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PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS

OF THE ROMANIAN ACADEMY

ISSN 0250 3638

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EXISTENCE AND UNICITY

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PREPRINT No.9/1994

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May, 1994

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Dilation operators in excessive structures; Existence and unicity

by N. Boboc and Gh. Bucur

Introduction

Let $\mathcal{V} = (V_\alpha)_{\alpha \geq 0}$ be a proper submarkovian resolvent of kernels on a measurable space $(X, \mathcal{B})$ such that the set $\mathcal{E}_\mathcal{V}$ of all $\mathcal{V}$-excessive functions on X which are finite $\mathcal{V}$-a.s. is min-stable, contains the positive constant functions and generates $\mathcal{B}$. We suppose that X is a Lusin space and that it is semisaturated with respect to $\mathcal{V}$ (i.e any $\mathcal{V}$-excessive measure on X, dominated by an $\mathcal{V}$-excessive measure on X of the form $\mu \circ V$ is also of the same form). The above conditions are equivalent with the fact that there exists a right process on $(X, \mathcal{B})$ for which $\mathcal{V}$ is the associated resolvent.

In the paper ([4],[5]) we consider two submarkovian resolvents $\mathcal{V} = (V_\alpha)_{\alpha \geq 0}$, $\mathcal{W} = (W_\alpha)_{\alpha \geq 0}$ on $(X, \mathcal{B})$ which possesses a reference measure and such that the absorbent points with respect to $\mathcal{V}$ and $\mathcal{W}$ are the same. If $\mathcal{E}_\mathcal{W} \subset \mathcal{E}_\mathcal{V}$, if any $s \in \mathcal{E}_\mathcal{V}$ is lower semicontinuous with respect to the fine topology generated by $\mathcal{E}_\mathcal{W}$ and if for any $A \in \mathcal{B}$ and any positive $\mathcal{B}$-measurable function f we have

$$\mathcal{V}_{B^A} f \leq \mathcal{W}_{B^A} f$$

(where $\mathcal{V}_{B^A}$ (resp. $\mathcal{W}_{B^A}$) is the balayage on A with respect to $\mathcal{E}_\mathcal{V}$ (resp. $\mathcal{E}_\mathcal{W}$)) then there exists an other submarkovian resolvent $\overline{\mathcal{V}} = (\overline{V}_\alpha)_{\alpha \geq 0}$ such that $\mathcal{E}_\mathcal{V} = \mathcal{E}_{\overline{\mathcal{V}}}$ and such that

$$\overline{V}_\alpha f \leq W_\alpha f \quad \forall\, \alpha \geq 0 \text{ and } f \geq 0, \quad \mathcal{B}-\text{measurable}.$$

In fact there exists a kernel Q on $(X, \mathcal{B})$ such that $Q(\mathcal{E}_\mathcal{W}) \subset \mathcal{E}_\mathcal{W}$,

$$s \wedge (Qs + t - Qt + Qf) \in \mathcal{E}_\mathcal{W}$$

for any $s, t \in \mathcal{E}_\mathcal{W}$ and any positive $\mathcal{B}$-measurable function f on X and such that

$$Wf = \overline{V} f + QWf \quad \forall\, f \geq 0, \quad \mathcal{B}-\text{measurable}.$$

This type of kernels was firstly considered by G. Mokobodzki ([11]) in conection with the subordination in excessive structures. Moreover if Q is such a kernel for which

$$Wf \text{ bounded} \Longrightarrow \inf_n Q^n Wf = 0$$

then for any $u \in \mathcal{E}_\mathcal{V}$ we have

$$u \in \mathcal{E}_\mathcal{W} \Longleftrightarrow Qu \preceq_{\mathcal{E}_\mathcal{V}} u.$$

This last problem was considered recently by R. K. Getoor and M. J. Sharpe ([10]) in the frame of the right processes without reference measure.

In this paper we deal with the above problems in the general frame of H-cones. We give two H-cones S, T such that S is an H-subcone of T (i.e. $S \subset T$ and for any $M \subset S$ we have $\wedge_S M = \wedge_T M$ and respectively $\vee_S M = \vee_T M$ if moreover M is increasing and dominated in S). Here S (resp. T) is instead of $\mathcal{E}_\mathcal{W}$ (resp. $\mathcal{E}_\mathcal{V}$) in the preceding considerations. A map $Q : S \longrightarrow S$ is called a (S,T)-dilation operator on S if

1) $s_1, s_2 \in S,\ s_1 \leq s_2 \Longrightarrow Qs_1 \preceq_T Qs_2 \preceq_T s_2$

2) $s \in S,\ u \in T,\ Qs \preceq_T u \leq s \Longrightarrow u \in S$.

It is proved that if Q verifies 1) then the property 2) is equivalent with each of the following properties:

3) $s_1 \wedge (Qs_1 + s_2 - Q_2 + Qf) \in S \quad \forall\, s_1, s_2 \in S,\ f \in (S-S)_+$

4) $s_1, s_2 \in S,\ s_1 - Qs_1 \leq s_2 - Qs_2 \Longrightarrow s_1 - Qs_1 + Qs_2 \in S;\ s \in S,\ t \in T \Longrightarrow \exists s' \in S; (s - Qs) \wedge t = s' - Qs'$

If B is a balayage on T then the operator $B^{\#}$ on S defined by

$$B^{\#}s = \wedge\{s' \in S \mid s' \geq Bs\}$$

is a balayage on S. If Q is a (S,T)-dilation operator then we have

5) $B^{\#}f - Bf = QB^{\#}f - B(QB^{\#}f) \quad \forall\, f \in (S-S)_+$

and therefore

6) $Bf \leq B^{\#}f \quad \forall\, f \in (S-S)_+$.

Under suplimentary conditions (which are quite natural) we proved that if Q verifies 1) then 2) $\Longleftrightarrow$ 5). Moreover if the property 6) holds then there exists a minimal (S,T)-dilation operator P on S (i.e an (S,T)-dilation operator on S such that any other (S,T)-dilation operator Q verifies the relation

$$Ps \preceq_T Qs \qquad \forall\, s \in S.$$

We consider also the problem when there is a unique (S,T)-dilation operator. It is shown for instance that the unicity holds in one of the following situations:

a) T contains sufficiently many quasi-continuous elements, does not exists absorbent balayages on T and there exists a balayage B on S which is a (S,T)-dilation operator.

b) $S = \mathcal{E}_{\mathcal{W}}$ where $\mathcal{W} = (W_\alpha)_{\alpha>0}$ is a proper submarkovian resolvent on $(X,\mathcal{B})$ as in the begining of this introduction such that there is no fine open sets of the form $\{x\}$ and T is the H-cone of all α-excessive functions.

Finally we consider the problem when given a (S,T)-dilation operator Q on S we can extend it to a map $\tilde{Q} : D(Q) \longrightarrow T$ defined on a solid subcone $D(Q)$ of T such that $S \subset D(Q)$ and such that we have, for any $u \in D(Q)$, the relation

$$u \in S(Q) \Longleftrightarrow \tilde{Q}u \preceq_T u.$$

If $\mathcal{V}$, $\mathcal{W}$ are two proper submarkovian resolvents on $(X,\mathcal{B})$ which have all properties from the begining of this introduction except the existence of reference measure, we can apply the above considerations to the H-cone $Exc_{\mathcal{V}}$ and $Exc_{\mathcal{W}}$ of excessive measures on $(X,\mathcal{B})$ associated with $\mathcal{V}$ and $\mathcal{W}$ respectively.

1. Localizable dilation operators in H-cones

Let S be an H-cone. We recall ([4], [5]) that a map $P : S \longrightarrow S$ is called a localizable dilation operator ($l.d$-operator) on S if P is additive, increasing, contractive, continuous in order from below and if for any $s, t \in S$ and any $f \in (S-S)_+$ we have

$$s \wedge (Ps + t - Pt + Pf) \in S.$$

It is known that if P is an $l.d$-operator on S then the convex cone S_P of $S-S$ given by

$$S_P := \{s - Ps \mid s \in S\}$$

endowed with the natural order relation from $S-S$ is also an H-cone. Moreover we have

$$t \in S,\ u \in S_P \Longrightarrow t \wedge u \in S_P$$

$$f \in (S-S)_+,\ u \in S_P \Longrightarrow Pf \wedge u \in S_P$$

and for any $s \in S$ there exists a unique $s_0 \in S$ such that $s - Ps = s_0 - Ps_0$ and if $t \in S$ is such that $s_0 - Ps_0 \leq t - Pt$ then $s_0 \leq t$. (See [5])

Concerning the lattice operations on the H-cone S_P we remember that for any subset A of S_P (resp. any upper directed and dominated subset A in S_P) we have

$$\bigwedge_{S_P} A = \bigwedge_{S-S} A \quad \text{(resp. } \bigvee_{S_P} A = \bigvee_{S-S} A).$$

Remark. If $\mathcal{V} = (V_\alpha)_{\alpha > \alpha}$ is a proper submarkovian resolvent on a measurable space $(X, \mathcal{B})$ and there exists a reference measure then the cone $\mathcal{E}_\mathcal{V}$ of all $\mathcal{V}$-excessive functions on X which are finite $\mathcal{V}$-a.s. is an H-cone. In this case, supposing that X is semisaturated (i.e any H-integrale on $\mathcal{E}_\mathcal{V}$ dominated by a measure on X is also represented as a measure on X) then the above notion of localizable dilation operator in nothing else then a kernel of subordination on $\mathcal{E}_\mathcal{V}$ is the sense of Mokobodzki [11].

We remember that for any $f \in (S-S)_+$ the map $B_f : S \longrightarrow S$ given by $B_f^S s = B_f s = \wedge\{t \in S \mid t \geq s \wedge (nf),\ (\forall)\, n \in \mathrm{N}\} = \vee_{n \in \mathrm{N}} R(s \wedge (nf))$ is a balayage on S.

Proposition 1.1. Let P be a $l.d$-operator on S. The following assertions are equivalent:

a) For any $s, t \in S$ such that

$$\{u \in S_P \mid u \leq s\} = \{u \in S_P \mid u \leq t\}$$

we have $s = t$.

b) For any $s, t \in S$ such that

$$\{u \in S_P \mid u \leq s\} \subset \{u \in S_P \mid u \leq t\}$$

we have $s \leq t$.

c) For any $f \in (S-S)_+$ such that $B_f \leq P$ we have $f = 0$.

Proof. Obviously a) $\Longleftrightarrow$ b)

b) $\Longrightarrow$ c). Let $s, t \in S$ be such that $s \leq t$ and let $f = t - s$. If we suppose that $B_f \leq P$ then from the relations

$$B_f u \leq Pu \leq u \qquad (\forall)\, u \in S$$

we get

$$B_f u = B_f B_f u \leq B_f P u \leq B_f u; \quad B_f u = B_f P u$$

$$B_f u = B_f B_f u \leq P B_f u \leq B_f u; \quad B_f u = P B_f u$$

and therefore, using the definition of B_f, we have

$$v \in S, \ v \geq Pu \wedge (nf) \ \ (\forall)\, n \in \mathbf{N} \Longrightarrow v \geq u \wedge (nf) \qquad (\forall)\, n \in \mathbf{N}$$

Particularly, taking $v = Pu$,

$$(Pu) \wedge (nf) = u \wedge (nf) \qquad (\forall)\, n \in \mathbf{N},$$

$$(u - Pu) \wedge f = 0, \ (u - Pu) \wedge t = (u - Pu) \wedge [(t - s) + s] \leq$$

$$\leq (u - Pu) \wedge (t - s) + (u - Pu) \wedge s = (u - Pu) \wedge s$$

Hence if $u - Pu \leq t$ we get $u - Pu \leq s$ and therefore, using the hypothesis we have $s = t$, $f = 0$.

c) $\Longrightarrow$ b). Let $s_0, t_0 \in S$ be such that for any $u \in S$ we have

$$u - Pu \leq s_0 \Longrightarrow u - Pu \leq t_0$$

or equivalently

$$u - Pu \leq s_0 \Longrightarrow u - Pu \leq v_0, \quad \text{where } v_0 := s_0 \wedge t_0.$$

Hence, since for any $u \in S$ we have $(u - Pu) \wedge s_0 \in S_P$ we deduce

$$(u - Pu) \wedge s_0 \leq (u - Pu) \wedge v_0, \ (u - Pu) \wedge (s_0 - v_0) = 0$$

for any $u \in S$. Hence

$$0 \leq (u - Pu) \wedge n(s_0 - v_0) \leq n[(u - Pu) \wedge (s_0 - v_0)] = 0,$$

$$u \wedge n(s_0 - v_0) \leq Pu \wedge n(s_0 - v_0) + (u - Pu) \wedge n(s_0 - v_0) = Pu \wedge n(s_0 - v_0)$$

for any $u \in S$ and any $n \in \mathbf{N}$. We get

$$B_{(s_0 - v_0)}(u) \leq B_{(s_0 - v_0)}(Pu) \leq Pu, \quad B_{(s_0 - v_0)} \leq P$$

and using the hypothesis $s_0 - v_0 = 0$, $s_0 \leq t_0$.

Remark. We remember ([3], [5]) that any balayage B on S is a *l.d*-operator on S and we denote by S_B the set $S_B = \{s - Bs \mid s \in S\}$.

