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Vasile Dragan, Aristide Halanay and Adrian Stoica

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Vasile Dragan *
Aristide Halanay **
Adrian Stoica ***

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* Institute of Mathematics of the Romanian Academy, P.O.Box 1-764, RO-70700 Bucharest, Romania.

** Faculty of Mathematics, St-r. Academiei,no.14,RO-70109, Bucharest,Romania.

*** Departament of Aerospace Engineering,Polytechnic,University of Bucharest Str.Splaiul Independentei,no.313,RO-77206,Bucharest,Romania.

TWO-BLOCK NEHARI AND H^∞ PROBLEMS

Vasile Dragan

Institute of Mathematics of the Romanian Academy,

P.O.Box 1-764,RO-70700,Bucharest,Romania

Aristide Halanay

Faculty of Mathematics,Str.Academiei,no.14,RO-70109,

Bucharest,Romania

Adrian Stoica

Department of Aerospace Engineering,Polytechnic

University of Bucharest,Str.Splaiul Independentei,no.313,

RO-77206,Bucharest,Romania

Abstract. A γ-procedure to compute an optimal distance in the two-block Nehari problem is described.Explicit formulae for an optimal solution to the two-block Nehari problem in terms of the optimal distance value are given.Similar formulae are obtained for a two-block H^∞ approximation problem known as the DF(disturbance feedforward) problem.

1. Introduction

In control design the so-called DF problem is a two-block H^∞ optimization problem which reduces to a two-block Nehari problem.In fact there are also other design problems which can be solved via two-block Nehari problem.

State space solutions for the two-block Nehari problem have been described in [10],[11].If we try to perform efectively the state space construction for the suboptimal solution to the H^∞ problem an ill-conditioned computation appears when approaching the optimum of the same nature as the one mentioned by Habets[9] and Gahinet[6] in connection with the robust controller.One of the aims of the present paper is to remove this ill-conditioning by using a singular perturbation approach as in [4],[5].

As a main result,this procedure leads to explicit formulae for an optimal solution to the two-block Nehari problem depending on the optimal value γ_0 evidentiating thus the fact that the optimal value is attained for a finite dimensional linear time-invariant system.Explicit formulae for the optimal solution to the DF problem are also obtained.

On the other hand there are known formulae for an optimal value of H^∞-norm in terms of some Hankel and Toeplitz operators[13],[16].In the present paper,starting with a state-space construction of the suboptimal solution it is shown that this optimal value solves a specific transcendental equation which may simply be solved approximatively by an iterative procedure(γ-procedure).The γ-procedure proposed in this paper has been performed for an example considered in [1] leading to the same results.For the same example,state-space formulae for the solution are also given.

2. The two-block Nehari problem

Consider the two-block Nehari optimal problem consisting in computation of the optimal norm:

$$\inf_{G \in RH_{\perp}^{\infty}} \left\| \begin{matrix} G_1(s) - G(s) \\ G_2(s) \end{matrix} \right\|_{\infty} := \gamma_0 \quad ; \quad G_1, G_2 \in RH^{\infty} \tag{1}$$

The suboptimal Nehari problem associated to $\gamma > \gamma_0$ involves determining $G \in RH_{\perp}^{\infty}$ for which:

$$\left\| \begin{matrix} G_1(s) - G(s) \\ G_2(s) \end{matrix} \right\|_{\infty} < \gamma \tag{2}$$

In [10],[11] the following solution to the suboptimal Nehari problem has been proposed in a slightly different form:

Theorem 1 *Let* $\left(A, B, \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \right)$ *a realization of* $\begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix}$*;then the suboptimal Nehari problem (2) has a solution if and only if* $\gamma > \|G_2\|_{\infty}$ *and* $\gamma^2 > \rho(QR(\gamma))$*,where* $\rho(.)$ *denotes the spectral radius of (.),Q is the positive-semidefinite solution of the Lyapunov equation:*

$$A^T Q + QA + C_1^T C_1 + C_2^T C_2 = 0 \tag{3}$$

and $R(\gamma)$ *is the positive-semidefinite stabilizing solution to the Riccati equation:*

$$AR + RA^T + (RC_2^T + BD_2^T)(\gamma^2 I - D_2 D_2^T)^{-1}(C_2 R + D_2 B^T) + BB^T = 0 \tag{4}$$

