(0)}$ for each $i \in I$

(2) For all $n \geq 1$ and $t := (i_1, \ldots, i_n) \in I^n$

$$\hat{\varphi}(j_{i_{\pi(1)}}(a_{\pi(1)}) \cdots j_{i_{\pi(p)}}(a_{\pi(p)})) = \varphi^{(0)}(a_{\pi(1)}) \cdots \varphi^{(0)}(a_{\pi(p)})$$

if $a_{\pi(k)} \in \operatorname{Ker} \varphi^{(\chi^{(n)}_{\pi(k)}(t))}$, $\pi(k) \in \{1, \ldots, n\}$ $(1 \leq k \leq p)$,

for all $p \geq 1$ and $(i_{\pi(1)}, \ldots, i_{\pi(p)}) \in D_p(\{i_1, \ldots, i_n\})$. ∎

$(\hat{A}, \hat{\varphi})$ can be called a <u>reduced free power with amalgamation</u> of $(A, \varphi^{(o)})$ <u>over</u> B <u>with respect to</u> Ψ <u>via</u> χ.

Let be also in the following $B = \mathbb{C}\cdot 1$.

<u>Remark 2.1.</u> Let be $n \geq 1$ and $t, t' \in I^n$ such that $t = (i_1, \ldots, i_n) \sim t' = (i'_1, \ldots, i'_n)$. Suppose $\chi^{(n)}(t) = \chi^{(n)}(t')$. Then it is easy to see that

$$\hat{\varphi}(j_{i_1}(a_1) \cdots j_{i_n}(a_n)) = \hat{\varphi}(j_{i'_1}(a_1) \cdots j_{i'_n}(a_n))$$

for all $a_k \in A$ $(1 \leq k \leq n)$.

Moreover, if for example $\chi^{(n)}(t) = \chi^{(n)}(t')$ for all $t, t' \in I^n$ such that $t \sim t'$, then for all $\tau \in \mathfrak{S}_n$ (the symmetric group of order n) and $(i_1, \ldots, i_n) \sim (i'_1, \ldots, i'_n)$ of I^n, one has

$$\hat{\varphi}(j_{i_{\tau(1)}}(a_1) \cdots j_{i_{\tau(n)}}(a_n)) = \hat{\varphi}(j_{i'_{\tau(1)}}(a_1) \cdots j_{i'_{\tau(n)}}(a_n))$$

for all $a_k \in A$ $(1 \leq k \leq n)$.

Therefore, if $a_1, \dots, a_n \in A$ are fixed, the value of the expectation $\hat{\varphi}(j_{i_1}(a_1) \cdots j_{i_n}(a_n))$ is the same for all $(i_1, \dots, i_n)$ belonging to the same equivalence class relative to $\sim$, under the hypothesis $\chi^{(n)}(t) = \chi^{(n)}(t')$ for $t, t' \in [(i_1, \dots, i_n)]$. Under this hypothesis, the value of the expectation $\hat{\varphi}(j_{i_1}(a_1) \cdots j_{i_n}(a_n))$ depends only on that equivalence class i.e. only on the corresponding partition. So that, if $\chi^{(n)}(t) = \chi^{(n)}(t')$ for all $t, t' \in [(i_1, \dots, i_n)]$, one can denote

$$\hat{\varphi}(j_{i_1}(a_1) \cdots j_{i_n}(a_n)) =: \hat{\varphi}(\pi^t; a_1, \dots, a_n) \quad ,$$

$\pi^t \in P(\{1, \dots, n\})$ being the partition corresponding to the equivalence class $[t]$ which has the representative $(i_1, \dots, i_n)$.

($[t]$ is denoting the equivalence class of $t \in I^n$).

Let now us take in the following $I = \mathbb{N}$ and consider

$(\hat{A}, \hat{\varphi}) = \overset{(\chi)}{\underset{i \in \mathbb{N}}{\ast}} \Psi(A_i, \varphi_i)$ given by the above assertion.

