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OF OPEN SUBSPACES

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1 Introduction

By the work of M. Peternell ([5], Satz 2.3) it is known that if Ω_i are q_i-complete open subsets of a reduced complex space X, $i = 1, 2$, then their union $\Omega_1 \cup \Omega_2$ is $(q_1 + q_2)$-complete. The proof relies heavily on a technical criterion (viz. [5], Satz 2.2, p. 558-563).

The aim of this note is to give elementary direct short proofs of these theorems by effectively constructing special exhaustion functions. That will follow by composing the given exhaustion function with suitable real-valued convex functions of one real variable. As an application of the above mentioned criterion it is shown that any complex space which is an increasing union of Stein open subsets is always 2-complete.

On the other hand, our method gives similar results when Ω_i are q_i-concave, $i = 1, 2$. An example of two 1-concave open subsets of a complex manifold whose intersection is not 1-concave is shown. This is in contrast with the 1-complete analogous situation (The set-up is chosen so that 1-complete spaces corresponds to Stein spaces).

2 Preliminaries

All complex spaces are assumed to be reduced and with countable topology.

Let X be a complex space, φ a class C^2 real-valued function defined on X and $q \geq 1$ an integer. Then φ is said to be q-convex at a point $x_0 \in X$ if there are:

- an open neighborhood U of x_0, a biholomorphic map ι of U onto an analytic subset of an open set D of some $\mathbf{C}^N$, $z_0 := \iota(x_0)$ and
- a C^2-extension $\widehat{\varphi} : D \longrightarrow \mathbf{R}$ of φ, i.e. $\widehat{\varphi} \circ \iota = \varphi_{|U}$ such that the Levi form $L(\widehat{\varphi}, z_0)$ of $\widehat{\varphi}$ at z_0 has at least $(N - q + 1)$-positive eigenvalues or, equivalently that there is a complex vector subspace $E \subseteq \mathbf{C}^N$ with dim $E \geq N - q + 1$ such that the Levi form $L(\widehat{\varphi}, z_0)$ is positive definite when restricted to E.

It can be easily seen that q-convexity at x_0 does not depend on the chosen local embedding $\iota : U \longrightarrow D$.

φ is said to be q-convex on a set $W \subset X$ if it is q-convex at any point of W.

A complex space X is said to be q-complete (resp. q-convex) if there exists a C^2-exhaustion function $\varphi : X \longrightarrow \mathbf{R}$ which is q-convex on the whole space X (resp. outside a compact subset of X).

X is said to be q-concave if there is a C^2 function $\varphi : X \longrightarrow (0, \infty)$, q-convex outside a compact set and which exhausts X from below, i.e. the set $\{ x \in X \mid \varphi(x) > \epsilon \}$ is relatively compact in X for any $\epsilon > 0$.

Remark 2.1 Sometimes it is worthwhile to have easier criteria of q-complete, q-convex and q-concave spaces. In order to do this we say that a class C^2 function $\varphi : X \longrightarrow \mathbf{R}$ is *tangentially q-convex* at x_o if there exists:

- a local chart $\iota : U \longrightarrow D \subset \mathbf{C}^N$, $U \ni x_0$, $z_0 := \iota(x_0)$ and
- a C^2-extension $\widehat{\varphi} : D \longrightarrow \mathbf{R}$ of φ, i.e. $\widehat{\varphi} \circ \iota = \varphi_{|U}$ such that the restriction of the Levi form $L(\widehat{\varphi}, z_0)$ of $\widehat{\varphi}$ at z_0 to the holomorphically tangent space

$$H_{z_0}(\widehat{\varphi}) := \{ \xi \in \mathbf{C}^N \mid < \partial\widehat{\varphi}(z_0), \xi > := \sum_{i=1}^{N} \frac{\partial \widehat{\varphi}}{\partial z_j}(z_o)\xi_j = 0 \}$$

has at most $(q-1)$-nonpositive eigenvalues.

It is straightforward that φ is tangentially q-convex at x_0 if and only if there exists a sufficiently large constant $c_0 > 0$ such that $\exp(c\varphi)$ is q-convex at x_0 for any $c \geq c_0$. Consequently tangential q-convexity does not depend on the chosen local embedding.

φ is said to be tangentially q-convex on a set $W \subset X$ if it is so at any point of W. Note that if $\varphi : X \longrightarrow \mathbf{R}$ is tangentially q-convex on W and $\chi : \mathbf{R} \longrightarrow \mathbf{R}$ is an arbitrary strictly increasing smooth function, then $\chi(\varphi)$ is also tangentially q-convex on W.

Now we have

Proposition 2.1 *In the above definitions of q-complete (resp. q-convex, q-concave) spaces we may replace the q-convexity of the above exhaustion function on the corresponding set by its tangential q-convexity.*

Proof: We carry out the proof only in the q-concave case (The other cases are treated in a similar way). By standard arguments, this will follow from Remark 2.1 and from the following

Lemma 2.1 *For any continuous function* $\mu : (0,\infty) \longrightarrow (0,\infty)$ *there exists a smooth strictly increasing convex function* $\lambda : (0,\infty) \longrightarrow (0,\infty)$ *such that* $\lambda''/\lambda' > \mu$ *and* $\lambda(t) \searrow 0$ *as* $t \searrow 0$.

