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ABSTRACT. We study the consequences in the probabilistic structure of the process which follow after the perturbation of the integral part of the infinitesimal generator of a Markov process. Introducing or eliminating jumps of a process leads to addition or subtraction of an integral part to the infinitesimal generator. Moreover, we construct pure jump processes with trajectories of bounded variation generated by a class of Lévy kernels on a complete metric space.

INTRODUCTION. A Markov process may be modified by eliminating the jumps greater as a strictly positive number ε. Immediately after the moment of a jump eradication,the process is continued with the same law as before.This corresponds to a killing at the moment of the first jump greater as ε, completed with a resurrection. The appropriate transformation of the infinitesimal generator is the following: if the initial process has the Lévy kernel $N(x, dy)$ then the infinitesimal generator of the modified process is obtained by subtracting the operator $\widetilde{N^\varepsilon}$ defined by

$$\widetilde{N^\varepsilon}u(x) := \int (u(y) - u(x))\, \chi_{\{|x-y|>\varepsilon\}} N(x, dy)$$

from the original generator (Theorem 1.6). The question is whether we obtain a continuous paths process when ε tends to zero?

We prove (Corollary 2.3) that when the initial process has the infinitesimal generator $L + \widetilde{N}$, where L is a second order elliptic differential operator and $\widetilde{N}$ is given by

$$\widetilde{N}u(x) = \int (u(y) - u(x))\, N(x, dy)$$

(i.e. N is a first order Lévy kernel in the sense of [2]) then it is possible to pass to the limit and the limit process is the diffusion induced by L.

Similarly, it is possible to introduce jumps in the evolution of a process. The corresponding modification for the infinitesimal operator is the addition of an operator like the above $\widetilde{N^\varepsilon}$ (Theorem 1.8). Starting with a diffusion generated by an elliptic operator L and a first order Lévy kernel N, we prove that the process with generator $L + \widetilde{N}$ may be obtained as a limit of the processes generated by $L + \widetilde{N^\varepsilon}$.

In the general case of a second order Lévy kernel the approximation procedure should be modified. This time, at each approximation step the killing and the resurrection should be accompanied by a drift modification.

The above ideas suggest that a pure jump process can be constructed starting from a given kernel. In Section 3 we show that, under certain smoothness conditions, a first order Lévy kernel on a complete metric space generates a Markov process of pure jumps. This result is well known for bounded kernels (see e.g. [3]), when the process is regular step. In our case the obtained process has pure jump trajectories of bounded variation. The method was inspired by the treatment in $\mathbb{R}^n$ for stochastic differential equations with jumps (cf.[5]) and is available for Lévy kernels which are represented on a measurable space by a map satisfying Lipschitz regularity conditions. The main tool is a Poisson point process which generates the jumps.

1 Modification of jumps

Let E be a locally compact space with countable base and $X=(\Omega,\mathcal{F},\mathcal{F}_t,X_t,\theta_t,P^x)$ be a standard (Markov) process with state space E. Denote by Δ the point at infinity, ζ the lifetime of the process X, $\mathcal{E}$ the σ-algebra of Borel sets of E and $\mathcal{E}^*$ its universally completion. We refere to [1] for basic facts and notions concerning Markov processes and to [4] for the stochastic integral calculus. Let N be the *Lévy kernel associated with* X in the following sense: $N(x,dy)$ is a positive measure on $E\setminus\{x\}$ and $N(x,E\setminus V)<\infty$ for any $x\in E$ and any neighborhood V of x; the function $x\longmapsto N(x,A\setminus\{x\})$ is $\mathcal{E}^*$-measurable for any $A\in\mathcal{E}^*$; for any bounded positive function $f\in(\mathcal{E}^*\otimes\mathcal{E}^*)_{b+}$ with $f=0$ on $\Delta_E:=\{(y,y)/y\in E\}$, the following formula holds:

$$(1.1)\qquad E^x\left[\sum_{s\leq t} f(X_{s-},X_s)\right]=E^x\left[\int_0^t\int f(X_s,y)\,N(X_s,dy)\,ds\right],$$

for any $x\in X, t\geq 0$. Such a kernel $N(x,dy)$ is exactly the Lévy measure of the process X with respect to the additive functional $A_t:=t\wedge\zeta$ from [8].

We remark that the set of values of s considered in the left side sum is at most countable, namely the jump moments of the sample paths. We can regard the jumps of the standard process X as a point process. More precisely, this point process takes values in the space $(E\times E)\cup\{\delta\}$, where δ is a fictitious point attached to $E\times E$. For every $\omega\in\Omega$ we put $J(\omega):=\{s>0/X_{s-}(\omega)\neq X_s(\omega)\}$, $n(\omega)(s):=(X_{s-}(\omega),X_s(\omega))$ if $s\in J(\omega)$ and $n(\omega)(s):=\delta$ if $s\in\mathbb{R}_+\setminus J(\omega)$. In this way we have defined a point process of class (QL) in the sense of [4]. To see this, for any $\Lambda\in\mathcal{E}\otimes\mathcal{E}$ and $t\geq 0$ we put

$$\widehat{n}((0,t]\times\Lambda):=\int_0^t\int\chi_\Lambda(X_s,y)N(X_s,dy).$$

Let V be a neighborhood of the diagonal Δ_E of $E \times E$, $\Lambda := E \times E \setminus V$ and let us define the real valued function h on E by:

$$h(x) := \int \chi_\Lambda(x,y) N(x,dy) \quad , x \in E.$$

If $A_k := \{h < k\}, k \in \mathbb{N}$ and $B_k := \Lambda \cap (A_k \times E)$ then $\bigcup_{k \in \mathbb{N}} B_k = \Lambda$ and $\int \chi_{B_k}(x,y) N(x,dy) \leq k$, $(\forall) k \in \mathbb{N}$. Therefore for all $t \geq 0$ and $k \in \mathbb{N}$ we have

$$\widehat{n}((0,t] \times B_k) = \int_0^t \int \chi_{B_k}(X_s, y)\, N(X_s, dy)\, ds \leq kt.$$

If we put $\widetilde{n} := n - \widehat{n}$ it follows that the stochastic integrals with respect to $\widetilde{n}$ are martingales. Indeed, let us put

$$A_f(t) := \sum_{s \in J_t} f(X_{s-}, X_s) \quad , \quad \widehat{A}_f(t) := \int_0^t \int f(X_s, y)\, N(X_s, dy)\, ds,$$

where $f \in (\mathcal{E} \otimes \mathcal{E})_{b+}$, $\operatorname{supp} f \subset B_k$ for some k and $J_t = J_t(\omega) := (0,t] \cap J(\omega)$, $\omega \in \Omega$. If $r < t$ then using the Markov property and (1.1) we have:

$$E^x\left[A_f(t) - \widehat{A}_f(t) | \mathcal{F}_r\right] - \left(A_f(r) - \widehat{A}_f(r)\right) =$$

$$E^x\left[\sum_{s \in J_t \setminus J_r} f(X_{s-}, X_s) | \mathcal{F}_r\right] - E^x\left[\int_r^t \int f(X_s, y)\, N(X_s, dy)\, ds | \mathcal{F}_r\right] =$$

$$E^x\left[A_f(t-r) \circ \theta_r | \mathcal{F}_r\right] - E^x\left[\widehat{A}_f(t-r) \circ \theta_r | \mathcal{F}_r\right] =$$

$$E^{X_r}\left[A_f(t-r) - \widehat{A}_f(t-r)\right] = 0.$$

As a consequence (cf.[4], Theorem 3.1) we have the following:

Lemma 1.1. *If $\varphi = \varphi(s,(x,y),\omega)$ is an $\mathcal{F}_t$-predictable function such that*

$$E^x\left[\int_0^t |\varphi(s,(X_{s-},y),\cdot)|\, N(X_s, dy)\, ds\right] < \infty,$$

$$E^x\left[\int_0^t |\varphi(s,(X_{s-},y),\cdot)|^2 N(X_s, dy)\, ds\right] < \infty,$$

for all $t > 0$, then

$$\widetilde{n}(\varphi)(t) := \sum_{s \in J_t} \varphi(s,(X_s, X_s),\cdot) - \int_0^t \varphi(s,(X_{s-},y),\cdot)\, N(X_s, dy)\, ds$$

is a square integrable martingale.

