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Quasi-Boundedness and Subtractivity; Applications to Excessive Measures

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ABSTRACT. We study the quasi-boundedness and subtractivity in a general frame of cones of potentials (more precisely in H-cones). Particularly we show that the subtractive elements are strongly related to the existence of recurrent balayages. In the special case of excessive measures we improve results of P.J. Fitzsimmons and R.K. Getoor from [13], obtained with probabilistic methods.

Key words: quasi-boundedness, subtractivity, excessive measures, recurrent balayages, quasi-continuity, H-cones.
AMS subject classification: 31D05, 60J45.

Introduction

In a recent paper ([13]) P.J. Fitzsimmons and R.K. Getoor characterized the quasi-boundedness and subtractivity for excessive measures. Although these notions and the obtained results have a pure analytic aspect, they use essentially in the proofs probabilistic tools (Kuznetsov measures, random measures). We underline that a similar goal was already acheaved (see [7] and [8]) with analytic methods in the presence of a reference measure.

The starting point for us was to give an analytic treatement for the general situation (without reference measure). In this paper we present a new approach for the study of quasi-boundedness and subtractivity which allows us to avoid the probabilistic arguments, to clarify and improve resuts from [13]. Our method is available for general H-cones and consequently for the excessive measures as well as for Dirichlet spaces.

Let Exc be the convex cone of all excessive measures associated to a proper submarkovian resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on a Lusin measurable space $(X, \mathcal{X})$. If $m \in Exc$ we denote by Exc_m the convex cone of all exessive measures ξ such that $\xi \ll m$ (i.e. ξ is absolutely continuous with respect to m). Recall that an element $\xi \in Exc_m$ is called m-quasi-bounded ($\xi \in Q_{bd}(m)$) if $\xi = \sum_{k\in\mathbb{N}} \xi_k$ with $\xi_k \leq m$ for all $k \in \mathbb{N}$. We show (Theorem 3.1) that if $\xi \in Exc_m$, $\xi = \mu \circ U$ then $\xi \in Q_{bd}(m)$ iff μ does not charge any m-polar set which is ρ-negligible, where $m = h + \rho \circ U$. In fact this is a first important result from [13]. We give the following refinement:

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$\xi \in Q_{bd}(m)$ iff μ does not charge any m-polar set which is semi-polar and ρ-negligible. Particularly if μ does not charge the semi-polar sets then ξ and any minorant of ξ belongs to $Q_{bd}(m)$ for any m such that $\xi \in Exc_m$.

We distinguish now a special class of quasi-bounded elements. An excessive measure ξ is called universally quasi-bounded in Exc_m ($\xi \in Q_{bd}(Exc_m)$) if $\xi \in Q_{bd}(m')$ for any $m' \in Exc$ such that $m' \ll m$ and $m \ll m'$. We show that if $\xi = \mu \circ U \in Exc_m$ then $\xi \in Q_{bd}(Exc_m)$ iff μ does not charge the m-polar sets (or only the m-polar sets which are semi-polar). Recall that an element $\xi \in Exc_m$ is called subtractive in Exc_m ($\xi \in Sub(Exc_m)$) if any majorant of ξ from Exc_m is a specific majorant. If $\xi \in Exc_m$ has no specific minorants from $Q_{bd}(Exc_m)$ then it is subtractive in Exc_m. Moreover if $\xi = \mu \circ U$ then μ is carried by a m-polar set (or even by a m-polar set which is semi-polar). It remains to characterize the elements $\xi = \mu \circ U \in Exc_m$ which are simultaneously universally quasi-bounded and subtractive in Exc_m. We prove that $\xi \in Q_{bd}(Exc_m) \cap Sub(Exc_m)$ iff μ is carried by a basic set $A \subset X$ with the following property:

(*) *For any measurable (or only Ray compact) subset M of A we have $B^M 1 = 0$ m-a.s. on $X \setminus M$.*

If A is such a set then its subset $A_o := \{x \in A / \{x\}$ is fine open$\}$ is universally measurable and $A \setminus A_o$ is m-polar (Theorem 4.4 and 4.5). Therefore μ is always carried by A_o. Also for any Ray compact subset K of A we have:

$$B^{K\setminus\{x\}}1(x) = 0 \quad m-\text{a.s. (in } x\text{) on} K.$$

Using the above results we obtain immediately the following Riesz decomposition from [13] for any $\xi \in Sub(Exc_m)$:

$$\xi = h + \mu \circ U + \nu \circ U$$

where h is harmonic, μ is carried by a m-polar set and ν is carried by a set A satisfying the above property (*).

If A is a basic set we denote by $(B^A)^*$ the operator on Exc given by the duality relation $L((B^A)^*\xi, s) = L(\xi, B^A s)$, for any $\mathcal{U}$-excessive function s. Obviously $(B^A)^*$ is a balayage on Exc (i.e. it is additive, increasing, continuous in order from below, contractive in order and idempotent). If moreover A verifies (*) then $(B^A)^*$ is recurrent on Exc_m (i.e. $\xi, \eta \in Exc_m, \xi \leq \eta \Longrightarrow (B^A)^*\xi \prec \eta$). Conversely any recurrent balayage on Exc_m is of the above form (Theorem 4.4). In fact if $\xi \in Exc_m$ then $\xi \in Q_{bd}(Exc_m) \cap Sub(Exc_m)$ iff there exists a recurrent balayage B on Exc_m such that $B\xi = \xi$. We obtain in Section 1 this last result in the general frame of H-cones. Therefore the above description of the m-recurrent balayages becomes the crucial point which allowed us to deduce the characterization of subtractivity and quasi-boundedness for the special case of excessive measures. In the first section we develop the above topics in an H-cone (which may be considered as an abstract setting for Exc_m) and we prove results similar to those which hold for excessive measures. We also show that the covex cone of all universally quasi-bounded elements in an H-cone is increasingly dense (Theorem 1.5) and that a quasi-bounded subtractive element is necessarily quasi-continuous (Theorem 1.8).

The general case when the resolvent $\mathcal{U}$ is not proper can be reduced easily to the case when $\mathcal{U}$ is proper; see Final remark.

In the second section we give some complements on excessive measures and excessive functions.

1 Quasi-Bounded and Subtractive Elements in H-Cones

In this section S will be an H-cone. We refer to [10] for basic results concerning the H-cones. Recall that a *balayage* on S is a map $B:S \longrightarrow S$ which is additive, increasing, contractive (i.e. $Bs \leq s$ for all $s \in S$) idempotent and continuous in order from below (i.e. for any increasing family $(s_i)_{i\in I} \subset S$ such that $\bigvee\limits_{i\in I} s_i{=}s \in S$ we have $\bigvee\limits_{i\in I} Bs_i{=}Bs$; $\vee$, $\wedge$ are the lattice operations in S). We denote by B' the complement of the balayage B i.e. the smallest balayage T on S such that $B \vee T{=}I$. If $S-S$ denotes the vector lattice generated by S then for any $f \in (S-S)_+$ the balayage B_f on S is defined by

$$B_f = \bigvee_{n\in\mathbb{N}} R(s \wedge nf), \quad (\forall)\, s \in S$$

and we note that $B_f(Rf){=}Rf$, where $Rf{:=} \bigwedge\{t \in S / f \leq t\}$. A balayage B on S is called *absorbent* if $Bs{\prec}\, s$ for all $s \in S$ ($\prec$ is the specific order on S). The balayage B is called *recurrent* (cf. [9]) if $Bs{\prec}\, t$ for all s,$t \in S$ with $s \leq t$. For any $x \in S$ we denote by S_x the set of all $s \in S$ such that $\bigvee\limits_{n\in\mathbb{N}} (s \wedge nx){=}s$. Obviously S_x is a natural solid convex subcone of S and for any family $(s_i)_{i\in I}$ from S_x which is dominated in S its supremum in S belongs to S_x. Therefore S_x is also an H-cone. An element $x \in S$ is called *weak unit* in S if $\bigvee\limits_{n\in\mathbb{N}} (s \wedge nx){=}s$ for all $s \in S$.

From now on in this section we suppose that S possesses a weak unit and we denote by x a fixed weak unit in S.

Definition. An element $s \in S$ is called *x-quasi-bounded* if there exists a sequence $(s_n)_{n\in\mathbb{N}}$ in S such that $s{=}\sum\limits_{n\in\mathbb{N}} s_n$ and $s_n \leq x$ for all $n \in \mathbb{N}$.

We denote by $Q_{bd}(x)$ the set of all x-quasi-bounded elements of S. It is easy to see that $Q_{bd}(x)$ is a natural solid convex subcone of S and a specific band in S.

Proposition 1.1. *For any $s \in S$ the following assertions are equivalent:*

1) $s \in Q_{bd}(x)$.

2) $S^x_\infty(s) := \bigwedge\limits_{n\in\mathbb{N}} R(s - nx) = 0$.

3) $\bigwedge\limits_{n\in\mathbb{N}} B_n s = 0$, *for any decreasing sequence $(B_n)_{n\in\mathbb{N}}$ of balayages on S with* $\bigwedge\limits_{n\in\mathbb{N}} B_n x = 0$.

*Proof.*1)$\Rightarrow$2) is immediate.