Proposition 1.2. Let P be a *l.d*-operator on S and let B be a balayage on S such that $B \leq P$. Then the map Q on S_B defined by

$$Q(s - Ps) = Ps - Bs = Ps - BPs$$

is a $l.d$-operator on S_B and $S_P = (S_B)_Q$.

Proof. Let B be a balayage on S such that $B \leq P$. From

$$s \in S \Longrightarrow BPs \leq Bs = B^2 s \leq BPs; \; B^2 s \leq PBs \leq Bs$$

it follows that

$$PBs = Bs = BPs \quad (\forall)\, s \in S.$$

We have

$$Q(s - Ps) = Ps - Bs \leq s - Bs \quad (\forall)\, s \in S$$

Obviously Q is additive. If $s_1, s_2 \in S$ are such that

$$s_1 - Bs_1 \leq s_2 - Bs_2$$

we deduce

$$Ps_1 - Bs_1 = Ps_1 - PBs_1 \leq Ps_2 - PBs_2 = Ps_2 - Bs_2$$

i.e the map Q is increasing. If the family $(s_i - Bs_i)_{i \in I}$ increases to $s - Bs$, without loss of generality we may suppose that the family $(s_i)_{i \in I}$ increases to s. In this case the family $(Ps_i - Bs_i)_{i \in I}$ is increasing and since $(Ps_i)_i \uparrow Ps$, $(Bs_i)_i \uparrow Bs$ we get

$$\bigvee_{i \in I} Q(s_i - Bs_i) = Q(s - Bs)$$

i.e Q is continuous in order from below.

Let now $u = s - Bs$, $v = t - Bt$, $f = w_1 - w_2 \in (S_B - S_B)_+$ where $s, t \in S$. We have $f \in (S - S)_+$, $v - Qv = t - Pt$ and

$$u \wedge (Qu + v - Qv + Qf) = s \wedge (Ps + t - Pt + Pf) - Bs.$$

Since P is a $l.d$-operator on S it follows that the element

$$r := s \wedge (Ps + t - Pt + Pf)$$

belongs to S and $Br = Bs$. Hence

$$u \wedge (Qu + v - Qv + Qf) = r - Br \in S_B$$

and therefore Q is a $l.d$-operator on S_B. From the relations

$$s - Ps = (s - Bs) - (Ps - Bs) = (s - Bs) - Q(s - Bs) \quad (\forall)\, s \in S$$

we deduce the equality

$$S_P = (S_B)_Q.$$

Corollary 1.3. Let P be a $l.d$-operator on S and let B be the balayage on S defined by

$$B = \bigvee \{B_f \mid f \in (S - S)_+, B_f \leq P\}$$

Then the map $Q : S_B \longrightarrow S_B$ given by

$$Q(s - Bs) = Ps - Bs$$

is a *l.d*-operator on S_B such that there is no function $g \in (S_B - S_B)_+$, $g \not\equiv 0$ such that $B_g^{S_B} \leq Q$.

Proof. We consider $g \in (S_B - S_B)_+$ and suppose that $B_g^{S_B} \leq Q$. Obviously $g \in (S - S)_+$ and using ([3], [6]) we deduce

$$(B_g^S \vee B)(s) = B_g^{S_B}(s - Bs) + Bs \qquad (\forall)\ s \in S$$

From the definition of B it follows that $B_g \leq B$ and $B_g \vee B = B$.
Hence $B_g^{S_B}(s - Bs) = 0 \qquad (\forall)\ s \in S$.

Remark. From Proposition 1.2 and Corollary 1.3 we see that for any l.d-operator P on S we have $S_P = (S_B)_Q$ where B is a balayage on S and Q is a *l.d*-operator on S_B which verifies one of the assertions from Proposition 1.1.

In the sequel we suppose that P is a (l.d)-operator on S which verifies one of the equivalent assertions a)-c) from Proposition 1.1.

We recall ([4]) that if T is an H-cone then a convex subcone S of T is termed an *H-subcone of T* if S, endowed with the natural order relation of T is an H-cone and for any subset (resp. any upper directed and dominated subset) A of S we have

$$\bigwedge_S A = \bigwedge_T A \quad (\text{resp. } \bigvee_S A = \bigvee_T A)$$

If S is an H-subcone of T then for any balayage B on T the map $B^{\#} : S \longrightarrow S$ defined by

$$B^{\#}s := \wedge\{t \in S \mid t \geq Bs\}$$

is a balayage on S and we have $BB^{\#}s = Bs$ for any $s \in S$ (see [4]).

Definition. An H-cone S is called a *complete H-cone* if any subset A of S such that

$$\bigwedge_{n \in \mathbb{N}^*} \left(\bigvee_{a \in A} \left(p \wedge \left(\frac{a}{n} \right) \right) \right) = 0 \qquad (\forall)\ p \in S.$$

is bounded.

In A_1 we show that: *for any H-cone S there exists a unique* (up to an isomorphism) *complete H-cone $\overline{S}$ such that S is a solid and increasingly dense convex subcone of $\overline{S}$ which is termed the completion of S.*

Theorem 1.4. There exists an order preserving embeding $\theta : S \longrightarrow \overline{S}_P$ such that

$$\theta(t) - \theta(Pt) = t - Pt \qquad (\forall)\ t \in S.$$

More precisely for any $s \in S$ we have

$$\theta(s) = \bigvee_{\overline{S}_P}\{u \in S_P \mid u \leq s\}$$

and $\theta(S)$ is an H-subcone of $\overline{S}_P$.

Proof. First we remark that for any $s \in S$ the set $A := \{a \in S_P \mid a \leq s\}$ is bounded in $\overline{S}_P$. Indeed, for any $v \in S_P$ we have

$$\bigwedge_{n \in \mathbb{N}^*} \left(\bigvee_{a \in A} v \wedge \left(\frac{a}{n}\right) \right) \leq \bigwedge_{n \in \mathbb{N}^*} \left(v \wedge \frac{s}{n} \right) \leq \bigwedge_{n \in \mathbb{N}^*} \frac{s}{n} = 0$$

Since S_P is increasingly dense in $\overline{S}_P$ then for any element $w \in \overline{S}_P$ we consider an increasing family $(v_i)_i$ in S_P such that $w = \bigvee_i v_i$.
Let us denote

$$w_0 = \bigwedge_{n \in \mathbb{N}^*} \left(\bigvee_{a \in A} \left(\frac{a}{n} \wedge w \right) \right)$$

The set A is upper directed because for any $t \in S$ we have $(t - Pt) \wedge s \in S_P$.
Hence

$$w_0 \wedge v_i = \bigwedge_{n \in \mathbb{N}^*} \left(\bigvee_{a \in A} \left(\frac{a}{n} \wedge w \wedge v_i \right) \right) = \bigwedge_{n \in \mathbb{N}^*} \left(\bigvee_{a \in A} \left(\frac{a}{n} \wedge v_i \right) \right) = 0$$

and therefore

$$w_0 = w_0 \wedge w = \bigvee_i (w_0 \wedge v_i) = 0.$$

The H-cone $\overline{S}_P$ being complete we deduce that A is bounded. We put, for any $s \in S$,

$$\theta(s) = \bigvee_{\overline{S}_P} \{u \in S_P \mid u \leq s\}$$

Since, for any $s \in S$, the family $\{u \in S_P \mid u \leq s\}$ is upper directed it follows that for any $s_1, s_2 \in S$ we have

$$\theta(s_1 + s_2) \geq \theta(s_1) + \theta(s_2).$$

If $u \in S_P$ and $u \leq s_1 + s_2$ then we have

$$u \wedge s_1 \in S_P, \quad u \wedge s_2 \in S_P, \quad u \leq u \wedge s_1 + u \wedge s_2$$

$$u \wedge s_1 \leq \theta(s_1), \quad u \wedge s_2 \leq \theta(s_2), \quad u \leq \theta(s_1) + \theta(s_2)$$

i.e $\theta(s_1 + s_2) \leq \theta(s_1) + \theta(s_2)$. Obviously θ is increasing.

Suppose now that $s_1, s_2 \in S$ are such that $\theta(s_1) \leq \theta(s_2)$. Then for any $u \in S_P$, $u \leq s_1$ we have

$$u \leq \theta(s_1) \leq \theta(s_2) = \bigvee_{\overline{S}_P} \{v | v \in S_P, v \leq s_2\},$$

$$u = \bigvee_{\overline{S}_P} \{u \wedge v | v \in S_P, v \leq s_2\} = \bigvee_{S_P} \{u \wedge v | v \in S_P, v \leq s_2\} =$$

$$= \bigvee_{S} \{u \wedge v | v \in S_P, v \leq s_2\} \leq s_2.$$

Hence $s_1 \leq s_2$.

We show now that for any $s \in S$ we have

$$\theta(s) - \theta(Ps) = s - Ps$$

or equivalently

$$s - Ps + \theta(Ps) = \theta(s)$$

Indeed, if $u \in S_P$ is such that $u \leq Ps$ then we have

$$s - Ps + u \in S_P, \quad s - Ps + u \leq s \text{ and } s - Ps + u \leq \theta(s)$$

Hence u being arbitrary we get

$$s - Ps + \theta(Ps) \leq \theta(s)$$

Let now $u \in S_P$ be such that $u \leq s$. Then we have

$$u \leq s - Ps + Ps, \quad u \leq (s - Ps) + u \wedge Ps$$

Since $u \wedge Ps$ belongs to S_U we get

$$u \leq (s - Ps) + \theta(Ps)$$

and therefore, u being arbitrary, we obtain

$$\theta(s) \leq s - Ps + \theta(Ps); \quad s - Ps = \theta(s) - \theta(Ps).$$

To prove that $\theta(S)$ is an H-subcone of $\overline{S}_P$ we consider now $A \subset S$ arbitrary. Obviously we have

$$\theta(\bigwedge_S A) \leq \bigwedge_{\overline{S}_P} \theta(A)$$

Conversely if $u \in S_P$ is such that $u \leq \theta(s)$ for any $s \in A$ then we have $u \leq s$ for all $s \in A$ and therefore

$$u \leq \bigwedge_S A, \quad u \leq \theta(\bigwedge_S A).$$

Since S_P is increasingly dense in $\overline{S}_P$ we deduce

$$\bigwedge_{\overline{S}_P} \theta(A) = \theta(\bigwedge_S A).$$

Let now A be an upper directed and dominated subset of S. Obviously we have

$$\theta(\bigvee_S A) \geq \bigvee_{\overline{S}_P} \theta(A)$$

For the converse inequality we consider an element $u \in S_P$ such that $u \leq \theta(\bigvee_S A)$. We have

$$u \leq \bigvee_S A, \quad u = \bigvee_{S-S} \{u \wedge s \mid s \in A\}$$

Since $u \wedge s \in S_P$ and $u \wedge s \leq s$ we get

$$u = \bigvee_{S_P} \{s \wedge u \mid s \in A\} \leq \bigvee_{\overline{S}_P} \{\theta(s) \mid s \in A\}.$$

The element $u \in S_P$ being arbitrary and S_P being increasingly dense in $\overline{S}_P$ we get

$$\theta(\bigvee_S A) \leq \bigvee_{\overline{S}_P} \theta(A), \quad \theta(\bigvee_S A) = \bigvee_{\overline{S}_P} \theta(A).$$

Remark. 1. In the sequel we identify S with its image $\theta(S)$ in $\overline{S}_P$. In this way S becomes an H-subcone of $\overline{S}_P$.

2. For any balayage B on $\overline{S}_P$ (or equivalently on S_P) we denote by $B^\#$ the balayage on S associated with B by

$$B^\# s = \bigwedge\{t \in S \mid t \geq Bs\} \qquad (\forall) s \in S.$$

3. For any $f \in (S - S)_+$ we have $Pf \in \overline{S}_P$. The assertion follows from the fact that $Pf \in (S - S)_+$ and $u \wedge Pf \in S_P$ for any $u \in S_P$.

If $f \in (S - S)_+$ we shall denote by B_f the balayage on $\overline{S}_P$ defined by

$$B_f u = \bigwedge\{v \in \overline{S}_P \mid v \geq v \wedge (nf) \quad (\forall)\, n \in \mathbf{N}\}$$

and we remark that in this case we have

$$B_f^\# s = \bigwedge\{t \in S \mid t \geq s \wedge (nf) \quad (\forall)\, n \in \mathbf{N}\}.$$

Theorem 1.5. For any balayage B on $\overline{S}_P$ and any $s \in S$ we have

$$B^\# s - Bs = PB^\# s - BPB^\# s.$$

Proof. For any $s \in S$ we put

$$Ls := Ps + B(s - Ps)$$

Since $B(s - Ps) \in S_P$, $B(s - Ps) \leq s - Ps$ we get

$$Ps + B(s - Ps) = s \wedge (Ps + B(s - Ps)) \in S$$

On the other hand for any $t \in S$ such that

$$t - Pt \leq s = (s - Ps) + Ps$$

there exists $s_1, s_2 \in S$ such that

$$t - Pt = (s_1 - Ps_1) + (s_2 - Ps_2),$$

$$s_1 - Ps_1 \leq s - Ps, \; s_2 - Ps_2 \leq Ps.$$

Hence

$$B(t - Bt) = B(s_1 - Ps_1) + B(s_2 - Ps_2) \leq B(s - Ps) + (s_2 - Ps_2)$$

and therefore

$$B(t - Pt) \leq B(s - Ps) + Ps,$$

$$Bs = \bigvee\{B(u - Pu) | u \in S, u - Pu \leq s\} \leq$$

$$\leq B(s - Ps) + Ps = Ls$$

From the above considerations we have

$$s \geq Ls \geq B^{\#}s, \quad L(B^{\#}s) = B^{\#}s$$

and therefore

$$B^{\#}s - Bs = P(B^{\#}s) - BP(B^{\#}s)$$

Corollary 1.6. For any balayage B on $\overline{S}_P$ and any $f \in (S - S)_+$ we have

$$Bf \leq B^{\#}f.$$

2. (S,T)-dilation operators

In this section we suppose that T is an H-cone and S is an H-subcone of T.