In the assumptions above,a solution to the suboptimal Nehari problem (2) has the realization:

$$G(s) := \left(-[A - W(\gamma) C_1^T C_1]^T, -(QB + C_2^T D_2), C_1 W(\gamma), D_1 \right) \tag{5}$$

where:

$$W(\gamma) := R(\gamma)[\gamma^2 I - QR(\gamma)]^{-1} \tag{6}$$

□

An alternative proof for the necessity part in the theorem will be given in the Appendix,the reverse

Proposition 1 *If $\gamma > \|G_2\|_\infty$ and $\gamma^2 > \rho(QR(\gamma))$ then $W(\gamma)$ is the positive-semidefinite stabilizing solution to the game-theoretic Riccati equation:*

$$\left[A + BD_2^T(\gamma^2 I - D_2D_2^T)^{-1}C_2 + B(\gamma^2 I - D_2^TD_2)^{-1}B^TQ\right]W(\gamma) + W(\gamma)\left[A + BD_2^T(\gamma^2 I - D_2D_2^T)^{-1}C_2 + B(\gamma^2 I - D_2^TD_2)^{-1}B^TQ\right]^T + W(\gamma)\left[-C^TC + \gamma^{-2}QBB^TQ + (\gamma^{-1}QBD_2^T + \gamma C_2^T)(\gamma^2 I - D_2D_2^T)^{-1} \cdot (\gamma^{-1}D_2B^TQ + \gamma C_2)\right]W(\gamma) + B(\gamma^2 I - D_2^TD_2)^{-1}B^T = 0 \tag{7}$$

where $C := [C_1\ C_2]$.

Proof. From (4) we deduce that:

$$\begin{bmatrix} \hat{A}^T & C_2^T(\gamma^2 I - D_2D_2^T)^{-1}C_2 \\ -\hat{B} & -\hat{A} \end{bmatrix}\begin{bmatrix} I \\ R(\gamma) \end{bmatrix} = \begin{bmatrix} I \\ R(\gamma) \end{bmatrix}\left[\hat{A} + C_2^T(\gamma^2 I - D_2D_2^T)^{-1}C_2R(\gamma)\right]$$

where we denoted:

$$\hat{A} := A + BD_2^T(\gamma^2 I - D_2D_2^T)^{-1}C_2$$

$$\hat{B} := BD_2^T(\gamma^2 I - D_2D_2^T)^{-1}D_2B^T + BB^T$$

Consider now the similarity transformation:

$$T := \begin{bmatrix} \gamma^2 I & -Q \\ 0 & I \end{bmatrix}$$

for which one obtains:

$$T\begin{bmatrix} \hat{A}^T & C_2^T(\gamma^2 I - D_2D_2^T)^{-1}C_2 \\ -\hat{B} & -\hat{A} \end{bmatrix}T^{-1}T\begin{bmatrix} I \\ R(\gamma) \end{bmatrix} = T\begin{bmatrix} I \\ R(\gamma) \end{bmatrix}\left[\hat{A} + C_2^T(\gamma^2 I - D_2D_2^T)^{-1}C_2R(\gamma)\right]$$

and therefore:

$$\begin{bmatrix} \hat{A}^T + \gamma^{-2}Q\hat{B} & \hat{A}^TQ + \gamma^{-2}Q\hat{B}Q + \gamma^2C_2^T(\gamma^2 I - D_2D_2^T)^{-1}C_2 + Q\hat{A} \\ -\gamma^{-2}\hat{B} & -\gamma^{-2}\hat{B}Q - \hat{A} \end{bmatrix}\begin{bmatrix} \gamma^2 I - QR(\gamma) \\ R(\gamma) \end{bmatrix} = \begin{bmatrix} \gamma^2 I - QR(\gamma) \\ R(\gamma) \end{bmatrix}\left[\hat{A} + C_2^T(\gamma^2 I - D_2D_2^T)^{-1}C_2R(\gamma)\right]$$

Using (3) and the fact that $\hat{A} + C_2^T(\gamma^2 I - D_2D_2^T)^{-1}C_2R(\gamma)$ is hurwitz since $R(\gamma)$ is a stabilizing solution for (4),we deduce after some direct calculations that $W(\gamma)$ is a stabilizing solution for (7).

