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On the rate of relative Veronese submodules

Annetta Aramova, Serban Bărcănescu and Jürgen Herzog

Introduction

In this paper we study Veronese submodules of graded modules defined over a homogeneous k-algebra. Let k be a field. A graded ring $R = \bigoplus_{i\geq 0} R_i$ with $R_0 = k$ is called homogeneous if R is finitely generated over k by its elements of degree 1, and for any integer $d \geq 1$, the subring $R^{(d)} = \bigoplus_{i\geq 0} R_{id}$ of R is called the d-th *Veronese subring of* R. Observe that $R^{(d)}$ is again a homogeneous k-algebra with graduation $(R^{(d)})_i = R_{id}$ for all $i \geq 0$.

In their article [3] Bărcănescu and Manolache proved that all Veronese subrings of a polynomial ring are Koszul algebras, where for a homogeneous k-algebra R this means that the residue field k of R has a linear resolution, that is, has a free resolution as R-module whose maps are given by matrices of linear forms. This result was later generalized by Backelin [1]: For any homogeneous k-algebra R he introduced a numerical invariant, called the *rate of* R which measures how much R deviates from being Koszul ($\operatorname{rate} R \geq 1$, with equality if and only if R is Koszul), and showed that

$$\operatorname{rate} R^{(d)} \leq \lceil \operatorname{rate} R/d \rceil .$$

Here $\lceil a \rceil$ denote the upper integer part of a real number a, that is, the smallest integer $\geq a$.

This result implies in particular that the d-th Veronese subring is Koszul when $d \geq \operatorname{rate} R$. In a recent paper [5] Eisenbud, Reeves and Totaro gave a bound c in terms of the regularity of the defining ideal of R such that $R^{(d)}$ is Koszul for all $d \geq c$; see Section 2 where we will use their result in a particular case.

The purpose of this paper is to extend these results partly to relative Veronese submodules. Let M be a graded R-module, and $d > 0$ an integer. For $j = 0, \ldots, d-1$

we let $M_j^{(d)}$ be the graded $R^{(d)}$-module with homogeneous components $(M_j^{(d)})_i = M_{id+j}$. Note that M as an $R^{(d)}$-module decomposes into the direct sum $\bigoplus_{j=0}^{d-1} M_j^{(d)}$. The modules $M_j^{(d)}$ are called the *relative Veronese submodules* of M. One may ask whether for a graded R-module M there is an integer c such that for all $d \geq c$ the d-th Veronese submodules of M all have a linear resolution, or equivalently, M as an $R^{(d)}$-module has a linear resolution. This is indeed the case, and as for the residue class field it can be quantatively controlled.

Let M be a finitely generated graded R-module. Then $\mathrm{Tor}_i(k, M)$ is a finitely generated graded k-vector space, and we set

$$t_i(M) = \sup\{j : \mathrm{Tor}_i(k, M)_j \neq 0\}.$$

Note that $t_i(M)$ is the highest shift in the i-th position of the minimal free homogeneous resolution of M. We define

$$\mathrm{rate}_R\, M = \sup_{i \geq 1}\{t_i(M)/i\}.$$

Recall that Backelin's rate which we henceforth denote by 'Rate' is defined as follows:

$$\mathrm{Rate}\, R = \sup_{i \geq 2}\{(t_i(k) - 1)/(i - 1)\}.$$

A comparison with our rate shows that $\mathrm{Rate}\, R = \mathrm{rate}_R\, \mathfrak{m}(1)$ where $\mathfrak{m}(1)$ is the graded maximal ideal of R, shifted by one.

The main result of this paper is the following

Theorem. *Let R be a homogeneous k-algebra. Then there exists a constant c, only depending on R, such that for all finitely generated R-modules M with generators of degree 0, one has*

$$\mathrm{rate}_{R^{(d)}}\, M \leq \lceil \mathrm{rate}_R\, M/d \rceil \quad \textit{for all} \quad d \geq c.$$

Moreover, if R is the polynomial ring then $c = 1$.

Unfortunately we do not know whether $c = 1$ for all homogeneous k-algebras, as we expect.