Proof: First choose a smooth bijection $\delta : (0,\infty) \longrightarrow (0,\infty)$ with $\exp(\delta) - \delta' > \tilde{\mu}$ where $\tilde{\mu} : (0,\infty) \longrightarrow (0,\infty)$ is given by

$$\tilde{\mu}(t) := \frac{\mu(1/t) + 2t}{t^2},\ t > 0.$$

Then set $\alpha : (0,\infty) \longrightarrow \mathbf{R}$ by $\alpha(t) := \int_1^t \exp(\delta(s))\, ds$, $t > 0$. Finally put $\lambda(t) := \exp(-\alpha(1/t))$, $t > 0$. Straightforward computations gives us the lemma, whence the proposition.

3 Elementary lemmas

Here we collect some special convex functions of one real variable.

Lemma 3.1 *Let* $\epsilon : [0,\infty) \longrightarrow (0,\infty)$ *be a continuous function. Then there exists a smooth strictly increasing convex function* $\lambda : [0,\infty) \longrightarrow (0,\infty)$ *such that*

$$\lambda' \exp(-\lambda) < \epsilon \text{ and } \lambda > 1/\epsilon.$$

Proof: Define succesively $\lambda : [0,\infty) \longrightarrow (0,\infty)$ by

(1) $$\lambda := -\log F$$

where $F : [0,\infty) \longrightarrow (0,1)$ is a smooth function with

(2) $$F(t) := \int_t^\infty f(s)\, ds, \ t \geq 0$$

and finally $f : [0,\infty) \longrightarrow (0,\infty)$ is constructed by

(3) $$f(s) := \exp(-u(s)), \ s \geq 0.$$

Here $u : [0,\infty) \longrightarrow (0,\infty)$ is a smooth rapidly increasing strictly convex function to be chosen later in proof.

That F and λ are well defined may be easily achieved by a suitable choice of u (sufficiently large). Now in order to verify the lemma we note that

(4) $$\lambda' \exp(-\lambda) = (-\exp(-\lambda))' = -F' = f = \exp(-u).$$

Therefore by choosing u large enough we get $\lambda' \exp(-\lambda) < \epsilon$. Also from (4) it follows that $\lambda' > 0$. Now condition $\lambda > 1/\epsilon$ is equivalent to $F < \exp(-1/\epsilon)$ which is fulfilled as soon as $u > \beta - \log \beta'$ where $\beta : [0,\infty) \longrightarrow (0,\infty)$ is a smooth strictly increasing function so that $\beta > 1/\epsilon$ and, then by integration.

Condition $\lambda'' > 0$ is equivalent to $f' \cdot F + f^2 > 0$ which in turn means $A > 0$ where

$$A(t) := f(t)^2 + f'(y) \int_t^\infty f(s)\, ds, \ t \geq 0.$$

But this is true as

$$A(t) = \exp(-u(t)) \int_t^\infty (u'(s) - u'(t)) \exp(-u(s))\, ds > 0$$

and since u' is strictly increasing.

Lemma 3.2 *Let ϵ_1, $\epsilon_2 : [0,\infty) \longrightarrow (0,\infty)$ be arbitrary continuous functions. Then there exists a smooth rapidly increasing convex function $\lambda : [0,\infty) \longrightarrow (0,\infty)$ such that*

$$\lambda' \exp(-\lambda) < \epsilon_1 \textit{ and } \lambda'' \exp(-\lambda) < \epsilon_2.$$

Proof: Consider a smooth rapidly increasing convex function $u : [0, \infty) \longrightarrow (0, \infty)$ to be chosen later in proof. As in Lemma 3.1 set

$$(5) \qquad g : [0, \infty) \longrightarrow (0, \infty), \ g := \exp(-u) \ \text{ and}$$

$$(6) \qquad v : [0, \infty) \longrightarrow (0, \infty), \ v(t) := -\log \int_t^\infty g(s)\, ds, \ t \geq 0.$$

We impose $\int_0^\infty g(s)\, ds = \int_0^\infty \exp(-u(s))\, ds < 1$. It evidently holds if u is large enough. Now we observe that $v' = g \cdot \exp(v)$. As in the proof of Lemma 3.1 we get

$$(7) \qquad g^2 + g' \exp(-v) > 0.$$

Now set $\lambda := 2v$. To check the hypothesis we proceed as follows:

a) First we have $\lambda' \exp(-\lambda) = 2v' \exp(-2v) = 2g \exp(-v)$. Hence $v' > 0$ and $\lambda' \exp(-\lambda) \leq 2g$ since $v > 0$.

b) Second $\lambda'' \exp(-\lambda) = 2v'' \exp(-v) = 2(g' \exp(-v) + g^2) > 0$ by (7).

In particular $\lambda'' > 0$ and $\lambda'' \exp(-\lambda) \leq 2(|g'| + g^2)$. Now choose u according to Lemma 3.1 so that $|g'| = u' \exp(-u)$ is small enough. The lemma follows.

These lemmas readily imply the subsequent two lemmas.