Definition. If $Q : S \longrightarrow S$ is a $(l.d)$-operator on S such that

$$s_1, s_2 \in S, \; s_1 \leq s_2 \Longrightarrow Qs_1 \preceq_T Qs_2 \preceq_T s_2$$

and such that the set

$$S_Q := \{s - Qs | s \in S\}$$

is a solid subset of T with respect to the natural order of T then Q will be termed an (S,T)*-dilation operator.*

Remark 1. If S is an H-cone and P is an $l.d$-operator on S which verifies one of the equivalent properties a)-c) from Proposition 1.1 then P is an (S,T)-dilation operator where T is the completion of the H-cone S_P. Moreover in this case S_P is increasingly dense on T.

Remark 2. If Q is an (S,T)-dilation operator then Q is an (S,T_0)-dilation operator where T_0 is the smallest naturally solid subcone of T such that $S \subset T_0$

Theorem 2.1. Let $Q : S \longrightarrow S$ be an additive, increasing an continuous in order from below map such that

$$s_1, s_2 \in S, \; s_1 \leq s_2 \Longrightarrow Qs_1 \preceq_T Qs_2 \preceq_T s_2.$$

Then the following assertions are equivalent:

a) Q is an (S,T)-dilation operator.

b) For any element $u \in T$ such that there exists $s \in S$ for which $Qs \preceq_T u \leq s$ we have $u \in S$.

c) The set S_Q defined by

$$S_Q = \{s - Qs | s \in S\}$$

is a solid subcone of T (w.r. to the natural order of T) and for any $s_1, s_2 \in S$ we have

$$s_1 - Qs_1 \leq s_2 - Qs_2 \Longrightarrow s_1 - Qs_1 + Qs_2 \in S$$

d) There exists an (S,T)-dilation operator Q_1 such that

$$Q_1 s \preceq_T Qs \qquad (\forall)\, s \in S$$

Proof. a) $\Longrightarrow$ b). If $u \in T$ and there exists $s \in S$ such that $Qs \preceq_T u \leq s$ then the element $u - Qs$ belongs to T and we have $u - Qs \leq s - Qs$. Since S_Q is a solid part of T with respect to the natural order then there exists $s' \in S$ with

$$s' - Qs' = u - \mathbf{Q}s \leq s - Qs.$$

Since Q is a localizable dilation operator on S we deduce

$$u = s' - Qs' + Qs = s \wedge (Qs + s' - Qs') \in S$$

b) $\Longrightarrow$ a). Let $s_1, s_2 \in S$ and $f \in (S - S)_+$ be arbitrary and let u be the element of $S - S$ defined by

$$u := s_2 \wedge (Qs_2 + s_1 - Qs_1 + Qf).$$

From hypothesis we have $u \in T$ and

$$Qs_2 \preceq_T u \leq s_2$$

i.e $u \in S$ and therefore Q is localizable dilation operator on S. It remains to show that the set S_Q is solid in T.

Let $u \in T$ and $s \in S$ be such that $u \leq s - Qs$. Let us consider the element $s_0 \in S$ defined by

$$s_0 = \bigwedge \{s' \in S \mid u \leq s' - Qs'\}$$

Obviously we have $u + Qs_0 \leq s'$ for any $s' \in S$ such that $u \leq s' - Qs'$.
Since S is an H-subcone of T we get

$$u + Qs_0 \leq \bigwedge \{s' \in S \mid u \leq s' - Qs'\} = s_0, \; u \leq s_0 - Qs_0.$$

Obviously we have $u + Qs_0 \in T$ and

$$Qs_0 \preceq_T u + Qs_0 \leq s_0$$

Hence the element $s'_0 := u + Qs_0$ belongs to S and we have

$$s'_0 \leq s_0, \;\; u = s'_0 - Qs_0 \leq s'_0 - Qs'_0$$

The last inequality implies that $s_0' \geq s_0$ and therefore $s_0' = s_0$. Hence $u = s_0 - Qs_0$.

The relations a) $\Longrightarrow$ c) and a) $\Longrightarrow$ d) are obvious.

c) $\Longrightarrow$ b). Let $u \in T$ and $s \in S$ be such that

$$Qs \preceq_T u \leq s$$

The element $t := u - Qs$ belongs to T and $t = u - Qs \leq s - Qs$.
Hence there exists $s_1 \in S$ such that $t = s_1 - Qs_1$. From the hypothesis and using the inequality

$$s_1 - Qs_1 \leq s - Qs$$

we get

$$u = t + Qs = s_1 - Qs_1 + Qs \in S.$$

d) $\Longrightarrow$ *b*). Let Q_1 be an (S,T)-dilation operator such that

$$Q_1 s \preceq_T Qs \qquad (\forall)\ s \in S$$

and let $u \in T$, $s' \in S$ be such that

$$Qs' \preceq_T u \leq s'$$

We have then

$$Q_1 s' \preceq_T Qs' \preceq_T u \leq s'$$

and therefore $u \in S$.

For any balayage B on T we denote by $B^{\#}$ the balayage on S given by

$$B^{\#}s = \bigwedge \{s' \in S \mid s' \geq Bs\} \qquad (\forall)\ s \in S.$$

Obviously we have

$$Bs \leq B^{\#}s, \ \ B(B^{\#}s) = Bs \qquad (\forall)\ s \in S.$$

Theorem 2.2. If Q is an (S,T)-dilation operator then for any balayage B on T we have

$$B^{\#}s - Bs = QB^{\#}s - BQB^{\#}s \qquad (\forall)\ s \in S.$$

Proof. Let $s \in S$ and let B be a balayage on T. We put

$$u := Bs + QB^{\#}s - BQB^{\#}s$$

From the relations

$$BB^{\#}s = Bs, \ \ Bs - BQB^{\#}s = B(B^{\#}s - QB^{\#}s) \leq B^{\#}s - QB^{\#}s$$

and using the fact that Q is an (S,T)-dilation operator we deduce that $u \in S$ and $Bs \leq u \leq B^{\#}s$. Hence $u = B^{\#}s$ and therefore

$$B^{\#}s - Bs = QB^{\#}s - BQB^{\#}s.$$

Corollary 2.3. If there exists an (S,T)-dilation operator Q on S then for any $f \in (S-S)_+$ any balayage B on T we have

$$Bf \leq B^{\#} f.$$

Proposition 2.4. Let Q be an (S,T)-dilation operator and let B be a balayage on T. Then the following assertions are equivalent

a) $B(s - Qs) = 0 \qquad (\forall)\ s \in S$

b) $B^{\#} s \leq Qs \qquad (\forall)\ s \in S$

c) $B^{\#} s = QB^{\#} s \qquad (\forall)\ s \in S$

d) $B^{\#} s = B^{\#} Qs \qquad (\forall)\ s \in S$

Proof. a) $\Longrightarrow$ b). If $s \in S$ then the relation a) implies:

$$Bs = BQs \leq Qs.$$

Since $Qs \in S$ we deduce $B^{\#} s \leq Qs$.

b) $\Longrightarrow$ c) and c) $\Longrightarrow$ d) follow from the fact that for any $s \in S$ we have

$$B^{\#} s \leq Qs \leq s \Longrightarrow B^{\#} s = B^{\#}(B^{\#} s) \leq Q(B^{\#} s) \leq B^{\#} s$$

$$B^{\#} s = QB^{\#} s \leq Qs \Longrightarrow B^{\#} s = B^{\#}(B^{\#} s) \leq B^{\#} Qs \leq B^{\#} s.$$

The relation d) $\Longrightarrow$ a) may be obtained from the fact that for any $s \in S$ we have $Bs = BB^{\#} s$.

Theorem 2.5. Let Q be an (S,T)-dilation operator and let B_0 be the greatest balayage on T which vanishes on S_Q. Then the balayage B_0 is absorbent (i.e $B_0 t \preceq_T t$ for any $t \in T$) and for any $s', s'' \in S$ we have

$$s' \leq s'' \Longrightarrow B_0 s' \preceq_T B_0 s''.$$

Particularly, if there is no absorbent balayage B on T, $B \neq 0$ such that

$$s', s'' \in S, \quad s' \leq s'' \Longrightarrow Bs' \preceq_T Bs''$$

then S_Q is increasingly dense in T.

Proof. For any $s \in S$ we consider the balayage B_{s-Qs} on T defined by

$$B_{s-Qs}(u) := \bigvee \{u \wedge n(s - Qs) \mid n \in \mathrm{N}\}$$

Since $s - Qs \in T$ the complement ([3]) B'_{s-Qs} of the balayage B_{s-Qs} is absorbent ([1]). Moreover for any $s', s'' \in S$ we have

$$(s' - Qs') + (s'' - Qs'') = (s' + s'' - Q(s' + s'')),$$

$$B_{s'-Qs'} \leq B_{s'+s''-Q(s'+s'')}, \quad B_{s''-Qs''} \leq B_{s'+s''-Q(s'+s'')}$$

and therefore the family $(B'_{s-Qs})_{s\in S}$ of absorbent balayages on T is decreasing. Hence the map $B_0 : T \longrightarrow T$ defined by

$$B_0 u = \bigwedge_{s\in S} B'_{s-Qs} u = \bigwedge_{s\in S} B'_{s-Qs} u$$

is also an absorbent balayage on T and moreover we have $B_0(s-Qs)=0$ for all $s\in S$ and

$$s',s''\in S, s'\leq s''\Longrightarrow B_0s'=B_0(Qs')\preceq_T B_0(Qs'')=B_0s''\preceq_T s''$$

The fact that B_0 is the greatest balayage which vanishes on S_Q follows from the fact that for any such a balayage B on T and any $s\in S$ we have

$$B(B_{s-Qs}u)=\bigvee_{n\in\mathrm{N}} B(n(s-Qs)\wedge u)\leq \bigvee_{n\in\mathrm{N}} nB(s-Qs)=0$$

and therefore

$$B\wedge B_{s-Qs}=0$$

Hence, using ([1], [3]) we have

$$B=B\wedge I=B\wedge(B_{s-Qs}\vee B'_{s-Qs})=(B\wedge B_{s-Qs})\vee(B\wedge B'_{s-Qs})=$$

$$=B\wedge B'_{s-Qs}\leq B'_{s-Qs}$$

and therefore, the element $s\in S$ being arbitrary,

$$B\leq\bigwedge_{s\in S} B'_{s-Qs}=B_0.$$

From the preceding considerations we see that B_0 is in fact the complement of the balayage $\bigvee_{s\in S} B_{s-Qs}$. If there is no trivial absorbent balayage B on T such that

$$s',s''\in S,\quad s'\leq s''\Longrightarrow Bs'\preceq_T Bs''$$

it follows that $B_0=0$ and therefore $I=\bigvee_{s\in S} B_{s-Qs}$ which liedes to the conclusion that the convex subcone S_Q is increasingly dense in T.

Theorem 2.6. For any balayage B on T which verifies one of the equivalent properties a)-d) from Proposition 2.4 for a given (S,T)-dilation operator Q on S we have

$$s',s''\in S, s'\leq s''\Longrightarrow B^{\#}s'\preceq_T B^{\#}s''\preceq_T s''$$

Particularly if there is no trivial balayage L on S such that

$$s',s''\in S,\quad s'\leq s''\Longrightarrow Ls'\preceq_T Ls''\preceq_T s''$$

then S_Q is increasingly dense in T.

Proof. From Proposition 2.4 we deduce

$$s',s''\in S, s'\leq s''\Longrightarrow B^{\#}s'=Q(B^{\#}s')\preceq_T Q(B^{\#}s'')\preceq B^{\#}s''$$

$$s\in S\Longrightarrow B^{\#}s=Q(B^{\#}s)\preceq_T Qs\preceq_T s.$$

The last part of the proof follows from Theorem 2.5.

3. Existence of (S, T)-dilation operators

In this section S and T are two H-cones such that S is a convex H-subcone of T. For any balayage B on T we denote by $B^\#$ the balayage on S defined by

$$B^\# s := \bigwedge \{t \in S \mid t \geq Bs\}$$

We want to construct, under some suplimentary conditions, an (S, T)-dilation operator on S.

Theorem 3.1. The following assertions are equivalent:

1) for any balayage B on T and any $f \in (S - S)_+$ we have

$$Bf \leq B^\# f$$

2) for any balayage B on T and any $f \in (S - S)_+$ there exists an element $t \in T$ such that:

$$B^\# f - Bf = t - Bt$$

3) for any balayage B on T, any $f \in (S - S)_+$ and any element $s \in S$ such that $f \leq s$ we have

$$O \leq B^\# f - Bf \preceq_T s$$

4) for any finite family $(f_i)_{i \in I}$, $f_i \in (S - S)_+$, any finite family $(B_i)_{i \in I}$ of balayages on T and any $s \in S$ such that $\sum\limits_{i \in I} B_i^\# f_i \leq s$ we have

$$O \leq \sum_{i \in I} (B_i^\# f_i - B_i f_i) \preceq_T s$$

Proof. The relations 4) $\Longrightarrow$ 3) $\Longrightarrow$ 1) and 2) $\Longrightarrow$ 1) are obvious.