2)$\Rightarrow$3). Let $(B_n)_{n\in\mathbb{N}}$ be a decreasing sequence of balayages on S such that

$\bigwedge_{n\in\mathbb{N}} B_n x = 0$. If we put $s_n := s - R(s-nx)$ then $s_n \in S$ and $s_n \leq nx$. We deduce that for any $m, n \in \mathbb{N}$ we have $B_m s = B_m s_n + B_m R(s-nx) \leq nB_m x + R(s-nx)$ and therefore $\bigwedge_{m\in\mathbb{N}} B_m s \leq R(s-nx)$ for all $n \in \mathbb{N}$, $\bigwedge_{m\in\mathbb{N}} B_m s = 0$.
3)$\Rightarrow$1). If for any $n \in \mathbb{N}$ we define $B_n := B_{(s-nx)_+}$ we get $B_{n+1} \leq B_n$, $B_n x \leq \frac{1}{n} s$ and therefore $\bigwedge_{n\in\mathbb{N}} B_n x = 0$. On the other hand we have $R(s-nx) = B_n R(s-nx) \leq B_n s$. Since $\bigwedge_{n\in\mathbb{N}} B_n s = 0$, it follows that $\bigwedge_{n\in\mathbb{N}} R(s-nx) = 0$.

Remark. *For the equivalence* 1)$\Leftrightarrow$2) *see also* [1].

If M is a subset of S we denote by $M^\perp$ the orthogonal of M with respect to the specific order i.e.

$$M^\perp := \{s \in S / s \curlywedge t = 0 \text{ for all } t \in M\}.$$

Proposition 1.2. *The following assertions are equivalent for any element $s \in S$:*
1) $s \in Q_{bd}(x)^\perp$.
2) $s = S^x_\infty(s)$.
3) $B_{(s-nx)_+} s = s$, *for all* $n \in \mathbb{N}$.
4) *There exists a decreasing sequence of balayages* $(B_n)_{n\in\mathbb{N}}$ *on* S *such that* $\bigwedge_{n\in\mathbb{N}} B_n x = 0$ *and* $B_n s = s$ *for all* $n \in \mathbb{N}$.

Proof. 1)$\Rightarrow$2). If we put $t_n := R(s-nx)$ then we have $t_n \prec s$, $s - t_n \leq nx$ and therefore $s - t_n = 0$ for all $n \in \mathbb{N}$, $s = \bigwedge_{n\in\mathbb{N}} t_n = S^x_\infty(s)$.
2)$\Rightarrow$3). We have $B_{(s-nx)_+} R(s-nx) = R(s-nx)$. Since $R(R(s-nx)-mx))$ $= R(s-(n+m)x)$ it follows that $S^x_\infty(s) = S^x_\infty(R(s-nx)) \prec R(s-nx))$ and therefore $B_{(s-nx)_+}(S^x_\infty(s)) = S^x_\infty(s)$. By hypothesis 2) we get $B_{(s-nx)_+} s = s$ for all $n \in \mathbb{N}$.
3)$\Rightarrow$4). Since $B_{(s-nx)_+} x \leq \frac{1}{n} s$ for all $n \in \mathbb{N}$ it follows that $(B_{(s-nx)_+})_{n\in\mathbb{N}}$ is a decreasing sequence of balayages on S such that $\bigwedge_{n\in\mathbb{N}} B_{(s-nx)_+} x = 0$.
4)$\Rightarrow$1). Let $t \in Q_{bd}(x)$ be such that $t \prec s$ and $t \leq kx$. If $(B_n)_{n\in\mathbb{N}}$ is a sequence of balayages as in 4) then we deduce that $B_n t = t$ for all $n \in \mathbb{N}$ and $\bigwedge_{n\in\mathbb{N}} B_n t \leq k(\bigwedge_{n\in\mathbb{N}} B_n x) = 0$. We conclude that $t = 0$.

Definition. An element $s \in S$ is called *universally quasi-bounded* in S if, for any weak unit y in S, s is y-quasi-bounded. We denote by $Q_{bd}(S)$ the set of all universally quasi-bounded elements of S. Since we have $Q_{bd}(S) = \bigcap\{Q_{bd}(y) / y \in S, S_y = S\}$ it follows that $Q_{bd}(S)$ is a natural solid convex subcone of S and a specific band in S.

Definition. An element $s \in S$ is called *subtractive* in S if

$$t \in S, s \leq t \Longrightarrow s \prec t.$$

The set of all subtractive elements of S is denoted by $Sub(S)$. It is easy to see that $Sub(S)$ is a convex subcone of S which is a specific band in S.

We recall the following results from [7]:

Proposition 1.3. *For any weak unit $y \in S$ we have $Q_{bd}(y)^\perp \subset Sub(S)$.*

Proof. Follows from Theorem 2.2 in [7] and Proposition 1.2.

Corollary 1.4. *The following inclusions hold:*

$$Sub(S)^{\perp} \subset Q_{bd}(S) \quad , \quad Q_{bd}(S)^{\perp} \subset Sub(S).$$

Theorem 1.5. *Let S be an H-cone possessing a weak unit. Then the convex cone of all universally quasi-bounded elements of S is increasingly dense in S (i.e. for any $s \in S$ we have $s = \bigvee\{t \in Q_{bd}(S)/t \leq s\}$).*

Proof. Let us put for any $s \in S$ $Bs := \bigvee\{t \in Q_{bd}(S)/t \leq s\}$. Note that for any $t \in Q_{bd}(S)$ we have $B_t s \leq Bs$ and in addition $Bs = \bigvee\{B_t s/ t \in Q_{bd}(S)\}$. The assertion stated by the theorem is equivalent with $B=I$. If B' is the complement of B then from Proposition 1.1. in [2] we have $B' = \bigwedge\{B'_t/t \in Q_{bd}(S)\}$ and therefore B' is an absorbent balayage on S. We want to show that $B' = 0$. First we remark that B' is a recurrent balayage on S. To prove this assertion, by Corollary 1.4, it is sufficient to show that for any $s \in S$ with $s = B's$ we have $s \in Q_{bd}(S)^{\perp}$i.e. $s \curlywedge t = 0$ for all $t \in Q_{bd}(S)$. Indeed, from $s \curlywedge t \in Q_{bd}(S)$ we get $B(s \curlywedge t) = s \curlywedge t$ and since $s \curlywedge t \prec s$ it follows that $B'(s \curlywedge t) = s \curlywedge t$ for all $t \in Q_{bd}(S)$. Since B' is absorbent, by Theorem 2.1 in [2] we get $B'B = 0$ and therefore $B'(s \curlywedge t) = B'B(s \curlywedge t) = 0$, $s \curlywedge t = 0$. Therefore B' is recurrent. Particularly we have $B's \in Sub(S)$ for any $s \in S$. On the other hand $B's \wedge ny \nearrow B's$ for any weak unit y in S and therefore the sequence $(B'(B's \wedge ny)_{n\in\mathbb{N}}$ increases in the specific order to $B's$. It follows that $R(B's - ny) \leq R(B's - B'(B's \wedge ny)) = B's - B'(B's \wedge ny) \searrow 0$. Therefore $B's \in Q_{bd}(S)$. From the preceding considerations we have also $B's \in Q_{bd}(S)^{\perp}$ and we conclude that $B's = 0$, completing the proof.

We recall that an element $s \in S$ is called *quasi-continuous* if for any increasing family $(s_i)_{i\in I}$ in S such that $\bigvee\limits_{i\in I} s_i = s$ we have $\bigwedge\limits_{i\in I} R(s - s_i) = 0$.

The origine of quasi-bounded elements turns back to the probabilistic notion of *regular potential* (see e.g. [6]). In the frame of cone of potentials such type of elements were considered by G. Mokobodzki (see [15]).

We denote by $Q_c(S)$ the set of all quasi-continuous elements of S and we note that $Q_c(S) \subset Q_{bd}(S)$. It is known (cf [4]) that if the dual S^* of S separates S then an element $s \in S$ is quasi-continuous iff for any decreasing family $(\mu_i)_{i\in I}$ in S^* such that $\inf\limits_{i\in I} \mu_i(s) < \infty$ we have $(\bigwedge\limits_{i\in I} \mu_i)(s) = \inf\limits_{i\in I} \mu_i(s)$. The next results give an analogous characterization for quasi-boundedness.

Proposition 1.6. *Suppose that $Q_c(S^*)$ separates S. Then $s \in Q_{bd}(x)$ iff for any decreasing sequence $(\mu_n)_{n\in\mathbb{N}}$ in S^* such that $\inf\limits_{n\in\mathbb{N}} \mu_n(x) = 0$ and $\inf\limits_{n\in\mathbb{N}} \mu_n(s) < \infty$ we have $\inf\limits_{n\in\mathbb{N}} \mu_n(s) = 0$.*

Proof. If $s \in Q_{bd}(x)$ then we have $s = \sum\limits_{n\in\mathbb{N}} s_n$ with $s_n \leq x$. Further if $(\mu_n)_{n\in\mathbb{N}}$ is a decreasing sequence with $\inf\limits_{n\in\mathbb{N}} \mu_n(x) = 0$ and $\inf\limits_{n\in\mathbb{N}} \mu_n(s) < \infty$ then we get $\inf\limits_{n\in\mathbb{N}} \mu_n(\sum\limits_{k=0}^{m} s_k) = 0$, for all $m \in \mathbb{N}$ and there exists $n_o \in \mathbb{N}$ with $\mu_{n_o}(s) < \infty$. Therefore $\inf\limits_{n\in\mathbb{N}} \mu_n(s) \leq \inf\limits_{m\in\mathbb{N}} \inf\limits_{n\in\mathbb{N}} [\mu_n(\sum\limits_{k\leq m} s_k) + \mu_{n_o}(\sum\limits_{k>m} s_k)] = \inf\limits_{m\in\mathbb{N}} \mu_{n_o}(\sum\limits_{k>m} s_k) = 0$.