Lemma 3.3 *Consider Ω be an open set of some complex space X and an arbitrary C^2 exhaustion function $\varphi : \Omega \longrightarrow [0, \infty)$. Then there is a smooth rapidly increasing convex function $\lambda : [0, \infty) \longrightarrow (0, \infty)$ such that the continuous function $\tilde{\varphi} : X \longrightarrow [0, \infty)$ defined by*

$$\tilde{\varphi}(x) := \begin{cases} \exp(-\lambda(\varphi(x))), & x \in \Omega; \\ 0, & \text{otherwise} \end{cases}$$

is of class C^2 on X.

Lemma 3.4 *Let Ω be an open set of some complex space X and $\varphi : \Omega \longrightarrow (0,\infty)$ a C^2 function which is exhaustive from below. Then there is a smooth rapidly increasing convex function $\lambda : (0,\infty) \longrightarrow (0,\infty)$ with $\lambda(t) \to 0$ as $t \searrow 0$ and such that the continuous function $\tilde{\varphi} : X \longrightarrow [0,\infty)$ defined by*

$$\tilde{\varphi}(x) := \begin{cases} \lambda(\varphi(x)), & x \in \Omega; \\ 0, & x \in X \setminus \Omega \end{cases}$$

is of class C^2 on X.

4 The results

Here we prove the following (see also [5], Satz 2.3).

Theorem 4.1 *Let Ω_1 and Ω_2 be open subsets of a complex space X which are q_1-complete, resp. q_2-complete. Then $\Omega_1 \cup \Omega_2$ is $(q_1 + q_2)$-complete.*

Proof: Let $\Omega := \Omega_1 \cup \Omega_2$. By Lemma 3.3 there are q_i-convex exhaustion functions $\varphi_i : \Omega_i \longrightarrow (0,\infty)$, $i = 1, 2$, such that the continuous functions defined by

$$\tilde{\varphi}_i(x) := \begin{cases} \exp(-\varphi_i(x)), & x \in \Omega_i; \\ 0, & x \in X \setminus \Omega_i \end{cases}$$

are of class C^2 on X. Now define $\Phi : \Omega \longrightarrow (0,\infty)$ as follows

$$\Phi := \frac{1}{\tilde{\varphi}_1 + \tilde{\varphi}_2}.$$

Then Φ is exhaustive and of class C^2. It remains to check its $(q_1 + q_2)$-convexity. Indeed, on $\left(\Omega_1 \setminus \overline{\Omega}_2\right) \cup \left(\Omega_2 \setminus \overline{\Omega}_1\right)$ this is obvious, since there $\Phi \equiv \exp(\varphi_1)$ or $\Phi \equiv \exp(\varphi_2)$, respectively, and $q_1, q_2 \leq q_1 + q_2$.

At points x_0 from $(\Omega_1 \cap \partial\Omega_2) \cup (\Omega_2 \cap \partial\Omega_1)$, say $x_0 \in \Omega_1 \cap \partial\Omega_2$ we can write in a small neighborhood of x_0, $\Phi = \exp(\varphi_1) + \theta$, with

$$\theta := \frac{\tilde{\varphi}_2 \exp(2\varphi_1)}{1 + \tilde{\varphi}_2 \exp(\varphi_1)}$$

is of class C^2 and plurisubharmonic at x_0. Consequently Φ is q_1-convex at x_0. Similarly at points from $\Omega_2 \cap \partial\Omega_1$.

Now fix an arbitrary point $x_0 \in \Omega_1 \cap \Omega_2$. Since the question is local around x_0, by working in local extensions, we may assume, without any loss of generality, that X is an open subset of some $\mathbf{C}^N$. For the sake of simplicity denote

$$a_i := \exp(-\varphi_i(x_0)),\ b_i :=< \partial\varphi_i(x_0), \xi >$$

where $\xi \in T_{x_0}X = \mathbf{C}^N$, $i = 1, 2$. Hence $a_i > 0$ and $b_i \in \mathbf{C}$, $i = 12$. On the other hand, the Levi form of Φ at x_0, computed in direction ξ has the expression

$$(\clubsuit) \qquad L(\Phi, x_0)\xi = \frac{1}{(a_1 + a_2)^2} \cdot A + \frac{1}{(a_1 + a_2)^3} \cdot B$$

where

$$A := (a_1 + a_2)\left(a_1 L(\varphi_1, x_0)\xi + a_2 L(\varphi_2, x_0)\xi\right) \text{ and}$$
$$B := 2|a_1 b_1 + a_2 b_2|^2 - (a_1 + a_2)(a_1|b_1|^2 + a_2|b_2|^2).$$

Straightforward computations give $B = |a_1 b_1 + a_2 b_2|^2 - a_1 a_2 |b_1 - b_2|^2$. Now let $E_i \subseteq \mathbf{C}^N$ complex vector subspaces, $\dim E_i \geq N - q_i + 1$ such that $L(\varphi_i, x_0)|_{E_i}$ is positive definite, $i = 1, 2$. Define

$$F := \{\xi \in \mathbf{C}^N \mid < \partial\varphi_1(x_0), \xi > = < \partial\varphi_2(x_0), \xi >\}.$$

Finally put $E := E_1 \cap E_2 \cap F$. Hence $\dim E \geq N - (q_1 + q_2) + 1$. Now for any $\xi \in E \setminus \{0\}$ we simply get that

$$A > 0 \text{ and } B = |a_1 b_1 + a_2 b_2|^2 \geq 0.$$

Thus $L(\Phi, x_0)\xi > 0$. Hence Φ is $(q_1 + q_2)$-convex at x_0, whence the theorem.