Remark 2.2. Let $r \geq 1$ and $N \in \mathbb{N}$, $N \geq 1$ be fixed. Take $a_1, \dots, a_n \in A$. If $\chi^{(r)}(t) = \chi^{(r)}(t')$ for all $t \sim t'$ of $\{1, 2, \dots, N\}^r$, then

$$\hat{\varphi}\Big(\Big(\sum_{i_1=1}^{N} j_{i_1}(a_1)\Big) \cdots \Big(\sum_{i_n=1}^{N} j_{i_n}(a_n)\Big)\Big) = \sum_{i_1, \dots, i_n = 1}^{N} \hat{\varphi}(j_{i_1}(a_1) \cdots j_{i_n}(a_n))$$

$$= \sum_{p=1}^{n} A_N^p \sum_{(\pi_1, \dots, \pi_p) \in P(\{1, \dots, n\})} \hat{\varphi}(\pi_1, \dots, \pi_p; a_1, \dots, a_n) \; ,$$

by collecting these terms corresponding to the same equivalence class. The equivalence class corresponding to $(\pi_1, \dots, \pi_p)$ contains exactly $A_N^p = \frac{N!}{(N-p)!}$ representatives.

In view of 1. , the proof of the following theorem becomes clear in the present context.

Theorem 2.3. (Limit theorem)

For each $N \geq 1$ let m elements $a_{N,k} \in A$, $1 \leq k \leq m$, be given and consider $S_{N,k} := \sum_{i=1}^{N} j_i(a_{N,k})$.

For all $r \geq 1$ and $\sigma(1), \ldots, \sigma(r) \in \{1, \ldots, m\}$ let suppose that

$$\lim_{N\to\infty} N \cdot \varphi^{(n)}(a_{N,\sigma(1)} \cdots a_{N,\sigma(r)}) =: Q^{(n)}(\sigma(1), \ldots, \sigma(r)) \qquad (*)$$

exists for each $n \in \mathbb{N}$, and $\chi^{(r)}(t) = \chi^{(r)}(t')$ for each $t \sim t'$ of N^r.

Then for all $r \geq 1$ and $\sigma(1), \ldots, \sigma(r) \in \{1, \ldots, m\}$:

$$\lim_{N\to\infty} \hat{\varphi}(S_{N,\sigma(1)} \cdots S_{N,\sigma(r)}) = \sum_{p=1}^{r} \sum_{\pi=(\pi_1,\ldots,\pi_p)\in P_{nc}(\{1,\ldots,r\})} \prod_{\ell=1}^{p} Q_\pi(\pi_\ell)$$

where

$$Q_\pi(\pi_\ell) := Q^{(\tilde{\chi}^{(r)}_{\pi_\ell}(t))}(\sigma(\pi_\ell(1)), \ldots, \sigma(\pi_\ell(r_\ell))) \quad ,$$

for $\pi_\ell = (\pi_\ell(1), \ldots, \pi_\ell(r_\ell))$, and $\tilde{\chi}^{(r)}_{\pi_\ell}(t) = \begin{cases} \chi^{(r)}_{\pi_\ell(r_\ell)}(t) & \text{if } \pi_\ell \text{ is inner} \\ 0 & \text{otherwise} \end{cases}$

$(\ell = \overline{1,p})$, $t \in I^r$ being a representative of the equivalence class corresponding to $\pi = (\pi_1, \ldots, \pi_p)$.

Note 2.4. The condition $(*)$ can be replaced by a weaker one :

For all $r \geq 1$ and $\sigma(1), \ldots, \sigma(r) \in \{1, \ldots m\}$,

$$\lim_{N\to\infty} N \cdot \varphi^{(0)}(a_{N,\sigma(1)} \cdots a_{N,\sigma(r)}) =: Q^{(0)}(\sigma(1), \ldots, \sigma(r))$$

and

$$\lim_{N\to\infty} N \cdot \varphi^{(\chi^{(r)}_j(t))}(a_{N,\sigma(1)} \cdots a_{N,\sigma(r)}) =: Q^{(\chi^{(r)}_j(t))}(\sigma(1), \ldots, \sigma(r))$$

exist for each $t \in I^r$ and $j = \overline{1,r}$.

Like in [8] and [3], one can easily derive analogues of a central limit theorem and a Poisson limit theorem, by specializing Theorem 2.3.