Conversely, let $(B_n)_{n\in\mathbb{N}}$ be a decreasing sequence of balayages on S with $\bigwedge_{n\in\mathbb{N}} B_n x = 0$ and let $\mu \in Q_c(S^*)$ be such that $\mu(s+x) < \infty$. Then we have $0 = \mu(\bigwedge_{n\in\mathbb{N}} B_n x) = \inf_{n\in\mathbb{N}} \mu(B_n x) = \inf_{n\in\mathbb{N}} (\mu \circ B_n)(x)$. By hypothesis we deduce that $\inf_{n\in\mathbb{N}} (\mu \circ B_n)(s) = 0$ and therefore $\mu(\bigwedge_{n\in\mathbb{N}} B_n s) = \inf_{n\in\mathbb{N}} \mu(B_n s) = 0$. Since $Q_c(S^*)$ separates S we get $\bigwedge_{n\in\mathbb{N}} B_n s = 0$. From Proposition 1.1 it follows that $s \in Q_{bd}(x)$.

Remark. *If S is solide in S^{**} then the above hypothesis "$Q_c(S^*)$ separates S" coincides with the fact that "S^* separates S and $Q_c(S^*)$ is increasingly dense in S^*".* Indeed, let us denote by B_c the balayage on S^* (see [4]) defined by $B_c\mu := \bigvee\{\nu \in Q_c(S^*)/\nu \le \mu\}$. We show that $B_c = I$. If we denote by B_c^* the dual of B_c and $B_c \neq I$ then there exists $s \in S$ such that $s \neq B_c^* s$. Since $Q_c(S^*)$ separates S it follows that there exists $\mu \in Q_c(S^*)$ with $\mu(B_c^* s) < \mu(s)$. Therefore $B_c\mu \neq \mu$ which is a contradiction.

Proposition 1.7. *Suppose that $Q_c(S^*)$ separates S, $Q_c(S)$ is increasingly dense in S and such that for any dominated family $(s_i)_{i\in I}$ in S there exists a sequence $(i_n)_{n\in\mathbb{N}} \subset I$ with $\bigvee_{i\in I} s_i = \bigvee_{n\in\mathbb{N}} s_{i_n}$. Then the following assertions are equivalent:*

1) $s \in Q_{bd}(S)$.

2) For any decreasing sequence $(\mu_n)_{n\in\mathbb{N}}$ in S^* such that $\bigwedge_{n\in\mathbb{N}} \mu_n = 0$ and $\inf_{n\in\mathbb{N}} \mu_n(s) < \infty$ we have $\inf_{n\in\mathbb{N}} \mu_n(s) = 0$.

Proof. 2)$\Rightarrow$1). Follows from Proposition 1.6.

1)$\Rightarrow$2). Let $(\mu_n)_{n\in\mathbb{N}}$ be a decreasing sequence in S^* such that $\bigwedge_{n\in\mathbb{N}} \mu_n = 0$ and $\inf_{n\in\mathbb{N}} \mu_n(s) < \infty$. By hypothesis we may construct a weak unit $u \in Q_c(S)$ such that $\inf_{n\in\mathbb{N}} \mu_n(u) = 0$. We get $(\bigwedge_{n\in\mathbb{N}} \mu_n)(u) = \inf_{n\in\mathbb{N}} \mu_n(u)$. Using the fact that $s \in Q_{bd}(u)$ and from Proposition 1.6 it follows that $\inf_{n\in\mathbb{N}} \mu_n(s) = 0$.

Remark. *The hypothesis from Proposition 1.7 are satisfied if $S = Exc_m$.*

Theorem 1.8. *Suppose that for any dominated family $(s_i)_{i\in I}$ in S there exists a sequence $(i_n)_{n\in\mathbb{N}} \subset I$ with $\bigvee_{i\in I} s_i = \bigvee_{n\in\mathbb{N}} s_{i_n}$. Then $Sub(S) \cap Q_{bd}(S) = Sub(S) \cap Q_c(S)$.*

Proof. The inclusion $Sub(S) \cap Q_c(S) \subset Sub(S) \cap Q_{bd}(S)$ is immediate. Let $s \in Sub(S) \cap Q_{bd}(S)$ and $(s_i)_{i\in I}$ be an increasing family in S such that $\bigvee_{i\in I} s_i = s$ and $(i_n)_{n\in\mathbb{N}}$ an increasing sequence with $\bigvee_{i\in I} s_i = \bigvee_{n\in\mathbb{N}} s_{i_n}$. It will be sufficient to show that $\bigvee_{n\in\mathbb{N}} R(s - s_{i_n}) = 0$. Since $R(s - s_{i_n}) \le s - s_{i_n} \curlywedge s$, it suffices to prove that $\bigvee_{n\in\mathbb{N}} s_{i_n} \curlywedge s = s$. If we set $t_n := s_{i_n} - s_{i_n} \curlywedge s$ then $s \curlywedge t_n + s_{i_n} \curlywedge s \le s_{i_n}$ and therefore, since $s \in Sub(S)$ we get $s \curlywedge t_n + s_{i_n} \curlywedge s \prec s_{i_n}$, $s \curlywedge t_n + s_{i_n} \curlywedge s \prec s$, $s \curlywedge t_n + s_{i_n} \curlywedge s \prec s_{i_n} \curlywedge s$. Hence $s \curlywedge t_n = 0$ for all $n \in \mathbb{N}$. Let us put now $s'' := \bigvee_{n\in\mathbb{N}} (s_{i_n} \curlywedge s) = \curlyvee\{s_{i_n} \curlywedge s / n \in \mathbb{N}\}$, $s' := s - s''$ and $t := \sum_{n\in\mathbb{N}} \frac{1}{2^n}(t_n \wedge s')$. We remark that $\bigvee_{n\in\mathbb{N}} (t_n \wedge s') = s'$ and therefore $B_t = B_{s'}$. On the other hand we have $(t_n \wedge s') \curlywedge s' \le t_n$, $(t_n \wedge s') \curlywedge s' \le s'$. Since $s' \in Sub(S)$ it follows that $(t_n \wedge s') \curlywedge s' \prec t_n \curlywedge s'$, $(t_n \wedge s') \curlywedge s' = 0$. If x is a weak unit in S then the element

$v := t + B_t'x$ is also a weak unit in S and therefore from $s' \in Sub(S) \cap Q_{bd}(S)$ we get $s' = \sum_{n\in\mathbb{N}} s_n'$ with $s_n' \prec v$. From the above considerations we deduce that $s' \curlywedge t = 0$ and consequently, for any $n \in \mathbb{N}$ we have $s_n' \prec B_t'x$, $s_n' = B_t'(s_n')$, $s' = B_t's'$. Since $(B_t)' = (B_{s'})'$ we conclude that $s' = (B_{s'})'(s') = (B_{s'})'(B_{s'}s') = 0$.

Remark. *If S has a quasi-continuous weak unit u (particularly if $S = Exc_m$) then the conclusion of Theorem 1.8 follows immediately.*

Indeed, if $s \in Q_{bd}(S) \cap Sub(S)$ then $s = \sum_{n\in\mathbb{N}} s_n$, $s_n \leq u$. for all $n \in \mathbb{N}$ and since $s_n \in Sub(S)$ we get $s_n \prec u \in Q_c(S)$ for all $n \in \mathbb{N}$, $s \in Q_c(S)$.

Proposition 1.9. *Let B be a balyage on S. Then B is recurrent iff $B(S) \subset Sub(S) \cap Q_c(S)$.*

Proof. Suppose that B is recurrent and let $s \in S$. If $t \in S$ is such that $Bs \leq t$ we get $Bs = B(Bs) \prec t$ and therefore $Bs \in Sub(S)$.Let now $(s_i)_{i\in I}$ be an increasing family in S such that $\bigvee_{i\in I} s_i = Bs$. It follows that $(Bs_i)_{i\in I}$ increases in the specific order to $B(Bs) = Bs$ and therefore $R(Bs - s_i) \leq R(Bs - Bs_i) = Bs - Bs_i$, $\bigwedge_{i\in I} R(Bs - s_i) \leq \bigvee_{i\in I}(Bs - Bs_i) = 0$. Hence $Bs \in Q_c(S)$. Conversely, suppose that $B(S) \subset Sub(S) \cap Q_c(S)$ and let $s, t \in S$ be such that $s \leq t$. We have $Bs \leq Bt \leq t$ and since Bs is subtractive we get $Bs \prec t$ which leads to the fact that B is recurrent.

Corollary 1.10. *Let L be a specific solid subcone of $Sub(S) \cap Q_c(S)$. Then the balayage B_L on S defined by*

$$B_L s := \bigvee\{t \in L / t \leq s\}, \quad s \in S$$

is a recurrent balayage on S. More precisely, B_L is the smallest recurrent balayage B on S such that $Bs = s$ for any $s \in L$. The map $L \mapsto B_L$ between the set of all specific bands of $Sub(S) \cap Q_c(S)$ and the set of all recurrent balayages on S is an order preserving bijection.

