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A BOUNDARY CONTROL PROBLEM IN FLUID MECHANICS

Anca Capatina* and Ruxandra Stavre†

Abstract. This paper deals with a boundary optimal control problem associated to the steady-state Navier-Stokes equations coupled with the heat equation. The most general type of boundary condition for the temperature is considered. The existence of a solution of this problem is proved. Moreover, for some values of the viscosity coefficient we also obtain the uniqueness. The control problem consists in minimizing the turbulence of the fluid, the control being the temperature of the surrounding medium. The existence of an optimal control is proved and necessary conditions of optimality are derived.

Key words. optimal control problem, Navier-Stokes equations, heat equation, turbulence

1. Introduction

The purpose of this paper is to study a boundary control problem associated with the stationary Navier-Stokes equations coupled with the heat equation. The boundary condition for the temperature is of oblique type (e.g.[3], p. 8):

$$\kappa\frac{\partial\tau}{\partial n}+\alpha(\tau-\theta)=0 \quad \text{on} \quad \Gamma \tag{1.1}$$

where κ, α are constants, θ is the temperature of the surrounding medium and Γ is the boundary of the flow region Ω. Equality (1.1) means that the heat flux across the boundary Γ is proportional to the difference between the temperature τ of the fluid and that of the surrounding medium, θ.

A similar problem has been studied in [1], with a less realistic boundary condition than (1.1).

Our aim is to characterize the controls $\theta=\theta_0$ which minimize the turbulence of the fluid, as measured by the functional:

$$J(\theta)=\frac{1}{2}\int_\Omega|\nabla\times \mathrm{v}(\theta)|^2dx \tag{1.2}$$

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where $\mathbf{v}(\theta)$ is the velocity of the fluid corresponding to the control θ .

The control of Navier-Stokes equations has been investigated in [2], [7], [8].

In §2 we give a variational formulation (VP) of the physical boundary-value problem (PP) and we prove that they are equivalent. The existence of a solution of (VP) can be obtained without any restriction on the size of the viscosity ν. It is known that, in general , we do not have uniqueness for problems of this type (see [9]). The main result of §2 is the uniqueness of the solution of (VP) for large enough values of the viscosity coefficient.

The existence of an optimal control is obtained in §3. The uniqueness result, proved in §2, allows us to derive the optimality conditions without approximating the control problem (CP) by a family of penalized problems as in [1] (Section 4).

2. The physical problem. Existence and uniqueness results

Let us consider a viscous, incompressible fluid, occupying a bounded domain $\Omega \subset \mathbf{R}^3$ with Lipschitz boundary Γ. Though the results and the methods are the same for the two-dimensional flow, we only consider, for simplicity, the three-dimensional case.

We seek for a vector function $\mathbf{v}$ representing the velocity of the fluid, a scalar function π -the pressure of the fluid and a scalar function τ - the temperature of the fluid, which are defined in Ω and satisfy the following system of Navier-Stokes and heat equations and the boundary conditions:

$$(PP)\begin{cases} -\nu \triangle \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\pi = \mathbf{f} + \mathbf{B}\tau & \text{in} \quad \Omega, \\ -\kappa \triangle \tau + \mathbf{v}\cdot\nabla\tau = g & \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{v} = 0 & \text{in} \quad \Omega, \\ \mathbf{v} = \mathbf{0} & \text{on} \quad \Gamma, \\ \kappa\dfrac{\partial\tau}{\partial n} + \alpha(\tau-\theta) = 0 & \text{on} \quad \Gamma \end{cases}$$

where $\nu > 0$ is the coefficient of the kinematic viscosity, $\mathbf{f}$ the body forces, $\kappa > 0$ the thermal conductivity, $\alpha > 0$ the heat-transfer coefficient corresponding to convection, θ the temperature of the surrounding medium, $\mathbf{B}$ a function given by the Boussinesq approximation, g an external heat source and $\mathbf{n}$ the outward unit normal to Γ.

The most general type of boundary condition for the temperature is $(PP)_5$ where:

$$\alpha = \alpha_c + \alpha_r(\tau+\theta)(\tau^2+\theta^2), \tag{2.1}$$

α_c and α_r denoting heat-transfer coefficients corresponding to convection and radiation, respectively; in our problem we neglected the radiation effects.

In order to give a variational formulation of problem (PP) we assume that:

$$\mathbf{f} \in (H^{-1}(\Omega))^3, \quad \mathbf{B} \in (L^\infty(\Omega))^3, \quad g \in L^{6/5}(\Omega), \quad \theta \in H^{-1/2}(\Gamma).$$

We denote by Y_0 the following separable Hilbert space (see [10]):

$$Y_0 = \{\mathbf{v} \in (H_0^1(\Omega))^3 / \operatorname{div}\mathbf{v} = 0 \quad \text{in} \quad \Omega\}. \tag{2.2}$$