Remark 4.1 *Suppose Ω_i are q_i-convex. Then $\Omega_1 \cup \Omega_2$ is $(q_1 + q_2)$-convex and $\Omega_1 \cap \Omega_2$ is $(q_1 + q_2 - 1)$-convex.*

The first part follows as in Theorem 4.1. For the second statement set $D := \Omega_1 \cap \Omega_2$ and define $\varphi : D \longrightarrow \mathbf{R}$ by $\varphi := \max\{\varphi_1|_D, \varphi_2|_D\}$ where $\varphi_i : \Omega_i \longrightarrow \mathbf{R}$ defines the q_i-convexity of Ω_i, $i = 1, 2$. Then approximate φ in the C^0-topology by smooth $(q_1 + q_2 - 1)$-convex functions (as done in [8]).

Note also that the q-complete analogon is trivial since then $\Omega_1 \cap \Omega_2$ can be viewed as an analytic subset of $\Omega_1 \times \Omega_2$ which is obviously $(q_1 + q_2 - 1)$-complete.

Theorem 4.2 *Let Ω_1 and Ω_2 be q_1-concave, resp. q_2-concave open subsets of a complex space X. Then $\Omega_1 \cup \Omega_2$ is (q_1+q_2-1)-concave and $\Omega_1 \cap \Omega_2$ is (q_1+q_2)-concave.*

Proof: Consider $\varphi_i : \Omega_i \longrightarrow (0, 1)$, C^2 functions which exhaust Ω_i from below and are q_i-convex on $\Omega_i \backslash K_i$ for some compact subset K_i of Ω_i, $i = 1, 2$.

a) Set $\Omega := \Omega_1 \cup \Omega_2$. By Lemma 3.4 we may assume that the trivial extensions $\tilde{\varphi}_i$ of φ_i to X, $\tilde{\varphi}_i \equiv 0$ on $X \setminus \Omega_i$ are of class C^2 on X for $i = 1, 2$. Then define $\varphi : \Omega \longrightarrow (0, \infty)$ as follows

$$\varphi := (\tilde{\varphi}_1 + \tilde{\varphi}_2)_{|\Omega}. \tag{8}$$

Hence φ is of class C^2 and exhausts Ω from below. Now we will show that φ is (q_1+q_2-1)-convex on $\Omega \backslash (K_1 \cup K_2)$. Indeed, by (8) it remains to check the (q_1+q_2-1)-convexity of φ at points $x_0 \in \Omega \backslash (K_1 \cup K_2)$ with $x_0 \in \partial\Omega_1 \cup \partial\Omega_2$, say $x_0 \in \partial\Omega_2$. Then locally $\varphi = \varphi_1 + \tilde{\varphi}_2$ with $\tilde{\varphi}_2(x_0) = 0 \leq \tilde{\varphi}_2$. Thus we have that $\tilde{\varphi}_2$ is plurisubharmonic at x_0; hence φ is q_1-convex at x_0. The $(q_1 + q_2 - 1)$-concavity of $\Omega_1 \cup \Omega_2$ follows.

b) Set $D := \Omega_1 \cap \Omega_2$ and define $\varphi : D \longrightarrow (0, \infty)$ by

$$\varphi := \varphi_1 \cdot \varphi_2 \cdot \exp(-\varphi_1 - \varphi_2). \tag{$\spadesuit$}$$

Since $\varphi < \min\{\varphi_1, \varphi_2\}$, φ exhausts D from below. Now in order to check the tangential $(q_1 + q_2)$-convexity of φ outside a suitable compact subset of D, we may assume, without any loss of generality, that Ω_1 and Ω_2 are open sets of some $\mathbb{C}^N$. Consequently let $a \in D$ and $\xi \in H_a(\varphi_1) \cap H_a(\varphi_2) \subseteq H_a(\varphi)$. We get

$$L(\varphi, a)\xi = \varphi(a) \left[\left(\frac{1}{\varphi_1(a)} - 1 \right) L(\varphi_1, a)\xi + \left(\frac{1}{\varphi_2(a)} - 1 \right) L(\varphi_2, a)\xi \right]. \tag{$\clubsuit$}$$

By ($\clubsuit$) it follows easily that φ is tangentially $(q_1 + q_2)$-convex at points from $\Omega_1 \cap \Omega_2 \setminus (K_1 \cup K_2)$. Indeed, let $E_i \subseteq \mathbb{C}^N$ be complex vector subspaces with $\dim E_i \geq N - q + 1$ such that $L(\varphi_i, a)|_{E_i}$ are positive definite, $i = 1, 2$.

Set $E := E_1 \cap E_2$, $H := H_a(\varphi)$ and $F := E \cap H_a(\varphi_1) \cap H_a(\varphi_2)$. Then codim ${}_H F \leq (q_1 + q_2 - 1)$ and, from (♣), taking into account that $\varphi_i(a) < 1$ the positivity of $L(\varphi, a)|_F$ follows. Hence φ is tangentially $(q_1 + q_2)$-convex at a.