1) $\Longrightarrow$ 2). Let $f \in (S - S)_+$ and let B be a balayages on T. For any balayages M on the H-cone T_B, where

$$T_B := \{t - Bt \mid t \in T\},$$

there exists a balayage B_1 on T, $B_1 \geq B$, such that

$$M(t - Bt) = B_1 t - Bt \qquad (\forall)\, t \in T$$

(see [6]).
Since the element $B^\# f - Bf$ belongs to $T_B - T_B$ it remains to show that $M(B^\# f - Bf) \leq B^\# f - Bf$. (see [3])

We have, using the hypothesis, $M(B^\# f - Bf) = B_1(B^\# f) - B_1 Bf \leq B_1^\# (B^\# f) - B_1 Bf = B^\# f - Bf$. Hence 1) $\Longleftrightarrow$ 2).

1) $\Longrightarrow$ 3). Let $f \in (S - S)_+$ and let B be a balayage on T. If $t \in S$ is such that $f \leq t$ we have

$$t - (B^\# f - Bf) - Bt = (t - Bt) - (B^\# t - Bt) + (B^\#(t - f) - B(t - f))$$

i.e the element $u := t - (B^{\#}f - Bf) - Bt$ belongs to $T_B - T_B$. We show now that $u \in T_B$. Indeed, if we consider a balayage M on T_B there exists a balayage B_1 on T, $B_1 \geq B$ such that

$$M(s - Bs) = B_1(s - Bs) \qquad (\forall)\ s \in T$$

and therefore

$$M(u) = B_1(u) = B_1(t - B^{\#}f) - B_1(Bt - Bf) \leq$$

$$\leq B_1^{\#}(t - B^{\#}f) - (Bt - Bf) = B_1^{\#}t - B^{\#}f + Bf - Bt \leq t - (B^{\#}f - Bf) - Bt,$$

$$Mu \leq u$$

Hence $u \in T_B$. On the other hand we have $u \leq t - Bt$ and therefore $u + Bt \in T$ i.e $t - (B^{\#}f - Bf) \in T$.

3) $\Longrightarrow$ 4). We proceed inductively and we suppose that for any system $(f_1, f_2, \ldots f_n)$ of elements of $(S - S)_+$ and any system $(B_1, B_2, \ldots, B_n)$ of balayages on T such that $\sum_{i=1}^{n} B_i^{\#} f_i \leq s$, where $s \in S$, we have

$$s - \sum_{i=1}^{n}(B_i^{\#} f_i - B_i f_i) \in T$$

Let now $\{f_1, f_2, \ldots, f_{n+1}\}$ be a subset of $(S - S)_+$ $(B_1, B_2, \ldots, B_{n+1})$ be a system of balayages on T such that $\sum_{i=1}^{n+1} B_i^{\#} f_i \leq s$ where s is an element of S. We have $s - \sum_{i=2}^{n+1}(B_i^{\#} f_i - B_i f_i) \in T$ and we want to show that $s - \sum_{i=1}^{n+1}(B_i^{\#} f_i - B_i f_i)$ is an element of T.

If we denote

$$f = s - \sum_{i=1}^{n+1}(B_i^{\#} f_i - B_i f_i), \quad u = s - \sum_{i=2}^{n+1}(B_i^{\#} f_i - B_i f_i)$$

and by Rf the reduite of f with respect to the H-cone T then we have

$$u \in T, \quad u = f + (B_1^{\#} f_1 - B_1 f_1), \quad u \geq f, \quad u \geq Rf.$$

On the other hand for any $\alpha \in (0, 1)$ we have

$$B_\alpha f \geq B_\alpha(\alpha Rf) = \alpha Rf$$

where B_α is the balayage on T defined by

$$t \in T, \quad B_\alpha t = \bigvee_{n \in \mathbb{N}} R(t \wedge ng), \quad g := (f - \alpha Rf)_+.$$

(see [2]). On the other hand we have

$$B_1(B_1^{\#} f_1 - B_1 f_1) = 0, B_1 f = B_1 u \geq B_1 R f \geq B_1(\alpha R f).$$

From the relations

$$B_\alpha f \geq B_\alpha(\alpha R f) = \alpha R f$$

$$B_1 f \geq B_1(\alpha R f)$$

and using the fact that for any $s \in T$ we have

$$(B_\alpha \vee B_1)s = (B_\alpha s) \vee (B_1 s)$$

it follows that

$$(B_\alpha \vee B_1) f \geq (B_\alpha \vee B_1)(\alpha R f) \geq \alpha R f$$

From the relation $O \leq u - B^{\#} f_1$ we deduce

$$(B_\alpha \vee B_1) f = (B_\alpha \vee B_1)(u - B_1^{\#} f_1) + (B_\alpha \vee B_1)(B_1 f_1) \leq$$

$$\leq (B_\alpha \vee B_1)^{\#}(u - B_1^{\#} f_1) + B_1 f_1 = (B_\alpha \vee B_1)^{\#} u - B_1^{\#} f_1 + B_1 f_1 \leq$$

$$\leq u - (B_1^{\#} f_1 - B_1 f_1) = f.$$

Hence for any $\alpha \in (0,1)$ we have

$$f \geq (B_\alpha \vee B_1) f \geq \alpha R f$$

and therefore $f = Rf \in T$.

Lemma 3.2. Suppose that

$$Bf \leq B^{\#} f$$

for any $f \in (S-S)_+$ and any balayage B on T and we denote by P_B the map $P_B : S \longrightarrow T$ defined by

$$P_B s := B^{\#} s - B^{\#} s \curlywedge_T B s.$$

Then we have

1. $P_B(s_1 + s_2) = P_B s_1 + P_B s_2$
2. $s_1, s_2 \in S,\ s_1 \leq s_2 \Longrightarrow P_B s_1 \preceq_T P_B s_2 \preceq_T s_2$
3. $P_B(B^{\#} s) = P_B s \qquad \forall s \in S$
4. P_B is continuous in order from below (i.e $s_i \uparrow s, s_1, s \in S \Longrightarrow P_B s_i \uparrow P_B s$).

For the proof see [4].

In the sequel, in this section we suppose that the pair (S, T) verifies the following two conditions:

a) Any increasing family of S, dominated in T is dominated in S.

b) For any $t_1, t_2 \in T$ such that

$$f \in (S-S)_+,\ f \leq t_1 \Longrightarrow f \leq t_2$$

then $t_1 \leq t_2$.

Remarks. 1) If S is complete then the property 1) is verified.

2) The property 2) is equivalent with the following one: for any $t \in T$ we have

$$t = \vee\{R(f) \mid f \in (S - S)_+, f \leq t\}$$

where Rf means the reduite of f in the H-cone T.

Lemma 3.3. Suppose that

$$Bf \leq B^\# f$$

for any $f \in (S - S)_+$ and any balayage B on T. Then if $u \in T$ is such that

$$f \in (S - S)_+, \;\; f \leq u \Longrightarrow P_B f \preceq_T u$$

for any balayage B on T then $u \in S$.

Proof. Let $f \in (S - S)_+$, $f \leq u$ and let B be a balayage on T. We show that

$$B^\# f \leq u.$$

Indeed, we have, by hypothesis $P_B f \preceq_T u$, and therefore there exists $v \in T$ with

$$u = v + P_B f.$$

Suppose that $f = s - t$ where $s, t \in S$. We have

$$B^\#(s - t) + v = B^\# s \curlywedge_T Bs - B^\# t \curlywedge_T Bt + u$$

and therefore

$$B(s - t) + Bv = B^\# s \curlywedge_T Bs - B^\# t \curlywedge_T Bt + Bu.$$

Hence we get

$$u + B(s - t) + Bv = v + B^\#(s - t) + Bu.$$

Since $Bv \leq v$ and $s - t \leq u$ we deduce

$$B(s - t) \leq Bu$$

and therefore

$$u \geq B^\#(s - t) = B^\# f$$

We show now that

$$u \geq R^S(f) := \wedge\{s' \in S \mid s' \geq f\}$$

It is known ([2]) that for any $\alpha \in (0, 1)$ we have

$$R^S(f) = B_h^S(R^S(f)) \leq \frac{1}{\alpha} B_h^S(f)$$

where $h = (f - \alpha R^S f)_+$ and B_h^S is the balayage on S defined by

$$B_h^S s' = \wedge\{s'' \in S \mid s'' \geq s' \wedge nh \qquad (\forall)\, n \in \mathbb{N}\}.$$

If B_h denote the following balayage on T

$$B_h t' := \wedge\{t'' \in T \mid t'' \geq t' \wedge nh \quad (\forall)\, n \in \mathbb{N}\}$$

then we have already remarked that $B_h^S = (B_h)^{\#}$ and therefore:

$$R^S(f) \leq \frac{1}{\alpha} B_h^S(f) = \frac{1}{\alpha}(B_h)^{\#}(f) \leq \frac{1}{\alpha} u,$$

$$\alpha R^S(f) \leq u.$$

Since $\alpha \in (0,1)$ is arbitrary we get $R^S(f) \leq u$.
If we denote, for any $g \in S - S$,

$$R^T(g) = \wedge\{t \in T \mid t \geq g\}$$

obviously we have $R^T(g) \leq R^S g$ and from the preceding considerations we deduce

$$u = \bigvee\{R^T(f) \mid f \in (S-S)_+,\ f \leq u\} \leq$$

$$\leq \bigvee\{R^S(f) \mid f \in (S-S)_+,\ f \leq u\} \leq u,$$

$$u = \bigvee\{R^S(f) \mid f \in (S-S)_+,\ f \leq u\} \in S.$$

The above result is an extension of a similar one ([4]) given in the case where S and T are standard H-cones of functions.

Notation. Suppose that $Bf \leq B^{\#} f$ for any $f \in (S-S)_+$ and any balayage B on T. In the following, for any $s \in S$, we shall denote by Ps the element of T given by

$$Ps := \curlyvee_T\{\sum_{i \in I} P_{B_i} s_i \mid \text{I finite},\ s_i \in S, \sum_{i \in I} s_i \leq s, B_i \text{ balayage on } T\}$$

Lemma 3.4. If $s \in S$ and $u \in T$ are such that

$$Ps \preceq_T u \leq s$$

then $u \in S$.

Proof. If $f \in (S-S)_+$ is such $f \leq u$ then for any balayage B on T we have

$$P_B f \preceq_T P_B s \preceq_T Ps \preceq_T u, \quad P_B f \leq u$$

The assertion follows now from the previous lemma.

Corollary 3.5. For any $s \in S$ we have $Ps \in S$ and

$$Ps \preceq_T s.$$

Proof. Indeed, from Theorem 3.1 we deduce

$$Ps \preceq_T s$$

and from Lemma 3.4 we get $Ps \in S$.

Theorem 3.6. The above map $P : S \longrightarrow S$ is an (S,T)-dilation operator on S. Moreover for any (S,T)-dilation operator Q on S we have

$$Ps \preceq_T Qs \quad (\forall)\ s \in S.$$

Proof. By definition and by Corollary 3.5 we have

$$s_1, s_2 \in S,\ \ s_1 \leq s_2 \Longrightarrow Ps_1 \preceq_T Ps_2 \preceq_T s_2.$$

From ([4], Theorem 2.8) it follows that P is additive and continuous in order from below. Using now Lemma 3.4 and Theorem 2.1 we deduce that P is (S,T)-dilation operator on S.

Since for any balayage B on S we have

$$B^{\#}s - Bs = QB^{\#}s - BQB^{\#}s \preceq_T QB^{\#}s \preceq_T Qs$$

we deduce, from the definition of P_B, that

$$P_B s \preceq_T Qs$$

and therefore $Ps \preceq_T Qs$.

Theorem 3.7. Let $Q : S \longrightarrow S$ be a map which is additive, continuous in order from below and such that

$$s_1, s_2 \in S,\ s_1 \leq s_2 \Longrightarrow Qs_1 \preceq_T Qs_2 \preceq_T s_2.$$

Then Q is an (S,T)-dilation operator iff for any balayage B on T and any $s \in S$ we have

$$B^{\#}s - Bs = QB^{\#}s - BQB^{\#}s.$$

Proof. If Q is an (S,T)-dilation operator then from Theorem 2.2 we have

$$B^{\#}s - Bs = QB^{\#}s - BQB^{\#}s$$

for any $s \in S$ and any balayage B on T. Conversely suppose that this formula holds for any $s \in S$ and any balayage B on T. We have for any balayage B on T and any $s \in S$,

$$B^{\#}s - Bs = QB^{\#}s - BQB^{\#}s \preceq_T QB^{\#}s \preceq_T Qs$$

and therefore

$$P_B s \preceq_T Qs, \qquad Ps \preceq Qs.$$

Using Theorem 2.1 the preceding inequality and Theorem 3.6 we deduce that Q is an (S,T)-dilation operator on S.