Suppose $(\mathbf{v},\pi,\tau) \in Y_0 \times L^2(\Omega) \times H^1(\Omega)$ satisfies (PP). Then, by Green's formula we easily obtain that $(\mathbf{v},\tau)$ is a solution of the following variational problem:

$$(VP)\begin{cases} (\mathbf{v},\tau) \in Y_0 \times H^1(\Omega), \\ \nu((\mathbf{v},\mathbf{z}))_0 + b_0(\mathbf{v},\mathbf{v},\mathbf{z}) = \langle \mathbf{f},\mathbf{z}\rangle + \int_\Omega (\mathbf{B}\cdot\mathbf{z})\tau dx \quad \forall \mathbf{z} \in Y_0, \\ \kappa((\tau,\eta)) + \alpha \int_\Gamma \tau\eta ds + b(\mathbf{v},\tau,\eta) = \int_\Omega g\eta dx + \alpha\langle \theta,\eta\rangle_\Gamma \quad \forall \eta \in H^1(\Omega) \end{cases}$$

where:

$$\begin{cases} ((\mathbf{v},\mathbf{z}))_0 = \int_\Omega \nabla\mathbf{v}\cdot\nabla\mathbf{z}dx & \forall \mathbf{v},\mathbf{z} \in Y_0, \\ ((\tau,\eta)) = \int_\Omega \nabla\tau\cdot\nabla\eta dx & \forall \tau,\eta \in H^1(\Omega), \\ b_0(\mathbf{v},\mathbf{w},\mathbf{z}) = \int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{w}\cdot\mathbf{z}dx & \forall \mathbf{v},\mathbf{w},\mathbf{z} \in Y_0, \\ b(\mathbf{v},\tau,\eta) = \int_\Omega (\mathbf{v}\cdot\nabla\tau)\eta dx & \forall \mathbf{v} \in Y_0,\ \forall \tau,\eta \in H^1(\Omega) \end{cases}$$

and the symbols $\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot\rangle_\Gamma$ denote the pairing between $((H^{-1}(\Omega))^3, (H_0^1(\Omega))^3)$ and $(H^{-1/2}(\Gamma), H^{1/2}(\Gamma))$, respectively.

Conversely, if $(\mathbf{v},\tau)$ is a solution of (VP) then , by using the same arguments as in [10], we obtain the existence of $\pi \in L^2(\Omega)$ such that $(\mathbf{v},\pi,\tau)$ satisfies $(PP)_1 - (PP)_4$. Moreover, by a Green's formula [4], it follows that $\dfrac{\partial\tau}{\partial n}$ is an element of $H^{-1/2}(\Gamma)$ and hence we get $(PP)_5$.

The previous considerations show that (PP) and (VP) are equivalent.

The following result can be easily proved by using the properties of b_0 and b (see [1]) and the inequality (e.g. [6], p.67):

$$((\tau,\tau)) + \int_\Gamma \tau^2 ds \geq C_1\|\tau\|^2_{H^1(\Omega)} \quad \forall \tau \in H^1(\Omega). \tag{2.3}$$

Lemma 2.1 *If $(\mathbf{v},\tau)$ is a solution of (VP) then we have:*

$$\begin{cases} \|\mathbf{v}\|_{(H_0^1(\Omega))^3} \leq \dfrac{1}{\nu}(\|\mathbf{f}\|_{(H^{-1}(\Omega))^3} \\ \qquad + C_2\|\mathbf{B}\|_{(L^\infty(\Omega))^3}\dfrac{C_5\|g\|_{L^{6/5}(\Omega)} + \alpha C_6\|\theta\|_{H^{-1/2}(\Gamma)}}{C_1 min(\kappa,\alpha)}), \\ \|\tau\|_{H^1(\Omega)} \leq \dfrac{1}{C_1 min(\kappa,\alpha)}(C_5\|g\|_{L^{6/5}(\Omega)} + \alpha C_6\|\theta\|_{H^{-1/2}(\Gamma)}) \end{cases} \tag{2.4}$$

where C_1, C_2, C_5 and C_6 are constants depending only on Ω.

We now establish the main result of this Section.

Theorem 2.2 *The problem (VP) has at least a solution $(\mathbf{v},\tau)$. Moreover, for any $\theta_1 > 0$ there exists $\nu_1 > 0$ such that for every $\nu > \nu_1$ and $\theta \in H^{-1/2}(\Gamma)$ with $\|\theta\|_{H^{-1/2}(\Gamma)} \leq \theta_1$, the problem (VP) has a unique solution.*

Proof. Let $\{\mathbf{w}_k\}_{k\in\mathrm{N}}$ be a Hilbertian basis of Y_0. Let Y_m be the space generated by the functions $\{\mathbf{w}_1, ..., \mathbf{w}_m\}$. For each $m \in \mathrm{N}$, $m \geq 1$ we define $F : Y_m \mapsto Y_m$ by $F(\mathbf{w}) = \mathbf{v}_m$ where $(\mathbf{v}_m, \tau_m)$ is the unique solution of:

$$(2.5)\begin{cases} (\mathbf{v}_m, \tau_m) \in Y_m \times H^1(\Omega), \\ \nu((\mathbf{v}_m, \mathbf{w}_k))_0 + b_0(\mathbf{w}, \mathbf{v}_m, \mathbf{w}_k) = \langle \mathbf{f}, \mathbf{w}_k\rangle + \int_\Omega (\mathbf{B}\cdot \mathbf{w}_k)\tau dx \,\forall k \in \{1, ..., m\}, \\ \kappa((\tau_m, \eta)) + \alpha \int_\Gamma \tau_m \eta ds + b(\mathbf{w}, \tau_m, \eta) = \int_\Omega g\eta dx + \alpha\langle \theta, \eta\rangle_\Gamma \forall \eta \in H^1(\Omega). \end{cases}$$

The existence and uniqueness of the solution of (2.5) is a consequence of the Lax-Milgram's theorem and of the inequality (2.3).