The theorem implies that the d-th Veronese submodules of any finitely generated graded R-module M (whose generators may have any degrees) all have linear resolutions for a large number d. Indeed, after a suitable shift of degrees, one may assume that all generators of M have positive degrees. Then for d_0 large enough M as an $R^{(d_0)}$-module is generated in degree 0. As the rate of any finitely generated module is finite (see 1.2), we have that $\mathrm{rate}_{R^{(d_0)}}\, M$ is a finite number. Thus if we apply our main theorem to the $R^{(d_0)}$-module M, we find another integer d_1 such that the $(R^{(d_0)})^{(d_1)}$-module M has a linear resolution. As $(R^{(d_0)})^{(d_1)} = R^{(d_0 d_1)}$ we may choose $d_0 d_1$ for d.

see 2.1. Their Hilbert series are easy to determine. Since for a graded module M with linear resolution, the Hilbert series $H_M(t)$ and the Poincarè series $P_M(t)$ of M are related by the equation

$$P_M(t) = H_M(-t)H_R(-t)^{-1},$$

1.2 allows us to compute the Poincarè series of the modules $R_j^{(d)}$; see 2.2.

In the last section we relate Backelin's rate of $R^{(d)}$ to the Castelnuovo-Mumford regularity of the defining ideal of R

1 The finiteness of the rate

Throughout this paper R is a homogeneous k-algebra, where k is an arbitrary field, and, unless otherwise stated all R-modules will be graded and finitely generated with all generators in degree 0.

In this section we will prove that the rate of any R-module is finite, following the simple arguments given by Avramov as quoted in Backelin's paper [1] where it is shown that Backelin's rate is finite.

Let M be a graded R-module. We set

$$P_M^R = \sum_{i,j} \dim \operatorname{Tor}_i(k, M)_j s^j t^i,$$

and call it the (bigraded) Poincarè series of M. Note that P_M^R is a formal power series in t with coeffients $c_i(s) \in \mathbb{Z}[s]$, and it is clear that

$$\operatorname{rate} M = \sup_{i\geq 1}\{\deg c_i(s)/i\}.$$

It is convinient to define the rate of an arbitrary power series $P = \sum_i c_i(s)t^i \in \mathbb{Z}[s][[t]]$ in the same way, namely as $\operatorname{rate} P = \sup_{i\geq 1}\{\deg c_i(s)/i\}$.

We shall use the following

Lemma 1.1. *Let* $P = \sum_i a(s)t^i$, $Q = \sum_i b_i(s)t^i$ *be elements in* $\mathbb{Z}[s][[t]]$ *with* $a_0(s), b_0(s) \in \mathbb{Z}$. *Then one has*
(a) $\operatorname{rate}(P + Q) \leq \max\{\operatorname{rate} P, \operatorname{rate} Q\}$;
(b) $\operatorname{rate}(PQ) \leq \max\{\operatorname{rate} P, \operatorname{rate} Q\}$;
(c) *if* P *is invertible, then* $\operatorname{rate} P = \operatorname{rate} P^{-1}$.

PROOF. (a) is trivial.

(b) Let $P = \sum_i a_i(s)t^i$, and $Q = \sum_i b_i(s)t^i$, and assume that $\operatorname{rate} P = c$ and $\operatorname{rate} Q = d$. The $\deg a_i(s) \leq ic$ and $\deg b_i(s) \leq id$ for all $i \geq 1$.

Now $PQ = \sum_i c_i(s)t^i$ with $c_i(s) = \sum_{j+k=i} a_j(s)b_k(s)$. Therefore

$$\begin{aligned}\deg c_i(s) &\leq \max_{j+k=i}\{\deg a_j(s) + \deg b_k(s)\} \leq \max_{j+k=i}\{jc + kd\}\\ &\leq (j + k)\max\{c, d\} = i\max\{c, d\}.\end{aligned}$$

Now $PQ = \sum_i c_i(s)t^i$ with $c_i(s) = \sum_{j+k=i} a_j(s)b_k(s)$. Therefore

$$\deg c_i(s) \le \max_{j+k=i}\{\deg a_j(s) + \deg b_k(s)\} \le \max_{j+k=i}\{jc + kd\}$$
$$\le (j+k)\max\{c,d\} = i\max\{c,d\}.$$

So rate $PQ \le \max\{c,d\}$.