In order to prove that $W(\gamma)$ is positive-semidefinite we consider the Cholesky factorization $R(\gamma) = S^T(\gamma)S(\gamma)$ and we obtain:

$$W(\gamma) = R(\gamma)[\gamma^2 I - QR(\gamma)]^{-1} = S^T(\gamma)[\gamma^2 I - S(\gamma) Q S^T(\gamma)]^{-1} S(\gamma)$$

Since $\gamma^2 > \rho(QR(\gamma))$,from the equality above we conclude that $W(\gamma) \geq 0$.

□

Remark 1 In the main body of this paper we shall assume that the systems G_1 and G_2 are minimal and hence Q and $R(\gamma)$ are positive-definite and also $W(\gamma)$ is positive-definite.

□

Remark 2 The game-theoretic Riccati equation (7) can be written in a Lyapunov equivalent form:

$$\left[A - W(\gamma) C_1^T C_1\right] W(\gamma) + W(\gamma)\left[A - W(\gamma) C_1^T C_1\right]^T + W(\gamma) C_1^T C_1 W(\gamma) + \{[I + W(\gamma) Q] B + W(\gamma) C_2^T D_2\}(\gamma^2 I - D_2^T D_2)^{-1}\left\{ B^T [I + Q W(\gamma)] + D_2^T C_2 W(\gamma)\right\} = 0 \qquad (8)$$

□

For the γ-procedure proposed in this paper,a crucial role is played by the dependence of $\rho(QR(\gamma))$ with respect to γ.

Lemma 1 *The function $\gamma \rightarrow \rho(QR(\gamma))$ is monotonically decreasing.*

Proof. Let $\gamma_1 > \gamma_2 > \|G_2\|_\infty$ and $R(\gamma_1)$ and $R(\gamma_2)$ the stabilizing solutions to the corresponding Riccati equations:

$$AR(\gamma_1) + R(\gamma_1) A^T + \left[R(\gamma_1) C_2^T + BD_2^T\right](\gamma_1^2 I - D_2 D_2^T)^{-1}\left[C_2 R(\gamma_1) + D_2 B^T\right] + BB^T = 0$$

$$AR(\gamma_2) + R(\gamma_2) A^T + \left[R(\gamma_2) C_2^T + BD_2^T\right](\gamma_2^2 I - D_2 D_2^T)^{-1}\left[C_2 R(\gamma_2) + D_2 B^T\right] + BB^T = 0$$

When substracting the two equations above one obtains after some direct calculations:

$$\left\{A + \left[R(\gamma_1) C_2^T + BD_2^T\right](\gamma_1^2 I - D_2 D_2^T)^{-1} C_2\right\}[R(\gamma_1) - R(\gamma_2)] + [R(\gamma_1) - R(\gamma_2)] \cdot \left\{A + \left[R(\gamma_1) C_2^T + BD_2^T\right](\gamma_1^2 I - D_2 D_2^T)^{-1} C_2\right\}^T - [R(\gamma_1) - R(\gamma_2)] C_2^T (\gamma_1^2 I - D_2 D_2^T)^{-1} C_2 [R(\gamma_1) - R(\gamma_2)] - \left[R(\gamma_2) C_2^T + BD_2^T\right]\left[(\gamma_2^2 I - D_2 D_2^T)^{-1} - (\gamma_1^2 I - D_2 D_2^T)^{-1}\right]\left[C_2 R(\gamma_2) + D_2 B^T\right] = 0$$

Since $\gamma_1 > \gamma_2$ and $A + [R(\gamma_1) C_2^T + BD_2^T)(\gamma_1^2 I - D_2 D_2^T)^{-1} C_2$ is stable we deduce from the Lyapunov equation above that $R(\gamma_1) - R(\gamma_2) < 0$. Then,using Proposition A1 from Appendix we conclude that $\rho(QR(\gamma_1)) < \rho(QR(\gamma_2))$.