It can be easily proved that the mapping F is continuous and, from (2.4), for all $\mathbf{w} \in Y_m$, it follows $F(\mathbf{w}) \in \bar{B}_r(0)$ where $B_r(0) \subset Y_m$ is the ball of radius

$$r = \frac{1}{\nu}(||\mathbf{f}||_{(H^{-1}(\Omega))^3} + C_2||\mathbf{B}||_{(L^\infty(\Omega))^3}\frac{C_5||g||_{L^{6/5}(\Omega)} + \alpha C_6||\theta||_{H^{-1/2}(\Gamma)}}{C_1 min(\kappa, \alpha)}).$$

By applying the Brower's theorem, it follows that F has a fixed point $\mathbf{v}_m$. Since the sequences $\{\mathbf{v}_m\}_{m\in\mathrm{N}}$, $\{\tau_m\}_{m\in\mathrm{N}}$ are bounded in $(H_0^1(\Omega))^3$ and $H^1(\Omega)$, respectively, we can pass to the limit, as in [10], in (2.5), with $\mathbf{w} = \mathbf{v}_m$, for a subsequence $\{\mathbf{v}_{m_p}, \tau_{m_p}\}_{p\in\mathrm{N}}$. The weak limit in $(H_0^1(\Omega))^3 \times H^1(\Omega)$ of this subsequence is a solution of (VP), i.e. the first statement of the theorem holds.

For proving the uniqueness of the solution of problem (VP) we consider $\theta_1 > 0$, $\theta \in H^{-1/2}(\Gamma)$ with $||\theta||_{H^{-1/2}(\Gamma)} \leq \theta_1$ and we define $S : Y_0 \mapsto Y_0$, $S(\mathbf{w}) = \mathbf{v}_w$, where $(\mathbf{v}_w, \tau_w)$ is the unique solution of:

$$(2.6)\begin{cases} (\mathbf{v}_w, \tau_w) \in Y_0 \times H^1(\Omega), \\ \nu((\mathbf{v}_w, \mathbf{z}))_0 + b_0(\mathbf{w}, \mathbf{v}_w, \mathbf{z}) = \langle \mathbf{f}, \mathbf{z}\rangle + \int_\Omega (\mathbf{B}\cdot \mathbf{z})\tau_w dx \quad \forall \mathbf{z} \in Y_0, \\ \kappa((\tau_w, \eta)) + \alpha \int_\Gamma \tau_w \eta ds + b(\mathbf{w}, \tau_w, \eta) = \int_\Omega g\eta dx + \alpha\langle \theta, \eta\rangle_\Gamma \quad \forall \eta \in H^1(\Omega). \end{cases}$$

We shall prove that there exists $\nu_1 > 0$ such that for every $\nu > \nu_1$, the mapping S is contraction.

For $\mathbf{w}_i \in Y_0$ we denote by $(\mathbf{v}_i, \tau_i)$ the corresponding solution of (2.6), $i = 1, 2$. By taking $\mathbf{z} = \mathbf{v}_i$ and $\eta = \tau_i$ in (2.6) for $\mathbf{w} = \mathbf{w}_i$ it follows the estimate (2.4) for $\mathbf{v} = \mathbf{v}_i$ and $\tau = \tau_i$, for $i = 1, 2$. Hence, by subtracting the equations (2.6) corresponding to $\mathbf{w} = \mathbf{w}_1$ and $\mathbf{w} = \mathbf{w}_2$, respectively, for $\mathbf{z} = \mathbf{v}_1 - \mathbf{v}_2$ and $\eta = \tau_1 - \tau_2$, we get:

$$(2.7) \qquad ||\mathbf{v}_1 - \mathbf{v}_2||_{(H_0^1(\Omega))^3} \leq C(\nu, \theta)||\mathbf{w}_1 - \mathbf{w}_2||_{(H_0^1(\Omega))^3},$$

where:

$$(2.8) \qquad C(\nu, \theta) = \frac{C_3}{\nu^2}(||\mathbf{f}||_{(H^{-1}(\Omega))^3} + \frac{C_2||\mathbf{B}||_{(L^\infty(\Omega))^3}}{C_1 min(\kappa, \alpha)}(C_5||g||_{L^{6/5}(\Omega)} + \alpha C_6||\theta||_{H^{-1/2}(\Gamma)})) + \frac{C_2 C_4||\mathbf{B}||_{(L^\infty(\Omega))^3}(C_5||g||_{L^{6/5}(\Omega)} + \alpha C_6||\theta||_{H^{-1/2}(\Gamma)})}{\nu C_1^2 min^2(\kappa, \alpha)}.$$

It follows immediatly that there exists $\nu_1 > 0$ such that for every $\nu > \nu_1$ we have $C(\nu, \theta) < 1$. Consequently S is contraction and, by applying the Banach's fixed point theorem, the uniqueness of the solution of (VP) is proved. □

In the sequel we shall consider $\theta_1 > 0$ an arbitrary constant and $\nu > \nu_1$, where ν_1 is given by Theorem 2.1.