On the other hand, set $L_1 := K_1 \cap \partial\Omega_2$ and $L_2 := K_2 \cap \partial\Omega_1$. Then L_1 and L_2 are compact subsets of Ω_1, Ω_2, respectively. Moreover $\tilde{\varphi}_2|_{L_1} = 0$ and $\tilde{\varphi}_1|_{L_2} = 0$. By a standard argument, from (♣), there are:

- a smooth strictly increasing convex function $\chi : (0,1) \longrightarrow (0,1)$ such that $\chi(t) \to 0$ as $t \searrow 0$, $\chi'(t)/\chi(t)$ is sufficiently large when t approaches zero by positive values, and
- open neighborhoods U_1 and U_2 of L_1, L_2, respectively,

such that, if we define $\Phi : \Omega_1 \cap \Omega_2 \longrightarrow (0, \infty)$ by $\Phi := \Phi_1 \cdot \Phi_2 \cdot \exp(-\Phi_1 - \Phi_2)$ where $\Phi_1 := \chi(\varphi_1)$, and $\Phi_2 := \chi(\varphi_2)$ then, according to (♣) (with φ_1, φ_2 and φ replaced by Φ_1, Φ_2 and Φ , respectively) Φ is tangentially (q_1+1)-convex on $U_1 \cap \Omega_2$ and tangentially (q_2+1)-convex on $U_2 \cap \Omega_1$. As $q_1+1, q_2+1 \leq q_1+q_2$, the tangential $(q_1 + q_2)$-convexity of Φ outside of a suitable compact subset of D follows. By Proposition 2.1, $\Omega_1 \cap \Omega_2$ will be $(q_1 + q_2)$-concave.

Example 4.1 *There exists two 1-concave open sets Ω_1 and Ω_2 of $\mathbf{P}^n$, $n \geq 2$, such that $\Omega_1 \cap \Omega_2$ is not 1-concave.*

In order to do this we say that a compact set K of a complex space X is a *special Stein compactum* if there is a Stein open neighborhood U of K so that K is holomorphically convex with respect to $\mathcal{O}(U)$.

Also we recall that compact set K of a complex space X is a *Stein compactum* if it has a fundamental system of Stein open neighborhoods.

It is straightforward that any compact subset K of $\mathbf{C}$ is a special Stein compactum. Indeed, take from each relatively compact connected component of $\mathbf{C} \setminus K$ an arbitrary point. One gets a discrete subset A of $\mathbf{C}$. Set $U := \mathbf{C} \setminus A$. Then U is an open neighborhood of K such that $U \setminus K$ does not have relatively compact (in U) connected components. Hence K is $\mathcal{O}(U)$-convex.

With this we have

Proposition 4.1 *Let K be a proper compact subset of some complex projective space $\mathbf{P}^n$. Then $\mathbf{P}^n \setminus K$ is 1-concave if and only if K is a special compactum in $\mathbf{P}^n$. In particular if K is not a Stein compactum, then $\mathbf{P}^n \setminus K$ is not 1-concave.*

Proof: First assume that $\mathbf{P}^n \setminus K$ is 1-concave. Then there exists a C^2-function $\varphi : \mathbf{P}^n \setminus K \longrightarrow (0, \infty)$ which is exhaustive from below, 1-convex on $\{ \varphi < \epsilon_0 \}$ for some $\epsilon_0 > 0$ small enough. Set $U := K \cup \{ \varphi < \epsilon_0 \}$. Hence U is a proper subset of $\mathbf{P}^n$ which is locally Stein.

By a classical result of Takeuchi, [7], U is Stein. Now $\{U_\epsilon\}_{0<\epsilon\leq\epsilon_0}$ where $U_\epsilon := K \cup \{ \varphi < \epsilon \}$ gives a fundamental system of Runge neighborhoods of K in U; hence K is $\mathcal{O}(U)$-convex.

Conversly, suppose that K is $\mathcal{O}(U)$-convex for some Stein neighborhood $U \subset \mathbf{P}^n$ of K.

We claim there is a smooth function $\theta : U \longrightarrow [0, \infty)$ which is plurisubharmonic on the whole U, 1-convex on $U \setminus K$ and $\theta^{-1}(0) = K$.

Indeed, let $\psi : U \longrightarrow (0, \infty)$ be a smooth 1-convex exhaustion function and $r > 0$ a suitable constant such that $K \subseteq \{ z \in U \mid \psi(z) \leq r \}$. Since K is $\mathcal{O}(U)$-convex, there is a sequence of holomorphic functions $f_i \in \mathcal{O}(U)$ with $\|f_i\| \leq 1$ on K and for any point $z_0 \in U \setminus K$ there exists an index $k \in \mathbf{N}$ such that $|f_k(z_0)| \geq 1 + r^2$.

Let $u : [0, \infty) \longrightarrow [0, \infty)$ be a smooth convex function such that $u^{-1}(0) = [0, 1 + r^2]$ and is strictly increasing on $[1 + r^2, \infty)$. Define $\theta : U \longrightarrow [0, \infty)$ by

$$\theta(z) := \sum \epsilon_i \cdot u(|f_i(z)|^2 + \psi(z)), \ z \in U$$

where ϵ_i is a sequence of positive numbers. If the sequence ϵ_i decreases fast enough to zero then θ has the required properties.