Remark 3 We have to stress that the stabilizing solution $R(\gamma)$ depends smoothly upon γ;there are several arguments in favor of this statement:we may reffer for instance to the way $R(\gamma)$ is obtained in a generalized Popov-Yakubovich theory[10].We may also reffer to the iterative procedures to obtain the solution to the Riccati equation;we may also reffer to a implicit function argument.

3. A γ-procedure

We shall describe an iterative procedure in order to determine γ_0 defined by (1).One of the main results of this paper is:

Theorem 2 *The transcendental equation:*

$$\gamma^2 = \rho(QR(\gamma)) \tag{9}$$

has a unique solution.

Proof Since it is a known fact that $\gamma_0^2=\rho(G_{1H}^*G_{1H}+G_{2T}^*G_{2T})$(see[13]) where G_{1H} denotes the Hankel operator associated with G_1 and G_{2T} is the Toeplitz operator associated to G_2,it follows that $\gamma_0>\|G_2\|_\infty$.

Assume now that (9) has no solution on $[\|G_2\|_\infty,\infty)$;then,since $\gamma^2-\rho(QR(\gamma))$ is continuous with respect to γ,from Lemma 1 it results that $\gamma^2-\rho(QR(\gamma))>0$ forall $\gamma\geq\|G_2\|_\infty$,therefore according to Theorem 1 it follows that $\gamma_0=\|G_2\|_\infty$ which contradicts the fact mentioned above, namely $\gamma_0>\|G_2\|_\infty$;it follows that equation (9) has a solution.

The uniqueness of this solution is a direct consequence of Lemma 1 and of the continuity of $\gamma^2-\rho(QR(\gamma))$ with respect to γ.

□

We give now the algorithm to compute γ_0 with an assigned level of tolerance $\varepsilon>0$.

1[st] Step Compute $\|G_2\|_\infty$ and set $\gamma=\|G_2\|_\infty$;

2[nd] Step Solve the Riccati equation (4).If $|\rho(QR(\gamma))-\gamma^2|<\varepsilon$ then set $\gamma_0=[\gamma^2+\rho(QR(\gamma))]^{1/2}$ and STOP;otherwise,go to 3;

3[rd] Step Set $\gamma\leftarrow[\gamma^2+\rho(QR(\gamma))]^{1/2}$ and return to 2.

4. A well-conditioned solution to the two-block Nehari problem

Let $\gamma>\|G_2\|_\infty$ and consider a balanced realization of $\begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix}$ with respect to $R(\gamma)$ and Q,that is $R(\gamma)$

and Q are diagonal and equal.Such a balanced realization can be obtained from any arbitrary minimal realization (A,B,C,D) of $\begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix}$ by performing the following procedure:

1^{st} *Step* Determine the solutions Q and $R(\gamma)$ of (3) and (4),respectively;

2^{nd} *Step* Perform a Cholesky factorization $Q=Z^TZ$;

3^{rd} *Step* Determine the singular value descomposition:

$$ZR(\gamma)Z^T = U(\gamma)\Sigma^2(\gamma)U^T(\gamma)$$

with $U(\gamma)$ orthogonal;

4^{th} *Step* Define $T(\gamma):=\Sigma^{-1/2}(\gamma)U^T(\gamma)Z$ and compute $T(\gamma)AT^{-1}(\gamma);T(\gamma)B$ and $CT^{-1}(\gamma)$.

Remark 4 In the balanced realization all matrices will depend upon γ and for $\gamma^2=\gamma_0^2+\varepsilon$ they will depend upon ε;this dependence is smooth around $\varepsilon=0$ because the dependence $R(\gamma)$ is smooth.