3. A boundary control problem

In this Section we look for a temperature θ of the surrounding medium, which minimize the turbulence of the fluid. For this purpose we consider the following control problem:

$$(CP) \begin{cases} \text{Find} \quad \theta_0 \in K \quad \text{such that} \\ J(\theta_0) = \min\{J(\theta) \,/\, \theta \in K\} \end{cases}$$

where:

$$K = \{\theta \in H^{-1/2}(\Gamma) \,/\, \|\theta\|_{H^{-1/2}(\Gamma)} \leq \theta_1\},$$

$$J : K \mapsto \mathbf{R}, \quad J(\theta) = \frac{1}{2}\int_\Omega |\nabla \times \mathrm{v}(\theta)|^2 dx,$$

$\mathrm{v}(\theta)$ being the first component of the unique solution of (VP) corresponding to the control θ.

Theorem 3.1 *The problem (CP) has at least one solution.*

Proof. We shall prove that J is lower semicontinuous with respect to the weak topology of $H^{-1/2}(\Gamma)$. Let $\{\theta_n\}_{n\in\mathrm{N}} \subset K$ be a weakly convergent sequence to some θ_0. Denoting by (v_n, τ_n) the unique solution of (VP) corresponding to θ_n, we obtain from (2.4) (for $\mathrm{v} = \mathrm{v}_n$, $\tau = \tau_n$ and $\theta = \theta_n$) and the boundedness of $\{\theta_n\}_{n\in\mathrm{N}}$ that the sequence $\{\mathrm{v}_n, \tau_n\}_{n\in\mathrm{N}}$ is bounded in $(H_0^1(\Omega))^3 \times H^1(\Omega)$. By extracting a subsequence with the weak limit denoted by (v^*, τ^*) and by passing to the limit into the corresponding (VP) we get, from the uniqueness of the solution of (VP) for θ_0, that $\mathrm{v}^* = \mathrm{v}(\theta_0)$, $\tau^* = \tau(\theta_0)$ and, hence:

$$\mathrm{v}_n \longrightarrow \mathrm{v}(\theta_0) \quad \text{weakly in} \quad (H_0^1(\Omega))^3, \quad \text{when} \quad n \longrightarrow \infty,$$

$$\tau_n \longrightarrow \tau(\theta_0) \quad \text{weakly in} \quad H^1(\Omega), \quad \text{when} \quad n \longrightarrow \infty.$$

It follows that:

$$\liminf_{n\longrightarrow\infty} \|\frac{\partial \mathrm{v}_n}{\partial x_i}\|^2_{(L^2(\Omega))^3} \geq \|\frac{\partial \mathrm{v}(\theta_0)}{\partial x_i}\|^2_{(L^2(\Omega))^3} \quad \text{for} \quad i = 1, 2, 3.$$

Thus, we get the weakly lower semicontinuity of J on $H^{-1/2}(\Gamma)$. Moreover, K is a bounded and weakly closed set in $H^{-1/2}(\Gamma)$. Therefore the assertion of the Theorem follows by applying a Weierstrass theorem (e.g. [5],p. 495). □

4. Optimality conditions

In the sequel we shall derive the necessary conditions of optimality. In order to obtain them, we shall prove that J is Gâteaux differentiable on the set of optimal controls. The necessary conditions of optimality will be deduced from:

$$J'(\theta_0)\cdot(\theta-\theta_0)\geq 0 \quad \text{for all} \quad \theta\in K, \tag{4.1}$$

where θ_0 is a solution of (CP).

Let $t\in(0,1)$, $\theta\in K$ and θ_0 be an optimal control. We shall compute

$$\lim_{t\searrow 0}\frac{J(\theta_0+t(\theta-\theta_0))-J(\theta_0)}{t}.$$

For this purpose, we denote by $(\mathrm{v}_{t\theta},\tau_{t\theta})$ and (v_0,τ_0) the solutions of (VP) corresponding to $\theta_0+t(\theta-\theta_0)$ and θ_0, respectively.

We first deduce some properties for $(\mathrm{v}_{t\theta},\tau_{t\theta})$.

