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KURAN'S REGULARITY CRITERION AND LOCALIZATION IN EXCESSIVE STRUCTURES

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ABSTRACT

We give a relation between the thinness of a measurable fine closed subset of a Lusin measurable space endowed with a submarkovian resolvent of kernels and the quasi-boundedness for the excessive measures associated with the same resolvent. We extend a classical result of Ü. Kuran and two recent generalizations of N. Suzuki and P.J. Fitzsimmons–R.K. Getoor. We use essentially the localization procedure for both excessive functions and excessive measures.

1. *Introduction and main result*

The frame in which we develop the subject of this paper is the excessive structure given by a proper submarkovian resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on a Lusin measurable space $(X, \mathcal{X})$ such that the set of all $\mathcal{U}$-excessive functions is min-stable, contains the positive constant functions and generates $\mathcal{X}$.

We explore the connection between the thinness of a measurable fine closed subset of X and the quasi-boundedness for $\mathcal{U}$-excessive measures. (We refer to **[11]** for basic facts and notations concerning the excessive measures; see also **[3]**.)

Ü. Kuran has proved in **[12]** that if D is a bounded open subset of $\mathbb{R}^n$ then a boundary point x for D is regular (with respect to the Dirichlet problem) if and only if the restriction to D of the Green potential with pole at x is quasi-bounded (i.e. a sum of a sequence of bounded positive harmonic functions on D). N. Suzuki has extended in **[14]** this regularity criterion to the case of a harmonic space for which there exists an adjoint structure of harmonic space. Recently P.J. Fitzsimmons and R.K. Getoor have obtained in **[10]** a general form of Kuran's criterion in the case of a (Borel) right process.

Our main result is the following:

Theorem 1.1. *Let D be a measurable fine open subset of X with respect to $\mathcal{U}$, $m = h + \rho \circ U$ be an $\mathcal{U}$-excessive measure on X (where $h \in Har$ and $\rho \circ U \in Pot$) and let ν be a finite positive measure on X. Then the following assertions hold:*

(i) *If ν is carried by a subset of X which is m-polar and ρ-negligible and if $\nu \circ U|_D$ is $m|_D$-quasi-bounded then ν is carried by the set of all non-thinness points of $X \setminus D$.*

(ii) *If $\nu \circ U$ is absolutely continuous of m and ν is carried by the set of all non-thinness points of $X \setminus D$ then $\nu \circ U|_D$ is $m|_D$-quasi-bounded.*

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(In the above theorem "$\nu \circ U|_D$ is $m|_D$-quasi-bounded" means that it is a countable sum of measures which are dominated by $m|_D$ and are excessive with respect to the resolvent on D having $U - B^{X\setminus D}U$ as initial kernel.)
Particularly, if $\nu=\varepsilon_x$ then we get the result of P.J. Fitzsimmons and R.K. Getoor from [10].

We underline that our treatment is purely analytic. The proof of this theorem (presented in Section 4) is obtained applying the results from [3] concerning the quasi-boundedness for excessive measures and realizing a localization procedure on D of both $\mathcal{U}$- excessive functions (Section 2) and $\mathcal{U}$-excessive measures (Section 3).

The localization uses the fact that the balayage operator $B^{X\setminus D}$ and its dual appear as subordination operators (see [8] and [13]).

The localization for excessive functions extends a similar one obtained in [6] for the case when there exists a reference measure and in [13] for the frame given by a bounded Ray resolvent. For a probabilist reader the first section is omissible if he consider that the given resolvent $\mathcal{U}$ is associated with a Markov process on X. Also in this case one can refer to [9] instead of [3].

2. *Localization in excessive functions*

In all this paper $\mathcal{U} = (U_\alpha)_{\alpha>0}$ will be a submarkovian resolvent of kernels on $(X, \mathcal{X})$ as in the previous section. We denote by $\mathcal{E}_\mathcal{U}$ (resp. $\mathcal{E}^*_\mathcal{U}$) the set of all $\mathcal{X}$-measurable (resp. universally measurable) $\mathcal{U}$-excessive functions on X which are finite $\mathcal{U}$-a.s. If $A \in \mathcal{X}$ and $s \in \mathcal{E}^*_\mathcal{U}$ then (cf. [2]) the function

$$R^A s := \inf\{t \in \mathcal{E}_\mathcal{U} / \, t \geq s \text{ on } A\} = \inf\{t \in \mathcal{E}^*_\mathcal{U} / \, t \geq s \text{ on } A\}$$

is universally measurable. We put $B^A s := \widehat{R^A s}$ (i.e. the excessive regularization of the $\mathcal{U}$-supermedian function $R^A s$) and we have $B^A s = R^A s$ on $X \setminus A$. The map $s \mapsto B^A s$ is called *the balayage operation on A with respect to $\mathcal{U}$.*

If $\alpha > 0$ we denote by $\mathcal{U}_\alpha$ the resolvent on X given by $\mathcal{U}_\alpha := (U_{\alpha+\beta})_{\beta>0}$ and by ${}^\alpha B^A$ the balayage operation on A with respect to $\mathcal{U}_\alpha$.