□

Without loosing the generality of the problem we shall use in the sequel the balanced realization $\left(\bar{A},\bar{B},\begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix},\begin{bmatrix} \bar{D}_1 \\ \bar{D}_2 \end{bmatrix}\right)$ of $\begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix}$ in the sense mentioned above,that is:

$$\bar{G}(\gamma)=\bar{R}(\gamma)=\Sigma(\gamma)=\begin{bmatrix} r_1(\gamma)I_1 & 0 \\ 0 & R_{22}(\gamma) \end{bmatrix} \tag{10}$$

where $r_1(\gamma)>..>r_p(\gamma);\Sigma_{22}:=diag(r_2(\gamma)I_2,..,r_p(\gamma)I_p)$ and I_k are $n_k x n_k$ unit matrices,$k=1,..,p$.

Let take $\gamma=\sqrt{\gamma_0^2+\varepsilon}$ where γ_0 is the solution of the equation $\gamma^2=\rho(QR(\gamma))$, therefore $W(\gamma)$ becomes:

$$\bar{W}(\gamma):=\bar{R}(\gamma)[\gamma^2 I-\bar{Q}(\gamma)\bar{R}(\gamma)]^{-1}=\begin{bmatrix} \frac{r_1(\gamma)}{\varepsilon}I_1 & 0 \\ 0 & W_{22}(\gamma) \end{bmatrix} \tag{11}$$

where:

$$W_{22}(\gamma):=R_{22}(\gamma)[\gamma^2 I-R_{22}^2(\gamma)]^{-1} \quad ; \quad \gamma=\sqrt{\gamma_0^2+\varepsilon} \tag{12}$$

Consider the following partitions of $\bar{A},\bar{B}$ and $\bar{C}$ conformally with (11):

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}; \tilde{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}; \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \tag{13}$$

With the notations above a solution to the optimal two-block Nehari problem is given by the following theorem:

Theorem 3 *Assume that $C_{11}{}^T C_{11}$ is nonsingular;let γ_0 satisfying $\gamma_0{}^2 = \rho(QR(\gamma_0))$;then γ_0 is the optimal Nehari distance and the system $G_o(s) := (A_o, B_o, C_o, D_o)$ with:*

$$\begin{aligned} A_o &:= C_{12}^T C_{11}(C_{11}^T C_{11})^{-1}[A_{21}^T - C_{11}^T C_{12} W_{22}(\gamma_0)] - A_{22}^T + C_{12}^T C_{12} W_{22}(\gamma_0) \\ B_o &:= C_{12}^T C_{11}(C_{11}^T C_{11})^{-1}(\gamma_0 B_1 + C_{21}^T D_2) - R_{22}(\gamma_0) B_2 - C_{22}^T D_2 \\ C_o &:= C_{11}(C_{11}^T C_{11})^{-1}[A_{21}^T - C_{11}^T C_{12} W_{22}(\gamma_0)] + C_{12} W_{22}(\gamma_0) \\ D_o &:= C_{11}(C_{11}^T C_{11})^{-1}(\gamma_0 B_1 + C_{21}^T D_2) + D_1 \end{aligned} \tag{14}$$

is an optimal solution to the two-block Nehari problem.

Proof We shall prove first that A_o is antistable.When writing the Lyapunov equation (8) for $\gamma = \gamma_0$ in the partitioned form corresponding to (13) and when taking into account that $r_1(\gamma_0) = \gamma_0$,one obtains:

- The block (1,1) of (8):

$$C_{11}^T C_{11} = (\gamma_o B_1 + C_{21}^T D_2)(\gamma_0^2 I - D_2{}^T D_2)^{-1}(\gamma_0 B_1{}^T + D_2{}^T C_{21}) \tag{15}$$

- The block (1,2) of (8):

$$\begin{aligned} &A_{21}^T - C_{11}^T C_{12} W_{22}(\gamma_0) + (\gamma_0 B_1 + C_{21}^T D_2)(\gamma_0^2 I - D_2{}^T D_2)^{-1} \cdot \\ &[\gamma_0^2 B_2{}^T + D_2{}^T C_{22} R_{22}(\gamma_0)][\gamma_0^2 I - R_{22}^2(\gamma_0)]^{-1} = 0 \end{aligned} \tag{16}$$