Lemma 4.1 *There holds:*

$$\begin{cases} \mathrm{v}_{t\theta}=\mathrm{v}_0+t(\mathrm{v}_t-\mathrm{v}_0), \\ \tau_{t\theta}=\tau_0+t(\tau_t-\tau_0) \end{cases} \tag{4.2}$$

where (v_t,τ_t) *is the unique solution of the following problem:*

$$\begin{cases} \nu((\mathrm{v}_t,\mathrm{z}))_0+(1-t)b_0(\mathrm{v}_0,\mathrm{v}_t,\mathrm{z})+(1-t)b_0(\mathrm{v}_t,\mathrm{v}_0,\mathrm{z})+tb_0(\mathrm{v}_t,\mathrm{v}_t,\mathrm{z}) \\ \quad =(1-t)b_0(\mathrm{v}_0,\mathrm{v}_0,\mathrm{z})+\langle\mathrm{f},\mathrm{z}\rangle+\int_\Omega(\mathrm{B}\cdot\mathrm{z})\tau_t dx\ \forall\mathrm{z}\in Y_0, \\ \kappa((\tau_t,\eta))+\alpha\int_\Gamma\tau_t\eta ds+(1-t)b(\mathrm{v}_0,\tau_t,\eta)+(1-t)b(\mathrm{v}_t,\tau_0,\eta) \\ \quad +tb(\mathrm{v}_t,\tau_t,\eta)=(1-t)b(\mathrm{v}_0,\tau_0,\eta)+\int_\Omega g\eta dx+\alpha\langle\theta,\eta\rangle_\Gamma\ \forall\eta\in H^1(\Omega). \end{cases} \tag{4.3}$$

Proof. We begin by establishing that (4.3) has a unique solution. We define the mapping $S_t: Y_0\mapsto Y_0$ which associates to every $\mathrm{w}\in Y_0$ the first component of the unique solution $(\mathrm{v}_{tw},\tau_{tw})$ of the problem:

$$\begin{cases} \nu((\mathrm{v}_{tw},\mathrm{z}))_0+(1-t)b_0(\mathrm{v}_0,\mathrm{v}_{tw},\mathrm{z})+tb_0(\mathrm{w},\mathrm{v}_{tw},\mathrm{z}) \\ \quad =(1-t)b_0(\mathrm{v}_0,\mathrm{v}_0,\mathrm{z})-(1-t)b_0(\mathrm{w},\mathrm{v}_0,\mathrm{z}) \\ \quad +\langle\mathrm{f},\mathrm{z}\rangle+\int_\Omega(\mathrm{B}\cdot\mathrm{z})\tau_{tw}dx \quad \forall\mathrm{z}\in Y_0, \\ \kappa((\tau_{tw},\eta))+\alpha\int_\Gamma\tau_{tw}\eta ds+(1-t)b(\mathrm{v}_0,\tau_{tw},\eta)+tb(\mathrm{w},\tau_{tw},\eta) \\ \quad =(1-t)b(\mathrm{v}_0,\tau_0,\eta)-(1-t)b(\mathrm{w},\tau_0,\eta) \\ \quad +\int_\Omega g\eta dx+\alpha\langle\theta,\eta\rangle_\Gamma \quad \forall\eta\in H^1(\Omega). \end{cases} \tag{4.4}$$

For proving that S_t is contraction, we take $\mathbf{w}_1, \mathbf{w}_2 \in Y_0$, we denote $(\mathbf{v}_{t_1}, \tau_{t_1}), (\mathbf{v}_{t_2}, \tau_{t_2})$ the corresponding solutions of (4.4) and, by subtracting these equalities for $\mathbf{z} = \mathbf{v}_{t_1} - \mathbf{v}_{t_2}$ and $\eta = \tau_{t_1} - \tau_{t_2}$, we get:

$$(4.5)\begin{cases} \|\tau_{t_1} - \tau_{t_2}\|_{H^1(\Omega)} \leq \dfrac{C_4}{C_1 min(\kappa, \alpha)} \|(1-t)\tau_0 + t\tau_{t_1}\|_{H^1(\Omega)} \|\mathbf{w}_1 - \mathbf{w}_2\|_{(H_0^1(\Omega))^3}, \\ \|\mathbf{v}_{t_1} - \mathbf{v}_{t_2}\|_{(H_0^1(\Omega))^3} \leq \dfrac{1}{\nu}(C_3\|(1-t)\mathbf{v}_0 + t\mathbf{v}_{t_1}\|_{(H_0^1(\Omega))^3} \\ \quad + \dfrac{C_2C_4}{C_1 min(\kappa, \alpha)} \|(1-t)\tau_0 + t\tau_{t_1}\|_{H^1(\Omega)}) \|\mathbf{w}_1 - \mathbf{w}_2\|_{(H_0^1(\Omega))^3}. \end{cases}$$

Multiplying by $(1-t)$ the problem $(VP)_0$ ((VP) for $\theta = \theta_0$) and by t the problem (4.4) for $\mathbf{w} = \mathbf{w}_1$ and adding them, it follows:

$$(4.6)\begin{cases} \|(1-t)\tau_0 + t\tau_{t_1}\|_{H^1(\Omega)} \leq \dfrac{1}{C_1 min(\kappa, \alpha)}(C_5\|g\|_{L^{6/5}(\Omega)} \\ \quad + C_6\alpha\|(1-t)\theta_0 + t\theta\|_{H^{-1/2}(\Gamma)}), \\ \|(1-t)\mathbf{v}_0 + t\mathbf{v}_{t_1}\|_{(H_0^1(\Omega))^3} \leq \dfrac{1}{\nu}(\|\mathbf{f}\|_{(H^{-1}(\Omega))^3} \\ \quad + \dfrac{C_2\|\mathbf{B}\|_{(L^\infty(\Omega))^3}}{C_1 min(\kappa, \alpha)}(C_5\|g\|_{L^{6/5}(\Omega)} + C_6\alpha\|(1-t)\theta_0 + t\theta\|_{H^{-1/2}(\Gamma)})). \end{cases}$$