Theorem 1.2. *Let* $X=(\Omega,\mathcal{F},\mathcal{F}_t,X_t,\theta_t,P^x)$ *be a standard process with state space* E*, Lévy kernel* N *and infinitesimal generator* W *(in the sense that the resolvent of* X *considered on* $\mathcal{E}_b^*$ *has the infinitesimal generator* W *with domain* $\mathcal{D}(W)\subseteq\mathcal{E}_b^*$*). Let* $\Lambda\subseteq E\times E\setminus\Delta_E$*, and define the kelnel* N' *by*

$$N'h(x):=\int\chi_\Lambda(x,y)h(y)N(x,dy),\quad (\forall)h\in\mathcal{E}_{b+}^*.$$

If we suppose that $\sup_{x\in E}N'1(x)<\infty$ *and put*

$$T:=\inf\{t>0/\,(X_{t-},X_t)\in\Lambda\},$$

then the following assertions hold:
(i) T *is a stopping time which is almost surely strictly positive and* $\lim_{k\to\infty}T_k\geq\zeta$*, where* $T_k,k\in\mathbb{N}$ *are the iterates of* T *(i.e.* $T_0=T$*,* $T_{k+1}=T_k+T\circ\theta_{T_k}$*).*
(ii) The process X' *obtained by killing* X *at the moment* T *has the infinitesimal generator* $W-N'$*. If moreover* X *is a Feller process and* $N'(C_o)\subseteq C_o$ *then* X' *is a Feller process too.*
Proof. By (1.1) applied for $f:=\chi_\Lambda$ we deduce:

$$E^x\left[\#\{s\leq t/\,(X_{s-},X_s)\in\Lambda\}\right]=E^x\left[\int_0^t N'1\left(X_s\right)ds\right]\leq t\sup_{x\in E}N'1(x)$$

As a consequence the process X has almost surely on the interval (0,t] at most a finite number of jumps in Λ. Consequently, almost surely $T>0$ and $\lim_{k\to\infty}T_k\geq\zeta$. Therefore the proof of the first assertion is complete.

Since T is a strictly positive terminal time, by killing X at the moment T we get a standard process X' with state space E. If we denote by $(U^\alpha)_{\alpha>0}$ (resp. $(U'^\alpha)_{\alpha>0}$) the resolvent of X (resp. X') then the following equality holds for all $\alpha>0$:

$$U^\alpha=U'^\alpha+P_T^\alpha U^\alpha.$$

On the other hand by Lemma 1.1 with

$$\varphi(s,(x,y),\omega)=e^{-\alpha s}f(x,y)\chi_\Lambda(x,y),\quad f\in(\mathcal{E}^*\otimes\mathcal{E}^*)_{b+}$$

we deduce that $\tilde{n}(\varphi)$ is a martingale. Therefore, taking f independent of x, i.e. $f(x,y)=f(y)$, we have $E^x\left[\tilde{n}(\varphi)(T)\right]=0$ or equivalently

$$E^x\left[e^{-\alpha T}f\left(X_T\right)\right]=E^x\left[\int_0^T e^{-\alpha s}N'f\left(X_s\right)ds\right].$$

Consequently,

$$P_T^\alpha f=U'^\alpha N'f.$$

It now follows that

$$U^\alpha=U'^\alpha+U'^\alpha N'U^\alpha \tag{1.2}$$

and by Lemma 1.4 below we conclude that $W - N'$ is the infinitesimal generator of $(U'^{\alpha})_{\alpha>0}$.

If X is Feller and $N'(C_\circ) \subseteq C_\circ$ then, as in the proof of Lemma 1.4, for sufficiently large α we have $U^{\alpha}(C_\circ)= U'^{\alpha}(C_\circ)$ and therefore X' results Feller.

Remark 1.3. *With the notation from the above proof, taking f independent of y, we deduce the following equality which will be used later:*

$$E^x\left[e^{-\alpha T} f(X_{T-})\right] = U'^{\alpha}(fN'1)(x), \quad (\forall) x \in E.$$

Lemma 1.4. *Let $(U^{\alpha})_{\alpha>0}$ and $(U'^{\alpha})_{\alpha>0}$ be two resolvents of bounded linear operators on the Banach space $\mathbf{B}$ such that $\|U^{\alpha}\|, \|U'^{\alpha}\| \leq \frac{1}{\alpha}$, for all α , and having the infinitesimal generators W and W' (with domains $\mathcal{D}(W)$ and $\mathcal{D}(W')$). If K is a bounded linear operator on $\mathbf{B}$, then the following assertions are equivalent:*
(i) $U^{\alpha} - U'^{\alpha} = U'^{\alpha} K U^{\alpha}$, for every $\alpha > 0$.
(ii) $\mathcal{D}(W) = \mathcal{D}(W')$ and $W = W' + K$.
Proof." *(i)⇒(ii)*" Since $U^{\alpha} = U'^{\alpha}(I + KU^{\alpha})$ and $I + KU^{\alpha}$ is invertible on $\mathbf{B}$ for large α it follows that $\mathcal{D}(W) = Im(U^{\alpha}) = Im(U'^{\alpha}) = \mathcal{D}(W')$. If $\alpha > 0$ then $(\alpha - W' - K)U^{\alpha} = (\alpha - W')U'^{\alpha}(I + KU^{\alpha}) - KU'^{\alpha} - KU'^{\alpha}KU^{\alpha} = I + K(U^{\alpha} - U'^{\alpha}) - K(U'^{\alpha}KU^{\alpha}) = I$. Hence $W = W' + K$.
" *(ii)⇒(i)*" We have: $U'^{\alpha}KU^{\alpha} = U'^{\alpha}((\alpha - W') - (\alpha - W))U^{\alpha} = U'^{\alpha}((\alpha - W')U^{\alpha} - I) = U^{\alpha} - U'^{\alpha}$, which completes the proof of the lemma.

Corollary 1.5. *Let X be a standard process with state space E, Lévy kernel N and infinitesimal generator W. If $\sup_{x \in E} N1(x) < \infty$ then the first jump of the process X defines a strictly positive stopping time and killing the process at the moment of the first jump we get a continuous paths standard process with infinitesimal generator $W - N$.*

The next theorem gives a probabilistic way to rebuild the process X starting from X'. We use the construction of resurrected processes. We recall the notation from [6]:

Let $X=(\Omega,\mathcal{F},\mathcal{F}_t,X_t,\theta_t,P^x)$ be a standard process and N a resurrection kernel. We set

$$\mathcal{W} := \{w = (\omega_0, \omega_1, \ldots) \in \Omega^{\mathbb{N}} / \quad \zeta(\omega_i) = 0 \Rightarrow \omega_{i+1} = \omega_{i+2} = \ldots = \delta\}$$

and for $w = (\omega_0, \omega_1, \ldots) \in \mathcal{W}$,

$$s_i(w) := \sum_{j=0}^{i} \zeta(\omega_j) \quad , \quad s_\infty(w) := \lim_{n\to\infty} s_n(w)$$

$$Y_t(w) := \begin{cases} X_t(\omega_0) \quad , & \text{if } t < s_0(w) \\ X_{t-s_i(w)}(\omega_{i+1}) \quad , & \text{if } s_i(w) \leq t < s_{i+1}(w) \end{cases}$$

$$\Theta_t(w) := \begin{cases} (\theta_t\omega_0, \omega_1, \ldots), & \text{if } t < s_0(w) \\ (\theta_{t-s_i(w)}\omega_{i+1}, \omega_{i+2}, \ldots), & \text{if } s_i(w) \leq t < s_{i+1}(w). \end{cases}$$

Then there exist on $\mathcal{W}$ a probability Π^x and an adequate filtration $(\mathcal{G}_t)$ such that $Y := (\mathcal{W}, \mathcal{G}, \mathcal{G}_t, Y_t, \Theta_t, \Pi^x)$ is a Markov process and killing Y at the moment s_0 it becomes equivalent with X under P^x. The kernel N gives the distribution of Y_{s_0} conditioned by the evolution up to s_0.