Proposition 2.1.([1], Theorem 1.10) *If $A \in \mathcal{X}$ and $s \in \mathcal{E}_\mathcal{U}$ then*

$$B^A s = {}^\alpha B^A s + \alpha W({}^\alpha B^A s)$$

where W is the kernel on X given by $Wf := Uf - B^A Uf$.

Proof. We may suppose that X is semi-saturated (i.e. any $\mathcal{U}$-excessive measure dominated by a potential is also a potential; see [4]). If $A \in \mathcal{X}$ is fine open then for any $x \in X$ there exists a finite measure μ_x on X carried by A such that $B^A s(x) = \mu_x(s)$ for any $s \in \mathcal{E}_\mathcal{U}$.

We begin with the particular case when $A \in \mathcal{X}$ is fine open. In this case for any $s \in \mathcal{E}_\mathcal{U}$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of bounded $\mathcal{X}$-measurable functions on X, $f_n = 0$ on $X \setminus A$ such that $U_\alpha f_n \nearrow {}^\alpha B^A s$. It follows that

$${}^\alpha B^A s + \alpha W({}^\alpha B^A s) = \sup_{n\in\mathbb{N}}(U_\alpha f_n + \alpha W U_\alpha f_n) = \sup_{n\in\mathbb{N}}(Uf_n - B^A(Uf_n - U_\alpha f_n)).$$

Since $B^A U f_n = U f_n$ we deduce that ${}^\alpha B^A s + \alpha W({}^\alpha B^A s) = B^A({}^\alpha B^A s)$. Because ${}^\alpha B^A s = s$ on the fine closure of A we get $B^A({}^\alpha B^A s) = B^A s$ and consequently ${}^\alpha B^A s + \alpha W({}^\alpha B^A s) = B^A s$.

Let now $A \in \mathcal{X}$ be arbitrary and let $s \in \mathcal{E}_{\mathcal{U}}$ be bounded. For any $x \in X \setminus A$ there exists (cf. [2]) a decreasing sequence $(G_n)_{n\in\mathbb{N}}$ of measurable fine open subsets of X such that $A \subset G_n$ and

$$B^A s(x) = \inf_{n\in\mathbb{N}} B^{G_n} s(x) \quad , \quad {}^{\alpha}B^A s(x) = \inf_{n\in\mathbb{N}} {}^{\alpha}B^{G_n} s(x)$$

$$W({}^{\alpha}B^A s)(x) = W({}^{\alpha}R^A s)(x) = \inf_{n\in\mathbb{N}} W({}^{\alpha}B^{G_n} s)(x).$$

From the preceding considerations we get now the desired equality on $X \setminus A$. Since $W1 = 0$ on the base of A it results that $B^A s = {}^{\alpha}B^A s + \alpha W(B^A s)$ on X excepting an $\mathcal{U}$-negligible set and finally the above equality holds everywhere, completing the proof.

Remark. *For any $x \in X$ and any finite $\mathcal{U}$-excessive function s on X the function $\alpha \mapsto {}^{\alpha}B^A s(x)$ is completely monotone on $[0, \infty)$.*
The assertion follows from the equality ${}^{\alpha}B^A s = (I - \alpha W_\alpha) B^A s$.

Let now fix a fine open $\mathcal{X}$-measurable subset D of X. For any $\alpha \geq 0$ we denote by W_α the kernel on $(X, \mathcal{X}^*)$ given by $W_\alpha f := U_\alpha f - {}^{\alpha}B^{X\setminus D} U_\alpha f$, for any positive measurable function f on X such that Uf is bounded, where as usual we have written U_o (resp. W_o) instead of U (resp. W) and $\mathcal{X}^*$ is the universal completion of $\mathcal{X}$.

Proposition 2.2. *The family $\mathcal{W} := (W_\alpha)_{\alpha>0}$ is a submarkovian resolvent on $(X, \mathcal{X}^*)$ having W as initial kernel such that $W_\alpha \leq U_\alpha$ for all $\alpha > 0$.*

Proof. If f is a positive measurable function on X such that Uf is bounded then from Proposition 2.1 we have $(I + \alpha W)({}^{\alpha}B^{X\setminus D} U_\alpha f) = B^{X\setminus D} U_\alpha f$ and therefore $(W_\alpha + \alpha W W_\alpha) f = (I + \alpha W)(U_\alpha f - {}^{\alpha}B^{X\setminus D} U_\alpha f) = (I + \alpha W) U_\alpha f - B^{X\setminus D} U_\alpha f = U_\alpha f + \alpha (U U_\alpha f - B^{X\setminus D} U U_\alpha f) - B^{X\setminus D} U_\alpha f = Uf - B^{X\setminus D} Uf = Wf$.

Remark. *For any finite functions $s, t, u, v \in \mathcal{E}^*_{\mathcal{U}}$ such that $u \leq v$ we have*

$$s \wedge (B^{X\setminus D} s + t - B^{X\setminus D} t + B^{X\setminus D} v - B^{X\setminus D} u) \in \mathcal{E}^*_{\mathcal{U}}.$$

Particularly if $s, t \in \mathcal{E}^*_{\mathcal{U}}$ are such that $s - B^{X\setminus D} s \leq t - B^{X\setminus D} t$ then $s - B^{X\setminus D} s + B^{X\setminus D} t \in \mathcal{E}^*_{\mathcal{U}}$.
The assertion follows from Proposition 2.2 and from [**13**].