From (4.5) and (4.6) we obtain:

$$(4.7) \qquad \|\mathbf{v}_{t_1} - \mathbf{v}_{t_2}\|_{(H_0^1(\Omega))^3} \leq C(\nu, (1-t)\theta_0 + t\theta)\|\mathbf{w}_1 - \mathbf{w}_2\|_{(H_0^1(\Omega))^3}.$$

From the proof of the Theorem 2.2 it follows that:

$$(4.8) \qquad C(\nu, \tilde{\theta}) \leq 1 \qquad \forall \tilde{\theta} \in K$$

where $C(\nu, \tilde{\theta})$ is defined by (2.8).

By using the convexity of K and the inequality (4.8) for $\tilde{\theta} = (1-t)\theta_0 + t\theta$, we conclude that S_t is contraction and, with the same arguments as in Theorem 2.2, the uniqueness of the solution of problem (4.3) is proved.

Computing $(1-t)\cdot(VP)_0 + t\cdot(4.3)$, we obtain that $(\mathbf{v}_0 + t(\mathbf{v}_t - \mathbf{v}_0), \tau_0 + t(\tau_t - \tau_0))$ is a solution of (VP) for θ replaced by $\theta_0 + t(\theta - \theta_0)$ and, from Theorem 2.2, the assertion of the Lemma follows. □

Lemma 4.2 *Let $(\mathbf{v}_t, \tau_t)$ be defined by (4.3). Then, as $t \longrightarrow 0$, we get:*

$$(4.9)\begin{cases} \mathbf{v}_t \longrightarrow \mathbf{v}_\theta^* \quad \text{weakly in } (H_0^1(\Omega))^3, \\ \tau_t \longrightarrow \tau_\theta^* \quad \text{weakly in } H^1(\Omega) \end{cases}$$

where $(\mathbf{v}_\theta^*, \tau_\theta^*)$ is the unique solution of:

$$(4.10)\begin{cases} \nu((\mathbf{v}_\theta^*, \mathbf{z}))_0 + b_0(\mathbf{v}_0, \mathbf{v}_\theta^*, \mathbf{z}) + b_0(\mathbf{v}_\theta^*, \mathbf{v}_0, \mathbf{z}) \\ \quad = b_0(\mathbf{v}_0, \mathbf{v}_0, \mathbf{z}) + \langle \mathbf{f}, \mathbf{z}\rangle + \int_\Omega (\mathbf{B}\cdot\mathbf{z})\tau_\theta^* dx \quad \forall \mathbf{z} \in Y_0, \\ \kappa((\tau_\theta^*, \eta)) + \alpha\int_\Gamma \tau_\theta^*\eta ds + b(\mathbf{v}_0, \tau_\theta^*, \eta) + b(\mathbf{v}_\theta^*, \tau_0, \eta) \\ \quad = b(\mathbf{v}_0, \tau_0, \eta) + \int_\Omega g\eta dx + \alpha\langle\theta, \eta\rangle_\Gamma \quad \forall \eta \in H^1(\Omega). \end{cases}$$

Proof. Existence and uniqueness of the solution of (4.10) can be obtained by using the same arguments as in Theorem 2.2. Next we shall prove that the sequence $\{\mathbf{v}_t, \tau_t\}_{t>0}$ is bounded in $(H_0^1(\Omega))^3 \times H^1(\Omega)$. Taking $\mathbf{z} = \mathbf{v}_t$ and $\eta = \tau_t$ in (4.3) we get the following estimations:

$$
(4.11)\quad \begin{cases}
\|\tau_t\|_{H^1(\Omega)} \leq \dfrac{1}{C_1 min(\kappa,\alpha)}(C_4\|\tau_0\|_{H^1(\Omega)}\|\mathbf{v}_t\|_{(H_0^1(\Omega))^3} \\[2mm]
\quad +C_4\|\tau_0\|_{H^1(\Omega)}\|\mathbf{v}_0\|_{(H_0^1(\Omega))^3} + C_5\|g\|_{L^{6/5}(\Omega)} + \alpha C_6\|\theta\|_{H^{-1/2}(\Gamma)}), \\[2mm]
\nu\|\mathbf{v}_t\|_{(H_0^1(\Omega))^3} \leq C_3\|\mathbf{v}_0\|_{(H_0^1(\Omega))^3}\|\mathbf{v}_t\|_{(H_0^1(\Omega))^3} + C_3\|\mathbf{v}_0\|^2_{(H_0^1(\Omega))^3} \\[2mm]
\quad +\|\mathbf{f}\|_{(H^{-1}(\Omega))^3} + \dfrac{C_2C_4\|\mathbf{B}\|_{(L^\infty(\Omega))^3}}{C_1 min(\kappa,\alpha)}\|\tau_0\|_{H^1(\Omega)}\|\mathbf{v}_t\|_{(H_0^1(\Omega))^3} \\[2mm]
\quad +\dfrac{C_2C_4\|\mathbf{B}\|_{(L^\infty(\Omega))^3}}{C_1 min(\kappa,\alpha)}\|\tau_0\|_{H^1(\Omega)}\|\mathbf{v}_0\|_{(H_0^1(\Omega))^3} \\[2mm]
\quad +\dfrac{C_2\|\mathbf{B}\|_{(L^\infty(\Omega))^3}}{C_1 min(\kappa,\alpha)}(C_5\|g\|_{L^{6/5}(\Omega)} + C_6\alpha\|\theta\|_{H^{-1/2}(\Gamma)}).
\end{cases}
$$