- The block (2,2) of (8):

$$\begin{aligned} &[A_{22} - W_{22}(\gamma_0) C_{12}^T C_{12}] W_{22}(\gamma_0) + W_{22}(\gamma_0)[A_{22} - W_{22}(\gamma_0) C_{12}^T C_{12}]^T + \\ &W_{22}(\gamma_0) C_{12}^T C_{12} W_{22}(\gamma_0) + [\gamma_0^2 I - R_{22}^2(\gamma_0)]^{-1}[\gamma_0^2 B_2 + R_{22}(\gamma_0) C_{22}^T D_2] \cdot \\ &(\gamma_0^2 I - D_2{}^T D_2)^{-1}[\gamma_0^2 B_2{}^T + D_2{}^T C_{22} R_{22}(\gamma_0)][\gamma_0^2 I - R_{22}^2(\gamma_0)]^{-1} = 0 \end{aligned} \tag{17}$$

From expression (14) of A_o and from (16) we deduce that:

$$[A_{22}-W_{22}(\gamma_0)C_{12}^TC_{12}]^T=-A_o+C_{12}^TC_{11}(C_{11}^TC_{11})^{-1}[A_{21}^T-C_{11}^TC_{12}W_{22}(\gamma_o)]=$$
$$-A_o-C_{12}^TC_{11}M(\gamma_0)N^T(\gamma_0)$$

where we have denoted:

$$\begin{aligned}M(\gamma_0)&:=(C_{11}^TC_{11})^{-1}(\gamma_0B_1+C_{21}^TD_2)(\gamma_0^2I-D_2^TD_2)^{-\frac{1}{2}}\\ N(\gamma_0)&:=[\gamma_0^2I-R_{22}^2(\gamma_0)]^{-1}[\gamma_0^2B_2+R_{22}(\gamma_0)C_{21}^TD_2](\gamma_0^2I-D_2^TD_2)^{-\frac{1}{2}}\end{aligned}\tag{18}$$

Therefore (18) becomes:

$$\begin{aligned}&-A_o^TW_{22}(\gamma_o)-W_{22}(\gamma_o)A_o-N(\gamma_0)M^T(\gamma_o)C_{11}^TC_{12}W_{22}(\gamma_o)-\\ &W_{22}(\gamma_o)C_{12}^TC_{11}M(\gamma_0)N^T(\gamma_0)+W_{22}(\gamma_0)C_{12}^TC_{12}W_{22}(\gamma_0)+N(\gamma_0)N(\gamma_0)^T=0\end{aligned}\tag{19}$$

With expression (14) for C_o and with notations (18) we also have:

$$C_o=-C_{11}M(\gamma_0)N^T(\gamma_0)+C_{12}W_{22}(\gamma_0)$$

therefore:

$$\begin{aligned}C_o^TC_o&=N(\gamma_0)M^T(\gamma_0)C_{11}^TC_{11}M(\gamma_0)N^T(\gamma_0)+W_{22}(\gamma_0)C_{12}^TC_{12}W_{22}(\gamma_0)-\\ &N(\gamma_0)M^T(\gamma_0)C_{11}^TC_{12}W_{22}(\gamma_0)-W_{22}(\gamma_0)C_{12}^TC_{11}M(\gamma_0)N^T(\gamma_0)\end{aligned}$$

When substituting the first two terms from the right side of the equation above into (19) we obtain:

$$\begin{aligned}&-A_o^TW_{22}(\gamma_0)-W_{22}(\gamma_0)A_o+C_o^TC_o-N(\gamma_0)M^T(\gamma_0)C_{11}^TC_{11}M(\gamma_0)N^T(\gamma_0)+\\ &N(\gamma_0)N^T(\gamma_0)=0\end{aligned}\tag{20}$$

Using (15) and (18) one can directly verify that (20) is equivalent with:

$$\begin{aligned}&-A_o^TW_{22}(\gamma_0)-W_{22}(\gamma_0)A_o+C_o^TC_o+\\ &N(\gamma_0)\{I-P(\gamma_0)[P^T(\gamma_0)P(\gamma_0)]^{-1}P^T(\gamma_0)\}N^T(\gamma_0)=0\end{aligned}\tag{21}$$

where $P(\gamma_0):=(\gamma_0^2I-D_2^TD_2)^{-1/2}(\gamma_0B_1^T+D_2^TC_{21})$.