(c) If P is invertible, then $a_0(s) = \pm 1$. We may assume that $a_0(s) = 1$, and write $P = 1 - R$ with $\deg_t R \ge 1$, say $R = \sum_{i\ge 1} c_i(s)t^i$. Then $P^{-1} = 1 + R + R^2 + \cdots$. Hence if $P^{-1} = \sum_{i\ge 0} d_i(s)t^i$, then $d_i(s)$ is the sum of all products

$$c_{i_1}(s)\cdots c_{i_k}(s), \quad k \ge 1, \quad \sum_j i_j = i, \quad i_j \ge 1.$$

Suppose rate $P = c$, then $\deg c_{i_1}(s)\cdots c_{i_k}(s) \le i_1c + \cdots + i_kc = ic$. This implies $\deg d_i(s) \le ic$, so that rate $P^{-1} \le$ rate P. Similarly rate P = rate$(P^{-1})^{-1} \le$ rate P^{-1}. □

Now we are ready to prove

Proposition 1.2. *Let $S \to R$ be a surjective homomorphism of graded rings. Then for any graded R-module M,*

$$\mathrm{rate}_R\, M \le \max\{\mathrm{rate}_S\, M, \mathrm{rate}_S\, R\}.$$

PROOF. The standard change of rings spectral sequence

$$\mathrm{Ext}^p_R(M, \mathrm{Ext}^q_S(R,k)) \Longrightarrow Ext^{p+q}_S(M,k)$$

respects the internal gradings of the Ext-groups, and thus provides the coefficient-wise inequality of formal power series

$$P^R_M \le P^S_M(1 + t - tP^S_R)^{-1}.$$

Hence 1.1 implies that $\mathrm{rate}_R\, M \le \max\{\mathrm{rate}_S\, M, \mathrm{rate}(1+t-tP^S_R)\}$. But

$$\mathrm{rate}(1+t-tP^S_R) = \sup_{i\ge 2}\{t^S_{i-1}(R)/i\} \le \sup_{i\ge 2}\{t^S_{i-1}(R)/(i-1)\} = \mathrm{rate}_S\, R,$$

as desired. □

The homogeneous k-algebra R has a presentation $\varepsilon\colon S \to R$, where S is a polynomial ring over R, and where ε is a surjective homomorphism of graded rings. Any S-module has finite projective dimension, and therefore the rate of any S-module is finite. Hence 1.2 implies

Corollary 1.3. $\mathrm{rate}_R\, M \le \mathrm{rate}_S\, M < \infty$ *for all R-modules M.*

2 The relative Veronese submodules of a polynomial ring

The purpose of this section is to show that all relative Veronese submodules of the polynomial ring $S = k[x_1, \ldots, x_n]$ have linear resolutions. In other words

Theorem 2.1. *For all integers $d \geq 1$, $\operatorname{rate}_{S^{(d)}} S = 1$.*

Proof. We will show that each $S_j^{(d)}$ has a linear $S^{(d)}$-resolution by suitably filtering these modules. The module $S_j^{(d)}$ is generated by all monomials $u_1, \ldots, u_m$ of S which are of degree j. Let us assume they are ordered in the degrevlex term order, that is, $u = x_1^{a_1} \cdots x_n^{a_n} < v = x_1^{b_1} \cdots x_n^{b_n}$ if and only if the first non-vanishing component of the vector

$$(\deg u - \deg v, b_n - a_n, \ldots, b_1 - a_1)$$

is negative.

Then for $i = 1, \ldots, m$ we set $U_i = S^{(d)}u_1 + \cdots + S^{(d)}u_i$, and thus we obtain a sequence of submodules

$$S_j^{(d)} = U_m \supset U_{m-1} \supset \cdots \supset U_1 \supset U_0 = 0$$

of $S_j^{(d)}$.

We claim that each of the successive quotients U_i/U_{i-1} has a linear $S^{(d)}$-resolution which in turn implies that $S_j^{(d)}$ itself has a linear $S^{(d)}$-resolution.