Proof. Follows from Proposition 1.9 and from [10],Proposition 2.2.10.

Corollary 1.11. *The map $B_o : S \to S$ defined by*

$$B_o s := \bigvee\{t \in Sub(S) \cap Q_c(S) / t \leq s\}, \quad s \in S$$

is the greatest recurrent balayage on S. Particularly, there are no non zero recurrent balayages on S iff $Sub(S) \cap Q_c(S) = 0$.

2 Complements on Excessive Functions and Excessive Measures

Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a submarkovian resolvent of kernels on a Lusin measurable space $(X, \mathcal{X})$ such that the initial.kernel $U = U_o$ is proper. We denote by $\mathcal{E}_\mathcal{U}$ the convex cone of all $\mathcal{X}$ -measurable $\mathcal{U}$- excessive functions on X which are finite $\mathcal{U}$-a.s. and we suppose that $\mathcal{E}_\mathcal{U}$ is min-stable, $1 \in \mathcal{E}_\mathcal{U}$ and $\sigma(\mathcal{E}_\mathcal{U}) = \mathcal{X}$. *From now on in this paper, without other special mentions, $\mathcal{U} = (U_\alpha)_{\alpha>0}$ will be such a resolvent.*

We denote by $Exc = Exc_{\mathcal{U}}$ the convex cone of all $\mathcal{U}$-excessive measures on X, i.e. the set of all σ-finite measures m on X for which $m(\alpha U_\alpha) \leq m$ for all $\alpha > 0$. Recall (cf. [11]; see also [14]) that Exc is an H-cone (with respect to the usual order relation in the set of all positive measures on X) which does not possesses a weak unit in general; the existence of a weak unit in Exc is equivalent with the existence of a reference measure. For any $m \in Exc$ we denote by Exc_m (as in Section 1) the natural solid subcone of Exc defined by $Exc_m := \{\xi \in Exc / \bigvee_{k \in \mathbb{N}} (\xi \wedge km) = \xi\}$. It is easy to see that $Exc_m = \{\xi \in Exc/\xi$ is absolutely continuous with respect to $m\}$ and that Exc_m is an H-cone for which m is a weak unit. Recall that a *potential* is an excessive measure of the form $\mu \circ U$, where μ is a positive measure on X and Pot denotes the set of all potentials. The set Har is by definition the othogonal of Pot and the elements of Har are called *harmonic.* The convex cones Pot and Har are specific bands in Exc. Generally Pot is not solid in Exc with respect to the natural order but it is increasingly dense in Exc. The energy functional associated to $\mathcal{U}$ is the map $L : Exc \times \mathcal{E}_{\mathcal{U}} \rightarrow \overline{\mathbb{R}}_+$ defined by $L(\xi, s) := \sup\{\mu(s)/\mu \circ U \in Pot, \mu \circ U \leq \xi\}$.

Definition. The set X is called *semi-saturated* if Pot is solid in Exc with respect to the natural order. We say that X is *saturated* if any $m \in Exc$ such that $L(m, 1) < \infty$ is a potential. Note that (cf. [5]) if X is saturated then X is semi-saturated.

Definition. Let $(X_1, \mathcal{X}_1)$ be a measurable space such that $X \subset X_1$, $X \in \mathcal{X}_1$ and $\mathcal{X}_1|_X = \mathcal{X}$. A submarkovian resolvent $\mathcal{U}^1 = (U^1_\alpha)_{\alpha>0}$ on $(X_1, \mathcal{X}_1)$ is called an *extension* of $\mathcal{U} = (U_\alpha)_{\alpha>0}$ if

a) $U^1_\alpha(\chi_{X_1 \setminus X}) = 0$, for all $\alpha > 0$;

b) $U^1_\alpha(f)_{|X} = U_\alpha(f_{|X})$, for all $\alpha > 0$ and f positive $\mathcal{X}_1$-measurable on X_1;

c) $\mathcal{E}_{\mathcal{U}^1}$ is min-stable, $1 \in \mathcal{E}_{\mathcal{U}^1}$ and $\sigma(\mathcal{E}_{\mathcal{U}^1}) = \mathcal{X}_1$.

Remark. 1. In the above definition a) and b) may be replaced with the following conditions a') and b'):

a') $U^1_o(\chi_{X_1 \setminus X}) = 0$;

b') $U^1_o(f)_{|X} = U_o(f_{|X})$, for any f positive $\mathcal{X}_1$-measurable function on X_1.

2. If $\mathcal{U}^1 = (U^1_\alpha)_{\alpha>0}$ is an extension of $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $(X_1, \mathcal{X}_1)$ then any $\mathcal{U}^1$-excessive measure is carried by X and a measure on X will be $\mathcal{U}$-excessive iff it is $\mathcal{U}^1$-excessive. Therefore $Exc_{\mathcal{U}} = Exc_{\mathcal{U}^1}$. The following result is proved in [5]:

Theorem 2.1. *There exists a Lusin measurable space $(X_1, \mathcal{X}_1)$ such that $X \subset X_1$, $X \in \mathcal{X}_1$ and $\mathcal{X}_1|_X = \mathcal{X}$ and a submarkovian resolvent $\mathcal{U}^1 = (U^1_\alpha)_{\alpha>0}$ on $(X_1, \mathcal{X}_1)$ which is an extension of $\mathcal{U}$ on $(X_1, \mathcal{X}_1)$ such that X_1 is saturated with respect to $\mathcal{U}^1$.*

We recall now some considerations concerning the Ray topology. We suppose that the initial kernel U of $\mathcal{U}$ is bounded. A *Ray cone* associated with $\mathcal{U}$ will be a subcone $\mathcal{R}$ of $(\mathcal{E}_{\mathcal{U}})_b$ (:=the set of all bounded $\mathcal{U}$-excessive functions) which is min-stable and separable in the uniform norm, separates the point of X and moreover $1 \in \mathcal{R}$, $U((\mathcal{R} - \mathcal{R})_+) \subset \mathcal{R}$, $U_\alpha(\mathcal{R}) \subset \mathcal{R}$, $\alpha > 0$ and $\sigma(\mathcal{R}) = \mathcal{X}$. The topology on X generated by a Ray cone is called *Ray topology.*

Let us denote by $\mathcal{E}^*_{\mathcal{U}}$ the convex cone of all universally measurable $\mathcal{U}$-excessive functions which are finite $\mathcal{U}$-a.s. We note that the fine topologies on X induced by

$\mathcal{E}_\mathcal{U}$ and $\mathcal{E}^*_\mathcal{U}$ coincide. If $A \in \mathcal{X}$ and $s \in \mathcal{E}^*_\mathcal{U}$ then (cf. [4]) $R^A s$, the reduit of s on A ($R^A s= \inf\{t \in \mathcal{E}_\mathcal{U}/t \geq s$ on $A\}= \inf\{t \in \mathcal{E}^*_\mathcal{U}/t \geq s$ on $A\}$) is universally measurable and $\mu(R^A s)= \inf\{\mu(R^G)/G$ fine open, $G \in \mathcal{X}, G \supset A\}$ for any bounded $s \in \mathcal{E}^*_\mathcal{U}$ and any measure μ with $\mu(s) < \infty$. This result may also be deduced from [12] in the special case when X is locally compact and U is a Hunt kernel on X. We put $B^A s := \widehat{R^A s}$ (i.e. the $\mathcal{U}$-excessive regularization of the $\mathcal{U}$-supermedian function $R^A s$) and we have $B^A s= R^A s$ on $X \setminus A$ and $\mu(B^A s)= \sup\{\mu(B^K s)/K$ Ray compact, $K \subset A\}$ for any $s \in \mathcal{E}^*_\mathcal{U}$ and any measure μ with $\mu(s) < \infty$. As usual the set $A \in X$ is called: *thin* at the point $x \in X$ if there exists $s \in \mathcal{E}_\mathcal{U}$ such that $B^A s(x) < s(x)$; *totally thin* if it is thin at any point of X; *semi-polar* if it is a countable union of totally thin sets; *polar* (resp. *m-polar*, where $m \in Exc$) if $B^A 1 = 0$ (resp. $m(B^A 1) = 0$). A fine closed subbasic set is termed *basic*.

If $A \in \mathcal{X}$ then the map $B^A : \mathcal{E}^*_\mathcal{U} \to \mathcal{E}^*_\mathcal{U}$ is additive, increasing, σ-continuous in order from below (i.e. $s_n \nearrow s \Rightarrow B^A s_n \nearrow B^A s$) and dominated by the identity. Therefore for any $\xi \in Exc$ the functional $s \mapsto L(\xi, B^A s)$, $s \in \mathcal{E}^*_\mathcal{U}$ defines the unique excessive measure $(B^A)^*\xi$ such that $L((B^A)^*\xi, s)= L(\xi, B^A s)$ for all $s \in \mathcal{E}^*_\mathcal{U}$. Moreover if A is subbasic then the map $(B^A)^* : Exc \to Exc$ is a balayage on Exc and its restriction to Exc_m is a balayage on Exc_m for any $m \in Exc$. If for any $A \in \mathcal{X}$ we put $A^* := \{x \in A/ \liminf_{n\to\infty} nU_n(\chi_A) = 1\}$ then A^* is a subbasic set, $B^{A^*}s \in \mathcal{E}_\mathcal{U}$ for any $s \in \mathcal{E}_\mathcal{U}$ and in addition for any $\xi \in Exc$ we have $L(\xi, B^{A^*}s)= L({}^*B^A\xi, s)$, where ${}^*B^A\xi := \bigwedge\{\eta \in Exc/\xi|_A \leq \eta\}$.