Since $I-P(\gamma_0)[P^T(\gamma_0)P(\gamma_0)]^{-1}P^T(\gamma_0)\geq 0$ we deduce that:

$$-A_o^TW_{22}(\gamma_0)-W_{22}(\gamma_0)A_o+C_o^TC_o\leq 0\tag{22}$$

From the expressions (14) of A_o and C_o it follows that $-A_o+C_{12}^TC_o=A_{22}^T$.It is known from [7] that when performing a balancing transformation to a stable system,the block A_{22} corresponding to the balanced realization,is stable too (this result is given in [7] for the antistable case but it also remains valid in the stable case;the result was proved for the balancing with respect to Gramians but it can be directly applied for the system $\left(A,[B \mid (R(\gamma_0)C_2^T + BD_2^T)(\gamma_0^2 I - D_2D_2^T)^{-\frac{1}{2}}], \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}\right)$ for which the Gramians are just the solutions Q and $R(\gamma_0)$ of (3) and (4),respectively).

Since A_{22}^T is stable it follows that the pair $(C_o, -A_o)$ is detectable;therefore,because $W_{22}(\gamma_o)>0$ we conclude from (22) that A_o is antistable.

We shall prove now that:

$$\left\| \begin{matrix} G_1(s) - G_o(s) \\ G_2(s) \end{matrix} \right\|_{-} \leq \gamma_0 \tag{23}$$

We have the following realization:

$$\begin{bmatrix} G_1(s) - G_o(s) \\ G_2(s) \end{bmatrix} := (A_d, B_d, C_d, D_d)$$

where:

$$A_d = \begin{bmatrix} \bar{A} & 0 \\ 0 & A_o \end{bmatrix}; B_d = \begin{bmatrix} \bar{B} \\ B_o \end{bmatrix}; C_d = \begin{bmatrix} \bar{C}_1 & -C_o \\ \bar{C}_2 & 0 \end{bmatrix}; D_d = \begin{bmatrix} D_1 - D_o \\ D_2 \end{bmatrix} \tag{24}$$

Consider the Riccati equation:

$$A_d^T\Pi + \Pi A_d + (\Pi B_d + C_d^T D_d)(\gamma_0^2 I - D_d^T D_d)^{-1}(B_d^T \Pi + D_d^T C_d) + C_d^T C_d = 0 \tag{25}$$

We shall prove that (25) is verified by:

$$\Pi = \begin{bmatrix} Q & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ [0 \ I] & -W_{22}(\gamma_0) \end{bmatrix} \tag{26}$$

where the dimension of I equals the number of columns of A_{12}.

When writing (25) in the partitioned form corresponding to (24) and (26) one obtains that the block (1,1) of (25) is just (3);the block (1,2) of (25) vanishes because of the expressions of A_o and C_o and the block (2,2) coincides with (21).Then we conclude that Π verifies (25).

Consider now the adjoint system of G_d,i.e.:

$$\begin{aligned} \dot{x} &= -A_d^T x - C_d^T u \\ y &= B_d^T x + D_d^T u \end{aligned} \tag{27}$$