Now construct a smooth function $\varphi : \mathbf{P}^n \setminus K \longrightarrow (0, \infty)$ such that $\varphi|_{U\setminus K} \equiv \theta|_{U\setminus K}$. This φ gives the desired 1-concavity of $\mathbf{P}^n \setminus K$, whence the proposition.

Similarly one has

Proposition 4.2 *Let K be a special Stein compactum of a q-concave complex space X. Then $X \setminus K$ is again q-concave.*

Proof: This is quite easy. Indeed, by the proof of Proposition 4.1 there exists an open neighborhood U of K and $\varphi_1 : U \longrightarrow [0, \infty)$, a smooth plurisubhatmonic function with $K = \{\varphi_1 = 0\}$ and 1-convex on $U \setminus K$. Let also $\varphi_2 : X \longrightarrow (0, \infty)$ define the q-concavity of X and L the exceptional compact set so that φ_2 is q-convex on $X \setminus L$. Consider $\rho \in C_0^\infty(X)$, $0 \leq \rho \leq 1$ with Supp $\rho \subset U$ and $\rho \equiv 1$ on an open neighborhood V of K. Hence $\overline{V}$ is a compact subset of U.

Define $\varphi : X \setminus K \longrightarrow (0, \infty)$ by

$$\varphi := \rho\varphi_1 + (1 - \rho)\varphi_2.$$

We check that φ defines the q-concavity of $X \setminus K$. In order to do this, fix $c > 0$ small enough such that

$$L \cup \mathrm{Supp}\ \rho \subseteq \{\varphi_2 \geq c\}.$$

Then φ is q-convex on $(V \setminus K) \cup \{x \in X \mid \varphi_2(x) < c\}$ whose complement with respect to $X \setminus K$ is the compact set $\{\varphi_2 \geq c\} \setminus V$.

Now let $\epsilon > 0$ be arbitrary. To see that the set $\{x \in X \setminus K \mid \varphi(x) \geq \epsilon\}$ is compact it sufficces to check the compacity of its trace on the following covering of $X \setminus K$ made up from the following three closed sets (the closure is taken in $X \setminus K$):

$$\{x \in X \mid \varphi_2(x) \geq c\},\ \{\varphi_2 \geq c\} \setminus V, \text{ and } \overline{V} \setminus K.$$

But this is straightforward. Thus $X \setminus K$ is q-concave and this conclude the proof of the proposition.

Now we produce the above mentioned example. Let B denote the closed unit disc in $\mathbf{C}$. Consider K_1 and K_2 compact subsets of $\mathbf{C}^n$, $n \geq 2$, which are special Stein compacta such that $K_1 \cup K_2$ is not a Stein compactum.

For instance, take $n \geq 2$ and set

$$K_1 := \{(z, w) \mid |z| \leq 2,\ 1 \leq |w| \leq 2\} \times B^{n-2} \text{ and}$$
$$K_2 := \{(z, w) \mid |z| \leq 1,\ |w| \leq 2\} \times B^{n-2}.$$

Then K_1 and K_2 are special Stein compacta, but $K_1 \cup K_2$ is not a Stein compactum.

Consider $\mathbf{C}^n$ canonically embedded in $\mathbf{P}^n$ and set $\Omega_1 := \mathbf{P}^n \setminus K_1$, $\Omega_2 := \mathbf{P}^n \setminus K_2$. Then, by Proposition 4.1, Ω_1 and Ω_2 are 1-concave; nevertheless $\Omega_1 \cap \Omega_2 = \mathbf{P}^n \setminus (K_1 \cup K_2)$ is not 1-concave.

5 Applications

First we give a simple proof of the following criterion of q-completeness due to M. Peternell (see [5], Satz 2.2, p. 558-563).

Theorem 5.1 *Let X be a complex space, f and g exhaustion functions and Ω an open neighborhood of the set $\{f = g\}$ such that $f|_{\Omega \cup \{f<g\}}$ is p-convex and $g|_{\Omega \cup \{g<f\}}$ is q-convex. Then X is $(p+q)$-complete.*

Proof: By taking exponentials we may arrange that f and g are positive. Set $h := \min\{f, g\}$. Then h is continuous and exhaustive.

Set

$$\Lambda := \left\{ \lambda \in C^\infty(\mathbf{R}_+^*, \mathbf{R}_+^*) \mid \lambda' > 0,\ \lambda'' \geq 0 \right\}$$

and for any $\lambda \in \Lambda$ define

$$F_\lambda := \frac{1}{\exp(-\lambda(f)) + \exp(-\lambda(g))}.$$

Then, for any $\lambda \in \Lambda$, F_λ is exhaustive and $(p+q)$-convex on Ω (see the proof of Theorem 4.1). Consider a continuous function $\epsilon : X \longrightarrow (0, \infty)$ such that $\{x \in X \mid |f(x) - g(x)| \leq 2\epsilon(x)\} \subset \Omega$ and then define two closed subsets of X, $A := \{x \in X \mid f(x) - g(x) \leq -2\epsilon(x)\}$ and $B := \{x \in X \mid g(x) - f(x) \leq -2\epsilon(x)\}$. Then $A \cup B \cup \Omega = X$.

The theorem will follow from the next

Claim. There exists $\lambda \in \Lambda$ such that F_λ is p-convex on A and q-convex on B.