Definition. We denote by $\widetilde{D}$ the set of all points $x \in X$ such that $X \setminus D$ is thin at x that is

$$\widetilde{D} := \{x \in X / \text{ there exists } s \in \mathcal{E}_{\mathcal{U}} \text{ with } B^{X\setminus D} s(x) < s(x)\}.$$

Remark. (a) *Since D is fine open we have $D \subset \widetilde{D}$.*
(b) *Because there exists a measurable function h on X, $0 < h \leq 1$, such that Uh is bounded it follows, using Hunt's approximation theorem, that*

$$\widetilde{D} = \{x \in X / B^{X\setminus D} Uh(x) < Uh(x)\}$$

and therefore the set $\widetilde{D}$ is universally measurable and fine open.

From the definition of the kernel W_α it follows that if Y is an universally measurable subset of X such that $D \subset Y$ then the map $g \mapsto W_\alpha(\overline{g})|_Y$ defined for any universally measurable function g on Y (where $\overline{g}$ is a measurable extension of g on X) is a kernel on $(Y, \mathcal{Y}^*)$ denoted by W_α^Y. Moreover the family $\mathcal{W}^Y := (W_\alpha^Y)_{\alpha>0}$ is a submarkovian resolvent on $(Y, \mathcal{Y}^*)$ (cf. Propositon 2.2) such that a positive universally measurable function on Y will be $\mathcal{W}^Y$-excessive if and only if it can be extended (uniquely) to a $\mathcal{W}$-excessive function on X. We say simply "$\mathcal{W}$-excessive on Y" instead of "$\mathcal{W}^Y$-excessive". We apply the above considerations especially to the set D or $\widetilde{D}$. The set $\widetilde{D}$ is distinguished with the following property: $\widetilde{D}$ is the greatest universally measurable subset Y of X such that there exists a $\mathcal{W}$-excessive function on Y which is strictly positive.

The next lemma was considered essentially in [**13**]; see also [7].

Lemma 2.3.(Mokobodzki) *Let C be a cone of potentials such that for any sequence $(s_n)_{n\in\mathbb{N}}$ in C which is increasing and dominated with respect to the specific order there exists its specific least upper bound and let $P : C \longrightarrow C$ be a map which is additive, increasing and contractive (that is $Ps \leq s$ for all $s \in C$). Then for any $s \in C$ there exists $s' \in C$, $s' \prec s$, with $s' - Ps' = s - Ps$ and such that $s' \leq t$ for any $t \in C$ for which $s - Ps \leq t - Pt$.* (We have denoted by $\prec$ the specific order on C.)

Proof. For any $s \in C$ we define inductively the sequences $(s_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ in C such that $s_o = r_o = s$,

$$s_{n+1} := R(r_n - Pr_n) \quad \text{and} \quad r_{n+1} := r_n - s_{n+1}$$

where R is the reduit operator with respect to C. We put $s' := \sum_{n\geq 1} s_n$ and $r := \bigwedge_{n\in\mathbb{N}} r_n$. Since $(r_n)_{n\in\mathbb{N}}$ is specifically decreasing and since $r_{n+1} \leq Pr_n$ we get $r = \curlywedge\{r_n / n \in \mathbb{N}\}$ and

$$Pr \leq r \leq \textstyle\bigwedge_{n\in\mathbb{N}} Pr_n = P(\bigwedge_{n\in\mathbb{N}} r_n) = Pr.$$

Hence $s' \prec s$ and $s' - Ps' = s - Ps$.

We show now that if $t, u \in C$, $t \prec R(u - Pu)$ and $t = Pt$ then $t = 0$. Indeed, from $t \prec R((u - nt) - P(u - nt))$ we get inductively that $nt \prec u$. Hence $t = 0$. Further we remark that if $t \in C$, $t \prec s'$ and $Pt = t$ then $t = 0$. Indeed, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in C such that $t = \sum_{n\in\mathbb{N}} t_n$ and $t_n \prec s_n$. Since $Pt_n = t_n$ and $s_n = R(r_{n-1} - Pr_{n-1})$ we deduce from the preceding considerations that $t_n = 0$ for all $n \in \mathbb{N}$ and therefore $t = 0$.

Let $t \in C$ be such that $s' - Ps' \leq t - Pt$. From

$$s' - t \leq Ps' - Pt \leq P(R(s' - t)) \leq R(s' - t)$$

it follows that $R(s' - t) = P(R(s' - t))$ and since $R(s' - t) \prec s'$ we get $R(s' - t) = 0$ and we conclude that $s' \leq t$.