Corollary 2.5. (Central limit theorem). Let m elements $a_k \in A$, $1 \leq k \leq m$, be given and consider $S_{N,k} := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} j_i(a_k)$,for each $N \geq 1$ and $1 \leq k \leq m$.

For all $1 \leq k, \sigma(1), \sigma(2) \leq m$ let suppose that

$$\varphi^{(n)}(a_k) = 0$$

and

$$\varphi^{(n)}(a_{\sigma(1)} a_{\sigma(2)}) = Q^{(n)}(\sigma(1), \sigma(2))$$

for each $n \in \mathbb{N}$.

Then for all $r \geq 1$ and $\sigma(1), \ldots, \sigma(r) \in \{1, \ldots, m\}$:

$$\lim_{N \to \infty} \hat{\varphi}(S_{N,\sigma(1)} \cdots S_{N,\sigma(r)}) = \begin{cases} \sum\limits_{(\pi_1, \ldots, \pi_{r/2}) \in P_{nc}(\{1, \ldots, r\})} \prod\limits_{l=1}^{r/2} Q_\pi(\pi_l) & \text{if } r \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

where $Q_\pi(\pi_l) = Q^{(\tilde{\chi}^{(n)}_{\pi_l}(t))}(\sigma(\pi_l(1)), \sigma(\pi_l(2)))$, for $\pi_l = (\pi_l(1), \pi_l(2))$,

$$\tilde{\chi}^{(r)}_{\pi_l}(t) = \begin{cases} \chi^{(r)}_{\pi_l(2)}(t) & \text{if } \pi_l \text{ is inner} \\ 0 & \text{otherwise} \end{cases} , \quad (l = 1, r/2) , \quad t \in I^r$$

being a representative of the equivalence class corresponding to $\pi = (\pi_1, \ldots, \pi_{r/2})$.

Corollary 2.6. (Poisson limit theorem) Let $a_N \in A$ for $N \in \mathbb{N}$, $N \geq 1$ be given and consider $S_N := \sum_{i=1}^{N} j_i(a_N)$.

For all $r \geq 1$ suppose

$$\lim_{N \to \infty} N \cdot \varphi^{(n)}((a_N)^r) = \alpha^{(n)}$$

independent of r , for each $n \in N$.

Then for all $r \geq 1$:

$$\lim_{N \to \infty} \hat{\varphi}((S_N)^r) = \sum_{p=1}^{r} \sum_{(\pi_1, \ldots, \pi_p) \in P_{nc}(\{1, \ldots, r\})} \prod_{l=1}^{p} \alpha^{(\tilde{\chi}^{(r)}_{\pi_l}(t))}$$

where

$$\tilde{\chi}^{(r)}_{\pi_\ell}(t) = \begin{cases} \chi^{(r)}_{\pi_\ell(r_\ell)}(t) & \text{if } \pi_\ell \text{ is inner} \\ 0 & \text{otherwise} \end{cases}$$

for $\pi_\ell = (\pi_\ell(1),\ldots, \pi_\ell(r_\ell))$, $\ell=\overline{1,p}$, $t\in I^r$ being a representative of the equivalence class corresponding to $(\pi_1,\ldots,\pi_p)$.

Note 2.8. a) The conditions of these two corrollaries can be replaced by a similar manner to the precedent note.

b) In particular, if for all $r\geqslant 1$ suppose

$$\lim_{N\to\infty} N\cdot\varphi^{(0)}((a_N)^r) = \alpha$$

and

$$\lim_{N\to\infty} N\cdot\varphi^{(\chi_j^{(r)}(t))}((a_N)^r) = \beta$$

independent of r, χ, $t\in I^r$ and $j=\overline{1,r}$, then Corollary 2.6. asserts for all $r\geqslant 1$:

$$\lim_{N\to\infty} \hat{\varphi}((S_N)^r) = \sum_{p=1}^{r} \sum_{\pi\in P_{nc}(\{1,\ldots,r\})} \alpha^{o(\pi)} \beta^{i(\pi)}$$

where $o(\pi)$ is the number of outer sets in π and $i(\pi)$ is the number of inner sets in π.