4. On the unicity of (S,T)-dilation operator

In this section S and T will be H-cones such that S is an H-subcone of T and the following are fulfiled:

a) Any increasing family in S, dominated in T is also dominated in S

b) For any $t_1, t_2 \in T$ such that

$$\{f \in (S-S)_+ \mid f \leq t_1\} \subset \{f \in (S-S)_+ \mid f \leq t_2\}$$

we have $t_1 \leq t_2$

c) For any $f \in (S-S)_+$ and any balayage B on T we have

$$Bf \leq B^{\#} f$$

In the preceding section we have proved that in the above conditions there exists (S,T)-dilation operators on S. Moreover there exists an (S,T)-dilation operator P on S such that for any (S,T)-dilation operator Q on S we have $Ps \leq Qs$ (or more precisely $Ps \preceq_T Qs$) for all $s \in S$. This remarkable (S,T)-dilation operator on S will be termed the *minimal (S,T)-dilation operator on S*.

In this section we deal with the unicity problem for the family of (S,T)-dilation operators on S.

For the simplicity reasons we suppose that the H-cone T contains sufficiently many quasi-continuous elements.

We remember that an element $u \in T$ is termed quasicontinuous if for any increasing family $(u_i)_i$ of T such that $\bigvee_{i \in I} u_i = u$ we have $\bigwedge_{i \in I} R(u - u_i) = 0$ where R means the reduite operator on T. We say that T contains sufficiently many quasi-continuous elements if any element of S is the suppemum of the family of its quasi-continuous minorants.

Lemma 4.1. Let C be an H-cone which contains sufficiently many quasi-continuous elements and let $\varphi : C \longrightarrow C$ be an additive, increasing, continuous in order from below map such that

$$s_1, s_2 \in C,\ s_1 \leq s_2 \Longrightarrow \varphi(s_1) \preceq \varphi(s_2) \preceq s_2.$$

Then there exists a recurrent balayage ([7]) B on C such that

$$\varphi(s) \preceq Bs \qquad (\forall)\, s \in C.$$

Proof. We denote

$$C_0 := \{s \in C \mid \varphi(s) = 0\}$$

$$C_1 := \{t \in C \mid t \curlywedge s = 0 \quad (\forall)\, s \in C_0\}$$

First we show that if $s_1 \in C_1$ and $s_2 \in C$ are such that $s_1 \leq s_2$ then $s_1 \preceq s_2$. Indeed, let us put

$$u = s_1 \quad s_2,\ s_1' = s_1 - u,\ s_2' = s_2 - u.$$

We have $s'_1 \in C_1$, $s'_1 \quad s'_2 = 0$, $s'_1 \leq s'_2$. Since $\varphi(s'_1) \preceq s'_1$, $\varphi(s'_1) \preceq \varphi(s'_2) \preceq s'_2$ we deduce $\varphi(s'_1) = 0$ and therefore $s'_1 \in C_0 \cap C_1$, $s'_1 = 0$. Hence $s_1 = u \preceq s_2$.

Now, for any quasi-continuous element $s \in C$ we put

$$Bs = \bigvee\{t \mid t \in C_1,\ t \leq s\} = \bigvee\{t \mid t \in C_1 \mid t \preceq s\}$$

It is easy to see that B is additive, increasing and continuous in order from below. Since $Bs \preceq s$ then Bs is also quasi-continuous and $BBs = Bs$. The map

$$s \longrightarrow \bigvee\{Bt \mid t \leq s,\ t \text{ quasi-continuous}\} = \tilde{B}s$$

is a balayage on C which extends the above map B and we have

$$s_1, s_2 \in C,\ s_1 \leq s_2 \Longrightarrow \tilde{B}s_1 \preceq \tilde{B}s_2 \preceq s_2,$$

and therefore $\tilde{B}$ is a recurrent balayage on C.

Moreover, for any quasi-continuous element s of C we have

$$s - Bs \in C_0,\ \varphi(s) = \varphi(Bs)$$

and therefore $\varphi(s) \preceq Bs = \tilde{B}s \preceq s$.

Theorem 4.2. Suppose that there is no recurrent balayage on T different from zero. Then for any balayage B on S which is an (S,T)-dilation operator we have $B = Q$ for any (S,T)-dilation operator Q in S with

$$Bs \leq Qs \quad \forall s \in S.$$

Proof. Let Q be a (S,T)-dilation operator on S and B be a balayage on S such that

$$s \in S \Longrightarrow Bs \leq Qs.$$

Then we have

$$Bs = B^2 s \leq Q(Bs) \preceq_T Bs$$

and therefore $Bs = Q(Bs)$ for any $s \in S$.

We consider now the map $M : T \longrightarrow T$ defined by

$$Mu = \bigvee\{Q(s - Bs) \mid s - Bs \leq u,\ s \in S\}$$

Since

$$s, t \in S, s - Bs \leq t - Bt \Longrightarrow Q(s - Bs) \preceq_T Q(t - Bt) = Qt - Bt \preceq_T t - Bt$$

we deduce

$$u, v \in T, u \leq v \Longrightarrow Mu \preceq_T Mv \preceq_T v.$$

From Lemma 4.1 it follows that there exists a recurrent balayage L on T such that

$$Mu \preceq_T Lu.$$

Using the hypothesis we get $L = 0$ and therefore $M = 0$,

$$s \in S \Longrightarrow Qs = QBs = Bs.$$

Remark. If B is a balayage on S such that $S_B := \{s - Bs \mid s \in S\}$ is solid in T then it follows that B is a (S,T)-dilation operator on S. Therefore the preceding theorem shows that if there is no recurrent balayage on T different from zero and B is a balayage on S such that S_B is solid in T then any (S,T)-dilation operator Q on S which dominates B coincides with B.

Corollary 4.3. If the minimal (S,T)-dilation operator P on S is a balayage on S and there is no recurrent balayages on T then any (S,T)-dilation operator on S is equal with P.

The following example show that the above theorem fails if we drop the supplementary condition abote T.

Example 1. We consider an H-cone S which is recurrent (i.e the natural order coincides with the specific order in S) and for any $\alpha \in [0,1]$ we denote by P_α the map $P_\alpha : S \longrightarrow S$ defined by $P_\alpha S = \alpha \cdot S$. It is easy to see that the pair (S,S) verifies the conditions from the beginning of this section and that P_α is a (S,S)-dilation operator on S for any $\alpha \in [0,1]$.

We remark that $P_0 = 0$ is the minimal (S,S)-dilation operator on S and that P_0 and P_1 are balayages on S. In this example $S_{P_\alpha} = S$ for any $\alpha \in [0,1)$ and $S_{P_1} = \{0\}$.

The following example shows that the above corollary holds even if the minimal (S,T)-dilation operator is not a balayage on S.

Example 2. Let S be the H-cone of all positive, increasing and lower semicontinuous real functions on the open interval $(-1,1)$ of $\mathbb{R}$. We consider the map $B^{\{0\}} : S \longrightarrow S$ where for any subset $A \subset (-1,1)$ and any $s \in S$ we have

$$B^A s = \bigwedge\{t \in S | t \geq s \text{ on } A\}.$$

It is known that $B^{\{0\}}$ is a localizable dilation operator on S and $S_{B^{\{0\}}}$ is solid and increasingly dense in the set T of all positive, real functions on $(-1,1)$ such that their restrictions to $(-1,0]$ and $(0,1)$ are increasing and lower semicontinuous. We show that any (S,T)-dilation operator on S coincides with $B^{\{0\}}$.

Indeed, we consider the H-cone S_1 of the restrictions to $(-1,0]$ of all $s \in S$ and the map $T : S_1 \longrightarrow S_1$ defined by

$$Tt = Q(\bar{t})/_{(-1,0]}$$

where $\bar{t}$ is equal t on $(-1,0]$ and equal $t(0)$ on $(0,1)$. Obviously T verifies the conditions from Lemma 4.1. Since there is no recurrent balayages on S_1 different from zero we get $Tt = 0$ for any $t \in S_1$ and therefore

$$Q(\bar{t}) \mid_{(-1,0]} = 0 \qquad \forall t \in S_1.$$

Let now $s \in S$ and let $s_1 := s|_{(-1,0]}$. We have

$$s \leq \bar{s}_1 + B^{(0,1)} s,$$

$$Qs \preceq_T Q(\bar{s}_1) + Q(B^{(0,1)}s).$$

Since $Q(\bar{s}_1)|_{(-1,0]} = 0$, $B^{(0,1)}s|_{(-1,0)} = 0$ we deduce

$$Qs|_{(-1,0)} = 0$$

We consider now the H-cone S_2 given by

$$S_2 := \{s \in S \mid s|_{(-1,0]} = 0\}$$

It is easy to see, using the preceding consideration that the map $s \longrightarrow Qs$ verifies the conditions from Lemma 4.1 with respect to the H-cone S_2 and therefore, since there is no recurrent balayages on S_2 different from zero, we get

$$Qs = 0 \qquad \forall s \in S_2.$$

Because $B^{(-1,0)}s \preceq_S s$ for any $s \in S$ we get

$$s - B^{(-1,0)}s \in S_2$$

and therefore

$$Q(s) + Q(B^{(-1,0)}s).$$

Hence for any $s, t \in S$ we have

$$s = t \text{ on } (-1,0] \Longrightarrow Qs = Qt$$

and therefore

$$s \in S \Longrightarrow Qs = Q(B^{(-1,0)}s) \preceq_T B^{(-1,0)}s.$$

Since $B^{(-1,0)}s = B^{\{0\}}s$ on $(0,1)$ and $Qs = 0$ on $(-1,0]$ we have

$$Qs \preceq_T B^{\{0\}}s.$$

Let now P be the minimal (S,T)-dilation operator on S. It remains to show that

$$B^{\{0\}}s = Ps \qquad \forall\, s \in S$$

Indeed if $s \in S$ is a continuous function then there exists $u \in S$, $u \leq s$ such that $s - B^{\{0\}}s = u - Pu$. From $B^{\{0\}}s = 0$ on $(-1,0]$ and $Pu \leq B^{\{0\}}u$ it follows that

$$s = u \text{ on } (-1,0].$$

and therefore $Pu = Ps$. Hence

$$s - u = B^{\{0\}}s - Ps, \qquad Ps \preceq_T B^{\{0\}}s$$

and therefore $s - u \in S$. Because s is continuous we deduce that u and $B^{\{0\}}s - Ps$ are also continuous. From the fact that $B^{\{0\}}s - Ps = 0$ on $(-1,0]$ and that $B^{\{0\}}s$ is constante on $(0,1)$ we deduce that $B^{\{0\}}s - Ps$ is also constante on $(0,1)$ and therefore being continuous is equal to zero. Hence

$$B^{(0)}s = Ps.$$

Theorem 4.4. Suppose that there is no absorbent balayages B on T different from zero such that

$$s_1, s_2 \in S, \; s_1 \leq s_2 \Longrightarrow Bs_1 \preceq_T Bs_2.$$

and that there exists a balayage B_0 on S which is a (S,T)-dilation operator on S. Then B_0 is the only (S,T)-dilation operator on S.

Proof. From hypothesis it follows that there is no recurrent balayages on T different from zero and therefore, using Theorem 4.2, any (S,T)-dilation operator Q on S such that $Qs \geq B_0 s$ for any $s \in S$, is equal with B_0.

To finish the proof we show that B_0 is the minimal (S,T)-dilation operator on S.

Let P be the minimal (S,T)-dilation operator on S and let s be an arbitrary element of S. We have $s - Bs \leq s - Ps$ and since the set S_B is increasingly dense in T we can choose an increasing family $(s_i - Bs_i)_{i \in I}$ such that $\bigvee_{i\in I}(s_i - Bs_i) = s - Ps$. On the other hand the set S_P being a solid subcone of T we deduce that for any $i \in I$ there exists $t_i \in S$ such that $t_i - Pt_i = s_i - Bs_i$ and moreover taking, for every $i \in I$, the smallest element t_i of S with the above property then we deduce (see [5]) that the family $(t_i)_{i\in I}$ is increasing and dominated by s. If we denote $t = \bigvee_{i\in I} t_i$ we get

$$(Pt_i)_i \uparrow Pt, \;\; (t_i - Pt_i)_i \uparrow t - Pt = s - Ps.$$

Using the fact that $s_i - Bs_i = t_i - Pt_i$ we get $Bt_i = BPt_i$ for any $i \in I$ and therefore, passing to the limite $Bt = BPt$ or equivalently $Bs = BPs$. Hence $Bs \leq Ps$ i.e $Bs = Ps$.

Let now $\mathcal{V} = (V_\alpha)_{\alpha \geq 0}$ be a submarkovian resolvent on a measurable space $(X, \mathcal{B})$ such that its initial kernel $V_0 = V$ is bounded and absolutely continuous with respect to a finite measure μ.

It is know that in this case the convex cone $\mathcal{E} = \mathcal{E}_\mathcal{V}$ of all $\mathcal{V}$-excessive functions on X which are finite $\mathcal{V}$-a.s is an H-cone. Further we suppose that $\mathcal{E}_\mathcal{V}$ separates the points of X containes the positive constant functions, is min-stable and generates σ-algebra $\mathcal{B}$.