From (2.4) for $\theta = \theta_0$ and from (4.11) it follows:

$$
(4.12)\qquad (1 - C(\nu,\theta_0))\|\mathbf{v}_t\|_{(H_0^1(\Omega))^3} \leq C',
$$

with C' independent on t.

Combining (4.12) with (4.11), we also obtain the boundedness of $\{\tau_t\}_{t>0}$; hence, we can extract a subsequence $\{\mathbf{v}_{t_k}, \tau_{t_k}\}_{k\in\mathbf{N}}$ weakly convergent in $(H_0^1(\Omega))^3 \times H^1(\Omega)$ to some $(\mathbf{v}', \tau')$. By passing to the limit with $k \longrightarrow \infty$ in (4.3) for this subsequence, we get that $(\mathbf{v}', \tau')$ is a solution of (4.10). By the uniqueness stated before, the proof of the Lemma is complete. □

Lemma 4.3 *The cost functional defined by (1.2) is Gâteaux differentiable on the set of optimal controls, and:*

$$
(4.13)\qquad J'(\theta_0)\cdot(\theta-\theta_0) = \int_\Omega (\nabla\times\mathbf{v}(\theta_0))\cdot(\nabla\times(\mathbf{v}^*_\theta - \mathbf{v}(\theta_0)))dx \quad \forall\theta\in K,
$$

where $\mathbf{v}^*_\theta$ *is defined by (4.10).*

Proof. We have:

$$
\lim_{t\searrow 0}\frac{J(\theta_0+t(\theta-\theta_0)) - J(\theta_0)}{t} = \frac{1}{2}\lim_{t\searrow 0}\frac{\int_\Omega(|\nabla\times\mathbf{v}_{t\theta}|^2 - |\nabla\times\mathbf{v}_0|^2)dx}{t}
$$

$$
= \frac{1}{2}\lim_{t\searrow 0}(2\int_\Omega(\nabla\times\mathbf{v}_0)\cdot(\nabla\times(\mathbf{v}_t-\mathbf{v}_0))dx + t\int_\Omega|\nabla\times(\mathbf{v}_t-\mathbf{v}_0)|^2dx)
$$

$$
= \int_\Omega(\nabla\times\mathbf{v}_0)\cdot(\nabla\times(\mathbf{v}^*_\theta-\mathbf{v}_0))dx.
$$

In the above computation we used (4.2) and (4.9). □

The rest of this Section is devoted to the proof of the optimality conditions for (CP).

Theorem 4.4 *Let θ_0 be a solution for (CP). Then there exists the unique elements $(\mathrm{v}_0, \tau_0), (\mathrm{p}_0, q_0) \in Y_0 \times H^1(\Omega)$ which satisfy:*

$$(4.14)\quad \begin{cases} \nu((\mathrm{v}_0, \mathrm{z}))_0 + b_0(\mathrm{v}_0, \mathrm{v}_0, \mathrm{z}) = \langle \mathrm{f}, \mathrm{z}\rangle + \int_\Omega (\mathbf{B}\cdot \mathrm{z})\tau_0 dx \quad \forall \mathrm{z} \in Y_0, \\ \kappa((\tau_0, \eta)) + \alpha \int_\Gamma \tau_0 \eta ds + b(\mathrm{v}_0, \tau_0, \eta) \\ \qquad = \int_\Omega g\eta dx + \alpha\langle \theta_0, \eta\rangle_\Gamma \quad \forall \eta \in H^1(\Omega), \end{cases}$$

$$(4.15)\quad \begin{cases} \nu((\mathrm{p}_0, \mathrm{z}))_0 - b_0(\mathrm{v}_0, \mathrm{p}_0, \mathrm{z}) + b_0(\mathrm{z}, \mathrm{v}_0, \mathrm{p}_0) \\ \qquad = b(\mathrm{z}, q_0, \tau_0) + \int_\Omega (\nabla \times \mathrm{v}_0)\cdot(\nabla\times \mathrm{z})dx \quad \forall \mathrm{z} \in Y_0, \\ \kappa((q_0, \eta)) + \alpha \int_\Gamma q_0 \eta ds - b(\mathrm{v}_0, q_0, \eta) \\ \qquad = \int_\Omega (\mathbf{B}\cdot \mathrm{p}_0)\eta dx \quad \forall \eta \in H^1(\Omega), \end{cases}$$

$$(4.16)\quad \alpha\langle \theta - \theta_0, q_0\rangle_\Gamma \geq 0 \;\forall \theta \in K.$$