Note 2.9. a) Let be $r\geqslant 1$ and $\sigma(1),\ldots,\sigma(r)\in\{1,\ldots,m\}$ be fixed. Consider $\pi=(\pi_1,\ldots,\pi_p)\in P_{nc}(\{1,\ldots,r\})$ and $[t]$ the equivalence class corresponding to π. Fix $\ell=\overline{1,p}$. $Q_\pi(\pi_\ell)$ depends only on π_ℓ if and only if $\chi^{(r)}_{\pi_\ell(r_\ell)}(t)$ does not depend on $[t]$ i.e. $\chi^{(r)}_{\pi_\ell(r_\ell)}=\chi(\pi_\ell(r_\ell))$ on the reunion of all equivalence classes corresponding to non-crossing elements in N^r (in particular, if $\chi_j^{(r)}=\chi(j)$ for all $j=\overline{1,r}$ on the above reunion).

b) Let be $n\geqslant 1$. Let n maps $\chi_j : I^j \longrightarrow \Lambda$, $j=\overline{1,n}$, and $\tau\in\mathfrak{S}_n$ (the symmetric group of order n) be given. For each $j=\overline{1,n}$, take $\pi^{(j)}=(\pi_1^{(j)},\pi_2^{(j)})\in P(\{1,\ldots,n\})$ such that $\tau(j)\in\{\#\pi_1^{(j)}, \#\pi_2^{(j)}\}$.

Consider $\chi^{(n)}: I^n \longrightarrow \Lambda^n$ by

$$\chi_j^{(n)}(i_1,\dots,i_n) := \chi_{\tau(j)}\left(i_{\pi_\ell^{(j)}(1)},\dots,i_{\pi_\ell^{(j)}(\tau(j))}\right)$$

if $\tau(j) = \#\pi_\ell^{(j)}$ $(\ell\in\{1,2\})$, for each $j=\overline{1,n}$.

If $\chi_j(t_j) = \chi_j(t_j')$ for all $t_j \sim t_j'$ in I^j , for each $j=\overline{1,n}$, then $\chi^{(n)}(t) = \chi^{(n)}(t')$ for all $t\sim t'$ in I^n .

This is a method to produce $\chi^{(n)}$ for Limit theorem.

3. Let us describe a little the set of the central limit distributions in view of Corollary 2.5.

A connection between this limit distribution set, the universal continued fractions of the Stieltjes type and the classical moment problem can be established.

The way is the equivalence between the characteristic series of certain labelled paths in the plane and the universal Stieltjes-Jacobi continued fractions. This equivalence holds in the algebra $\mathbb{C}\langle\langle X\rangle\rangle$ of formal series over a non-commutative alphabet X with complex coefficients (see [5]).

Let us recall some facts from [5] .

In N^* , the free monoid with basis N, consider the set of words of the form $y=y_0y_1\dots y_r$ where $y_0 = 0$ and $y_k - y_{k-1} \in\{-1,1\}$ for each $k=\overline{1,r}$ (so that $y_1=1$).

Each word of this form in N^* can be characterized as a word $u=u_1u_2\dots u_r$ over the alphabet $\{\alpha,\beta\}$ with $\alpha=(1,+1)$, $\beta=(1,-1)$, and can be geometrically represented as a sequence of points in the plane $M_0M_1\dots M_r$ such that $M_k=(k,y_k)$ with $y_0=0$ and $y_k-y_{k-1}\in\{-1,1\}$ $(k=\overline{1,r})$.

According to [5], the words $y=y_0y_1\dots y_r$ of the above form can be called positive paths without level steps (they consist of only two type of "steps": rises and falls). The number r is called the length of φ, denoted $|\varphi|$.

Let us introduce the labelling operation.

Consider an arbitrary alphabet of non-commutative indeterminates $X = \{\alpha_n, n \geq 0\} \cup \{\beta_{n+1}; n \geq 0\}$.