Theorem 1.6. *With the hypothesis and notation from Theorem 1.2 we have:*
(i) If for all $\omega \in \Omega$ we put

$$R(\omega, dy) := \frac{\chi_{\{T<\zeta\}}(\omega)}{N'1\,(X_{T-}(\omega))} N'\,(X_{T-}(\omega), dy) + \chi_{\{T\geq\zeta\}}(\omega)\varepsilon_\Delta(dy)$$

then R is a resurrection kernel for the process X' and the resurrected process is equivalent with X.
(ii) If for all $\omega \in \Omega$ we define

$$Q(\omega, dy) := \chi_{\{T<\zeta\}}(\omega)\varepsilon_{X_{T-}(\omega)}(dy) + \chi_{\{T\geq\zeta\}}(\omega)\varepsilon_\Delta(dy)$$

then Q is a resurrection kernel for the process X' and the resurrected process has the infinitesimal generator $W - N' + N'1$ and $\mathcal{D}(W)=\mathcal{D}(W - N' + N'1)$. If in addition X is a Feller process, $N'(C_\circ) \subseteq C_\circ$ and $N'1$ is a continuous function then the resurrected process is also Feller.

Proof. Note that the expresion defining R makes sense even if $X_{T-}(\omega) \in [N'1 = 0]$ because in this case the first term vanishes. Recall that $(U^\alpha)_{\alpha>0}$ (resp. $(U'^\alpha)_{\alpha>0}$) denotes the resolvent of X (resp.X'). Also , we denote by Y=$(\mathcal{W}, \mathcal{G}$,$\mathcal{G}_t$,$Y_t$, Θ_t,$\Pi^x)$ the resurrected process with the kernel R and by $(V^\alpha)_{\alpha>0}$ its resolvent. To prove the first assertion it suffices to show that $U^\alpha = V^\alpha$, for all $\alpha > 0$. By Lemma 1.7 below we have for any $g \in \mathcal{E}^*_{b+}$:

$$E^x_\Pi\left[e^{-\alpha s_0} g\left(Y_{s_0}\right)\right] = E'^x\left[e^{-\alpha\zeta'} R(\cdot, g)\right] = E^x\left[e^{-\alpha T} R(\cdot, g)\right] = E^x\left[e^{-\alpha T}\frac{N'g\left(X_{T-}\right)}{N'1\left(X_{T-}\right)}\right].$$

From Remark 1.3 we get now that

$$E^x_\Pi\left[e^{-\alpha s_0} g\left(Y_{s_0}\right)\right] = U'^\alpha N'g(x), \quad (\forall) x \in E$$

and therefore, for any $f \in \mathcal{E}^*_{b+}$ and $x \in E$,

$$V^\alpha f(x) = E^x_\Pi\left[\int_0^{s_\infty} e^{-\alpha t} f\left(Y_t\right) dt\right] = E^x_\Pi\left[\int_0^{s_0} e^{-\alpha t} f\left(Y_t\right) dt\right] + E^x_\Pi\left[e^{-\alpha s_0} V^\alpha f\left(Y_{s_0}\right)\right]$$

$$= U'^\alpha f(x) + U'^\alpha N' V^\alpha f(x).$$

Hence for α sufficiently large we have $V^\alpha = (I - U'^\alpha N')^{-1} U'^\alpha$. Using also (1.2) we deduce $V^\alpha = U^\alpha$.

We prove now assertion (ii). This time we denote by Y=$(\mathcal{W}, \mathcal{G}$,$\mathcal{G}_t$,$Y_t$, Θ_t,$\Pi^x)$ the process obtained resurrecting X' with the kernel Q and by $(V^\alpha)_{\alpha>0}$ its resolvent.

If $g \in \mathcal{E}^*_{b+}$, by Lemma 1.7 we get

$$E^x_\Pi\left[e^{-\alpha s_0} g\left(Y_{s_0}\right)\right] = E'^x_\Pi\left[e^{-\zeta'} Q(\cdot, g)\right] =$$

$$E^x\left[e^{-\alpha T} Q(\cdot, g)\right] = E^x\left[e^{-\alpha T} g\left(X_{T-}\right)\right].$$

By Remark 1.3 we deduce

$$E^x_\Pi\left[e^{-\alpha s_0} g\left(Y_{s_0}\right)\right] = U'^\alpha(gN'1)(x)$$

and further, for any $f \in \mathcal{E}^*_{b+}$, seting $g := V^\alpha f$ we get

$$V^\alpha f(x) = U'^\alpha f(x) + E^x_\Pi\left[e^{-\alpha s_0} g\left(Y_{s_0}\right)\right] = U'^\alpha f(x) + U'^\alpha(N'1 \cdot V^\alpha f).$$

The assertion follows now by Theorem 1.2 and Lemma 1.4.

Lemma 1.7. *Let $X=(\Omega,\mathcal{F},\mathcal{F}_t,X_t,\theta_t,P^x)$ be a standard process with state space E and $Y=(\mathcal{W},\mathcal{G},\mathcal{G}_t,Y_t,\Theta_t,\Pi^x)$ be the process obtained by resurrecting X with the resurrection kernel R. Then for any $f \in (\mathcal{E}^* \otimes \mathcal{E}^*)_{b+}$, $\alpha > 0$ and $x \in E$ we have:*

$$E^x_\Pi\left[e^{-\alpha s_0} f\left(Y_{s_0-}, Y_{s_0}\right)\right] = E^x\left[e^{-\alpha\zeta} \int R(\cdot, dy) f\left(X_{\zeta-}(\cdot), y\right)\right].$$

Proof. Following [6] we have

$$E^x_\Pi\left[e^{-\alpha s_0} f\left(Y_{s_0-}, Y_{s_0}\right)\right] =$$

$$\int \Pi^x\left(d\omega_0\right) \int R\left(\omega_{0-}, dy\right) \int \Pi^y\left(d\omega_1\right) e^{-\alpha\zeta(\omega_0)} f\left(X_{\zeta-}\left(\omega_0\right), X_0\left(\omega_1\right)\right) =$$

$$\int \Pi^x\left(d\omega_0\right) e^{-\alpha\zeta(\omega_0)} \int R\left(\omega_0, dy\right) f\left(X_{\zeta-}\left(\omega_0\right), y\right).$$