Let us denote by J_i the colon ideal $S^{(d)}u_1 + \cdots + S^{(d)}u_{i-1} :_{S^{(d)}} u_i$. Then U_i/U_{i-1} is isomorphic to $S^{(d)}/J_i$. Observe that J_i is generated by a part of the degree 1 generators of the k-algebra $S^{(d)}$. Indeed, first note that if l is the smallest number such that x_l divides u_i, then $L_i = Su_1 + \cdots + Su_{i-1} :_S u_i = Sx_{l+1} + \cdots + Sx_n$. Secondly note that $L_i \cap S^{(d)} = J_i$, so that J_i is generated by all monomials of degree d which are divisible by one of the variables $x_{l+1}, \ldots, x_n$.

Now we present $S^{(d)}$ as the factor ring $T_d/V_d(I)$, where $T_d = k[z_1, \ldots, z_r]$ is a polynomial ring in $r = \operatorname{emb}\dim S^{(d)}$ many variables. We may assume that the last variables $z_l, \ldots, z_r$ are mapped to the generators of J_i, and choose the degrevlex order on the monomials in the z_i. By [5, Theorem 2] we know that the ideal of initial forms of $V_d(I)$ is generated by monomials of degree 2. Therefore all hypotheses of [4, Corollary 2.5] are satisfied, and it follows that $S^{(d)}/J_i$ has a linear $S^{(d)}$-resolution, as we wanted to show. □

Corollary 2.2. *For the Poincaré series of the relative Veronese submodules of the polynomial ring $S = k[x_1, \ldots, x_n]$ we have*

$$P_{S_j^{(d)}}(t) = \frac{\sum_{i \geq 0} (-1)^i a_{id+j}^{(d)} t^i}{\sum_{i \geq 0} (-1)^i a_{id}^{(d)} t^i}$$

with

$$a_r^{(d)} = \sum_{l=0}^{n} (-1)^l \binom{n}{l} \binom{n + r - ld - 1}{n - 1}.$$

Here, $a_r^{(d)} = 0$ *for* $r > r(n-1)$, *so that* $P_{S_j^{(d)}}(t)$ *is the quotient of two polynomials of degree at most* n.

PROOF. Since the resolution of $S_j^{(d)}$ is linear, its Poincaré series can be computed by means of Hilbert series, that is, $P_{S_j^{(d)}}(t) = H_{S_j^{(d)}}(-t)H_{S^{(d)}}(-t)^{-1}$. Since all the modules $S_j^{(d)}$ are Cohen-Macaulay of maximal dimension, we get $H_{S_j^{(d)}}(t) = H_{T_j^{(d)}}(t)/(1-t)^n$ with $T_j^{(d)} = S_j^{(d)}/(x_1^d,\ldots,x_n^d)S_j^{(d)}$ where $T = S/(x_1^d,\ldots,x_n^d)S$. Therefore

$$P_{S_j^{(d)}}(t) = H_{T_j^{(d)}}(-t)/H_{T_0^{(d)}}(-t),$$

and it remains to compute $H_{T_j^{(d)}}(t)$.

The explicit formula for $H_{T_0^{(d)}}(t)$ is already given in [2]. The general case is just as simple: We set $b_r^{(d)} = \dim_k T_r$ for all $r \geq 0$. Then $H_{T_j^{(d)}}(t) = \sum_{i\geq 0} b_{id+j}^{(d)} t^i$, and the asserted formula for the Poincaré series follows once we have shown that $b_r^{(d)} = a_r^{(d)}$ for all r. But the Hilbert function of T (which gives us the $b_r^{(d)}$) is easily computed from the Koszul complex

$$0 \longrightarrow S(-nd) \longrightarrow \cdots \longrightarrow S(-2d)^{\binom{n}{2}} \longrightarrow S(-d)^n \longrightarrow S \longrightarrow T \longrightarrow 0.$$

associated with the regular sequence $x_1^d,\ldots,x_n^d$, which yields a homogeneous free S-resolution of T. So the desired conclusion follows. Moreover, since $T_r = 0$ for $r > n(d-1)$, we see that the polynomials $H_{T_j^{(d)}}(-t)$ have degree at most n. □

3 Proof of the main theorem

Recall that we assert that for any finitely generated graded R-module M with generators in degree 0, one has

$$\operatorname{rate}_{R^{(d)}} M \leq \lceil \operatorname{rate}_R M/d \rceil \quad \text{for all} \quad d \geq c,$$

where c is a constant only depending on R.