If $\varphi = \varphi_0 \varphi_1 \cdots \varphi_r$ is of the above form, then the labelling of φ denoted $\lambda(\varphi)$, is defined as a word of X^*, the free monoid with the basis X, by

$$\lambda(\varphi) = v_1 v_2 \cdots v_r$$

where

$$v_k = \begin{cases} \alpha_{\varphi_{k-1}} & \text{if } \varphi_k - \varphi_{k-1} = 1 \\ \beta_{\varphi_{k-1}} & \text{if } \varphi_k - \varphi_{k-1} = -1 \end{cases} \qquad (k=\overline{1,r})$$

Examine now the following set of positive paths without level steps :

$$\mathcal{P}^+ = \{\varphi = \varphi_0 \varphi_1 \cdots \varphi_r ;\ \varphi_0 = 0 = \varphi_r ,\ \varphi_k - \varphi_{k-1} \in \{-1,1\}\ (k=\overline{1,r})\}.$$

For $\varphi \in \mathcal{P}^+$ it is easy to observe that $\varphi_{r-1}=1$, $r=|\varphi|$ is even, $r \geq 2$, and also $\#\{k=\overline{1,r} ; \varphi_k - \varphi_{k-1} = 1\} = \#\{k=\overline{1,r}; \varphi_k - \varphi_{k-1} = -1\} = \frac{r}{2}$.

Therefore one can also consider the following set of labelled paths

$$\mathcal{P} := \lambda(\mathcal{P}^+) \quad (\subset X^*) .$$

(card($\mathcal{P} \cap X^{2n}$) is the n-th Catalan number)

The characteristic series of $\mathcal{P}$ is $\mathrm{char}(\mathcal{P}) = \sum_{\varphi \in \mathcal{P}^+} \lambda(\varphi)$ which is considered as an element of the monoid algebra of X^* i.e. as a formal series over the non-commutative alphabet X with coefficients in the complex field, usually denoted $\mathbb{C}\langle\langle X \rangle\rangle$.

In view of the basic equivalence theorem in [5], the series char($\wp$) appears as the non-commutative analogue of the Stieltjes type continued fraction

$$S(X,z):=\cfrac{1}{1-\cfrac{a_0b_1z^2}{1-\cfrac{a_1b_2z^2}{1-\dots}}}$$

i.e.

$$\mathrm{char}(\wp)\equiv S(X,z)$$

where $s \equiv s'$ means s and s' are equivalent modulo the commutativity of the indeterminates in X .

There exist also an one-to-one correspondence between the set of partitions $\pi=(\pi_1,\dots,\pi_p) \in P(\{1,\dots,2p\})$ where $\#\pi_\ell = 2$ for each $\ell=\overline{1,p}$ and the set of paths in $\wp^+$ which have the length $2p$, i.e. $\wp^+ \cap X^{2p}$:

If $\pi = (\pi_1,\dots,\pi_p) \in P(\{1,\dots,2p\})$, $\#\pi_\ell = 2$ and $\pi_\ell=(\pi_\ell(1),\pi_\ell(2))$ for $\ell=\overline{1,p}$, then the path corresponding to π is $\varphi^\pi=\varphi_0\varphi_1\dots\varphi_{2p}$ where

$$\varphi_0=0 \quad \text{and} \quad \varphi_k=\begin{cases}\varphi_{k-1}+1 & \text{if } k\in\{\pi_\ell(1);\ \ell=\overline{1,p}\}\\ \varphi_{k-1}-1 & \text{if } k\in\{\pi_\ell(2);\ \ell=\overline{1,p}\}\end{cases}$$

(It is easy to see, in this context $\pi \in P_{nc}(\{1,\dots,2p)$ if and only if $\varphi_{\pi_\ell(1)}-\varphi_{\pi_\ell(2)}=1$. Moreover, if π_ℓ is outer in π , then $\varphi_{\pi_\ell(2)}=0$, $\varphi_{\pi_\ell(1)}=1$).

The above correspondence is also onto : if $\varphi=\varphi_0\varphi_1\dots\varphi_{2p}\in\wp^+\cap X^{2p}$ then the partition corresponding to φ is $\pi^\varphi=(\pi_1,\dots,\pi_p)\in P(\{1,\dots,2p\})$ where $\pi_\ell=(\pi_\ell(1),\pi_\ell(2))$ $\ell=\overline{1,p}$, are given by the equality

$$\{1,\dots,2p\}=\{k=\overline{1,2p}\ ;\ \varphi_k-\varphi_{k-1}=1\}\cup\{k=\overline{1,2p}\ ;\ \varphi_k-\varphi_{k-1}=-1\} ,$$

taking

$$\{k=\overline{1,2p}\ ;\ \gamma_k-\gamma_{k-1}=1\}=(\pi_1(1),\ldots,\pi_p(1)) \quad \text{and} \quad \{k=\overline{1,2p}\ ;\ \gamma_k-\gamma_{k-1}=-1\}=$$

$=(\pi_1(2),\ldots,\pi_p(2))$.