3 Quasi-Bounded Excessive Measures

Theorem 3.1. *Let $m \in Exc$, $m = h + \rho \circ U$ with $h \in Har$ and $\xi = \mu \circ U \in Exc_m$. Then the following assertions are equivalent:*
1) The measure ξ is m-quasi-bounded.
2) $\mu(M) = 0$ for any m-polar subset M of X with $\rho(M) = 0$.
3) $\mu(M) = 0$ for any m-polar subset M of X such that M is semi-polar and $\rho(M) = 0$.

Proof. Let q be a positive $\mathcal{X}$- measurable function on X, $0 < q \leq 1$, such that Uq is bounded and $m(q) < \infty$. Let W be the kernel on $(X, \mathcal{X})$ given by $Wf := \frac{1}{Uq}U(qf)$ and let $\mathcal{W} = (W_\alpha)_{\alpha>0}$ be the submarkovian resolvent for which W is its initial kernel. We have $\mathcal{E}_\mathcal{W} = \frac{1}{Uq}\mathcal{E}_\mathcal{U}$, $Exc_\mathcal{W}= q \cdot Exc_\mathcal{U}$ and ${}^\mathcal{W}L(q \cdot \xi, \frac{s}{Uq})= {}^\mathcal{U}L(\xi, s)$ for any $s \in \mathcal{E}_\mathcal{U}$ and $\xi \in Exc_\mathcal{U}$, where ${}^\mathcal{U}L(\cdot,\cdot)$ (resp. ${}^\mathcal{W}L(\cdot,\cdot)$) denotes the energy functional associated with $\mathcal{U}$ (resp. $\mathcal{W}$). Particularly ${}^\mathcal{W}L(q\cdot m, 1)= {}^\mathcal{U}L(m, Uq) < \infty$. By Theorem 2.1 there exists a Lusin measurable space $(X', \mathcal{X}')$ such that $X \subset X'$, $X \in \mathcal{X}'$ and $\mathcal{X}'|_X = \mathcal{X}$ and a submarkovian resolvent $\mathcal{W}'= (W'_\alpha)_{\alpha>0}$ on $(X', \mathcal{X}')$ which is an extension of $\mathcal{W}$ on $(X', \mathcal{X}')$ and X' is saturated with respect to $\mathcal{W}'$. Therefore there exist two measures ρ' and ρ'' on X' such that $q \cdot (\rho \circ U)= \rho' \circ W'$ and $q \cdot h= \rho'' \circ W'$. It is easy to see that ρ' is the measure on X given by $\rho' := \frac{1}{Uq} \cdot \rho$ and $\rho''(X) = 0$.
1)$\Rightarrow$2). Suppose that $\xi \in Q_{bd}(m)$ and let M be a m-polar subset of X with respect

to $\mathcal{U}$ such that $\rho(M)=0$. From the above considerations we get ${}^{\mathcal{W}}L(q\cdot m, {}'B^M 1)$ $= {}^{\mathcal{U}}L(m, B^M Uq)$, where ${}'B^M$ denotes the balayage on M with respect to $\mathcal{E}_{\mathcal{W}'}$. It follows that M is also $q\cdot m$-polar (with respect to $\mathcal{W}'$). Moreover we have $\rho'(M)=\rho''(M)$. From [4] we deduce that there exists a decreasing sequence $(G_n)_{n\in\mathbb{N}}$ of fine open subsets of X such that $M\subset G_n$, $G_n\in\mathcal{X}$ and $\inf\limits_{n\in\mathbb{N}}(\rho'+\rho'')({}'B^{G_n}1)=$ $(\rho'+\rho'')({}'R^M 1)=(\rho'+\rho'')({}'B^M 1)={}^{\mathcal{W}}L(q\cdot m,{}'B^M 1)={}^{\mathcal{U}}L(m,B^M Uq)$. Since M is m-polar we get $\inf\limits_{n\in\mathbb{N}}{}^{\mathcal{U}}L(m,B^{G_n}Uq)=\inf\limits_{n\in\mathbb{N}}{}^{\mathcal{W}}L(q\cdot m,{}'B^{G_n}1)=\inf\limits_{n\in\mathbb{N}}(\rho'+\rho'')({}'B^{G_n}1)=0$. It follows $\inf\limits_{n\in\mathbb{N}}L({}^*B^{G_n}m,Uq)=0$, $\bigwedge\limits_{n\in\mathbb{N}}{}^*B^{G_n}m=0$, where we recall that ${}^*B^{G_n}m$ $:=\bigwedge\{\xi\in Exc/m|_{G_n}\le\xi\}$ and from Theorem 1.3 in [3] we have $L({}^*B^{G_n},s)=$ $L(m,B^{G_n}s)$ for all $s\in\mathcal{E}_{\mathcal{U}}$. On the other hand if we put $\mu':=\mu|_M$ and $\xi':=\mu'\circ U$ then for any $s\in\mathcal{E}_{\mathcal{U}}$ we have: $L({}^*B^{G_n}\xi',s)=L(\xi',B^{G_n}s)=\mu'(B^{G_n}s)=\mu'(s)=L(\xi',s)$. From Proposition 1.2 we deduce that $\xi'\in Q_{bd}(m)^{\perp}$ and therefore $\xi'=0$.
3)$\Rightarrow$1). Suppose now that $\mu(M)=0$ for any m-polar subset M of X such that M is semi-polar and $\rho(M)=0$. It is sufficient to show that: $\xi\in Q_{bd}(m)^{\perp}\Rightarrow\xi=0$. If $\xi\in Q_{bd}(m)^{\perp}$ then from Proposition 1.2 it follows that $B_{(\xi-km)_+}\xi=\xi$ for all $k\in\mathbb{N}$. We need the following lemma:

Lemma. *Let $m\in Exc$ and $\xi_1,\xi_2\in Exc_m$, $\xi_i=f_i\cdot m$, $i=1,2$. Then*

$$B_{(\xi_1-\xi_2)_+}\eta={}^*B^{[f_1>f_2]}\eta,\quad(\forall)\eta\in Exc_m.$$

Proof of Lemma. Let us put $F:=[f_1>f_2]$. Then for all $j\in\mathbb{N}$ we have ${}^*B^F\eta|_F=\eta|_F\ge\eta\wedge[j(\xi_1-\xi_2)_+]$. It follows that ${}^*B^F\eta\ge R(\eta\wedge[j(\xi_1-\xi_2)_+])$ and therefore ${}^*B^F\eta\ge\bigvee\limits_{j\in\mathbb{N}}R(\eta\wedge[j(\xi_1-\xi_2)_+])=B_{(\xi_1-\xi_2)_+}\eta$. On the other hand if $\eta=g\cdot m$ we get $\bigvee\limits_{j\in\mathbb{N}}(\eta\wedge[j(\xi_1-\xi_2)_+])\ge\bigvee\limits_{j\in\mathbb{N}}(g\cdot m\wedge[j(f_1-f_2)_+\cdot m])=(g\chi_F)\cdot m=\eta|_F$ and we conclude that $B_{(\xi_1-\xi_2)_+}\eta=\bigvee\limits_{j\in\mathbb{N}}R(\eta\wedge[j(\xi_1-\xi_2)_+])\ge\eta|_F$, $B_{(\xi_1-\xi_2)_+}\eta\ge{}^*B^F\eta$ which completes the proof of Lemma.

If we write $\xi=f\cdot m$ then from the above Lemma we have $B_{(\xi-km)_+}\eta={}^*B^{[f>k]}\eta$ for all $\eta\in Exc_m$. From Theorem 1.4 in [3], for any $s\in\mathcal{E}_{\mathcal{U}}$ and $\eta\in Exc$ we have $L({}^*B^{[f>k]}\eta,s)=L(s,B^{[f>k]^*}s)$, where for a set $A\in\mathcal{X}$ we put $A^*:=$ $\{x\in A/\liminf\limits_{n\to\infty}nU_n(\chi_A)=1\}$. (Recall that the set A^* is subbasic and $B^{A^*}s\in$ $\mathcal{E}_{\mathcal{U}}$.) For any $k\in\mathbb{N}$ we denote by M_k the fine closure of the set $[f>k]^*$ and let $M:=\bigcap\limits_{k\in\mathbb{N}}M_k$. Obviously ${}^*B^{M_k}m\le\frac{1}{k}{}^*B^{M_k}\xi\le\frac{1}{k}\xi$ for all $k\in\mathbb{N}$ and therefore $\bigwedge\limits_{k\in\mathbb{N}}{}^*B^{M_k}m=0$. Since for any f with $m(f)<\infty$ we have $L(m,B^M Uf)\le$ $\inf\limits_{k\in\mathbb{N}}L(m,B^{M_k}Uf)=\inf\limits_{k\in\mathbb{N}}L({}^*B^{M_k}m,Uf)=\inf\limits_{k\in\mathbb{N}}{}^*B^{M_k}m(f)=(\bigwedge\limits_{k\in\mathbb{N}}{}^*B^{M_k}m)(f)=0$ we deduce that $L(m,B^M Uq)=0$ which means that M is m-polar. From $m(B^M 1)=0$ it follows that the measure m is carried by the absorbent set $A:=\{x\in\mathcal{X}/t(x)=0\}$ where $t\in\mathcal{E}_{\mathcal{U}}$ is such that $t\le B^M 1$ and $t=B^M 1$ m-a.s. Since $\xi\in Exc_m$ we deduce that ξ is also carried by A and $\mu(B^M 1)=L(\xi,B^M 1)=0$. Hence μ is carried by A. If we put $M_k':=M_k\cap A$ then we have ${}^*B^{M_k'}\xi={}^*B^{M_k}\xi=\xi$ for all $k\in\mathbb{N}$. It follows that $\mu(B^{M_k'}Uq)=L(\xi,B^{M_k'}Uq)=L({}^*B^{M_k'}\xi,Uq)=L(\xi,Uq)=\mu(Uq)$