It is know also ([11]) that if $\alpha > 0$ then the kernel $Q := \alpha V_\alpha$ is a $(\mathcal{E}, \mathcal{E}_\alpha)$-dilation operator on $\mathcal{E}$ where $\mathcal{E}_\alpha$ is the H-cone of all α-excessive functions on X with respect to $\mathcal{V}$. In this case we have

$$Vf - \alpha V_\alpha V f = V_\alpha f$$

and therefore the H-cone

$$\mathcal{E}_Q = \{ s - \alpha V_\alpha s \mid s \in \mathcal{E} \}$$

is a solid and increasingly dense subcone of $\mathcal{E}_\alpha$.

Proposition 4.5. If there is no absorbent points of X with respect to $\mathcal{E}$ then any $(\mathcal{E}, \mathcal{E}_\alpha)$-dilation operator Q on $\mathcal{E}$ such that

$$\alpha V_\alpha s \preceq_{\mathcal{E}_\alpha} Qs \quad \forall s \in \mathcal{E}$$

coincides with αV_α.

Proof. Let Q be a $(\mathcal{E}, \mathcal{E}_\alpha)$-dilation opeator on $\mathcal{E}$ such that

$$\alpha V_\alpha s \preceq_{\mathcal{E}_\alpha} Qs \quad \forall s \in \mathcal{E}$$

or equivalently

$$\alpha V_\alpha V f \preceq_{\mathcal{E}_\alpha} QVf. \quad \forall f \in \mathcal{F}_b$$

We have

$$Wf := Vf - QVf \preceq_{\mathcal{E}_\alpha} Vf - \alpha V_\alpha V f = V_\alpha f$$

for any $f \in \mathcal{F}_b$ and therefore there exists $g \in \mathcal{F}_b$, $g \leq 1$ such that

$$Wf = V_\alpha(g \cdot f). \quad \forall f \in \mathcal{F}_b$$

Hence

$$QVf = Vf - V_\alpha(g \cdot f) = V_\alpha(f + \alpha Vf - g \cdot f) = V_\alpha((1-g)f + \alpha Vf) = \alpha V_\alpha(\epsilon f + Vf)$$

where $\epsilon = \frac{1-g}{\alpha}$. On the other hand if $f_1, f_2 \in \mathcal{F}_b$ we have

$$Vf_1 \leq Vf_2 \Longrightarrow QVf_1 \preceq_{\mathcal{E}_\alpha} QVf_2.$$

or equivalently

$$Vf_1 \leq Vf_2 \Longrightarrow \epsilon f_1 + Vf_1 \leq \epsilon f_2 + Vf_2 \quad \mathcal{V} - \text{a.s}$$

We want to show that the set $A := [\epsilon \geq r > 0\}$ is $\mathcal{V}$-negligible for any $r > 0$. In the contrary case let $A^* = \{x \in A \mid \lim_{\alpha \to \infty} \alpha V_\alpha(1_A)(x) = 1\}$.

It is known that A^* is a sub-basic subset of X and for any $s \in \mathcal{E}$, there exists a sequence $(f_n)_n$ in $\mathcal{F}_b$, $f_n = 0$ on $X \backslash A^*$ such that $(Vf_n)_n$ is increasing and

$$B^{A^*} s = \sup_n Vf_n$$

We show that any $x \in A^*$ is an absorbent point. We consider $x \in A^*$ and $(U_n)_n$ an decreasing sequence of natural open neighborhoods of x such that $\cap_n U_n = \{x\}$. We take $f_n := 1_{X \backslash U_n \cap A^*}$. Then for any $g \in \mathcal{F}_b$ such that $g = 0$ on $X \backslash U_n \cap A^*$, $Vg \leq Vf_n$ we have

$$\epsilon g + Vg \leq \epsilon f_n + Vf_n.$$

and therefore

$$\epsilon g + Vg \leq Vf_n \quad \text{with } U_n \cap A^*.$$

Let $(g_m)_m$ be a sequence such that $g_m = 0$ on $X \backslash U_m \cap A^*$ and $Vg_m \uparrow B^{U_n \cap A^*} Vf_n$. Since

$$Vg_m \uparrow Vf_n \quad \text{on } U_n \cap A^*$$

and since

$$\epsilon g_m + Vg_m \leq Vf_n \qquad \forall m \in \mathrm{N}$$

we deduce that

$$g_m \longrightarrow 0, \quad g_m \leq \frac{Vf_n}{r}$$

and therefore $Vg_m \longrightarrow 0$, $Vf_n = 0$ on $U_n \cap A^*$. Hence

$$\{x\} = \left[\left(\sum_n \frac{1}{2^n} Vf_n\right)(x) = 0\right]$$

i.e $\{x\}$ is absorbent.

From the hypothesis we get that A^* is $\mathcal{V}$-negligible for any $r > 0$ and so $Q = \alpha V_\alpha$.

Theorem 4.6. If there is no fine open singleton subset of X with respect to $\mathcal{E}$ then any $(\mathcal{E}, \mathcal{E}_\alpha)$-dilation operator on $\mathcal{E}$ coincides with αV_α.

Proof. Let P be the minimal $(\mathcal{E}, \mathcal{E}_\alpha)$-dilation operator on $\mathcal{E}$. We want to show that $P = Q$. We have

$$PVf \preceq_{\mathcal{E}_\alpha} QVf \preceq_{\mathcal{E}_\alpha} Vf \qquad \forall f \in \mathcal{F}_b.$$

and

$$Vf = V_\alpha(f + \alpha Vf), \; QVf = \alpha V_\alpha Vf$$

Since $PVf \in \mathcal{E} \subset \mathcal{E}_\alpha$ there exists $g_f \in \mathcal{F}_b$ such that

$$PVf = V_\alpha(g_f), \qquad g_f \leq \alpha Vf.$$

On the other hand the kernel W on $(X, \mathcal{B})$ given by

$$Wf = Vf - PVf, \qquad f \in \mathcal{F}_b$$

verifies the complete maxim principle, $Wf \in \mathcal{E}_\alpha$ and

$$Wf = V_\alpha(f + \alpha Vf - g_f)$$

But the kernel V_α verifies also the complete maximum principle and

$$W1 = V_\alpha(1 + \alpha V1 - g_1).$$

We deduce that

$$Wf = V_\alpha((1 + \alpha V1 - g_1) \cdot f) \qquad \forall f \in \mathcal{F}_b$$

and therefore for any $f \in \mathcal{F}_b$ we have

$$f + \alpha Vf - g_f = (1 + \alpha V1 - g_1) \cdot f, \;\; \mathcal{V} - \text{a.s.}$$

or equivalently

$$g_f = \alpha Vf - (\alpha V1 - g_1) \cdot f, \;\; \mathcal{V} - \text{a.s.}$$

Hence if we put $\epsilon = V1 - \frac{1}{\alpha} g_1$ we get

$$0 \leq \epsilon,$$

$$\epsilon f \leq Vf \qquad \mathcal{V}-\text{a.s}, \ \forall f \in \mathcal{F}_b$$

$$PVf = \alpha V_\alpha(Vf - \epsilon f)$$

On the other hand we have, for any $f_1, f_2 \in \mathcal{F}_b$ such that $Vf_1 \leq Vf_2$,

$$PVf_1 \preceq_{\mathcal{E}_\alpha} PVf_2$$

or equivalently

$$\alpha V_\alpha(Vf_1 - \epsilon f_1) \preceq_{\mathcal{E}_\alpha} \alpha V_\alpha(Vf_2 - \epsilon f_2).$$

This relation is equivalent also with the following

$$Vf_1 - \epsilon f_1 \leq Vf_2 - \epsilon f_2 \qquad \mathcal{V}-\text{a.s.}$$

Since there exists a reference measure μ on X with respect to $\mathcal{V}$ and since $\mathcal{E}$ generates $\mathcal{B}$ it follows that $\mathcal{B}$ is countable generated and therefore there exists a $\mathcal{V}$-negligible subset M of X, $M \in \mathcal{B}$ for which we have

$$\epsilon f \leq Vf \text{ on } X \setminus M, \qquad \forall f \in \mathcal{F}_b$$

We want to show that

$$\epsilon = 0 \qquad \mathcal{V}-\text{a.s}$$

Indeed let $T = \{x \in X \setminus M \mid \epsilon(x) > 0\}$. For any $x \in T$ we have

$$Vf(x) \geq \epsilon(x) f(x) \qquad \forall f \in \mathcal{F}_b$$

and therefore

$$V(\mathbf{1}_{\{x\}}) \geq \epsilon(x).$$

Let now $g \in \mathcal{F}_b$ be such that $Vg \leq V1$ and such that $g(x) = 0$. We have

$$P(Vg) \preceq_{\mathcal{E}_\alpha} P(V\mathbf{1}_x).$$

or equivalently

$$Vg - \epsilon g \leq V\mathbf{1}_x - \epsilon \mathbf{1}_x \qquad \mathcal{V}-\text{a.s.}$$

Hence

$$\epsilon(x) \leq V\mathbf{1}_x(x) - Vg(x).$$

Since

$$B^{X\setminus\{x\}} V\mathbf{1}_x = \sup\{Vg \mid g \in \mathcal{F}_b, \ Vg \leq V\mathbf{1}_{\{x\}}, \ g(x) = 0\}$$

it follows

$$\epsilon(x) \leq V\mathbf{1}_x(x) - B^{X\setminus\{x\}} V\mathbf{1}_x(x).$$

But

$$V\mathbf{1}_x = B^{X\setminus\{x\}} V\mathbf{1}_x \text{ on } X \setminus \{x\}.$$

i.e. $\{x\}$ is fine open with respect to $\mathcal{E}$. From the hypothesis we deduce that $\epsilon = 0$ and therefore $P = \alpha V_\alpha$.

The fact that any $(\mathcal{E}, \mathcal{E}_\alpha)$-dilation operator Q on $\mathcal{E}$ coincides with αV_α follows now from the preceding proposition.

Remark. The assertion of the above Theorem fails if instead of "there is no fine open singleton in X" we put "there is no absorbent points in X with respect to $\mathcal{E}$". We consider $X = \{1, 2\}$ and

$$Vf(x) = f(x) + \frac{1}{2}(f(x) + f(y)) \qquad \forall\, x, y \in X.$$

The associated resolvent will be $\mathcal{V} = (V_\alpha)_{\alpha \geq 0}$ where

$$V_\alpha f(x) = f(x) + \frac{1}{(1+\alpha)(2\alpha+1)} \left(\frac{f(x) + f(y)}{2} \right)$$

We have

$$\mathcal{E} = \mathcal{E}_\mathcal{V} = \{(x_1, x_2) \mid x_1, x_2 \geq 0,\ x_1 \leq 5x_2,\ x_2 \leq 5x_1\}$$

$$\mathcal{E}_1 = \mathcal{E}_{\mathcal{V}_1} = \{(x_1, x_2) \mid x_1, x_2 \geq 0 \mid x_1 \leq 25x_2,\ x_2 \leq 25x_1\}$$

and the map $Q : \mathcal{E} \longrightarrow \mathcal{E}$ defined by

$$QVf(x) = \frac{1}{2} f(x) + \frac{5}{4} \frac{f(x) + f(y)}{2}.$$

is a $(\mathcal{E}, \mathcal{E}_1)$-dilation operator on $\mathcal{E}$ such that

$$QVf \preceq_{\mathcal{E}_{\mathcal{V}_1}} V_1 V f$$

and $Q \neq V_1$.

5. The compression operator associated with a (S, T)-dilation operator

In this section S, T are two H-cones as in the preceding sections and Q is a given (S, T)-dilation operator on S. We intend to extend Q to a map $\overline{Q} : D(\overline{Q}) \longrightarrow T$ where $D(\overline{Q})$ a solid convex subcone of T, containing S such that $\overline{Q}$ is additive, increasing, continuous in order from below and such that

$$u, v \in D(\overline{Q}),\ \ u \leq v \Longrightarrow \overline{Q}u \preceq_T \overline{Q}v.$$

and to show that for $u \in D(\overline{Q})$ we have

$$u \in S \Longleftrightarrow \overline{Q}u \preceq_T u.$$

Proposition 5.1. Let us denote by $\overline{Q}$ the operator on $S_Q := \{s - Qs \mid s \in S\}$ defined by

$$\overline{Q}(s - Qs) = Qs - Q^2 s.$$

Then $\overline{Q}$ is additive, increasing, continuous in order from below and

$$s - Qs \leq t - Qt \Longrightarrow \overline{Q}(s - Qs) \preceq_T \overline{Q}(t - Qt).$$

Proof. By definition we get immediately that Q is additive and that

$$s - Qs \le t - Qt \Longrightarrow \overline{Q}(s - Qs) \preceq_T \overline{Q}(t - Qt).$$

Let now $(s_i - Qs_i)_{i \in I}$ be an increasing family in S_Q such that

$$\bigvee_{i \in I} (s_i - Qs_i) = s - Qs.$$

We denote by s'_i (resp. s') the smallest element in S such that

$$s_i - Qs_i = s'_i - Qs'_i, \; s - Qs = s' - Qs'.$$

We know that

$$s_i - Qs_i \le s_j - Qs_j \Longrightarrow s'_i \le s'_j \le s'.$$

If we denote by $t' := \bigvee_{i \in I} s'_i$ we get $t \le s'$ and

$$\bigvee_{i \in I} (s_i - Qs_i) = t - Qt = s' - Qs'$$

Hence $s' \le t$, $s' = t$,

$$\overline{Q}(s - Qs) = \overline{Q}(s' - Qs') = Qs' - Q^2 s',$$

$$Qs'_i \uparrow Qs; \quad Q^2 s'_i \uparrow Q^2 s'.$$

From

$$Q(s'_i - Qs'_i) + Q^2 s'_i = Qs'_i$$

we get

$$\bigvee_{i \in I} Q(s'_i - Qs'_i) + \bigvee_{i \in I} Q^2 s'_i = \bigvee_{i \in I} Qs'_i,$$

$$\bigvee_{i \in I} Q(s'_i - Qs'_i) + Q^2 s' + Qs',$$

$$\bigvee_{i \in I} \overline{Q}(s_i - Qs_i) = \bigvee_{i \in I} Q(s'_i - Qs'_i) = Q(s' - Qs') = \overline{Q}(s - Qs).$$

Corollary 5.2. If we denote

$$D(\overline{Q}) := \{u \in T \mid \exists\, v \in T, \;\; s - Qs \le u \Longrightarrow \overline{Q}(s - Qs) \le v\}$$

then $D(\overline{Q})$ is a solid convex cone in T, $S_Q \subset D(\overline{Q})$ and

$$u \longrightarrow \bigvee_{s - Qs \le u} \overline{Q}(s - Qs)$$

is a map from $D(\overline{Q})$ in T which is additive increasing, continuous in order from below coincides with $\overline{Q}$ on S_Q. We denote also by $\overline{Q}$ this map and we have

a) $u_1, u_2 \in D(\overline{Q}), \;\; u_1 \le u_2 \Longrightarrow \overline{Q}(u_1) \preceq_T \overline{Q}(u_2)$

b) $S \subset D(\overline{Q})$ and $\overline{Q}s \preceq_T s, \ \forall s \in S$.