In consequence, the labelling corrercponding to $\gamma^{\pi}=\gamma_0\gamma_1\ldots\gamma_{2p}$ (the path corresponding to $\pi\in P(\{1,\ldots,2p\})$, with $\pi_\ell=(\pi_\ell(1),\pi_\ell(2))$, for $\ell=\overline{1,p}$) is :

$$\lambda(\gamma^{\pi})=v_1v_2\ldots v_{2p}$$

where $v_{\pi_\ell(1)}=\alpha_{\gamma_{\pi_\ell(1)-1}}$ and $v_{\pi_\ell(2)}=\beta_{\gamma_{\pi_\ell(2)-1}}$ $(\ell=\overline{1,p})$.

In particular, for each $\ell=\overline{1,p}$ to the partition of $\{\pi_\ell(1),\pi_\ell(2)\}$ (π_ℓ), consisting of only one set π_ℓ with $\pi_\ell=(\pi_\ell(1),\pi_\ell(2))$ it corresponds the following path of length 2 : $\gamma^{\pi_\ell}=\gamma_0\gamma_{\pi_\ell(1)}\gamma_{\pi_\ell(2)}$ and its labelling is $\lambda(\gamma^{\pi_\ell})=v_{\pi_\ell(1)}v_{\pi_\ell(2)}$.

Remark 3.1. If $[v_i,v_j]=0$ for $i\neq j$ i.e. $v_1,v_2,\ldots,v_{2p}$ commute, then $\lambda(\gamma^{\pi})=\lambda(\gamma^{\pi_1})\,\lambda(\gamma^{\pi_2})\cdots\lambda(\gamma^{\pi_p})$ in X^*. $\square$

Coming back to Central limit theorem, take now $a\in A$ with $\varphi^{(n)}(a)=0$, $\varphi^{(n)}(a^2)=Q^{(n)}(1,1)$ for each $n\in\mathbb{N}$.

For all $r\geq 1$ denote

$$\mu_r=\begin{cases}\displaystyle\sum_{\substack{(\pi_1,\ldots,\pi_p)\in P_{nc}(\{1,\ldots,r\})\\ \#\pi_\ell=2}}\prod_{\ell=1}^{p}Q_\pi(\pi_\ell) & \text{if } r=2p\\ 0 & \text{otherwise}\end{cases}$$

where

$$Q_\pi(\pi_\ell)=Q^{(\tilde{\chi}^{(n)}_{\pi_\ell}(t))}(1,1) \quad \text{for } \pi_\ell=(\pi_\ell(1),\pi_\ell(2)),$$

$$\tilde{\chi}^{(n)}_{\pi_\ell}(t)=\begin{cases}\chi^{(n)}_{\pi_\ell(2)}(t) & \text{if } \pi_\ell \text{ is inner}\\ 0 & \text{otherwise}\end{cases} \qquad (\ell=\overline{1,p}),\ t\in I^n,$$

[t] corresponding to $\pi=(\pi_1,\ldots,\pi_p)$.

Remark 3.2. Suppose the alphabet X consists of the commutative variables $\alpha_n = \sqrt{\varphi^{(n)}(\alpha^2)} = \beta_{n+1}$, $n \geq 0$ belonging to the multiplicative semigroup $\mathbb{R}^+$.

Consider $\pi = (\pi_1, \dots, \pi_p) \in P_{nc}(\{1, \dots, 2p\})$, $\pi_\ell = (\pi_\ell(1), \pi_\ell(2))$ $(\ell = \overline{1,p})$, and also $[t]$ and $\gamma^\pi = \gamma_0 \gamma_1 \dots \gamma_{2p}$, respectively, the equivalence class and the path corresponding to π .