and therefore $\mu(Uq - B^{M'_k}Uq) = 0$ for all $k \in \mathbb{N}$. Since M'_k is a basic set we deduce that μ is carried by M'_k for any $k \in \mathbb{N}$ and therefore μ is carried by $M \cap A$. From $B^{M\cap A}Uq = 0$ on A and $B^A Uq < Uq$ on $X \setminus A$ we get that $M \cap A$ is totally thin. To deduce that $\xi = 0$ it remains to show that $\rho(M) = 0$. Indeed we have $\rho(\chi_M Uq) \leq \rho(B^{M_k}Uq) \leq L(m, B^{M_k}Uq) = L({}^*B^{M_k}m, Uq) = {}^*B^{M_k}m(q)$ and therefore, since $\bigwedge_{k\in\mathbb{N}} {}^*B^{M_k}m = 0$, we get $\rho(M) = 0$.

Corollary 3.2. *Let $m \in Exc$ and $\xi = \mu \circ U \in Exc_m$. Then the following assertions are equivalent:*

1) ξ is universally quasi-bounded in Exc_m.

2) The measure μ does not charge any m-polar subset of X.

3) The measure μ does not charge any m-polar subset of X which is semi-polar.

Proof. Let m_o be a weak unit in Exc_m which is quasi-continuous in Exc. We deduce that $Q_{bd}(Exc_m) = Q_{bd}(m_o)$. On the other hand if $m_o = \rho_o \circ U + h$, where $h \in Har$, from [4], Theorem 3.3 we deduce that $\rho_o(M) = 0$ for any semi-polar subset M of X. The assertion follows now directly from Theorem 3.1.

Corollary 3.3. *Suppose that $m \in Har$ and let $\xi = \mu \circ U \in Exc_m$. Then ξ is universally quasi-bounded in Exc_m iff ξ is m-quasi-bounded.*

Remark. *The above characterizations for universally quasi-boundedness and m-quasi-boundedness are sharpened versions of those given in the abstract setting by Proposition 1.6 and Proposition 1.7.*

Corollary 3.4. *Let $m \in Exc$, $m = h + \rho \circ U$ with $h \in Har$ and $\xi = \mu \circ U \in Exc_m$. Then the following assertions are equivalent:*

1) $\xi \in Q_{bd}(Exc_m)^{\perp}$ (resp. $\xi \in Q_{bd}(m)^{\perp}$).

2) The measure μ is carried by a m-polar set (resp. a m-polar set which is ρ-negligible).

3) The measure μ is carried by a m-polar set which is semi-polar (resp. a m-polar set which is semi-polar and ρ-negligible).

Remark. *Let $m \in Exc$ and $\xi = \mu \circ U \in Exc_m$. If μ is carried by a m-polar set then ξ is subtractive in Exc_m. Particularly, if μ is carried by a polar set then ξ is subtractive in Exc.*

4 Recurrent Balayages on Exc

In this section we characterize the recurrent balayages on Exc_m. The Ray topology which is considered on X is the topology generated by a Ray cone associated with a bounded kernel of the form $q \cdot U$, $0 < q \leq 1$, as in Section 2.

Let $A \subset X$ be $\mathcal{X}$-measurable. Recall that if $\xi \in Exc$ then $(B^A)^*\xi$ is by definition the unique excessive measure such that $L((B^A)^*\xi, s) = L(\xi, B^A s)$ for all $s \in \mathcal{E}^*_U$. Moreover if A is subbasic and $m \in Exc$ then the map $(B^A)^* : Exc_m \to Exc_m$ is a balayage on Exc_m.

Proposition 4.1. *Let $m \in Exc$, $\xi = \mu \circ U \in Exc_m$ and $A \in \mathcal{X}$ be a subbasic set. Suppose that the balayage $(B^A)^*$ on Exc_m is absorbent. Then $(B^A)^*\xi = \mu|_{\overline{A}^f} \circ U$.*
($\overline{A}^f$ denotes the fine closure of A.)

Proof. Since $(B^A)^*\xi \prec \xi$ it follows that $(B^A)^*\xi = \nu \circ U$, where $\nu \leq \mu$. For any $s \in \mathcal{E}_U$ we have $\nu(s) = L((B^A)^*\xi, s) = L(\xi, B^A s) = \mu(B^A s) = L((B^A)^*\xi, B^A s) = \nu(B^A s)$. Therefore ν is carried by $\overline{A}^f$. We deduce that $(B^A)^*(\mu|_{X\setminus\overline{A}^f} \circ U) = 0$, $(B^A)^*(\mu|_{\overline{A}^f} \circ U) = \mu|_{\overline{A}^f} \circ U$ and therefore $(B^A)^*\xi = \mu|_{\overline{A}^f} \circ U$.

Proposition 4.2. *Let $m \in Exc$ and $A \in \mathcal{X}$ be a subbasic set. Then the following assertions are equivalent:*
1) $(B^A)^*$ is an absorbent balayage on Exc_m.
2) $B^A 1 = 0$ m-a.s. on $X \setminus \overline{A}^f$.

Proof. 1)$\Rightarrow$2). Let $q > 0$ be such that $0 < Uq \leq 1$ and put $m' := \frac{1}{Uq} \cdot m$. Since $m'|_{X\setminus\overline{A}^f} \circ U \in Exc_m$, by Proposition 4.1 we deduce that $(B^A)^*(m'|_{X\setminus\overline{A}^f} \circ U) = 0$. Particularly we get: $m'|_{X\setminus\overline{A}^f}(B^A 1) = L(m'|_{X\setminus\overline{A}^f} \circ U, B^A 1) = L((B^A)^*(m'|_{X\setminus\overline{A}^f} \circ U), 1) = 0$.
2)$\Rightarrow$1). It will be sufficient to show that for any $\xi \in Exc_m$ with $L(\xi, 1) < \infty$ and $s, t \in \mathcal{E}_U$, $s \leq t$ we have $L(\xi - (B^A)^*\xi, s) \leq L(\xi - (B^A)^*\xi, t)$. Since $L(\xi - (B^A)^*\xi, t) = L(,\xi, t) - L(\xi, B^A t)$, $L(\xi - (B^A)^*\xi, s) = L(\xi, s) - L(\xi, B^A s)$ and from
$$t + B^A s = s + B^A t = s + t \text{ on } \overline{A}^f,$$
$$t + B^A s = t \geq s = s + B^A t \quad m-\text{a.s. on } X \setminus \overline{A}^f$$
we get $s + B^A t \leq t + B^A s$ m-a.s. on X. Therefore $L(\xi, s + B^A t) \leq L(\xi, t + B^A s)$, $L(\xi - (B^A)^*\xi, s) \leq L(\xi - (B^A)^*\xi, t)$.

Proposition 4.3. *Let $m \in Exc$ and B be an absorbent balayage on Exc_m. Then there exists a basic set $A \in \mathcal{X}$ such that $B = (B^A)^*$. Moreover if $A \in \mathcal{X}$ is a basic set such that $B = (B^A)^*$ then there exists a fine clopen $\mathcal{X}$-measurable subset A_o of A with $B = (B^{A_o})^*$.*

Proof. Recall that (cf. [2], Proposition 1.5) we have $B = B_{m-B'm}$, where B' denotes the complement of the balayage B on Exc_m. Since the measure $m - B'm$ is absolutely continuous with respect to m we have $m - B'm = f \cdot m$ and therefore, from Lemma in the proof of Theorem 3.1 we get $B = B_{m-B'm} = {}^*B^{[f>0]}$. From Theorem 1.4 in [3] deduce that ${}^*B^{[f>0]}\xi = (B^A)^*\xi$ for all $\xi \in Exc$, where A is the basic set given by the fine closure of the subbasic set $[f > 0]^*$. Let now $A \in \mathcal{X}$ be a basic set such that $B = (B^A)^*$. Since B is absorbent, by Proposition 4.2 it follows $B^A 1 = 0$ m-a.s. on $X \setminus A$. Replacing the universally measurable excessive function $B^A 1$ with $s_o \in \mathcal{E}_U$, $s_o \geq B^A 1$, $B^A 1 = s_o$ m-a.s., we define $D := [s_o = 0]$ and $C := (X \setminus A) \setminus D$. Then C is $\mathcal{X}$-measurable, fine open and since m is excessive it follows that $m(\alpha U_\alpha(\chi_C)) \leq m(C) = 0$, $m(U(\chi_C)) = 0$. Hence $U(\chi_C) = 0$ m-a.s. on X. Since $U(\chi_C) > 0$ on C we deduce that $A \cap [U(\chi_C) = 0] = (X \setminus D) \cap [U(\chi_C) = 0]$ and therefore the set $A_o := [U(\chi_C) = 0] \cap A$ is fine clopen. On the other hand from $U(\chi_C) = 0$ m-a.s. we get $m(A \setminus A_o) = m(A \setminus [U(\chi_C) = 0]) = 0$. As a consequence for any $s \in \mathcal{E}_U$ we have $B^A s = B^{A_o} s$ m-a.s. and we conclude that $(B^A)^* = (B^{A_o})^*$.