Theorem 2.4. *Let $\mathcal{W} = (W_\alpha)_{\alpha>0}$ be the submarkovian resolvent on $(X, \mathcal{X}^*)$ having $W = U - B^{X\setminus D}U$ as initial kernel. Then the following assertions hold:*

(a) *For any measurable subset M of $\widetilde{D}$ such that $W(\chi_M) = 0$ we have $U(\chi_M) = 0$. If f is a positive universally measurable function on $\widetilde{D}$ such that $Uf < \infty$ and $s \in \mathcal{E}_{\mathcal{U}}^*$, $s < \infty$, is such that $Uf - B^{X\setminus D}Uf \leq s - B^{X\setminus D}s$ then $Uf \leq s$.*

(b) *Let f be a positive universally measurable function on $\widetilde{D}$ which is fine continuous with respect to $\mathcal{E}_{\mathcal{U}}$. Then f will be $\mathcal{W}$-excessive on $\widetilde{D}$ if and only if*

f *is* $\mathcal{W}$*-supermedian.*

(c) *The set of all functions on* $\widetilde{D}$ *of the form* $(s - B^{X\setminus D}s)|_{\widetilde{D}}$ *where* s *is* $\mathcal{U}$*-excessive and finite, is a solid and increasingly dense convex subcone of the set of all* $\mathcal{W}$*-excessive functions on* $\widetilde{D}$.

Proof. (a) Let M be a measurable subset of $\widetilde{D}$ with $W(\chi_M) = 0$. To show that $U(\chi_M) = 0$ we may suppose that M is a Ray compact subset of X. We consider the convex cone $\mathcal{T} := \{\bar{p} - \alpha\overline{T}(\chi_M) / \ p \in \mathcal{R}, \alpha \in \mathbb{R}_+\}$ where T is the bounded kernel on X of the form $Tg := U(hg)$ (h is a measurable function on X, $0 < h \leq 1$ and Uh is bounded), $\mathcal{R}$ is a Ray cone associated with the resolvent generated by T, $\overline{T}$ is the extension of T to the Ray compactification $\overline{X}$ of X associated with $\mathcal{R}$ and for any $p \in \mathcal{R}$, $\bar{p}$ is the continuous extension of p to $\overline{X}$. Since M is a compact subset of X with respect to the Ray topology generated by $\mathcal{R}$, it follows that $\overline{T}(\chi_M)$ is an upper semi-continuous function on $\overline{X}$. Hence $\mathcal{T}$ is a convex cone of lower semi-continuous functions on $\overline{X}$ which separates the points of $\overline{X}$. From

$$p \in \mathcal{R},\ \bar{p} - \alpha\overline{T}(\chi_M) \geq 0 \text{ on } M \Rightarrow \bar{p} - \alpha\overline{T}(\chi_M) \geq 0 \text{ on } \overline{X}$$

it follows that M is a closed boundary set with respect to $\mathcal{T}$ and therefore for any $x \in \overline{X}$ there exists a positive measure μ_x on $\overline{X}$ carried by M and such that $\mu_x \leq_{\mathcal{T}} \varepsilon_x$. Suppose now that there exists $x \in X$ for which $U(\chi_M)(x) \neq 0$. We deduce that $T(\chi_M) \neq 0$. In this case the Choquet boundary of $\overline{X}$ with respect to $\mathcal{T}$ is not empty (see [5]). Let $x \in \overline{X}$ be a point which belongs to this Choquet boundary. From the above considerations we have $\mu_x = \varepsilon_x$. Hence $x \in M$. On the other hand there exists a positive measure ν_x on $\overline{X}$ such that $\nu_x(\bar{p}) = B^{X\setminus D}p(x)$. We have by hypothesis $W(\chi_M) = 0$ and therefore

$$\overline{T}(\chi_M)(x) = B^{X\setminus D}T(\chi_M)(x) = \nu_x(\overline{T}(\chi_M)).$$

Since $\nu_x(\bar{p}) \leq \bar{p}(x)$ for any $p \in \mathcal{R}$ we get $\nu_x \leq_{\mathcal{T}} \varepsilon_x$ and therefore $\nu_x = \varepsilon_x$. This last equality contradicts the fact that $x \in M \subset \widetilde{D}$.

Let now f be a positive universally measurable function on $\widetilde{D}$ such that $Uf < \infty$. From Lemma 2.3 applied to $\mathcal{E}^*_{\mathcal{U}}$ and to the operator $B^{X\setminus D}$ instead of P we deduce that there exists $s_o \in \mathcal{E}^*_{\mathcal{U}}$, $s_o \prec Uf$ such that $s_o - B^{X\setminus D}s_o = Uf - B^{X\setminus D}Uf$ and such that $s_o \leq s$ for any finite function $s \in \mathcal{E}^*_{\mathcal{U}}$ for which $Uf - B^{X\setminus D}Uf \leq s - B^{X\setminus D}s$. From $s_o \prec Uf$ we deduce that there exists an universally measurable function g on $\widetilde{D}$ such that $g \leq f$ and $s_o = Ug$. Since

$$Uf - B^{X\setminus D}Uf = s_0 - B^{X\setminus D}s_0 = Ug - B^{X\setminus D}Ug$$

we get $W(f - g) = 0$ and therefore $Uf = Ug = s_o$.