Theorem 1.8. *Let $X=(\Omega,\mathcal{F},\mathcal{F}_t,X_t,\theta_t,P^x)$ be a standard process with state space E, N a bounded kernel on E and $M = (M_t)_{t\geq 0}$ the multiplicative functional of X defined by*

$$M_t := exp\left(-\int_0^t N1\left(X_s\right) ds\right).$$

We denote by $\widehat{X}=(\widehat{\Omega},\widehat{\mathcal{F}}, \widehat{\mathcal{F}}_t,\widehat{X}_t,\widehat{\theta}_t,\widehat{P}^x)$ the subprocess of X induced by M and by $Y=(\mathcal{W},\mathcal{G},\mathcal{G}_t,Y_t,\Theta_t,\Pi^x)$ the process obtained resurrecting $\widehat{X}$ with the kernel R given by

$$R(\widehat{\omega}, dy) := \frac{\chi_{\{\widehat{\zeta}<\zeta\}}(\widehat{\omega})}{N1\left(\widehat{X}_{\widehat{\zeta}-}(\widehat{\omega})\right)} N\left(\widehat{X}_{\widehat{\zeta}-}(\widehat{\omega}), dy\right) + \chi_{\{\widehat{\zeta}\geq\zeta\}}(\widehat{\omega})\varepsilon_\Delta(dy),$$

where $\widehat{\omega} \in \widehat{\Omega}$. Then the following assertions hold:
(i) If X has the infinitesimal generator W then Y has the infinitesimal generator $W - N1 + N$ and $\mathcal{D}(W) = \mathcal{D}(W - N1 + N)$.
(ii) If for all $t \geq 0$ and $\omega \in \Omega$ we put $J_t(\omega):=\{s \leq t / s = s_n(\omega)$ for some $n \geq 1\}$,

where $s_n, n \in IN$ are the iterates of s_0, then for all $f \in (\mathcal{E}^* \otimes \mathcal{E}^*)_{b+}$, $\alpha > 0$ and $x \in E$ we have:

$$E^x_\Pi \left[\sum_{s \in J_t} e^{-\alpha s} f(Y_{s-}, Y_s) \right] = E^x_\Pi \left[\int_0^t e^{-\alpha s} \int f(Y_s, y) N(Y_s, dy)\, ds \right]$$

Proof. We denote by $(U^\alpha)_\alpha$, $(\widehat{U}^\alpha)_\alpha$, $(V^\alpha)_\alpha$ the resolvents of the processes $X, \widehat{X}, Y$. It is easy to see that

$$U^\alpha(f) = \widehat{U}^\alpha(f) + \widehat{U}^\alpha(N1 \cdot U^\alpha(f)).$$

Moreover by Lemma 1.4 we have

$$U^\alpha(f) = \widehat{U}^\alpha(f) + U^\alpha(N1 \cdot \widehat{U}^\alpha(f)) \tag{1.3}$$

(see also ch.IV,(2.22) in [1]). We prove first that

$$E^x_\Pi \left[e^{-\alpha s_0} f(Y_{s_0 -}, Y_{s_0}) \right] = \widehat{U}^\alpha(f \circ N)(x) = E^x_\Pi \left[\int_0^{s_0} e^{-\alpha u} f \circ N(Y_u)\, du \right], \tag{1.4}$$

where $f \circ N(x) := \int f(x,y) N(x,dy)$, for all $x \in E$. Indeed, by Lemma 1.7 we get

$$E^x_\Pi \left[e^{-\alpha s_0} f(Y_{s_0 -}, Y_{s_0}) \right] = \widehat{E}^x \left[e^{-\alpha \widehat{\zeta}} \int R(\cdot, dy) f\left(\widehat{X}_{\widehat{\zeta}-}(\cdot), y \right) \right] =$$

$$\widehat{E}^x \left[e^{-\alpha \widehat{\zeta}} \frac{f \circ N}{N1} \left(\widehat{X}_{\widehat{\zeta}-} \right) ; \{ \widehat{\zeta} < \zeta \} \right] = E^x \left[\int_{(0,\zeta)} e^{-\alpha r} \frac{f \circ N}{N1} (X_{r-}) (-dM_r) \right] =$$

$$E^x \left[\int_{(0,\zeta)} e^{-\alpha r} f \circ N(X_{r-}) M_r dr \right] = \widehat{U}^\alpha(f \circ N)(x).$$

This establishes (1.4).

Taking in (1.4) $f(x,y) := f(y)$ we deduce that for any $f \in \mathcal{E}^*_{b+}$ and $\alpha > 0$ we have

$$E^x_\Pi \left[e^{-\alpha s_0} f(Y_{s_0}) \right] = \widehat{U}^\alpha N f(x)$$

and since $V^\alpha = \widehat{U}^\alpha f + E^x_\Pi \left[e^{-\alpha s_0} V^\alpha f(X_{s_0}) \right]$ we obtain

$$V^\alpha = \widehat{U}^\alpha + \widehat{U}^\alpha N V^\alpha \quad ,(\forall)\alpha > 0.$$

From (1.3) and Lemma 1.4 we get that the infinitesimal generator of the process $\widehat{X}$ is $W - N1$, $\mathcal{D}(W) = \mathcal{D}(W - N1)$. Again by Lemma 1.4 and the above relation between $(\widehat{U}^\alpha)_\alpha$ and $(V^\alpha)_\alpha$ the first assertion of Theorem 1.8 follows.

Let now $f \in (\mathcal{E}^* \otimes \mathcal{E}^*)_{b+}$ and $\alpha > 0$. Then by the strong Markov property of Y and (1.4) we have:

$$E^x_\Pi \left[\sum_{n=1}^{\infty} e^{-\alpha s_n} f\left(Y_{s_n -}, Y_{s_n} \right) \right] = \sum_{n=0}^{\infty} E^x_\Pi \left[e^{-\alpha s_n} e^{-\alpha s_0 \circ \Theta_{s_n}} f(Y_{s_0 -}, Y_{s_0}) \circ \Theta_{s_n} \right] =$$

$$\sum_{n=0}^{\infty} E_{\Pi}^{x}\left[e^{-\alpha s_n} E_{\Pi}^{Y_{s_n}}\left[e^{-\alpha s_0} f\left(Y_{s_0-}, Y_{s_0}\right)\right]\right] =$$

$$\sum_{n=0}^{\infty} E_{\Pi}^{x}\left[e^{-\alpha s_n} E_{\Pi}^{Y_{s_n}}\left[\int_0^{s_0} e^{-\alpha u} f \circ N\left(Y_u\right) du\right]\right] =$$

$$\sum_{n=0}^{\infty} E_{\Pi}^{x}\left[e^{-\alpha s_n}\left(\int_0^{s_0} e^{-\alpha u} f \circ N\left(Y_u\right) du\right) \circ \Theta_{s_n}'\right] =$$

$$\sum_{n=0}^{\infty} E_{\Pi}^{x}\left[\int_{s_n}^{s_n+s_0\circ\Theta_{s_n}} e^{-\alpha u} f \circ N\left(Y_u\right) du\right] = E_{\Pi}^{x}\left[\int_{s_0}^{s_\infty} e^{-\alpha u} f \circ N\left(Y_u\right) du\right].$$

Consequently we have

$$E_{\Pi}^{x}\left[\sum_{n=0}^{\infty} e^{-\alpha s_n} f\left(Y_{s_n-}, Y_{s_n}\right)\right] = E_{\Pi}^{x}\left[\int_0^{s_\infty} e^{-\alpha u} f \circ N\left(Y_u\right) du\right].$$

If for any $t \in (0, \infty]$ we put

$$\sum\nolimits_t := \sum_{s \in J_t} e^{-\alpha s} f\left(Y_{s-}, Y_s\right) \quad , \quad \sigma_t := \int_0^t e^{-\alpha s} f \circ N\left(Y_s\right) ds,$$

then we have already proved that

$$E_{\Pi}^{x}\left[\sum\nolimits_\infty\right] = E_{\Pi}^{x}\left[\sigma_\infty\right] < \infty.$$

By standard arguments follows now assertion (ii). Indeed, since $\sum_t = \sum_\infty - \sum_\infty \circ \Theta_t$ and $\sigma_t = \sigma_\infty - \sigma_\infty \circ \Theta_t$, for all $t \in (0,\infty)$,it follows: $E_{\Pi}^{x}\left[\sum_\infty \circ \Theta_t\right] = E_{\Pi}^{x}\left[E^{Y_t}\left[\sum_\infty\right]\right] = E_{\Pi}^{x}\left[E^{Y_t}\left[\sigma_\infty\right]\right] = E_{\Pi}^{x}\left[\sigma_\infty \circ \Theta_t\right]$ which leads to $E_{\Pi}^{x}\left[\sum_t\right] = E_{\Pi}^{x}\left[\sigma_t\right]$. Thus Theorem 1.8 is proved.