Theorem 4.4. *Suppose that X is semi-saturated. If $m \in Exc$ and $A \in \mathcal{X}$ is a basic set then the following assertions are equivalent:*

1) The balayage $(B^A)^*$ on Exc_m is recurrent.

2) For any Ray compact subset K of A we have $B^K 1 = 0$ m-a.s. on $X \setminus K$.

3) For any subset M of A, $M \in \mathcal{X}$, we have $B^M 1 = 0$ m-a.s. on $X \setminus M$.

Proof. It is known that $(B^A)^*$ will be recurrent iff any balayage B on Exc_m such that $B \leq (B^A)^*$ is absorbent (cf. [9]).

1$\Rightarrow$2). Let K be a Ray compact subset of A. Then there exists a decreasing sequence $(G_n)_{n\in\mathbb{N}}$ of Ray open sets such that $\overline{G_n} \subset G_{n+1}$ and $\bigcap_{n\in\mathbb{N}} G_n = K$. If we put $A_n := A \cap G_n$ then A_n is a subbasic set, $A_n \subset A$ and $B^K s \leq \bigwedge_{n\in\mathbb{N}} B^{A_n} s$ for any $s \in \mathcal{E}_{\mathcal{U}}$. Since $(B^{A_n})^* \leq (B^A)^*$ and by hypothesis $(B^A)^*$ is recurrent it follows that $(B^{A_n})^*$ is an absorbent balayage on Exc_m. Therefore, by Proposition 4.2, for any $n \in \mathbb{N}$ we get

$$B^K 1 \leq B^{A_n} 1 = 0 \quad m - \text{a.s. on } X \setminus \overline{A_n}^f$$

and we conclude that $B^K 1 = 0$ m-a.s. on $X \setminus K$.

2)$\Rightarrow$3). Let M be a subset of A, $M \in \mathcal{X}$ and let μ be a finite measure on $X \setminus M$ which is absolutely continuous with respect to m. Since $\mu(B^M 1) = \sup\{\mu(B^K 1)/K$ Ray compact, $K \subset M\}$, by hypothesis 2) we deduce that $B^M 1 = 0$ m-a.s. on $X \setminus M$.

3)$\Rightarrow$1). Let B be a balayage on Exc_m such that $B \leq (B^A)^*$. We want to show that B is absorbent. If $B = (B^M)^*$ where M is a basic subset of A then by assertion 3) and Proposition 4.2 we deduce that $(B^M)^*$ is an absorbent balayage on Exc_m. Let now B be arbitrary. Then there exists a decreasing family $(\varphi_i)_{i\in I}$ in $(Exc_m - Exc_m)_+$ such that $B\xi = \bigwedge_{i\in I} B_{\varphi_i}\xi$ for all $\xi \in Exc_m$. Obviously we may suppose that $(\varphi_i)_{i\in I}$ is decresing in $(Exc_m - Exc_m)_+$. If for any $i \in I$ we put $\varphi_i = f_i \cdot m$ then we have $B_{\varphi_i} = {}^*B^{[f_i>0]} = (B^{A_i})^*$, where A_i is the fine closure of the set $[f_i > 0]^*$. Since the balayage $(B^A)^*$ is absorbent we may suppose, by Proposition 4.3, that the set A is fine clopen. Let $i \in I$ be fixed and let $G \in \mathcal{X}$ be a fine open set such that $G \supset A_i$. The set $A \cap \overline{G}^f = \overline{A \cap G}^f$ is a basic subset of A and moreover $B \leq (B^{A\cap\overline{G}^f})^*$. Indeed let $\xi = \mu \circ U \in Exc_m$. Since X is semi-saturated we have $B\xi = \nu \circ U$ and from ${}^*B^G(B\xi) = B\xi = (B^A)^* B\xi$ we conclude that the measure ν is carried by $A \cap \overline{G}^f$. Therefore $B\xi = (B^{A\cap\overline{G}^f})^* B\xi \leq (B^{\overline{G}^f})^*\xi = (B^G)^*\xi$. Obviously we have $B\xi \leq \bigwedge_{i\in I}(\bigwedge\{(B^{\overline{A\cap G}^f})^*\xi/$ $G \supset A_i, G$ fine open, $G \in \mathcal{X}\}) \leq \bigwedge_{i\in I}(B^{A_i})^*\xi = B\xi$. From the first part of the proof we get that all the balayages $(B^{\overline{A\cap G}^f})^*$ are absorbent and as a consequence $(B^{\overline{A\cap G}^f})^*\xi \prec \xi$. We deduce that $B\xi \prec \xi$ for all $\xi \in Exc_m$ and therefore assertion 1) follows.

Theorem 4.5. *Suppose that X is semi-saturated. If $m \in Exc$ and A is a basic set such that the balayage $(B^A)^*$ on Exc_m is recurrent then the following assertions hold:*

1) For any Ray compact subset K of A we have

$$B^{K\setminus\{x\}} 1(x) = 0 \quad m\text{-a.s. (in } x\text{) on } K.$$

2) The fine open set $A_o := \{x \in A / \{x\}$ is fine open $\}$ is universally measurable, $A \setminus A_o$ is m-polar and $(B^A)^* = (B^{A_o})^*$.

Proof. Let q be a measurable function on X, $0<q\leq 1$ be such that $0<Uq\leq 1$. Then for any $\mathcal{X}$-measurable subset A_1 of A there exists an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of Ray compact subsets of $A\setminus A_1$ such that $\sup_{n\in\mathbb{N}} B^{K_n}Uq= B^{A\setminus A_1}Uq$ m-a.s. By Theorem 4.4 we have $B^{K_n}Uq=0$ m-a.s. on $X\setminus K_n$. Therefore $B^{A\setminus A_1}Uq=0$ m-a.s. on A_1 and since $B^{A_1}Uq\leq B^AUq\leq B^{A_1}Uq+B^{A\setminus A_1}Uq$ we get $B^{A_1}Uq=B^AUq=Uq$ m-a.s. on A_1. Let now fix a distance on X associated to the Ray topology and let K be a Ray compact subset of A. We consider a sequence $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of finite open coverings of K, $\mathcal{G}_n=(G_i^n)_{i\in I_n}$ such that: a) any G_i^n has the diameter smaller than $1/n$; b) for every G_i^n there are $j\in I_{n+1}$ and $j'\in I_{n-1}$ with $G_j^{n+1}\subset G_i^n\subset G_{j'}^{n-1}$. For any $n\in\mathbb{N}$ we put $s_n:=\bigwedge_{i\in I_n} B^{K\setminus G_i^n}1$. From Theorem 4.4 we have $s_n=0$ m-a.s. on X. On the other hand by construction we get $s_n\leq s_{n+1}$ and

$$B^{K\setminus B(x,1/n)}1(x)\leq s_n(x)\leq B^{K\setminus\{x\}}1(x)$$

for any $x\in K$ such that $B^KUq(x)=Uq(x)$ (i.e. for any $x\in K$ such that K is not thin at x), where $B(x,1/n)$ is the open ball of radius $1/n$ centered in x. If we put $T:=\{x\in K/B^KUq(x)=Uq(x)\}$ then for all $x\in T$ we have $\sup_{n\in\mathbb{N}} s_n(x)=B^{K\setminus\{x\}}1(x)$. Since from the preceding considerations we have $m(K\setminus T)=0$ and $\sup_{n\in\mathbb{N}} s_n=0$ we deduce that $B^{K\setminus\{x\}}1(x)=0$ m-a.s. on K. Let now $(M_n)_{n\in\mathbb{N}}$ be an increasing sequence of Ray compact subsets of A such that $m(A\setminus\bigcup_{n\in\mathbb{N}} M_n)=0$ and let $t_n\in\mathcal{E}_{\mathcal{U}}$ be such that $B^{M_n}Uq\leq t_n$ and $t_n=B^{M_n}Uq$ m-a.s. If we put $D_n:=\{x\in X/t_n(x)=0\}$ it follows that D_n is a fine clopen $\mathcal{X}$-measurable subset of X and moreover for any $n\in\mathbb{N}$ we have $m((X\setminus M_n)\setminus D_n)=0$. The measure m being excessive we deduce that $m(\alpha U_\alpha(\chi_{(X\setminus M_n)\setminus D_n})\leq m((X\setminus M_n)\setminus D_n)=0$ and consequently $U(\chi_{(X\setminus M_n)\setminus D_n})=0$ m-a.s. On the other hand since the set $(X\setminus M_n)\setminus D_n$ is fine open and $\mathcal{X}$-measurable we get $U(\chi_{(X\setminus M_n)\setminus D_n})>0$ on $(X\setminus M_n)\setminus D_n$. We deduce that the fine open set $T_n:=(X\setminus D_n)\cap[U(\chi_{(X\setminus M_n)\setminus D_n})=0]$ is $\mathcal{X}$-measurable, $T_n\subset M_n$ and $m(M_n\setminus T_n)=0$. If $x\in T_n$ and $\{x\}$ is not fine open then $T_n\setminus\{x\}$ is not thin at x and therefore: $B^{M_n\setminus\{x\}}1(x)\geq B^{T_n\setminus\{x\}}1(x)=1>0$. Hence we have the inclusion

$$T_n\setminus\{x\in T_n/\{x\}\text{ is fine open }\}\subset\{x\in M_n/B^{M_n\setminus\{x\}}1(x)>0\}.$$

It follows that $m(M_n\setminus\{x\in M_n/\{x\}\text{ is fine open }\})=0$. We conclude that $m(A\setminus A_o)=0$. Obviously $B^{A\setminus A_o}1=0$ m-a.s. on X and therefore $(B^A)^*=(B^{A_o})^*$.