Remark. The map $\overline{Q}$ is a compression operator on T. Hence the set

$$T(\overline{Q}) := \{u \in D(\overline{Q}) \mid \overline{Q}u \preceq_T u\}$$

is an H-subcone of S with $S \subset T(\overline{Q})$ (see [4]).

From now on we suppose that the set

$$S_\square := \{s \in S \mid \bigwedge_{\alpha \in \Omega} Q^\alpha s = 0\}$$

is increasingly dense in S where Ω is the first ordinal number which is not countable and where Q^α is defined inductively by $Q^0 s = s$ and

$$Q^\alpha s = Q(\bigwedge_{\beta<\alpha} Q^\beta s).$$

Remark. If S is a standard H-cone then there exists a balayage B on S such that

$$Bs = \bigwedge_{\alpha \in \Omega} Q^\alpha s$$

for any universally continuous element s of S and therefore $Bs = Q(Bs)$ for any $s \in S$ or equivalently

$$Bs \leq Qs.$$

for any $s \in S$. Hence the fact that $S_\square$ is increasingly dense in S follows from the fact that there is no balayages B on S, different from zero, dominated by Q.

Theorem 5.3. If $S_\square$ is increasingly dense in S then $\overline{Q}s = Qs$ and the set $\{s - Qs | s \in S_\square\}$ is increasingly dense in T.

Proof. We have inductively, for any $\alpha \in \Omega$ and any $s \in S_\square$,

$$s - Q^\alpha s = \sum_{\beta<\alpha} (Q^\beta s - Q(Q^\beta s))$$

and therefore

$$Qs - Q^{\alpha+1} s = \sum_{\beta<\alpha} [Q^{\beta+1} s - Q(Q^{\beta+1} s)].$$

Hence for any $s \in S$ there exists an increasing family $(s_i - Qs_i)_{i \in I}$ in S_Q such that $s_i \in S_\square$ and such that

$$\bigvee_{i \in I} (s_i - Qs_i) = s$$

$$\bigvee_{i \in I} Q(s_i - Qs_i) \geq Qs - Q^{\alpha+1} s \qquad \forall \alpha \in \Omega.$$

Since α is arbitrary in Ω we get

$$\bigvee_{i \in I} \overline{Q}(s_i - Qs_i) \geq Qs$$

$$\overline{Q}(s) \geq \bigvee_{i\in I} \overline{Q}(s_i - Qs_i) \geq Qs,$$

$$\overline{Q}s = Qs.$$

The equality $\overline{Q}s = Qs$ for any $s \in S$ follows from the fact that $S_\square$ is increasingly dense in S and from the fact that Q and $\overline{Q}$ are continuous in order from below on S and respectively $D(\overline{Q})$.
We have, for any $u \in T$,

$$u = \bigvee_{s\in S} (u \wedge s)$$

. Because S_Q is solid in T it will be sufficient to show that for any $s \in S$ there exists an increasing family $(s_i - Qs_i)_{i\in T}$, where $s_i \in S_\square$ such that $\bigvee_{i\in T}(s_i - Qs_i) = s$. From the first part of the proof it follows that this assertion is true if $s \in S_\square$. The general assertion follows from the fact that $S_\square$ is increasingly dense in S.

Theorem 5.4. Suppose that $S_\square$ is increasingly dense in S. Then for any $u \in D(\overline{Q})$ we have

$$u \in S \Longleftrightarrow \overline{Q}u \preceq_T u.$$

Proof. Let us denote

$$T(\overline{Q}) := \{u \in D(\overline{Q}) \mid \overline{Q}u \preceq_T u\}.$$

Suppose $u \in T(\overline{Q})$ and there is no $v \in T(\overline{Q})$ such that $u - v \in T(\overline{Q})$ and $\overline{Q}v = v$.Since $u - \overline{Q}u \in T$ then from Theorem 5.3 there exists an increasing family $(s_i - Qs_i)_{i\in I}$ where $s_i \in S_\square$ and

$$\bigvee_{i\in I}(s_i - Qs_i) = u - \overline{Q}u.$$

Since $\bigwedge_{\alpha\in\Omega} Q^\alpha s_i = 0$ it follows that

$$s_i - Qs_i \leq s_j - Qs_j \leq u - \overline{Q}u \Longrightarrow s_i \leq s_j \leq u.$$

If we put

$$s := \bigvee_{i\in I} s_i$$

we get $s \leq u$ and

$$s - Qs = u - \overline{Q}u$$

Since there is no $v \in T(\overline{Q})$, such that $u - v \in T(\overline{Q})$ and $\overline{Q}v = v$ then we get $u \leq s$ and therefore

$$u = s, \quad u \in S.$$

Suppose now that $u \in D(\overline{Q})$ is such that $\overline{Q}u = u$. For any $s \in S_\square$ we have $s \in T(\overline{Q})$ and therefore $u \wedge s \in T(\overline{Q})$. Since $\overline{Q}s = Qs$ it follows that

$$\bigwedge_{\alpha\in\Omega} \overline{Q}^\alpha s = 0$$

and therefore there is no $v \in T(Q)$ with $v \leq u \wedge s$ such that $\overline{Q}v = v$. Hence $u \wedge s$ is as in the first part of the proof an so $u \wedge s \in S$. On the other hand

$$u = \bigvee_{s \in S_\square} (u \wedge s)$$

and therefore $u \in S$.

A. Complete H-cones; The complection of an H-cone

In this section we develop the notion of complete H-cones and the procedure of completion of a given H-cone. In the frame of hyperharmonic cones the same problem was studied in [8] and [9].

Definition. Let S be an H-cone ([3]). A non empty subset α of S is called a *Cauchy family* if α is a solide subset of S (with respect to the natural order), α is upper directed and for any $s \in S$ we have

$$\bigwedge_{n \in \mathbb{N}^*} \left(\bigvee_{t \in \alpha} \left(\left(\frac{1}{n} t \right) \wedge s \right) \right) = 0$$

Remark. It is easy to see that for any $s \in S$ the set $\bar{s}$ given by

$$\bar{s} := \{ t \in S \mid t \leq s \}$$

is a Cauchy family on S. In this particular case the Cauchy family $\bar{s}$ is bounded and $s = \bigvee \bar{s}$.

Definition. An H-cone S is termed *complete* if any Cauchy family of S is bounded.

Theorem 6.1. The dual of any H-cone is a complete H-cone.

Proof. Let S^* be the dual of the H-cone S ([3]). Without loss of the generality we may suppose ([9]) that S^* separates S.

Let α be a Cauchy family in S^* and let μ be the functional on S defined by

$$\mu(s) := \sup\{ \nu(s) \mid \nu \in \alpha \}$$

Since α is upper directed it follows that μ is additive, increasing and continuous in order from below. On the other hand from ([8]) it follows that for any $n \in \mathbb{N}$, $\theta \in S^*$ and any $s \in S$ with $\theta(s) < \infty$ there exist $s_1^n, s_2^n \in S$ with

$$s_1^n + s_2^n = s; \ \left(\frac{1}{n} \mu \right)(s_1^n) + \theta(s_2^n) = \left(\left(\frac{1}{n} \mu \right) \wedge \theta \right)(s) = \bigvee_{\nu \in \alpha} \left(\left(\frac{1}{n} \nu \right) \wedge \theta \right)(s).$$

If $\theta \not\equiv 0$ and $s \in S$ is such that $\theta(s) > 0$ then there exists $t \in S$, $n \in \mathbb{N}$ such that

$$t \leq s, \ \ 0 < \theta(t) < \infty, \ \ \left(\frac{1}{n} \mu \right) \wedge \theta(t) < \theta(t).$$

Indeed, in the contrary case we have, for any $t \in S$, $t \leq s$ with $\theta(t) < \infty$

$$\theta(t) = \left(\left(\frac{1}{n} \mu \right) \wedge \theta \right)(t) = \bigvee_{\nu \in \alpha} \left(\left(\frac{1}{n} \nu \right) \wedge \theta \right)(t)$$

and therefore

$$R\left(\theta-\left(\frac{1}{n}\mu\right)\wedge\theta\right)(t)=\sup\{\left(\theta-\left(\frac{1}{n}\mu\right)\wedge\theta\right)(u)\mid u\in S, u\leq s\}=0$$

If we denote

$$r_n:=R\left(\theta-\left(\frac{1}{n}\mu\right)\wedge\theta\right)$$

we have $r_n\in S^*$, $(r_n)_n$ is increasing, $r_n\leq\theta$ and

$$\theta\leq\left(\frac{1}{n}\mu\right)\wedge\theta+r_n$$

From the fact that α is a Cauchy family we get

$$\bigwedge_{n\in\mathbb{N}^*}\left(\frac{1}{n}\mu\right)\wedge\theta=0,\quad \theta\leq\bigvee_{n\in\mathbb{N}}r_n,\quad r_n(t)=0\ (\forall)\ t\leq s \text{ with } \theta(t)<\infty$$

and therefore $\theta(s)=0$ which contradicts the hypothesis. Hence for any $s\in S$ and any $\theta\in S^*$ with $\theta(s)>0$ there exist $n\in\mathbb{N}^*$, $t\in S$, $t\leq s$ such that $\theta(t)<\infty$ and $\left(\frac{1}{n}\mu\right)\wedge\theta(t)<\theta(t)$.
Hence taking $t_1^n,t_2^n\in S$ such that

$$t_1^n+t_2^n=t,\ \left(\frac{1}{n}\mu\right)(t_1^n)+\theta(t_2^n)=\left(\frac{1}{n}\mu\right)\wedge\theta(t)<\theta(t)=\theta(t_1^n)+\theta(t_2^n)$$

we deduce the existence of $t_1^n\in S$ such that

$$t_1^n\leq s,\ 0<\theta(t_1^n),\ \mu(t_1^n)<n\theta(t_1^n)<\infty.$$

Let now $s\in S$ be arbitrary and let us put

$$A:=\{t\in S\mid t\leq s,\mu(t)<\infty\}$$

Obviously A is a solid and upper-directed subset of S and for any $t\in A$ and any $n\in\mathbb{N}$ we have $(nt)\wedge s\in A$. We denote

$$\tau=\bigvee A$$

and we have $(n\tau)\wedge s=\tau$ for any $n\in\mathbb{N}^*$. Hence $B_\tau s=\tau$ where B_τ is the balayage on S given by

$$B_\tau u=\bigvee_{n\in\mathbb{N}}((n\tau)\wedge u)$$

We want to show that $\tau=s$. If $(B_\tau)'$ is the complement of the balayage B_τ ([3], [1]) we have

$$(B_\tau)'u\preceq u,\quad (B_\tau)'B_\tau u=0,\quad (B_\tau)'u\vee B_\tau u=u$$

for any $u\in S$. To show that $\tau=s$ it will be sufficient to prove that $(B_\tau)'s=0$. Let now $s_0:=(B_\tau)'(s)$ and suppose that $s_0\neq0$. Then there exists $\nu\in S^*$ such that

$$0<\nu(s_0)<\infty.$$

If we put $\theta := (B'_\tau)^*\nu$ we get $\theta(s_0) = \nu(s_0)$ and therefore $0 < \theta(s_0) < \infty$. From the first part of the proof we find $t_0 \in S$, $t_0 \leq s_0$ and $n_0 \in \mathbf{N}^*$ with

$$\left(\frac{1}{n}\mu\right)(t_0) < \theta(t_0)$$

Obviously $t_0 \leq \tau$ and therefore $B_\tau t'_0 = t_0$. Since

$$B'_\tau t_0 \preceq t_0 = B_\tau(t_0)$$

it follows that:

$$B'_\tau t_0 = (B'_\tau t_0)\ \ B_\tau t_0,\ \ B_\tau B'_\tau t_0 = B'_\tau t_0$$

$$0 = B'_\tau B_\tau B'_\tau t_0 = B'_\tau t_0,\ B'_\tau t_0 = 0$$

Hence from the equality

$$\theta = (B'_\tau)^*\theta$$

we get the contradictory relation $\theta(t_0) > 0$.