2 Convergence of processes associated with integro - differential operators

In this section we consider Markov processes in $\mathbb{R}^d$ associated to integro-differential operators. A first treatment of integro-differential operators in connection with Feller semi-groups they generate was given by Bony, Courrège and Priouret [2]. The main tool in our approach is an à priori estimate of Schauder type obtained by Pragarauskas and Mikulevičius [7] (see (2.5) below).

Let L be a second order elliptic differential operator with Hölder coefficients in $\mathbb{R}^d$,

(2.1) $$L := \sum_{i,j=1}^{d} a^{ij}(x)\partial_i\partial_j + \sum_{i=1}^{d} a^i(x)\partial_i + a(x).$$

The matrix (a^{ij}) is assumed to be symetric and uniform elliptic i.e.

$$\sum_{i,j=1}^{d} a^{ij}(x)\xi^i\xi^j \geq K_1|\xi|^2, \quad (\forall) \quad x, \xi = \left(\xi^i\right) \in \mathbb{R}^d,$$

the coefficient a is nonpositive and all the coefficients have finite Hölder norms,

$$|a^{ij}|_{0,\alpha}, |a^i|_{0,\alpha}, |a|_{0,\alpha} \leq K_2, \quad (\forall)\, i, j.$$

Let $N = N(x, dy)$ be a positive kernel on $\mathbb{R}^d$ such that $N(x, \{x\}) = 0$, for all $x \in \mathbb{R}^d$ and

$$N(x, \mathbb{R}^d \setminus B(x, 1)) = 0, \quad x \in \mathbb{R}^d \tag{2.2}$$

$$\int_{B(x,r)} |y - x|^2 N(x, dy) \leq \rho(r), \quad (\forall) \quad x \in \mathbb{R}^d, r \in (0, 1], \tag{2.3}$$

$$\left|\int_{B(x,r)} u(y - x)|y - x|^2 N(x, dy) - \int_{B(x',r)} u(y - x')|y - x'|^2 N(x', dy)\right|$$
$$\leq \rho(r)|x - x'|^\alpha |u|_0 \quad , \quad (\forall) \quad x, x' \in \mathbb{R}^d, u \in \mathcal{E}_b(\mathbb{R}^d), r \in (0, 1] \tag{2.4}$$

where $\rho : (0, 1] \to (0, \infty)$ is a function such that $\lim_{r\to 0} \rho(r) = 0$ and $B(x, r)$ denotes the closed ball of center x and radius r. Note that condition (2.3) with $r = 1$ gives the usual finiteness required to a Lévy kernel, while as $r \to 0$ we get a uniform integrability condition.

For $u \in C^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we define

$$\widetilde{\widetilde{N}}u(x) := \int \left(u(y) - u(x) - \sum_{i=1}^{d} \partial_i u(x)(y^i - x^i)\right) N(x, dy).$$

The integro-differential operator

$$W := L + \widetilde{\widetilde{N}}$$

is a Waldenfels operator (see [2]) and N is called the Lévy kernel of W. Let $\mathcal{W} := W - \partial_t$ be the associated parabolic operator on $\mathbb{R} \times \mathbb{R}^d$. By a sign change, $t \mapsto -t$, this operator is of the type studied in [7]. Theorem 8 from [7] gives the following result:

Theorem 2.1. *If $t_o > 0$ and $f \in C_o^{0,\alpha}([0,t_o]\times\mathbb{R}^d)$ then there exists a unique function $u \in C_o^{2,\alpha}([0,t_o]\times\mathbb{R}^d)$ such that*

$$\mathcal{W}u = f \text{ and } u(0,\cdot) = 0.$$

Moreover there exists a constant $c = c(t_o, d, \alpha, K_1, K_2, \rho)$ *such that*

$$|u|_{2,\alpha} \le c|f|_{0,\alpha}. \tag{2.5}$$

(Here $C_o^{\beta,\alpha}$ denotes the closure of the space C_o^∞ of all infinite differentiable functions of compact support with respect to the Hölder norm $|\ \ |_{\beta,\alpha}$.)

We remark that $\mathcal{W}$ has the positive maximum principle because $a \le 0$. Namely, if $u \in C_o^2([0,t_o]\times\mathbb{R}^d)$ has a positive maximum at the point $(t,u) \in (0,t_o]\times\mathbb{R}^d$ then $\mathcal{W}u(t,x) \le 0$. This implies the following properties:

1°. If $u \in C_o^2([0,t_o]\times\mathbb{R}^d)$, $u_{|t=0} \le 0$ and $\mathcal{W}u \ge 0$ then $u \le 0$.

2°. If $u \in C_o^2([0,t_o]\times\mathbb{R}^d)$ satisfies $\mathcal{W}u = 0$ then $|u|_o \le |u_{|t=0}|_o$.

From Theorem 2.1 we get the following:

3°. If $f \in C_o^{2,\alpha}(\mathbb{R}^d)$ then there exists a unique function $u \in C_o^{2,\alpha}([0,\infty)\times\mathbb{R}^d)$ such that $\mathcal{W}u = 0$ and $u(0,\cdot) = f$. Moreover for each $t_o > 0$ there exists a constant c such that

$$|u_{|[0,t_o]\times\mathbb{R}^d}|_{2,\alpha} \le c|f|_{2,\alpha}. \tag{2.6}$$

We introduce the notation $P_t f(x) = u(t,x)$ for f and u related as in 3° above. For each $t > 0$ we have a linear operator $P_t : C_o^{2,\alpha} \longrightarrow C_o^{2,\alpha}$. According to 1° this operator is monotone (i.e. $f \ge 0$ implies $P_t f \ge 0$) and from 2° we deduce that it is a contraction (i.e. $|P_t f|_o \le |f|_o$). Obviously the family $(P_t)_{t>0}$ is a semi-group which admits a unique extension, denoted by the same symbol, to the space C_o of all continuous functions in $\mathbb{R}^d$ vanishing to infinity. It is easy to see that $(P_t)_{t>0}$ is a Feller semi-group whose infinitesimal operator extends W.

Let us put now

$$N_r(x,dy) := \chi_{B(x,r)}\cdot N(x,dy) \quad , \quad M_r(x,dy) := N(x,dy) - N_r(x,dy). \tag{2.7}$$

As $r \longrightarrow 0$, M_r approximate N. The operators $W_r := L + \widetilde{\widetilde{M_r}}$ and $W_r' := L + \widetilde{\widetilde{N_r}}$ are similar to W so that they generate Feller semi-groups $(P_t^r)_{t>0}$ and $(P_t'^r)_{t>0}$ respectively. Denote by $(Q_t)_{t>0}$ the semi-group generated by L. A straightforward computation leads to the following estimate

$$|\widetilde{\widetilde{N_r}}u|_{0,\alpha} \le c\rho(r)|u|_{2,\alpha} \quad , \quad (\forall) \quad u \in C^{2,\alpha}(\mathbb{R}^d). \tag{2.8}$$

This allows us to prove the following result:

Proposition 2.2. *For any $f \in C_o$ and $t > 0$ we have*

$$\lim_{r \to 0} |P_t^r f - P_t f|_o = 0 \quad , \quad \lim_{r \to 0} |P_t'^r f - Q_t f|_o = 0.$$

Proof. Let $f \in C_o^\infty$ and set $u(t,x) := P_t f(x)$, $v(t,x) := P_t^r f(x)$. Then we have $(L + \widetilde{\widetilde{M_r}} - \partial_t)(v - u) = \widetilde{\widetilde{N}}_r u$. Combining (2.5), (2.8) and (2.6) we get

$$|P_t^r f - P_t f|_{2,\alpha} \leq c\rho(r)|f|_{2,\alpha}.$$

Letting $r \to 0$ we get $\lim_{r \to 0} |P_t^r f - P_t f|_o = 0$. Since C_o^∞ is dense in C_o and the operators P_t^r and P_t are contractions, this relation extends to any function $f \in C_o$. The second convergence is checked similarly.