(b), (c) Suppose that f is $\mathcal{W}$-supermedian and there exists a finite function $u \in \mathcal{E}^*_{\mathcal{U}}$ for which $f \leq u - B^{X\setminus D}u$. From Hunt's approximation theorem there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of positive bounded universally measurable functions on $\widetilde{D}$ such that $Uf_n < \infty$, $(Wf_n)_{n\in\mathbb{N}}$ is increasing and

$$\sup_{n\in\mathbb{N}} Wf_n = \sup_{n\in\mathbb{N}} nW_nf =: \hat{f}.$$

Since

$$Uf_n - B^{X\setminus D}Uf_n = Wf_n \leq f \leq u - B^{X\setminus D}u$$

it follows from assertion (a) that $(Uf_n)_{n\in\mathbb{N}}$ is increasing and $Uf_n \leq u$ for all $n \in \mathbb{N}$. If we put $s := \sup_{n\in\mathbb{N}} Uf_n$ then $s \in \mathcal{E}^*_{\mathcal{U}}$, $s \leq u$ and $\hat{f} = s - B^{X\setminus D}s$ on $\widetilde{D}$. If f

is $\mathcal{W}$-excessive on $\widetilde{D}$ then we get $f = \hat{f} = s - B^{X\setminus D}s$ on $\widetilde{D}$. Generally we have $f = s - B^{X\setminus D}s$ $\mathcal{W}$-a.s. on $\widetilde{D}$ and therefore, from assertion (a) we deduce that

$$f = s - B^{X\setminus D}s \quad \mathcal{U}\text{-a.s. on } \widetilde{D}.$$

If f is $\mathcal{W}$-supermedian and fine continous (with respect to $\mathcal{E}_{\mathcal{U}}$) then we get $f = s - B^{X\setminus D}s$ on $\widetilde{D}$ and therefore $f = \hat{f}$ on $\widetilde{D}$ that is f is $\mathcal{W}$-excessive on $\widetilde{D}$.

Assertion (c) follows from the above considerations since for any finite function $s \in \mathcal{E}^*_{\mathcal{U}}$, the function $s - B^{X\setminus D}s$ is obviously fine continuous and $\mathcal{W}$-supermedian on $\widetilde{D}$ (we have $s - B^{X\setminus D}s = \lim_{n\to\infty}(Uf_n - B^{X\setminus D}Uf_n)$, where $Uf_n \nearrow s$).

To complete the proof of (b) we remark that $f = \sup_{n\in\mathbb{N}} \inf(f, nWh)$.

Remark. *The above theorem was proved in* [13] *for the case when $\mathcal{U}$ is a bounded Ray resolvent and $\mathcal{W}$ is the subordinated resolvent associated to a subordination operator P.*

Corollary 2.5. *Let D and $\mathcal{W}$ be as in Theorem 2.4. Then the following assertions hold:*

(a) *The function $s|_{\widetilde{D}}$ is $\mathcal{W}$-excessive on $\widetilde{D}$ for any $\mathcal{U}$-excessive function s.*

(b) *Any $\mathcal{W}$-excessive function on $\widetilde{D}$ is fine continuous (with respect to $\mathcal{E}_{\mathcal{U}}$).*

(c) *The set of all $\mathcal{W}$-excessive functions on $\widetilde{D}$ is min-stable. Particularly for any finite functions $s, t \in \mathcal{E}^*_{\mathcal{U}}$ there exist $u, v \in \mathcal{E}^*_{\mathcal{U}}$, $u, v < \infty$, such that*

$$\inf(s - B^{X\setminus D}s, t - B^{X\setminus D}t) = u - B^{X\setminus D}u\,,$$

$$\inf(s - B^{X\setminus D}s, t) = v - B^{X\setminus D}v.$$

(d) *For any $s, t \in \mathcal{E}^*_{\mathcal{U}}$, $t < \infty$ and $t \le s$, the function $(B^{X\setminus D}s - B^{X\setminus D}t)|_{\widetilde{D}}$ is $\mathcal{W}$-excessive on $\widetilde{D}$.*

Remark. *For any finite function $s \in \mathcal{E}^*_{\mathcal{U}}$ and any $x \in \widetilde{D}$ we have* $\inf_{\alpha>0} {}^{\alpha}B^{X\setminus D}s(x) = 0$.

The assertion follows from the fact that $B^{X\setminus D}s|_{\widetilde{D}}$ is $\mathcal{W}$-excessive on $\widetilde{D}$ and from the equality ${}^{\alpha}B^{X\setminus D}s = (I - \alpha W_\alpha)(B^{X\setminus D}s)$; see Proposition 2.1.

The following result gives a "polarity property" of the set $\widetilde{D} \setminus D$ with respect to the potential theory associated with $\mathcal{E}^*_{\mathcal{W}}$.