To see this we let F be a minimal graded free resolution of M as an R-module. Then the sequence

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is also an exact sequence of $R^{(d)}$-modules, yielding a convergent spectral sequence

$$\operatorname{Tor}_i^{R^{(d)}}(k, F_j) \Rightarrow \operatorname{Tor}_{i+j}^{R^{(d)}}(k, M).$$

This implies that $\operatorname{Tor}_l^{R^{(d)}}(k, M)$ is filtered by subquotients of the $\operatorname{Tor}_i^{R^{(d)}}(k, F_j)$, $i + j = l$. All is compatible with the internal gradings, so that we obtain a coefficient-wise inequality of power series rings

(1) $$P_M^{R^{(d)}} \leq \sum_j P_{F_j}^{R^{(d)}} t^j.$$

each F_j is a direct sum of modules $R(-a)$ for some $a \in \mathbb{N}$. Now $R(-a)$ as an $R^{(d)}$-module equals $\bigoplus_{j=0}^{d-1} R(-a)_j^{(d)}$, and $R(-a)_j^{(d)} = \bigoplus_{i\geq 0} R(-a)_{id+j} = \bigoplus_{i\geq 0} R_{id+j-a}$.

Let i_j be the smallest integer such that $i_j d \geq a-j$, i.e., $i_j = \lceil \frac{a-j}{d} \rceil$. Then $\bigoplus_{i\geq 0} R_{id+j-a} = \bigoplus_{i\geq 0} R_{(i-i_j)d+k_j}$, where $k_j = i_j d + j - a$. Since $0 \leq k_j \leq d-1$, we now get

$$(2)\qquad R(-a) = \bigoplus_{j=0}^{d-1} R_{k_j}^{(d)}(-\lceil (a-j)/d \rceil).$$

Let us first assume that R is the polynomial ring. Then, by 2.1, R has a linear $R^{(d)}$ resolution. Therefore (2) yields

$$(3)\qquad P_{F_j}^{R^{(d)}} = \sum_{i\geq 0} c_i(s)t^i \qquad \text{with} \qquad \deg c_i(s) \leq \lceil t_j(M)/d \rceil + i.$$

So we get

$$\begin{aligned}\operatorname{rate}(P_{F_j}^{R^{(d)}} t^j) &\leq \sup_{i\geq 0}\{\frac{\lceil t_j(M)/d\rceil + i}{j+i}\} = \begin{cases} \lceil t_j(M)/d\rceil / j & \text{if } \lceil t_j(M)/d\rceil \geq j \\ 1 & \text{if } \lceil t_j(M)/d\rceil < j \end{cases} \\ &\leq \max\{1, \frac{\lceil t_j(M)/d\rceil}{j}\}.\end{aligned}$$

Now inequality (1) implies

$$\operatorname{rate}_{R^{(d)}} M \leq \max\{1, \sup_{i\geq 1}\{\frac{\lceil t_i(M)/d\rceil}{i}\}\}.$$

Say, $\sup_i\{t_i(M)/i\} = b$, then $t_i(M) \leq ib$, and so $\lceil t_i(M)/d\rceil \leq \lceil ib/d\rceil$. Now since for any $c \in \mathbb{R}$, $c > 0$, and $i \in \mathbb{N}$ one has $\frac{1}{i}\lceil ic\rceil \leq \lceil c\rceil$, we get

$$\frac{\lceil t_i(M)/d\rceil}{i} \leq \frac{\lceil ib/d\rceil}{i} \leq \lceil b/d\rceil.$$

Therefore

$$\begin{aligned}\operatorname{rate}_{R^{(d)}} M &\leq \max\{1, \sup_{i\geq 1}\{\frac{\lceil t_i(M)/d\rceil}{i}\}\} \leq \max\{1, \lceil \frac{\sup_{i\geq 1}\{t_i(M)/i\}}{d}\rceil\} \\ &\leq \max\{1, \lceil \operatorname{rate}_R M/d\rceil\} \leq \lceil \operatorname{rate}_R M/d\rceil.\end{aligned}$$

Finally, if R is an arbitrary homogeneous k-algebra, we do not necessarily have that R has a linear $R^{(d)}$-resolution which is needed for (3). But if $R = S/I$ where S is a polynomial ring, then according to 1.2 and the first part of this proof we get

$$\operatorname{rate}_{R^{(d)}} R \leq \operatorname{rate}_{S^{(d)}} R \leq \lceil \operatorname{rate}_S R/d\rceil.$$

So that for $d \geq c = \operatorname{rate}_S R$, R has a linear $R^{(d)}$-resolution. Thus, for these d's, (3), and hence the rest of the proof is valid.