Now let us consider the case of a first order Lévy kernel. More precisely we suppose that N is a kernel on $\mathbb{R}^d$ such that $N(x, \{x\}) = 0$ for all $x \in \mathbb{R}^d$, (2.2) holds and the following conditions are satisfied:

(2.9)
$$\int_{B(x,r)} |y - x| N(x, dy) \leq \rho_\infty \quad , \quad x \in \mathbb{R}^d, r \in (0,1],$$

(2.10)
$$\left| \int_{B(x,r)} u(y-x)|y-x| N(x,dy) - \int_{B(x',r)} u(y-x')|y-x'| N(x',dy) \right|$$
$$\leq \rho(r)|x - x'|^\alpha |u|_0 \quad , \quad x, x' \in \mathbb{R}^d, u \in \mathcal{E}_b(\mathbb{R}^d)$$

where $\lim_{r \to 0} \rho(r) = 0$. We denote by $\widetilde{N}$ the operator

$$\widetilde{N}u(x) := \int (u(y) - u(x))\, N(x, dy), \quad (\forall) u \in C^1(\mathbb{R}^d).$$

An integro-differential operator W has the first order Lévy kernel N if it is of the form

$$W = L + \widetilde{N},$$

where L is a second order elliptic differential operator with Hölder coefficients as in (2.1). Such an operator W may also be written in the form $W = L' + \widetilde{\widetilde{N}}$, where $L' := L + \sum_{i=1}^d b^i \partial_i$, with $b^i(x) := \int (y^i - x^i) N(x, dy)$. Conditions assumed ensure that $b^i \in C^{0,\alpha}(\mathbb{R}^d)$ and N satisfies (2.3) and (2.4).Consequently the preceding results may be applied to W. In particular there is a Feller semi-group associated with W. If N_r and M_r are the kernels derived from N as in (2.7), the operators $W_r := L + \widetilde{M_r}$ and $W_r' := L + \widetilde{N}_r$ generate semi-groups $(P_t^r)_{t>0}$ and $(P_t'^r)_{t>0}$ that are Fellerian. Similar to (2.8) the following inequality holds:

$$|\widetilde{N}_r u|_{0,\alpha} \leq \rho(r) c |u|_{1,\alpha}, \quad (\forall) u \in C^{1,\alpha}(\mathbb{R}^d).$$

As a consequence we have a result analogous to Proposition 2.2 :

Proposition 2.2'. *The assertion from Proposition 2.2 holds true in the case of the first order Lévy kernels.*

Under the assumtion that N is a first order Lévy kernel, let us denote by X, Y, Y^r, Y'^r the processes having respectively the semi-groups $(Q_t)_{t>0}$, $(P_t)_{t>0}$, $(P_t^r)_{t>0}$, $(P_t'^r)_{t>0}$. Relation (1.1) is satisfied by the processes Y, Y^r and Y'^r with the kernels N, M_r and N_r (cf. Théorème 10 in [5]). The process Y^r can be constructed from X by killing with a multiplicative functional and resurrecting like in Theorem 1.8, so introducing jumps counted by the bounded kernel M_r. On the other hand the process Y'^r can be obtained from Y by eliminating the jumps larger than r, with the procedure from Theorem 1.6.

If N is not a first order Lévy kernel but just satisfies conditions (2.2)-(2.4) we still preserve the notation X, Y, Y^r, Y'^r for the processes associated with the semi-groups $(Q_t)_{t>0}$, $(P_t)_{t>0}$, $(P_t^r)_{t>0}$, $(P_t'^r)_{t>0}$. Then the probabilistic relations between X and Y^r or Y and Y'^r are a little bit more complicated. Since $\widetilde{\widetilde{M_r}} = \widetilde{M_r} - \sum_{i=1}^{d} b^i \partial_i$, the process Y^r is obtained from X first transforming it as in Theorem 1.8 with the kernel M_r and then introducing the effect of the drift $-\sum_{i=1}^{d} b^i \partial_i$. The probabilistic interpretation of the drift modification for processes with jumps is analogous to the Cameron-Martin-Girsanov transformation in the case of diffusions (see Théorème 25 in [5]). The process Y'^r can be constructed from Y by using Theorem 1.6 with the kernel M_r and then taking into account the influence of the drift $\sum_{i=1}^{d} b^i \partial_i$.
Propositions 2.2 and 2.2' imply the following conclusion:

Corollary 2.3. *If either N satisfies* (2.9), (2.10) *and $W = L + \widetilde{N}$ or N satisfies* (2.3) , (2.4) *and $W = L + \widetilde{\widetilde{N}}$ then the process Y is the limit in distribution of the processes Y^r and the diffusion X is the limit in distribution of the processes Y'^r , as r tends to zero.*
For the proof see Theorem 1.6.1 and 4.2.5 in [3]

3 Pure jump processes on metric spaces

Let (E, d) be a complete separable metric space. We denote by $\mathcal{E}$ the σ-algebra of all Borel measurable subsets of E. Let $(U, \mathcal{U})$ a measurable space on which a σ-finite measure n is fixed and let $\varphi : E \times U \longrightarrow E$ an $\mathcal{E} \otimes \mathcal{U}/\mathcal{E}$-measurable function. We suppose that the following conditions are satisfied:
(3.1) $\int_U d(x, \varphi(x,u))\mathrm{n}(du) \leq c$, for all $x \in E$;
(3.2) There exists an increasing sequence $(U_k)_{k\in\mathbb{N}} \subseteq \mathcal{U}$ with $\mathrm{n}(U_k) < \infty$ for all $k \in \mathbb{N}$ such that $\lim_{k\to\infty} a_k = 0$ where $a_k := \sup_{x\in E} \int_{U\setminus U_k} d(x, \varphi(x,u))\mathrm{n}(du)$;
(3.3) $d(\varphi(x,u), \varphi(x',u)) \leq d(x,x') + cd(x,x') \cdot d(x, \varphi(x,u))$, for all $x, x' \in E$, $u \in U$;
where c is a positive constant.

Example. Let (E, d) be a metrizable compact space, $o \in E$ a fixed point, n a Radon measure on $E \setminus \{o\}$ with $\int d(o,x)\mathrm{n}(dx) < \infty$ and a function $\varphi : E \times E \longrightarrow E$ such that $\varphi(x, o) = x$ and $d(\varphi(x,y), \varphi(x',y')) \leq d(x,x') + d(y,y')$ for all x, x',

$y, y' \in E$. If we take $U := E \setminus \{o\}$ then condition (3.1)-(3.3) are fulfiled.

If for all $f \in \mathcal{E}_b$ and $x \in E$ we put

$$Nf(x) := \int_U f(\varphi(x,u))\mathrm{n}(du)$$

then we obtain a kernel on E which is of first order by (3.1). As in Section 2 we associate to this *Lévy kernel* N the operator $\widetilde{N}$ defined by $\widetilde{N}f(x) := \int (f(y) - f(x))\, N(x, dy)$. It is well defined, following (3.1), at least in the case when f is Lipschitz continuous on E.

The following existence result for a Markov process generated by $\widetilde{N}$ will be a consequence of Theorem 3.4 below:

Theorem 3.1. *There exists a quasi-left-continuous, strong Markov process on E having cadlag trajectories and for which the ifinitesimal operator contains in its domain the Lipschitz continuous functions on E and coincides with $\widetilde{N}$ on these functions.*

In fact this process will be a jump process with trajectories of "bounded variation" (as it is suggested by the infinitesimal operator which is associated to a first order Lévy kernel). The next Lemma gives us the convenient notion of jump trajectory with bounded variation. The proof is left to the reader.