Theorem 2.6. *For any finite positive measure μ on D and any positive universally measurable function h on $\widetilde{D}$, $0 < h < 1$, such that Uh is bounded we have*

$$\inf\{\mu(t) / t \in \mathcal{E}^*_{\mathcal{W}}\,,\ Wh \le t \text{ on } \widetilde{D} \setminus D\} = 0.$$

Proof. Since μ is carried by D there exists a decreasing sequence $(G_n)_{n\in\mathbb{N}}$ of fine open subsets of X such that $X \setminus D \subset G_n$ and such that

$$\mu(B^{X\setminus D}Uh) = \inf_{n\in\mathbb{N}} \mu(B^{G_n}Uh).$$

We have $B^{G_n}Uh - B^{X\setminus D}Uh = B^{G_n}Uh - B^{X\setminus D}B^{G_n}Uh$ and therefore, by Theorem 2.4,

$$B^{G_n}Uh - B^{X\setminus D}Uh \in \mathcal{E}^*_{\mathcal{W}}.$$

From $B^{G_n}Uh - B^{X\setminus D}Uh \ge Wh$ on $\widetilde{D} \setminus D$ we conclude that

$$\inf\{\mu(t) / t \in \mathcal{E}^*_{\mathcal{W}}\,,\ Wh \le t \text{ on } \widetilde{D} \setminus D\} \le \inf_{n\in\mathbb{N}} \mu(B^{G_n}Uh - B^{X\setminus D}Uh) = 0.$$

3. *Localization in excessive measures*

For the resolvent $\mathcal{U}$ on X we denote by $Exc_{\mathcal{U}}$ the convex cone of all $\mathcal{U}$-excessive measures on X that is the set of all σ-finite measures m on X for which $m(\alpha U_\alpha) \le m$

for all $\alpha > 0$. If $A \in \mathcal{X}$ we denote by $(B^A)^*$ the operator on $Exc_{\mathcal{U}}$ defined by

$$L((B^A)^*\xi, s) = L(\xi, B^A s)$$

where $s \in \mathcal{E}^*_{\mathcal{U}}$, $\xi \in Exc_{\mathcal{U}}$ and $L : Exc_{\mathcal{U}} \times \mathcal{E}^*_{\mathcal{U}} \to \overline{\mathbb{R}}_+$ is the energy functional (see e.g. [11]). It is easy to see that if $\xi \in Exc_{\mathcal{U}}$ then $(B^A)^*\xi \le \xi$ and $\xi|_D, (\xi-(B^{X\setminus D})^*\xi)|_D$ are $\mathcal{W}$-excessive measures on D (that is $\mathcal{W}^D$-excessive measures; see Section 2 and note that $Exc_{\mathcal{W}^D} \equiv Exc_{\widetilde{\mathcal{W}^D}} \equiv Exc_{\mathcal{W}}$) where recall that D is a fine open $\mathcal{X}$-measurable subset of X.

Theorem 3.1. *For any $\xi, \eta \in Exc_{\mathcal{U}}$ and any $l \in (Exc_{\mathcal{U}} - Exc_{\mathcal{U}})_+$ we have*

$$\xi \wedge ((B^{X\setminus D})^*\xi + \eta - (B^{X\setminus D})^*\eta + (B^{X\setminus D})^* l) \in Exc_{\mathcal{U}}.$$

Particularly the set $\{\eta - (B^{X\setminus D})^*\eta / \eta \in Exc_{\mathcal{U}}\}$ is a solide subcone of $Exc_{\mathcal{W}}$ and for any $l \in (Exc_{\mathcal{U}} - Exc_{\mathcal{U}})_+$ we have $(B^{X\setminus D})^* l|_D \in Exc_{\mathcal{W}}$. Moreover for any $\xi \in Exc_{\mathcal{U}}$ there exists $\xi' \in Exc_{\mathcal{U}}$ such that $\xi - (B^{X\setminus D})^*\xi = \xi' - (B^{X\setminus D})^*\xi'$ and if for $\eta \in Exc_{\mathcal{U}}$ we have $\xi' - (B^{X\setminus D})^*\xi' \le \eta - (B^{X\setminus D})^*\eta$ then $\xi' \le \eta$.

Proof. First we suppose that there exists a $\mathcal{X}$-measurable fine open set $G \subset X$ such that $X\setminus D$ coincides with the fine closure of G. In this case we have $B^{X\setminus D} = B^G$ and $(B^{X\setminus D})^*$ is a balayage on the H-cone $Exc_{\mathcal{U}}$. Then the assertion follows from [8].

Suppose now that D is general and that ξ, η, l are of the form $\xi = \mu_1 \circ U$, $\eta = \mu_2 \circ U$, $l = \mu_3 \circ U - \mu_4 \circ U$, where μ_i are finite measures on X which does not charge the $\mathcal{U}$-negligible subsets of X. In this case if $s \in \mathcal{E}_{\mathcal{U}}$ is bounded then we have (cf. [2])

$$\mu_i(B^{X\setminus D}s) = \inf\{\mu_i(B^G s) / G \in \mathcal{X}, G \text{ fine open}, X \setminus D \subset G\}, \quad i = \overline{1,4}$$

and therefore $(B^{X\setminus D})^*(\mu_i \circ U) = \bigwedge\{(B^G)^*(\mu_i \circ U) / G \in \mathcal{X}, G \text{ fine open}, X \setminus D \subset G\}$. On the other hand if for any measurable fine open set G with $X \setminus D \subset G$ we put

$$\theta_G := \xi \wedge ((B^G)^*\xi + \eta - (B^G)^*\eta + (B^G)^* l),$$
$$\theta := \xi \wedge ((B^{X\setminus D})^*\xi + \eta - (B^{X\setminus D})^*\eta + (B^{X\setminus D})^* l)$$