Proof. Let w be an element of Y_0 and $(\mathrm{p}_w, q_w) \in Y_0 \times H^1(\Omega)$ the unique solution of:

$$(4.17)\quad \begin{cases} \nu((\mathrm{p}_w, \mathrm{z}))_0 - b_0(\mathrm{v}_0, \mathrm{p}_w, \mathrm{z}) = b(\mathrm{z}, q_w, \tau_0) - b_0(\mathrm{z}, \mathrm{v}_0, \mathrm{w}) \\ \qquad + \int_\Omega (\nabla\times \mathrm{v}_0)\cdot(\nabla\times \mathrm{z})dx \quad \forall \mathrm{z} \in Y_0, \\ \kappa((q_w, \eta)) + \alpha \int_\Gamma q_w \eta ds - b(\mathrm{v}_0, q_w, \eta) \\ \qquad = \int_\Omega (\mathbf{B}\cdot \mathrm{w})\eta dx \quad \forall \eta \in H^1(\Omega). \end{cases}$$

We define $S : Y_0 \mapsto Y_0$, $S(\mathrm{w}) = \mathrm{p}_w$. It can be easily proved that

$$(4.18)\quad \|S(\mathrm{w}_1) - S(\mathrm{w}_2)\|_{(H_0^1(\Omega))^3} \leq C(\nu, \theta_0)\|\mathrm{w}_1 - \mathrm{w}_2\|_{(H_0^1(\Omega))^3} \quad \forall \mathrm{w}_1, \mathrm{w}_2 \in Y_0.$$

As in Theorem 2.2, we obtain the existence and uniqueness of a fixed point of S, denoted by p_0. Let q_0 be the unique solution of $(4.17)_2$ corresponding to $\mathrm{w} = \mathrm{p}_0$. Hence, (p_0, q_0) is the unique solution of problem (4.15).

In the sequel, we shall prove (4.16). From (4.1), (4.10), (4.13), (4.14) and (4.15) we obtain:

$$0 \leq \int_\Omega (\nabla\times \mathrm{v}_0)\cdot(\nabla\times(\mathrm{v}_\theta^* - \mathrm{v}_0))dx = \nu((\mathrm{p}_0, \mathrm{v}_\theta^* - \mathrm{v}_0))_0$$

$$-b_0(\mathrm{v}_0, \mathrm{p}_0, \mathrm{v}_\theta^* - \mathrm{v}_0) + b_0(\mathrm{v}_\theta^* - \mathrm{v}_0, \mathrm{v}_0, \mathrm{p}_0) - b(\mathrm{v}_\theta^* - \mathrm{v}_0, q_0, \tau_0)$$

$$= \int_\Omega (\mathbf{B}\cdot \mathrm{p}_0)(\tau_\theta^* - \tau_0)dx - b(\mathrm{v}_\theta^* - \mathrm{v}_0, q_0, \tau_0)$$

$$= \kappa((q_0, \tau_\theta^* - \tau_0)) + \alpha \int_\Gamma q_0(\tau_\theta^* - \tau_0)ds - b(\mathrm{v}_0, q_0, \tau_\theta^* - \tau_0)$$
$$-b(\mathrm{v}_\theta^* - \mathrm{v}_0, q_0, \tau_0) = \alpha\langle \theta - \theta_0, q_0\rangle_\Gamma \quad \forall \theta \in K.$$

Hence, the Theorem is proved. □

The last result of this paper is a consequence of the above Theorem.

Corollary 4.5 *Let θ_0 be a solution of (CP). Then there exists the unique elements $(\mathrm{v}_0, \tau_0), (\mathrm{p}_0, q_0) \in Y_0 \times H^1(\Omega)$ and there exists $\pi_0, \lambda_0 \in L^2(\Omega)$ such that:*

$$(4.19) \qquad \begin{cases} -\nu \bigtriangleup \mathrm{v}_0 + (\mathrm{v}_0 \cdot \nabla)\mathrm{v}_0 + \nabla \pi_0 = \mathrm{f} + \mathbf{B}\tau_0 & in \quad \Omega, \\ -\kappa \bigtriangleup \tau_0 + \mathrm{v}_0 \cdot \nabla \tau_0 = g & in \quad \Omega, \\ \kappa \dfrac{\partial \tau_0}{\partial n} + \alpha(\tau_0 - \theta_0) = 0 & on \quad \Gamma, \end{cases}$$

$$(4.20) \qquad \begin{cases} -\nu \bigtriangleup \mathrm{p}_0 - (\mathrm{v}_0 \cdot \nabla)\mathrm{p}_0 + (\nabla \mathrm{v}_0)\mathrm{p}_0 + \nabla \lambda_0 & \\ \qquad = \tau_0 \nabla q_0 + \nabla \times (\nabla \times \mathrm{v}_0) & in \quad \Omega, \\ -\kappa \bigtriangleup q_0 - \mathrm{v}_0 \cdot \nabla q_0 = \mathbf{B} \cdot \mathrm{p}_0 & in \quad \Omega, \\ \kappa \dfrac{\partial q_0}{\partial n} + \alpha q_0 = 0 & on \quad \Gamma \end{cases}$$

and:

$$\alpha\langle \theta - \theta_0, q_0 \rangle_\Gamma \geq 0 \qquad \forall \theta \in K.$$

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