4 Rate and Castelnuovo-Mumford regularity

Let R be a homogeneous k-algebra, and present it as $R = S/I$ where S is a polynomial ring. In this section we want to relate the Backelin rate to the *Castelnuovo-Mumford regularity* $\operatorname{reg}(I)$ of I which is defined to be

$$\operatorname{reg}(I) = \max_i\{t_i(I) - i\}.$$

We have the following result

Theorem 4.1. *For all integers $d > 0$ one has*

$$\operatorname{Rate} R^{(d)} \leq \lceil \operatorname{reg}(I)/d \rceil.$$

Note that 4.1 is a sort of 'mixture' of the Backelin [1] and the Eisenbud-Reeves-Totaro [5] inequality.

PROOF OF 4.1. We first observe that $t_i(I) = t_{i+1}(R)$, so that $t_{i+1}(R) \leq \operatorname{reg}(I) + i$ for all $i \geq 0$. Therefore

$$\begin{aligned}\operatorname{rate}_S R &= \sup_{i\geq 1}\{t_i(R)/i\} \leq \sup_{i\geq 1}\{\frac{(\operatorname{reg}(I)-1)+i}{i}\} \\ &= \begin{cases} 1 & \text{if } \operatorname{reg}(I) = 0, \\ \operatorname{reg}(I) & \text{if } \operatorname{reg}(I) > 0. \end{cases}\end{aligned}$$

Hence

$$\operatorname{rate}_S R \leq \max\{1, \operatorname{reg}(I)\}.$$

Combining this with 1.2 and the main theorem we get

(1) $\operatorname{rate}_{R^{(d)}} M \leq \max\{\operatorname{rate}_{S^{(d)}} M, \operatorname{rate}_{S^{(d)}} R\} \leq \max\{\lceil \operatorname{rate}_S M/d\rceil, \lceil \operatorname{reg}(I)/d\rceil\}.$

Let $\mathfrak{n}$ denote the graded maximal ideal of $R^{(d)}$. Then $\mathfrak{n}(1)$ is the $(d-1)$-th relative Veronese submodule $\mathfrak{m}(1)^{(d)}_{d-1}$ of $\mathfrak{m}(1)$. Hence together with (1) we get

(2)
$$\begin{aligned}\operatorname{Rate} R^{(d)} &= \operatorname{rate}_{R^{(d)}} \mathfrak{n}(1) \leq \operatorname{rate}_{R^{(d)}} \mathfrak{m}(1) \\ &\leq \max\{\lceil (\operatorname{rate}_S \mathfrak{m}(1))/d\rceil, \lceil \operatorname{reg}(I)/d\rceil\}.\end{aligned}$$

From the exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow R \longrightarrow k \longrightarrow 0$$

we get an exact sequence of vector spaces

$$\operatorname{Tor}^S_{i+1}(k,k)_j \longrightarrow \operatorname{Tor}^S_i(k,\mathfrak{m})_j \longrightarrow \operatorname{Tor}^S_i(k,R)_j.$$

We have

$$\operatorname{Tor}^S_i(k,R)_j = 0 \qquad \text{for} \qquad j > t_i(R),$$

and

$$\mathrm{Tor}^S_{i+1}(k,k)_j = 0 \qquad \text{for} \qquad j > i+1.$$

This implies $t_i(\mathfrak{m}) \leq \max\{i+1, t_i(R)\}$ and hence $t_i(\mathfrak{m}(1)) \leq \max\{i, t_i(R) - 1\}$. Therefore

$$\mathrm{rate}_S\,\mathfrak{m}(1) \leq \max\{1, \sup_{i\geq 1}\{t_i(R) - 1)/i\} \leq \max\{1, \mathrm{rate}_S R\} \leq \max\{1, \mathrm{reg}(I)\}.$$

This together with (2) implies our assertion. □

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