Definition. Let S be an arbitrary H-cone. The *completion of* S is a complete H-cone $\overline{S}$ such that S is isomorphic with a solid and increasingly dense convex sub-cone of $\overline{S}$.

Remark. The completion of S is uniquely determined up to an isomorphism of H-cones.

Theorem 2. For any H-cone there exists its completion.

Proof. Let S be an H-cone and let us denote by C the set of all Cauchy family in S. For any $a, b \in C$ and $\alpha \in \mathbf{R}_+$ we put

$$a + b := \{s + t \mid s \in a, t \in b\}$$

$$\alpha a := \{\alpha s \mid s \in a\}$$

It is easy to see that $a + b$, $\alpha a \in C$ and the map

$$(a, b) \longrightarrow a + b$$

is a composition low on C which is comutative, associative and $\bar{o}$ is the neutral element of C with respect to this low. The following relations are obvious too:

$$1 \cdot a = a,\ \ \alpha(a + b) = \alpha a + \alpha b$$

$$(\alpha + \beta)a = \alpha a + \beta a,\ \ \alpha(\beta a) = (\alpha\beta)a$$

for all $\alpha, \beta \in \mathbf{R}_+$ and $a, b \in C$. For $s, t \in S$ and $\alpha \in \mathbf{R}_+$ we have also

$$\overline{s + t} = \bar{s} + \bar{t},\ \ \overline{\alpha s} = \alpha \cdot \bar{s}$$

$$s \leq t \Longleftrightarrow \bar{s} \subset \bar{t}$$

In $\mathcal{C}$ we consider the following relation

$$a \leq b \overset{\text{def}}{\Longleftrightarrow} ((\forall)\, s \in a \Longrightarrow s = \vee\{t \in b \mid t \leq s\})$$

Obviously we have

$$a \subset b \Longrightarrow a \leq b$$

and for any $s, t \in S$

$$s \leq t \Longleftrightarrow \bar{s} \leq \bar{t}$$

Moreover, if $a, b \in \mathcal{C}$ then

$$a \leq b \Longleftrightarrow (s \in a \Longrightarrow \bar{s} \leq b).$$

It is easy to verify that for any $a, b, c \in \mathcal{C}$ and $\alpha, \beta \in \mathbf{R}_+$ we have

$$a \leq b \Longrightarrow a + c \leq b + c$$

$$a \leq b \Longrightarrow \alpha a \leq \alpha b$$

We have also the relation

$$a + c \leq b + c \Longrightarrow a \leq b$$

Indeed, if we suppose $a + c \leq b + c$ then we deduce inductively that $a + nc \leq b + nc$ for any $n \in \mathbf{N}^*$ and therefore

$$a \leq b + \frac{1}{n} \cdot c \qquad (\forall)\, n \in \mathbf{N}^*$$

Hence for any $s \in a$ we have

$$s = \bigvee\{u + \frac{1}{n}v \mid u \in b, v \in c, u + \frac{1}{n}v \leq s\}.$$

Since the set

$$\{u + \frac{1}{n}v \mid u \in b, v \in c, u + \frac{1}{n}v \leq s\}$$

is upper directed we deduce

$$s = \bigvee\{s \wedge (u + \frac{1}{n}v) \mid u \in b, v \in c, u + \frac{1}{n}v \leq s\} \leq$$

$$\leq \bigvee\{s \wedge u \mid u \in b\} + \bigvee\{s \wedge \frac{1}{n}v \mid v \in c\}$$

and therefore the element c being a Cauchy family we get

$$\bigwedge_{n \in \mathbf{N}^*} (\bigvee_{v \in c} (s \wedge \frac{1}{n}v)) = 0, \; s = \bigvee\{s \wedge u \mid u \in b\}$$

Hence $a \leq b$.

We denote by "$\sim$" the equivalence relation on C given by

$$a \sim b \overset{\text{def}}{\Longleftrightarrow} a \leq b \text{ and } b \leq a$$

If $(a_i)_{i\in I}$ is a family in C then we consider the set

$$a := \{t \in S \mid \bar{t} \leq a_i \quad (\forall)\, i \in I\}$$

One can easely verify that a is a Cauchy family and

$$a \leq a_i \quad (\forall)\, i \in I$$

$$b \in C,\ b \leq a_i \quad (\forall)\, i \in I \Longrightarrow b \leq a.$$

The above element a of C will be denoted $\bigwedge_{i\in I} a_i$ and it represent the greatest minorant of the family $(a_i)_{i\in I}$ in the preordered set $(C, \leq)$. We have

$$\bigwedge_{i\in I} a_i + b \leq \bigwedge_{i\in I}(a_i + b) \quad (\forall)\, b \in C$$

and we shall prove that the converse inequality holds. Let for that t be an element of $\bigwedge_{i\in I}(a_i + b)$. For any $i \in I$ we have $\bar{t} \leq a_i + b$ and if we consider $u_0 \in S$ defined by

$$u_0 = \bigvee\{u \in b \mid u \leq t\}$$

then $\bar{u}_0 \leq b$ and we can show that

$$\overline{R(t - u_0)} \leq a_i \quad (\forall)\, i \in I$$

Indeed, since $\bar{t} \leq a_i + b$ for all $i \in I$ we deduce that for any $i \in I$ there exists an increasing family $(\alpha_\lambda^i + \beta_\lambda^i)_{\lambda\in\Lambda}$ with $\alpha_\lambda^i \in a_i$, $\beta_\lambda^i \in b$ with

$$t = \bigvee\{(\alpha_\lambda^i + \beta_\lambda^i) \mid \lambda \in \Lambda\}$$

Since $\beta_\lambda^i \leq u_0$ for all $i \in I$, $\lambda \in \Lambda$ we deduce

$$t - u_0 \leq \bigvee\{\alpha_\lambda^i + \beta_\lambda^i \mid \lambda \in \Lambda\} \leq \bigvee\{\alpha_i' \in a_i \mid \alpha_i' \leq t\} \quad (\forall)\, i \in I,$$

$$R(t - u_0) \leq \bigvee\{\alpha_i' \in a_i \mid \alpha_i' \leq t\} \quad (\forall)\, i \in I$$

$$\overline{R(t - u_0)} \leq a_i \quad (\forall)\, i \in I$$

If we consider now $n \in S$ such that

$$t = R(t - u_0) + r$$

then $r \leq u_0$ and therefore $\bar{r} \leq \bar{u} \leq b$. Hence

$$\bar{t} = \overline{R(t-u_0)} + \bar{r} \leq \bigwedge_{i\in I} a_i + b$$

Let now $(b_i)_{i\in I}$ be an increasing family in $\mathcal{C}$ which is dominated in $(\mathcal{C}, \leq)$. We put

$$b := \{t \in S \mid (\exists)\, i \in I;\ \bar{t} \leq b_i\}$$

It is easy to see that $b \in \mathcal{C}$ and

$$b_i \leq b \qquad (\forall)\, i \in I$$

$$c \in \mathcal{C},\ c \geq b_i\ (\forall)\, i \in I \Longrightarrow c \geq b$$

The above element b is denoted by $\bigvee_{i\in I} b_i$ and it is the smallest majorant of the family $(b_i)_{i\in I}$ in the preordered set $(\mathcal{C}, \leq)$. For any $c \in \mathcal{C}$ we have

$$\bigvee_{i\in I}(b_i + c) \leq (\bigvee_{i\in I} b_i) + c$$

Now we consider $a, b \in \mathcal{C}$ such that $b \leq a$. We put

$$R(a-b) := \bigwedge\{c \in \mathcal{C} \mid a \leq b + c\}$$

We have

$$R(a-b) \leq a;\ a \leq b + R(a-b).$$

We want to show that there exists $a' \in \mathcal{C}$ such that

$$a \sim R(a-b) + a'.$$

Let $s \in S$ be such that $\bar{s} \leq a$ and let $t \in S$ such that $\bar{t} \leq b$.
We put

$$r_{s,t} := R(s-t),\quad r'_{s,t} = s - r_{s,t}$$

The family $(r_{s,t})_{\bar{t}\leq b}$ is decreasing and $(r'_{s,t})_{\bar{t}\leq b}$ is increasing in S. We have

$$s = r_{s,t} + r'_{s,t},\ s \leq \bigwedge_{\bar{t}\leq b} r_{s,t} + \bigvee_{\bar{t}\leq b} r'_{s,t}$$

If we consider $r_{s,b}$ and $r'_{s,b}$ given by

$$r_{s,b} = \bigwedge_{t\in b} r_{s,t},\quad r'_{s,b} = \bigvee_{t\in B} r'_{s,t}$$

it follows that the family $(r_{s,b})_{\bar{s}\leq a}$ (resp. $(r'_{s,b})_{\bar{s}\leq a}$) is increasing (resp. decreasing) and we have

$$s = r_{s,b} + r'_{s,b} \qquad (\forall)\, s \in S, \bar{s} \leq a$$

Hence

$$\bar{s} = \bar{r}_{s,b} + \bar{r'}_{s,b} \quad (\forall)\ s \in S, \bar{s} \leq a$$

and therefore

$$\bigvee_{\bar{s}\leq a} \bar{s} = \bigvee_{\bar{s}\leq a} \bar{r}_{s,b} + \bigwedge_{\bar{s}\leq a} r'_{s,b}$$

Obviously we have

$$a \sim \bigvee_{\bar{s}\leq a} \bar{s}$$

and if we put $a' := \bigwedge_{\bar{s}\leq b} \bar{r'}_{s,b}$ we get

$$a \sim \bigvee_{\bar{s}\leq a} \bar{r}_{s,b} + a'$$

On the other hand we have

$$\bigvee r'_{s,b} = \bigvee_{\bar{t}\leq b} r'_{s,t} \leq b$$

and therefore

$$a \leq \bigvee_{\bar{s}\leq a} \bar{s} \leq \bigvee_{\bar{s}\leq a} \bar{r}_{s,b} + b, R(a-b) \geq \bigvee_{\bar{s}\leq a} \bar{r}_{s,b}.$$

Let now $c \in C$ be such that $a \leq b+c$. For any $s \in S$ with $\bar{s} \leq a$ we have $\bar{s} \leq b+c$ and therefore there exists two increasing families $(t_\lambda)_{\lambda\in\Lambda}$, $(u_\lambda)_{\lambda\in\Lambda}$ where $t_\lambda \in b$, $u_\lambda \in c$ and

$$s = \bigvee_{\lambda\in\Lambda} (t_\lambda + u_\lambda)$$

If we put $t := \bigvee_\lambda t_\lambda$, $u = \bigvee_\lambda u_\lambda$ we have

$$s = t + u,\ \bar{t} \leq b,\ \bar{u} \leq c$$

Hence

$$r'_{s,t} = s - R(s-t) = s - u = t; r'_{s,b} \leq t,\ \bar{r}_{s,b} \leq c,$$

$$\bigvee_{\bar{s}\leq a} \bar{r}_{s,b} \leq c,\ \bigvee_{\bar{s}\leq a} \bar{r}_{s,b} \leq R(a-b)$$

and therefore

$$\bigvee_{\bar{s}\leq a} \bar{r}_{s,b} \sim R(a-b),\ a \sim R(a-b) - a'.$$

From the above considerations we deduce that the quotient space $C/\sim$ is an H-cone with respect to the addition operation and multiplication with positive real numbers induced by the same operations from C. The map

$$s \longrightarrow \bar{s}$$

from S into $C/\sim$ is an order preserving morphism. Since for any $a \in C$ we have $a \sim \bigvee_{t\in a} \bar{t}$ it follows that S is increasingly dense in $C/\sim$. Obviously S is a solid subset of $C/\sim$. Let now θ be a Cauchy family in $C/\sim$. We put

$$A = \bigcup\{a | a \in \theta\}$$

Since the family $(a)_{a\in\theta}$ is upper directed it follows that A is upper directed in S. Since any $a \in C$ is solid in S we deduce that A is also solid in S. It remains only to show that for any $s \in S$ we have

$$\bigwedge_{n\in\mathbb{N}^*}\left(\bigvee_{t\in A} s \wedge \left(\frac{1}{n}t\right)\right) = 0$$

This assertion may be obtained from the fact that θ is a Cauchy family in $C/\sim$ in the following way

$$\overline{\bigwedge_{n\in\mathbb{N}}\left(\bigvee_{t\in A}\left(s\wedge\left(\frac{1}{n}t\right)\right)\right)} = \bigwedge_{n\in\mathbb{N}^*}\overline{\left(\bigvee_{t\in A}\left(s\wedge\left(\frac{1}{n}t\right)\right)\right)} = \bigwedge_{a\in\theta}\left(\bigvee\left(\bar{s}\wedge\left(\frac{1}{n}a\right)\right)\right) = 0$$

Remark. In [8] is presented a scheme for a completion of an H-cone in the cathegory of cones of hyperharmonics. Such a completion was realised in large in [9]. Using this type of completion one can construct also a completion of an H-cone in the cathegory of H-cones.

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