Lemma 3.2. *Let $f : [0, \infty) \longrightarrow E$ be a function which is right continuous and has left limits and let $D \subset (0, \infty)$ be at most countable such that*

$$\sum_{s \in D} d(f(s-), f(s)) < \infty$$

and if $0 \leq t_1 \leq t_2$ then

$$d(f(t_1), f(t_2)) \leq \sum_{s \in (t_1, t_2] \cap D} d(f(s-), f(s)).$$

The function f is then continuous at each point of $[0, \infty) \setminus D$ and constant on each open interval which does not contain points of D. Moreover if $g : E \longrightarrow \mathbb{R}$ is Lipschitz continuous then

$$g(f(t_2)) - g(f(t_1)) = \sum_{s \in (t_1, t_2] \cap D} (g(f(s)) - g(f(s-)))\,.$$

The starting point for the construction of a jump process is the Poisson point process (with characteristic measure n) which will generate the jumps. Therefore we begin with some considerations concerning the point functions.

Let $p : D_p \longrightarrow U$ be a point function, $D_p \subset (0, \infty)$ being at most countable, such that the following condition is satisfied:

(3.4) $\quad \#\{t \in D_p \cap (0, u] / p_t \in U_k\} < \infty$, for all $u > 0, k \in \mathbb{N}$,

where $p_t := p(t)$. For any $k \in \mathbb{N}$ we put $D_k := \{t \in D_p / p_t \in U_k\}$ and we define the following sequence (which depends on k):

$$\tau_0 := 0\,,\ \tau_{m+1} = \inf\{t > \tau_m / t \in D_k\}.$$

In fact, in our case we will have $\lim_{u\to\infty} \#((0,u]\cap D_k)=\infty$ and therefore τ_m will be finite for any $m\in\mathbb{N}$. If $\xi\in E$ we may define a trajectory as follows:
$Y_t^k(\xi,p):=\xi$ if $t\in[0,\tau_1)$, $Y_t^k(\xi,p):=\varphi(Y_{\tau_m}^k(\xi,p),p_{\tau_m})$ if $t\in[\tau_m,\tau_{m+1})$.

Lemma 3.3. *Let us suppose that:*
(3.5) there exists $\lim_{k\to\infty} Y_t^k(\xi,p)$ *uniformly on each compact interval* $[0,u]$, $u>0$.
Then the limit trajectory

$$Y_t(\xi,p):=\lim_{k\to\infty} Y_t^k(\xi,p)$$

has the following properties:
(3.6) $Y_0(\xi,p)=\xi$;
(3.7) the trajectory $t\mapsto Y_t(\xi,p)$ *is right continuous and has left limits;*
(3.8) $Y_t(\xi,p)=\varphi(Y_{t-}(\xi,p),p_t)$, *for all* $t\in D_p$;
(3.9) $d(Y_s(\xi,p),Y_t(\xi,p))\le\sum_{u\in D_p\cap(s,t]} d(Y_{u-}(\xi,p),Y_u(\xi,p))$ *if* $s\le t$.
The proof is obvious.

Let now p be a Poisson point process on a probability space $(\Omega,\mathcal{F},P)$ with values in $(U,\mathcal{U})$ having the characteristic measure n and $(\mathcal{F}_t)_t$ a filtration with respect to which p becomes an $(\mathcal{F}_t)_t$-adapted point process (see [4]). Recall that if for any $t,s\ge 0$ with $t+s\in D$ we put $\theta_t p_s := p_{t+s}$ then $\theta_t p$ is an $(\mathcal{F}_{s+t})_s$-adapted Poisson point process with the same characteristic measure as p. Since by hypothesis $\mathrm{n}(U_k)<\infty$ for all $k\in\mathbb{N}$, we deduce that condition (3.4) is verified a.s. by the Poisson point process p. In the sequel τ_m , $Y_t^k(\xi,p)$ and $Y_t(\xi,p)$ will appear naturally randomized. In this way $(\tau_m)_m$ will be a sequence of stopping times. For a random variable ξ, Y^k and Y become processes.

Theorem 3.4. *If* $\xi : U\longrightarrow E$ *is an* $\mathcal{F}_0$*-measurable random variable then condition (3.5) is satisfied in probability (where* $\xi=\xi(\omega)$ *and* $p=p(\omega)$, $\omega\in\Omega$*). Moreover the following assertions hold:*

a) If we define

$$X_t(\omega):=Y_t(\xi(\omega),p(\omega))\,,\ \omega\in\Omega$$

then a.s.

$$\text{(3.10)}\qquad \sum_{u\in D_p\cap[0,t]} d(X_{u-},X_u)<\infty\ ,\ (\forall)t\ge 0$$

and X_t *is the unique adapted process which satisfies a.s. (3.6)-(3.10).*

b) If we put $Z_t(\omega):=Y_t(\eta(\omega),p(\omega))$, $\omega\in\Omega$, *where* $\eta : U\longrightarrow E$ *is another* $\mathcal{F}_0$*-measurable random variable then*

$$E\left[\sup_{s\le t} d(X_s,Z_s)\right]\le e^{c^2 t}E\left[d(\xi,\eta)\right].$$

c) The following equality is satisfied a.s.

$$Y_{t+s}(x,p(\omega))=Y_s(Y_t(x,p(\omega)),\theta_t p(\omega))\,,\ x\in E.$$

Proof. We show that $(Y^k)_{k\in\mathbb{N}}$ is a Cauchy sequence. For any $j \geq k$ we introduce the notation:

$$q_{k,j}(t) := \sup_{s\leq t} d(Y_s^k(\xi,p), Y_s^j(\xi,p)),$$

$$r_{k,j}(t) := \sum_{s\in D_p\cap[0,t]} \chi_{U_k}(p_s) d\left(Y_{s-}^k(\xi,p), Y_{s-}^j(\xi,p)\right)\cdot d\left(Y_{s-}^k(\xi,p), \varphi(Y_{s-}^k(\xi,p),p_s)\right)$$

$$= \int_0^t \int_{U_k} d\left(Y_{s-}^k(\xi,p), Y_{s-}^j(\xi,p)\right)\cdot d\left(Y_{s-}^k(\xi,p), \varphi(Y_{s-}^k(\xi,p),u)\right) N_p(ds,du),$$

$$r'_{k,j}(t) := \sum_{s\in D_p\cap[0,t]} \chi_{U_j\setminus U_k}(p_s) d\left(Y_{s-}^j(\xi,p), Y_s^j(\xi,p)\right)$$

$$= \sum_{s\in D_p\cap[0,t]} \chi_{U_j\setminus U_k}(p_s) d\left(Y_{s-}^j(\xi,p), \varphi(Y_{s-}^j(\xi,p),p_s)\right)$$

$$= \int_0^t \int_{U_j\setminus U_k} d\left(Y_{s-}^j(\xi,p), \varphi(Y_{s-}^j(\xi,p),u)\right) N_p(ds,du).$$

If $s = \tau_{m+1}$ and $v = \tau_m$, where τ_m, $m \in \mathbb{N}$, are the stopping times related to U_k, then using also (3.3) we have:

$$d\left(Y_s^k(\xi,p), Y_s^j(\xi,p)\right) = d\left(\varphi(Y_{s-}^k(\xi,p),p_s), \varphi(Y_{s-}^j(\xi,p),p_s)\right)$$

$$\leq d\left(Y_{s-}^k(\xi,p), Y_{s-}^j(\xi,p)\right) + c d\left(Y_{s-}^k(\xi,p), Y_{s-}^j(\xi,p)\right)\cdot d\left(Y_{s-}^k(\xi,p), \varphi(Y_{s-}^k(\xi,p),p_s)\right);$$

$$d\left(Y_{s-}^k(\xi,p), Y_{s-}^j(\xi,p)\right)$$

$$\leq d\left(Y_v^k(\xi,p), Y_v^j(\xi,p)\right) + \sum_{v<t<s, t\in D_p} \chi_{U_j\setminus U_k}(p_t) d\left(Y_{t-}^j(\xi,p), Y_t^j(\xi,p)\right).$$