5 Quasi-Bounded Subtractive Excessive Measures

In this section we suppose that X is semi-saturated and we consider on X a Ray topology as in Section 4.

Theorem 5.1. *Let $m \in Exc$ and $\xi = \mu \circ U \in Exc_m$. Then the following assertions are equivalent:*
1) The measure ξ is subtractive and universally quasi-bounded in Exc_m.
2) There exists a basic set set $A \in \mathcal{X}$ such that μ is carried by A and for any Ray compact subset K of A we have $B^K 1 = 0$ m-a.s. on $X \setminus K$.
3) There exists a basic set set $A \in \mathcal{X}$ such that μ is carried by A and for any $\mathcal{X}$-measurable subset M of A we have $B^M 1 = 0$ m-a.s. on $X \setminus M$.
4) There exists $A \in \mathcal{X}$ such that μ is carried by A, $\{x\}$ is fine open for any $x \in A$ and $B^K 1 = 0$ m-a.s. on $X \setminus K$ for any Ray compact subset K of A.
5) There exists a basic set $A \in \mathcal{X}$ such that μ is carried by A and the balayage $(B^A)^*$ on Exc_m is recurrent.

Proof. 1)$\Rightarrow$2). By Corollary 1.10 there exists a recurrent balayage B on Exc_m such that $B\xi = \xi$. From Theorem 4.2 and Theorem 4.4 there exists a basic set $A \in \mathcal{X}$ such that $B = (B^A)^*$ and $B^K 1 = 0$ m-a.s. on $X \setminus K$ for any Ray compact subset K of A. Let $0 < f_o \leq 1$ be a function which is $\mathcal{X}$-measurable and such that Uf_o is bounded and $\mu(Uf_o) \leq \infty$. Since $L(\mu \circ U, Uf_o) = \mu(Uf_o)$ and $L((B^A)^*(\mu \circ U), Uf_o) = L(\mu \circ U, B^A Uf_o) = \mu(B^A Uf_o)$ we get $\mu(Uf_o) = \mu(B^A Uf_o)$ and therefore, A being a basic set, μ is carried by A.
2)$\Rightarrow$3). Follows from the fact that for any $M \in \mathcal{X}$, $M \subset A$, there exists an incresing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of M such that $B^{K_n}1 \nearrow B^M 1$ m-a.s. Since $B^{K_n}1 = 0$ m-a.s. on $X \setminus K_n$ we deduce that $B^M 1 = 0$ m-a.s. on $X \setminus \bigcup_{n\in\mathbb{N}} K_n$ and consequently $B^M 1 = 0$ m-a.s. on $X \setminus M$.
3)$\Rightarrow$5) Follows from Theorem 4.4.
5)$\Rightarrow$4). From Theorem 4.5 it follows that the set $A_o := \{x \in A / \{x\}$ is fine open$\}$ is universally measurable , $A \setminus A_o$ is m-polar and $(B^A)^* = (B^{A_o})^*$ on Exc_m. Since $(B^A)^*(\mu \circ U) = \mu \circ U$ and since $(B^A)^*$ is recurrent on Exc_m we deduce, using Proposition 1.9, that $\mu \circ U \in Q_{bd}(Exc_m)$ and therefore, by Corollary 3.2, μ does not charge the set $A \setminus A_o$. Hence μ is carried by A_o. From Theorem 4.4 it follows that $B^K 1 = 0$ m-a.s. on $X \setminus K$ for any Ray compact subset K of A_o. To obtain 4) we replace A_o with a subset of A_o which is $\mathcal{X}$-measurable and differs from A_o by a μ-negligible set.
4)$\Rightarrow$1). Let A be as in assertion 4). We remark that for any $s \in \mathcal{E}_\mathcal{U}$ we have $B^A s \in \mathcal{E}_\mathcal{U}$ and $B^A s = s$ on A. Also any Ray compact subset K of A is a basic set and by Theorem 4.4 the balayage $(B^K)^*$ on Exc_m is recurrent. Further there exists an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of Ray compacts subsets of A such that

$$B^A U f_o = \sup_{n\in\mathbb{N}} B^{K_n} U f_o \quad \nu - \text{a.s.}$$

where ν is a finite measure on X and $\nu \circ U$ is a generator of Exc_m. Hence $\bigvee_{n\in\mathbb{N}} (B^{K_n})^* = (B^A)^*$. Therefore (cf. [9]) it follows that the balayage $(B^A)^*$ is also recurrent on Exc_m. Since μ is carried by A we deduce that $(B^A)^*(\mu \circ U) = \mu \circ U$ and by Proposition 1.9 we conclude that $\mu \circ U \in Q_{bd}(Exc_m) \cap Sub(Exc_m)$.
Remark. *Let A be a subset of X which is $\mathcal{X}$-measurable and such that $\{x\}$ is fine open for any $x \in A$ and $B^K 1 = 0$ m-a.s. on $X \setminus K$ for any Ray compact*

subset K of A. Then

$$B^{K\setminus\{x\}}1(x) = 0 \quad m\text{-a.s. (in } x) \text{ on } K$$

for any Ray compact subset K of A.

This assertion follows immediately from Theorem 4.4 and Theorem 4.5 applied to any Ray compact subset of A.

Corollary 5.2. *Let $m \in Exc$ and $\xi = \mu \circ U \in Exc_m$. Then each of the assertions from Theorem 5.1 is equivalent with the following one:*

4') There exists $A \in \mathcal{X}$ such that μ is carried by A, $\{x\}$ is fine open for any $x \in A$ and such that for any Ray compact subset K of A we have:

$$-B^{K}1 = 0 \quad m\text{-a.s. on } X \setminus K,$$
$$-B^{K\setminus\{x\}}1(x) = 0 \quad m\text{-a.s. (in } x) \text{ on } K.$$

As a consequence of the above theorem we deduce the following Riesz decomposition obtained in [13] :

Corollary 5.3. *Let $m \in Exc$ and $\xi \in Exc_m$. Then $\xi \in Sub(Exc_m)$ iff ξ is of the form*

$$\xi = h + \mu \circ U + \nu \circ U$$

where $h \in Har$, μ is carried by a m-polar set (or even by a m-polar set which is semi-polar) and ν is carried by a set $A \in \mathcal{X}$ such that $\{x\}$ is fine open for any $x \in A$ and for any Ray compact subset K of A we have:

$$-B^{K}1 = 0 \quad m\text{-a.s. on } X \setminus K,$$
$$-B^{K\setminus\{x\}}1(x) = 0 \quad m\text{-a.s. (in } x) \text{ on } K.$$

Moreover the above decomposition is unique: $\mu \circ U \in Q_{bd}(Exc_m)^{\perp}$ and $\nu \circ U \in Q_{bd}(Exc_m) \cap Sub(Exc_m)$.

Corollary 5.4. *Suppose that there are no non zero recurrent balayages on Exc_m (this is the case when Exc_m is elliptic or if X has no fine isolated points which are no m-polar). Then $\xi \in Sub(Exc_m)$ iff $\xi = h + \mu \circ U$ where $h \in Har$ and μ is carried by a m-polar set.*

Remark. *The above result is an improvement of (2.28) in [13].*

Final remark. *All the assertions proved in this paper for $\mathcal{U}$-excessive measures hold even if the initial kernel U is not proper.*

Indeed if $m \in Exc$ then there exists (see [14]) a decomposition of X of the form $X = D \cup C$ where D, $C \in \mathcal{X}$, $D \cap C = \emptyset$ and C is absorbent and such that D is a dissipative subset of X and therefore the restriction of U to C is proper and for any $\xi \in Exc_m$ the measure $\xi|_D$ (resp. $\xi|_C$) is the dissipative (resp. conservative) component of ξ. It is easy to show that for any $\xi \in Exc_m$ the measure $\xi|_C$ belongs to $Q_{bd}(Exc_{m|_C}) \cap Sub(Exc_{m|_C})$ and the measure ξ belongs to $Q_{bd}(Exc_m)$ (resp. $Q_{bd}(Exc_m) \cap Sub(Exc_m)$) iff $\xi|_D$ belongs to $Q_{bd}(Exc_{m|_D})$) (resp. $Q_{bd}(Exc_{m|_D}) \cap Sub(Exc_{m|_D})$. On the other hand if $\xi = \mu \circ U$ then we have $\xi = \xi|_D$.

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