then we have $\theta_G \in Exc_{\mathcal{U}}$,

$$\theta_G + (B^G)^*\eta + (B^G)^*(\mu_4 \circ U)$$
$$= (\xi + (B^G)^*\eta + (B^G)^*(\mu_4 \circ U)) \wedge ((B^G)^*\xi + \eta + (B^G)^*(\mu_3 \circ U)),$$
$$\theta + (B^{X\setminus D})^*\eta + (B^{X\setminus D})^*(\mu_4 \circ U)$$
$$= (\xi + (B^{X\setminus D})^*\eta + (B^{X\setminus D})^*(\mu_4 \circ U)) \wedge ((B^{X\setminus D})^*\xi + \eta + (B^{X\setminus D})^*(\mu_3 \circ U)).$$

Using the above formula we deduce that the families of positive measures

$$(\xi + (B^G)^*\eta + (B^G)^*(\mu_4 \circ U))_G\ ,\ ((B^G)^*\xi + \eta + (B^G)^*(\mu_3 \circ U))_G\ ,$$
$$((B^G)^*\eta)_G\ ,\ ((B^G)^*(\mu_4 \circ U))_G$$

are decreasing respectively to

$$\xi + (B^{X\setminus D})^*\eta + (B^{X\setminus D})^*(\mu_4 \circ U)\ ,\ (B^{X\setminus D})^*\xi + \eta + (B^{X\setminus D})^*(\mu_3 \circ U)\ ,$$
$$(B^{X\setminus D})^*\eta\ ,\ (B^{X\setminus D})^*(\mu_4 \circ U).$$

Hence we get $\lim_G \theta_G(f) = \theta(f)$ for any positive bounded measurable function on X and therefore

$$\theta(\alpha U_\alpha f) = \lim_G \theta_G(\alpha U_\alpha f) \le \lim_G \theta_G(f) = \theta(f).$$

We conclude that $\theta \in Exc_{\mathcal{U}}$.

Let now $D, \xi, \eta, l = \lambda^1 - \lambda^2$ $(\xi, \eta, \lambda^1, \lambda^2 \in Exc_{\mathcal{U}})$ be general. We take sequences $(\mu^i_n)_{n\in\mathbb{N}}$, $i = \overline{1,4}$ of bounded measures on X which does not charge the $\mathcal{U}$-negligible subsets of X such that $\xi_n := \mu^1_n \circ U \nearrow \xi$, $\eta_n := \mu^2_n \circ U \nearrow \eta$, $\lambda^1_n := \mu^3_n \circ U \nearrow \lambda^1$,

$\lambda_n^2 := \mu_n^4 \circ U \nearrow \lambda^2$ and such that $\lambda_n^1 \geq \lambda_n^2$. From the preceding considerations we get

$$\xi_n \wedge ((B^{X\setminus D})^*\xi_n + \eta_n - (B^{X\setminus D})^*\eta_n + (B^{X\setminus D})^*(\lambda_n^1 - \lambda_n^2)) \in Exc_{\mathcal{U}}.$$

Leting $n \to \infty$ we conclude that $\xi \wedge ((B^{X\setminus D})^*\xi + \eta - (B^{X\setminus D})^*\eta + (B^{X\setminus D})^*(\lambda^1 - \lambda^2)) \in Exc_{\mathcal{U}}$. We deduce now that the map $(B^{X\setminus D})^*$ is a localizable dilation operator on the H-cone $Exc_{\mathcal{U}}$. Hence from [8] it follows that the set

$$F := \{\eta - (B^{X\setminus D})^*\eta / \eta \in Exc_{\mathcal{U}}\}$$

is an H-cone (with respect to the natural order relation between measures on D) such that for any $\xi_1, \xi_2 \in Exc_{\mathcal{U}}$, $\xi_2 \leq \xi_1$ and $\varphi \in F$ we have

$$\xi_1 \wedge \varphi \in F \;,\; (B^{X\setminus D})^*(\xi_1 - \xi_2) \wedge \varphi \in F.$$

Since F is increasingly dense in $Exc_{\mathcal{W}}$ we get also that F is solid in $Exc_{\mathcal{W}}$. From $\xi|_D = \bigvee\{\xi \wedge \varphi / \varphi \in F\}$ for all $\xi \in Exc_{\mathcal{U}}$, we deduce that

$$(B^{X\setminus D})^*(\xi_1 - \xi_2)|_D = \bigvee\{(B^{X\setminus D})^*(\xi_1 - \xi_2) \wedge \varphi / \varphi \in F\} \in Exc_{\mathcal{W}}.$$

The last assertion from theorem follows by Lemma 2.3.

Corollary 3.2. *If X is semi-saturated with respect to $\mathcal{U}$ then D is semi-saturated with respect to $\mathcal{W}^D$.* (X is semi-saturated means that any $\mathcal{U}$-excessive measure dominated by a potential is a potential.)