If $\widetilde{\chi}^{(n)}_{\pi_\ell}(t) = \mu_{\pi_\ell(2)}$, for each $\ell = \overline{1,p}$, then the labelling of γ^π is

$$\lambda(\gamma^\pi) = \prod_{\ell=1}^{p} Q_\pi(\pi_\ell) .$$

Therefore

$$\mu_{2p} \equiv \mathrm{char}\,(\lambda(\Phi^+) \cap X^{2p}) .$$

For the power series $\sum_{n \geq 0} \mu_n z^n$, the basic equivalence theorem in [5] implies then an expansion in the following Stieltjes type continued fraction.

$$\cfrac{1}{1 - \cfrac{\alpha_0^2 z^2}{1 - \cfrac{\alpha_1^2 z^2}{1 - \ddots}}}$$

(here $\mu_0 = 1$).▯

Some definitions are now necessary.

Denote, as usually, by $\mathbb{C}[X]$ the algebra of complex polynomials in one variable.

Call probability distributions all linear functionals $\mu : \mathbb{C}[X] \to \mathbb{C}$ with $\mu[1] = 1$. Such a μ is positive-definite if $\mu[P(x)] > 0$ for every polynomial $P(x)$ that is not identically zero and is non-negative for all real x ; say μ is symmetric if all of its moments of odd order are zero ($\mu[X^{2p+1}] = 0$ for all $p \in \mathbb{N}$). (A classical fact implies that such a positive-definite probability distribution is given by a solution of Hamburger's classical moment problem).

For a random variable w in $(\hat{A},\hat{\varphi})$, call the distribution of w the probability distribution $\mu_w : \mathbb{C}[X] \to \mathbb{C}$ given by $\mu_w[P] = \hat{\varphi}(P(w))$ for all $P \in \mathbb{C}[X]$.

In the space of linear functionals on $\mathbb{C}[X]$ sending 1 into 1, say that probability distributions μ_n converge to μ (denote $\mu_n \Rightarrow \mu$) if $\lim\limits_{n\to\infty} \mu_n[P] = \mu[P]$ for all $P \in \mathbb{C}[X]$.

For a sequence $(w_n)_{n\geq 0}$ of random variables in $(\hat{A},\hat{\varphi})$, say that $(w_n)_{n\geq 0}$ converges in distribution to a probability distribution $\mu : \mathbb{C}[X] \to \mathbb{C}$ if $\mu_{w_n} \Rightarrow \mu$.

Remark 3.3. Using classical facts (see [1], [4], [10]), it is not difficult to see then that central limit distribution set given by Corollary 2.5. is the set of all probability distributions, in the above sense, which are positive-definite and symmetric. In particular, any symmetric probability distribution on the real line with moments of all order is contained in the central limit distribution set.∎

Thus applying the method of [2], one can find operators possessing the combinatorics of the central limit theorem.

More precisely, for each symmetric probability distribution μ on the real axis with moments of all order, and such that the set of all polynomials is dense in $L^2(\mu)$, consider the orthonormal basis $(e_n)_{n\geq 0}$ in $L^2(\mu)$ consisting of the monic orthonormal polynomial sequence ([4]) corresponding to μ. Then the self-adjoint operator Z of multiplication by x with maximal domain in $L^2(\mu)$ has in this basis the matrix of the Stieltjes bi-diagonal form

$$\begin{pmatrix} 0 & g(0) & 0 & 0 & 0 & 0 & \cdots \\ g(0) & 0 & g(1) & 0 & 0 & 0 & \cdots \\ 0 & g(1) & 0 & g(2) & 0 & 0 & \cdots \\ 0 & 0 & g(2) & 0 & g(3) & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

where $g(j) := \langle e_j, Ze_{j+1} \rangle > 0$, $0 \leq j < \dim L^2(\mu)$.

So that Z is equal to $g(N)L + L^* g(N)$ on the space of all polynomials, where N and L are the number and the standard annihilation operators

$$N = \sum_j j|e_j\rangle\langle e_j| \qquad , \quad L = \sum_j |e_j\rangle\langle e_{j+1}|$$

in Dirac's notation.

The distribution of Z in the pure state e_0 is μ.

R E F E R E N C E S

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