Therefore we get $q_{k,j} \leq c\cdot r_{k,j}(t) + r'_{k,j}(t)$. Theorem 3.1 in [4] allows us to replace the integrals with $N_p(ds,du)$ by integrals with $ds\mathrm{n}(du)$, Y_{s-}^k and Y_{s-}^j being predictable processes. By (3.1) we obtain

$$E\left[r_{k,j}\right] = E\left[\int\int_{(0,t]\times U_k} d\left(Y_{s-}^k(\xi,p), Y_{s-}^j(\xi,p)\right) d\left(Y_{s-}^k(\xi,p), \varphi(Y_{s-}^k(\xi,p),u)\right) \mathrm{n}(du)ds\right]$$

$$\leq c\cdot E\left[\int_{(0,t]} d\left(Y_{s-}^k(\xi,p), Y_{s-}^j(\xi,p)\right) ds\right].$$

We conclude that $E[q_{k,j}(t)] \leq c^2 \int_0^t E[q_{k,j}(s)]ds + ta_k$ and by Gronwall's lemma we get

$$E[q_{k,j}(t)] \leq ta_k e^{c^2 t}.$$

Since by (3.2) we have $\lim_{j,k\to\infty} E[q_{k,j}(t)] = 0$ one can find a subsequence $(Y^{k_i}(\xi,p))_{i\in\mathbb{N}}$ which converges uniformly a.s. on the trajectories to $Y(\xi,p)$ i.e.

$$\lim_{i\to\infty} \sup_{s\leq t} d\left(Y_s^{k_i}(\xi,p), Y(\xi,p)\right) = 0\,,\ (\forall)t \geq 0\,a.s.$$

From Lemma 3.3 it follows that the process $X_t := Y_t(\xi,p)$ verifies conditions (3.6)-(3.9). To check (3.10) we compute

$$E\left[\sum_{s\in D_p\cap[0,t]} d(X_{s-},X_s)\right] = E\left[\int_U \int_0^t d(X_{s-},\varphi(X_s,u))\,\mathrm{n}(du)ds\right].$$

By (3.1) it results that the right hand term in the above relation is dominated by ct which implies the finiteness asserted in (3.10). Let us now prove the uniqueness. If $(Y_t)_t$ is another process verifying a.s. condition (3.6)-(3.10) then we define

$$q_k(t) := \sup_{s\leq t} d(Y_s^k(\xi,p),Y_s)\,,\ r_k'(t) := \sum_{s\in D_p\cap[0,t]} \chi_{U\setminus U_k}(p_s) d(Y_{s-},Y_s)\,,$$

$$r_k(t) := \sum_{s\in D_p\cap[0,t]} \chi_{U_k}(p_s) d\left(Y_{s-}^k(\xi,p),Y_{s-}\right)\cdot d\left(Y_{s-}^k(\xi,p),\varphi(Y_{s-}^k(\xi,p),p_s)\right).$$

As before we obtain $q_k \leq c\cdot r_k(t) + r_k'(t)$ and

$$E[q_k(t)] \leq c^2 \int_0^t E[q_k(s)]ds + ta_k.$$

Again from Gronwall's lemma we get

$$E[q_k(t)] \leq ta_k e^{c^2 t} \tag{3.11}$$

and in the limit we deduce $Y = X$ a.s.

Let us prove now the inequality from assertion b). For the approximation sequences $Y_t^k(\xi,p)$ and $Y_t^k(\eta,p)$ we have the estimate:

$$d\left(Y_t^k(\xi,p),Y_t^k(\eta,p)\right)$$

$$\leq c \sum_{s\in D_p\cap[0,t]} d\left(Y_{s-}^k(\xi,p),Y_{s-}^k(\eta,p)\right) d\left(Y_{s-}^k(\xi,p),\varphi(Y_{s-}^k(\xi,p),p_s)\right) + d(\xi,\eta).$$

By the method used above we obtain the desired inequality.

The equality from c) is verified by each of the processes $Y_t^k(x,p)$, Moreover we have:

$$d(Y_s^k(Y_t^k(x,p),\theta_t p), Y_s(Y_t(x,p),\theta_t p))$$
$$\leq d(Y_s(Y_t^k(x,p),\theta_t p), Y_s(Y_t(x,p),\theta_t p)) + d(Y_s^k(Y_t^k(x,p),\theta_t p), Y_s(Y_t^k(x,p),\theta_t p)).$$

From assertion b) and (3.11) we get now: $E\left[d(Y_s^k(Y_t^k(x,p),\theta_t p), Y_s(Y_t(x,p),\theta_t p))\right]$ $\leq e^{c^2 s} E\left[d(Y_t^k(x,p), Y_t(x,p)\right] + s e^{c^2 s} a_k \leq (t e^{c^2(s+t)} + s e^{c^2 s}) a_k$. When k tends to infinity we deduce a convergence on a subsequence and the relation from c) follows.

Proof of Theorem 3.1. Obviously the process starting from $x \in E$ will be given by Theorem 3.4 taking $\xi(\omega) = x$. We consider the canonical trajectory space for this process. More precisely let

$$\mathcal{W} := \{w : [0,\infty) \longrightarrow E / w \text{ is cadlag and satisfies (3.9) and (3.10) }\},$$

$$P^x := P \circ Y(x,p)^{-1}, \, x \in E \quad , \quad X_t(w) := w(t), \, w \in \mathcal{W}.$$

Let now $f \in \mathcal{C}_l$ (:= the real valued Lipschitz continuous functions on E). From Theorem 3.4 b) we deduce that the function $P_t f(x)$ on E defined by

$$P_t f(x) := E^x[f(X_t)] = E[f(Y_t(x,p))]$$

is Lipschitz continuous and using monotone class arguments it is $\mathcal{E}$-measurable for all $f \in \mathcal{E}$. The Markov property follows from

$$E[f(Y_{t+s}(x,p)/\mathcal{F}_t] = E[f(Y_s(Y_t(x,p),\theta_t p))/\mathcal{F}_t] = P_s f(X_t)$$

where we have used assertion c) from Theorem 3.4 as well as the fact that $\theta_t p$ is independent from $\mathcal{F}_t$ and identically distributed with p.

If $k \in \mathbb{N}$ let $\mathrm{n}_k := \chi_{U_k} \cdot \mathrm{n}$ and $\widetilde{N}^k$ be the corresponding operator

$$\widetilde{N}^k f(x) := \int_{U_k} [f(\varphi(y,u)) - f(x)] \mathrm{n}(du).$$

From Lemma 3.1 it follows that the process $X_t^k := Y_t^k(x,p)$ is the solution of the martingale problem associated to the bounded operator $\widetilde{N}^k$. By Theorem 4.4.1 in [3] we deduce that X_t^k is a Markov process and its semi-group has the infinitesimal generator $\widetilde{N}^k$. Moreover from (3.11) we get

$$\sup_{x \in E} |P_t f(x) - P_t^k f(x)| \leq K \cdot E[q_k(t)] \leq K t a_k e^{c^2 t},$$

where K is the Lipschitz constant of the function $f \in \mathcal{C}_l$. Moreover the following estimates hold for any $x \in E$:

$$\left|\frac{P_t f(x) - f(x)}{t} - \widetilde{N} f(x)\right| \leq \left|\frac{P_t f(x) - f(x)}{t} - \widetilde{N}^k f(x)\right| + \left|\frac{P_t f(x) - P_t^k f(x)}{t}\right| + K a_k.$$

The strong Markov property and the quasi-left-continuity follow now as in the Fellerian case (see ch.I (8.11) and (9.4) in [1]).

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