Proof. Let μ be a finite measure on D and $\theta \in Exc_{\mathcal{W}}$ be such that $\theta \leq \mu \circ W$. From Theorem 3.1 it follows that the set $\{\eta - (B^{X\setminus D})^*\eta / \eta \in Exc_{\mathcal{U}}\}$ is solid in $Exc_{\mathcal{W}}$ and therefore there exists $\xi \in Exc_{\mathcal{U}}$ with $\theta = \xi - (B^{X\setminus D})^*\xi$. Again from Theorem 3.1 we may suppose that ξ has the following property:

$$(\eta \in Exc_{\mathcal{U}} \text{ and } \xi - (B^{X\setminus D})^*\xi \leq \eta - (B^{X\setminus D})^*\eta) \Rightarrow \xi \leq \eta.$$

Because X is semi-saturated with respect to $\mathcal{U}$ there exists a measure ν on X such that $\xi = \nu \circ U$. From

$$\theta = \xi - (B^{X\setminus D})^*\xi = \nu \circ W = \nu|_{\widetilde{D}} \circ W$$

we deduce that $\xi \leq \nu|_{\widetilde{D}} \circ U$ and further $\xi = \nu|_{\widetilde{D}} \circ U$. To finish the proof it will be sufficient to show that $\nu|_{\widetilde{D}\setminus D} = 0$. Indeed, for any $t \in \mathcal{E}^*_{\mathcal{W}}$ we have

$$\nu|_{\widetilde{D}\setminus D}(t) = {}^{\mathcal{W}}L(\nu|_{\widetilde{D}\setminus D} \circ W, t) \leq {}^{\mathcal{W}}L(\mu \circ W, t) = \mu(t)$$

where ${}^{\mathcal{W}}L$ denotes the energy functional associated with $\mathcal{W}$. From Theorem 2.6 we get now

$$\nu|_{\widetilde{D}\setminus D}(1) \leq \inf\{\mu(t) / t \in \mathcal{E}^*_{\mathcal{W}} \;,\; 1 \leq t \text{ on } \widetilde{D} \setminus D\} = 0$$

and we conclude that $\nu|_{\widetilde{D}\setminus D} = 0$.

4. *Proof of main result*

Proof of Theorem 1.1. (i) We may suppose that ν is carried by $\widetilde{D}$. From $\nu \circ U = \nu \circ W + (B^{X\setminus D})^*(\nu \circ U)$, $\nu \circ W \leq \nu \circ U|_D$ it follows that $\nu \circ W$ is $m|_D$-quasi-bounded. Hence, using also Corollary 2.5 (c), there exists a sequence $(\nu_n)_{n\in\mathbb{N}}$ of positive measures on $\widetilde{D}$ such that

$$\nu = \textstyle\sum_{n\in\mathbb{N}} \nu_n$$

and such that $\nu_n \circ W \leq m|_D$ for all $n \in \mathbb{N}$. Since $R(\nu_n \circ W) \prec \nu_n \circ U$, where R is the reduit operator in $Exc_{\mathcal{U}}$, it follows that $R(\nu_n \circ W) = \nu_n' \circ U$, ν_n' being a positive measure on $\widetilde{D}$ with $\nu_n' \leq \nu_n$. Also we have $R(\nu_n \circ W) \leq m$ and therefore

$R(\nu_n \circ W)$ is m-quasi-bounded. On the other hand since ν'_n is carried by a subset of X which is m-polar and ρ-negligible we deduce by Corollary 3.4 in [**3**] (see also [9]) that $\nu_n \circ U$ is orthogonal on the m-quasi-bounded $\mathcal{W}$-excessive measures and consequently $\nu'_n \circ U = 0$ for all $n \in \mathbb{N}$. Hence $R(\nu_n \circ W) = 0$, $\nu_n \circ W = 0$ and therefore $\nu \circ W = 0$, $\nu = 0$.

(ii) Suppose now that $\nu \circ U$ is absolutely continuous with respect to m and ν is carried by $X \setminus \widetilde{D}$ or equivalently $\nu \circ W = 0$. It is easy to see that there exists an increasing sequence $(\nu_n \circ U)_{n\in\mathbb{N}}$ of m-quasi-bounded $\mathcal{U}$-excessive measures such that $\sup_{n\in\mathbb{N}} \nu_n \circ U = \nu \circ U$. From Theorem 3.1 it follows that the sequence $((B^{X\setminus D})^*(\nu_n \circ U)|_D)_{n\in\mathbb{N}}$ is specifically increasing in $Exc_{\mathcal{W}}$ to $(B^{X\setminus D})^*(\nu \circ U)|_D$. Since $\nu \circ W = 0$ we get $(B^{X\setminus D})^*(\nu \circ U) = \nu \circ U$ and therefore

$$\nu \circ U|_D = (B^{X\setminus D})^*(\nu \circ U)|_D$$
$$= (B^{X\setminus D})^*(\nu_o \circ U)|D + \textstyle\sum_{n\in\mathbb{N}}[(B^{X\setminus D})^*(\nu_{n+1} \circ U)|_D - (B^{X\setminus D})^*(\nu_n \circ U)|_D].$$

From the above considerations we conclude that $\nu \circ U|_D$ is $m|_D$-quasi-bounded, completing the proof.

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