In order to verify this, as in the proof of Lemma 3.1 we choose $\lambda \in \Lambda$ such there exists $u \in \Lambda$ with $\lambda' \exp(-\lambda) = \exp(-u)$. This u will be chosen leter in proof such that u and u' are sufficiently large.

Now fix an arbitrary point $x_0 \in A$ (the case $x_0 \in B$ is similar). Since the question is local around x_0, we may asume, without any loss of generality, that X is an open subset of some euclidean space $\mathbf{C}^N$. Computing the Levi form of F_λ (see (♣) in the prof of Theorem 4.1) one gets the inequality

$$\begin{aligned}(A_1 + A_2)^3 \cdot L(F_\lambda, x)\xi \quad & \geq (A_1 + A_2)(a_1 L(f, x)\xi + a_2 L(g, x)\xi) - \\ & -A_1 A_2 (a_1^2 \cdot |\partial f(x)\xi)|^2 + a_2^2 \cdot |\partial g(x)\xi)|^2)\end{aligned}$$

where $A_1 := \exp(-\lambda(f(x)))$, $A_2 := \exp(-\lambda(g(x)))$, $a_1 := \exp(-u(f(x)))$, and $a_2 := \exp(-u(f(x)))$ for $x \in X$, $\xi \in \mathbf{C}^N$.

Let also $E \subseteq \mathbf{C}^N$ be a complex vector subspace, $\dim E \geq N-p+1$, D a small ball $\ni x_0$ such that $f-g \leq \epsilon$ on D and, moreover there exists a constant $C_1 > 0$ such that

$$L(f,x)\xi \geq 3C_1 \cdot \|\xi\|^2$$

for any $x \in D$ and $\xi \in E$. Also there exists a constant $C_2 > 0$ such that:

$$|L(g,x)\xi| \leq C_2\|\xi\|^2 \text{ and}$$

$$|\partial f(x)\xi| \leq C_2\|\xi\|, \ |\partial g(x)\xi| \leq C_2\|\xi\|$$

for any $x \in D$ and $\xi \in \mathbf{C}^N$. Now we impose conditions on u such that

$$(\heartsuit) \qquad (A_1+A_2)(3a_1C_1 - a_2C_2) - A_1A_2(a_1^2+a_2^2)C_2^2 > 0.$$

But $(\heartsuit)$ will follow at once from $(\dagger)$ and $(\ddagger)$ below, by using the obvious inequalities $a_2 \leq a_1$ and $A_1 + A_2 > A_1 > A_1A_2$.

We can choose u such that:

$$(\dagger) \qquad a_1C_1 > a_2C_2 \text{ and}$$

$$(\ddagger) \qquad C_1 > a_1C_2^2$$

holds. Indeed, to get $(\dagger)$, by the mean value inequality one has

$$a_1/a_2 = \exp(-u(f(x)) + u(g(x))) \geq \exp(\epsilon(x) \cdot u'(h(x))), \ x \in D.$$

Hence, if u' is large enough on $h(D)$, $(\dagger)$ follows. Similarly, $(\ddagger)$ holds, as soon as u is large enough on $f(D)$. We do the same in case $x_0 \in B$.

Therefore, since f, g and h are exhaustive, by a standard argument, the claim follows. Thus the theorem.

The subsequent application was suggested to me by M. Colţoiu.

Theorem 5.2 *Let X be a complex space which is an increasing union of q-complete open subsets X_i, $i \in \mathbf{N}$. Then X is $2q$-complete.*

Proof: Without any loss of generality we may assume that X_i is relatively compact in X_{i+1} for any $i \in \mathbf{N}$. Let $\varphi_i : X_i \longrightarrow [0, \infty)$ be q-convex functions such that:

a) $\inf\{\varphi_{i+2}(x) \mid x \in \overline{X}_i\} > \sup\{\varphi_{i+1}(x) \mid x \in \overline{X}_i\}$ and

b) $\varphi_i(x) > i$ for any $x \in X_i$.

Consider $\Phi : X \longrightarrow [0, \infty)$ defined as follows:

$$\Phi(x) := \inf\{\, \varphi_i(x) \mid x \in X_i \,\}, \; x \in X.$$

By a), Φ is continuous (locally it is the minimum of two consecutive functions φ_i, φ_{i+1}). By b), we get that Φ is exhaustive.

Choose an exhaustion of X by relatively compact open subsets $\{D_i\}_{i \in \mathbf{N}}$ such that

$$\cdots \Subset D_i \Subset X_i \Subset D_{i+1} \Subset X_{i+1} \Subset \cdots$$

and

c) $\varphi_{i+1} < \varphi_i$ on $X_i \setminus D_i$, $i \in \mathbf{N}$.

This can be easily achieved as φ_i are exhaustive on X_i. Using conditions a) and c) we can define smooth functions $f, g : X \longrightarrow \mathbf{R}_+$ such that:

$$f \begin{cases} = \varphi_1 & \text{on } D_1; \\ > \varphi_i & \text{on } X_{i-1} \setminus D_{i-1}, \quad i \text{ even }, i \geq 2; \\ = \varphi_i & \text{on } D_i \setminus X_{i-2}, \quad i \text{ odd }, i \geq 3, \end{cases}$$

$$g \begin{cases} = \varphi_2 & \text{on } D_2; \\ > \varphi_i & \text{on } X_{i-1} \setminus D_{i-1}, \quad i \text{ odd }, i \geq 3; \\ = \varphi_i & \text{on } D_i \setminus X_{i-2}, \quad i \text{ even }, i \geq 4. \end{cases}$$

One can easily verify that $\Phi = \min\{f, g\}$; hence f and g are exhaustive on X. Now set

$$\Omega := D_1 \cup \bigcup_{i \in \mathbf{N}} (D_{i+1} \setminus \overline{X}_i).$$

It is straightforward that f, g and Ω fulfil the conditions of Theorem 5.1. Consequently X is $2q$-complete and the proof is completed.

Corollary 5.1 *Let X be a complex space which is an increasing union of Stein sets. Then X is 2-complete.*

Remark 5.1 It was shown by Fornæss ([2], [3]) that, in general, X is not a Stein space. In fact X is Stein if and only if $H^1(X, \mathcal{O})$ is separated ([4], [6]).

On the other hand, by Theorem B of Cartan one can easily obtain that X is cohomologically 2-complete, i.e. $H^i(X, \mathcal{F})$ vanishes for any $i \geq 2$ and any coherent sheaf $\mathcal{F}$ on X.

Corollary 5.2 *Let D be a q-Runge domain in a q-complete complex space X. Then D is $2q$-complete.*

Here we recall ([8], [9]) that an open subset D of a complex space X is said to be q-Runge in X if for any compact subset $K \subset D$ there is a q-convex exhaustion function $\varphi : X \longrightarrow \mathbf{R}$ (which may depend on K and, in particular, it gives the q-completeness of X) such that

$$K \subset \{x \in X \mid \varphi(x) < 0\} \Subset D.$$

Also note that D is always cohomologicaly $(q+1)$-complete, i.e. $H^i(X, \mathcal{F})$ vanishes for any $i \geq q+1$ and $\mathcal{F} \in \mathrm{Coh}(X)$.

Example 5.1 *Let D be a p-complete open subset of a q-complete complex space X. Then D is $(p+q)$-Runge in X.*

In order to prove this, let $\varphi_1 : D \longrightarrow \mathbf{R}$ be p-convex and exhaustive and $\psi_2 : X \longrightarrow \mathbf{R}$, q-convex and exhaustive. Then (by Lemma 3.3) there is $\lambda : \mathbf{R} \longrightarrow \mathbf{R}$ a smooth function, convex and rapidly increasing so that the function $\psi_1 : X \longrightarrow \mathbf{R}$,

$$\psi_1 := \begin{cases} -\exp(-\lambda(\varphi_1)), & \text{on } D; \\ 0, & \text{on } X \setminus D, \end{cases}$$

is of class C^2 on X, and (obviously) $(p+1)$-convex on D, plurisubharmonic on $X \setminus D$. Now set $\varphi_\epsilon := \epsilon\psi_1 + \psi_2$, $\epsilon > 0$. Then φ_ϵ is $(p+q)$-convex on X and it exhausts X. If $K \subset D$ is an arbitrary compact subset, then, with a sufficiently small $\epsilon > 0$, $K \subset \{x \in X \mid \varphi_\epsilon(x) < 0\} \Subset D$. Thus the example.

Within the same circle of ideas one has

Proposition 5.1 *Let X be a purely n-dimensional complex space. Assume that X is q-concave and p-convex. Then $p+q>n$.*

Indeed, let $\varphi, \psi : X \longrightarrow \mathbf{R}$ be smooth functions which define the q-concavity, resp. the p-convexity of X and K a compact set so that φ and ψ are q-convex, resp. p-convex on $X \setminus K$.

Choose $\epsilon_0 > 0$ small enough such that ψ and φ are p-convex, respectively q-convex on the set $\{\varphi \leq \epsilon_0\}$ and, moreover

$$\max_L \psi > \max_K \psi$$

where $L := \{\varphi \geq \epsilon_0\} \supset K$ is a compact set. Now consider $x_0 \in L$ at which ψ attains its maximum on L.

As in [1] there is an open neighborhood U of x_0 and an analytic subset $A \subset U$ whose irreducible components have all dimensions $\geq n-p$ with $A \cap \{\varphi \leq \epsilon_0\} = \{x_0\}$. Hence $A \subset L$. Also by shrinking U, if necessary, we may assume that ψ is q-convex on U. Hence $\psi|_A$ is q-convex and has a maximum at x_0. By the maximum principle for q-convex functions we get

$$q \geq 1 + \min \dim {}_{x_0} A_i$$

where (A_i) is the decomposition of the germ (A, x_0) into irreducible germs. Therefore $q \geq n-p+1$ or $p+q>n$, and this conclude the proof of the proposition.

Remark 5.2 We can replace the q-convexity of X by q-convexity with corners, i.e. there exists a continuous function $\psi : X \longrightarrow \mathbf{R}$ which is exhaustive and for a compact subset $K \subset X$ the following condition holds: for any point $x \in X \setminus K$ there are finitely many q-convex functions $\psi_1, \ldots, \psi_s$ defined on an open neighborhood $U \ni x$, $U \subset X \setminus K$ such that

$$\psi_{|U} = \max\{\psi_1, \ldots, \psi_s\}.$$

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