A direct calculation using (21) gives for an arbitrary $u \in L^2(-\infty,\infty)$:

$$\int_{-\infty}^{\infty} y^T y\,dt = \int_{-\infty}^{\infty} (x^T B_d + u^T D_d)(B_d^T x + D_d^T u)\,dt = -\int_{-\infty}^{\infty} \{x^T [A_d \Pi + \Pi A_d^T +$$

$$(\Pi C_d^T + B_d D_d^T)(\gamma_0^2 I - D_d D_d^T)^{-1}(C_d \Pi + D_d B_d^T)]x - u^T D_d B_d^T x -$$

$$x^T B_d D_d^T u - u^T D_d D_d^T u\}\,dt = -\int_{-\infty}^{\infty} [-\dot{x} \Pi x - u^T C_d \Pi x - x^T \Pi \dot{x} - x^T \Pi C_d^T u + \tag{28}$$

$$x^T (\Pi C_d^T + B_d D_d^T)(\gamma_0^2 I - D_d D_d^T)^{-1}(C_d \Pi + D_d B_d^T)x - u^T D_d B_d^T x -$$

$$x^T B_d D_d^T u - D_d D_d^T u]\,dt$$

Since A_d is dichotomic,the term:

$$\int_{-\infty}^{\infty} (\dot{x}^T \Pi x + x^T \Pi \dot{x})\,dt = \int_{-\infty}^{\infty} \frac{d}{dt}(x^T \Pi x)\,dt$$

vanishes,then from (28) it follows that:

$$\int_{-\infty}^{\infty} y^T y\,dt = -\int_{-\infty}^{\infty} [x^T (\Pi C_d^T + B_d D_d^T)(\gamma_0^2 I - D_d D_d^T)^{-1} - u](\gamma_0^2 I - D_d D_d^T) \cdot$$

$$[(\gamma_0^2 I - D_d D_d^T)^{-1}(C_d \Pi + D_d B_d^T)x - u]\,dt + \gamma_0^2 \int_{-\infty}^{\infty} u^T u\,dt$$

From the equality above we deduce that:

$$\int_{-\infty}^{\infty} y^T y\,dt \le \gamma_0^2 \int_{-\infty}^{\infty} u^T u\,dt$$

for all $u \in L^2(-\infty,\infty)$,therefore the L^∞-norm of $(-A_d^T, -C_d^T, B_d^T, D_d^T)$ is less or equal than γ_0.Since the L^∞-norm of a system equals the L^∞-norm of its adjoint,it follows that $\|G_d\|_\infty \le \gamma_0$.

We shall prove now by contradiction that in fact (23) we have an equality.Assume that there exists $\hat{G} \in RH_\perp^\infty$ and $\hat{\gamma}$ such that:

$$\left\| \begin{matrix} G_1(s) - \hat{G}(s) \\ G_2(s) \end{matrix} \right\|_\infty \le \hat{\gamma}$$

From Proposition 1 we deduce that $\rho(QR(\gamma_0)) < \rho(QR(\hat{\gamma}))$ therefore $\hat{\gamma}^2 - \rho(QR(\hat{\gamma})) < 0$ which contradicts the necessity part of Theorem 1. Therefore we conclude that we have in fact:

$$\left\| \begin{matrix} G_1(s) - G_o(s) \\ G_2(s) \end{matrix} \right\|_\infty = \gamma_0$$

and hence the theorem is completely proved.

□

Remark 5 If $C_{11}=0$,an optimal solution to the two-block Nehari problem (1) can be obtained using the suboptimal solution (5);with the partitions (10)÷(13) af A,B,C and W,in such situation no singularities appear for $\gamma \to \gamma_0$ and when taking $\gamma = \gamma_0$ one will obtain an optimal solution.If $C_{11}^T C_{11}$ is singular,by performing to it an orthogonal transformation,we shall obtain from (5) with $\gamma^2 = \gamma_0^2 + \varepsilon$ a singularly perturbed system which fast component with the dimension equal to the rank of $C_{11}^T C_{11}$,may be reduced according to the theory of singular perturbations;therefore if n denotes the order of G,then the dimension of the optimal Nehari approximation equals $n\text{-}rank(C_{11}^T C_{11})$.

□

Remark 6 The theorem proves that the optimal solution to the two-block distance problem may be obtained in a form of a finite dimensional time-invariant system that is in a form proper,rational transfer matrix function.The same conclusion follows from the construction in [8].

□

We have shown in Section 3 how one may compute γ_0 with an assigned level of tolerance;since the realization (14) of the optimal solution depends on γ_0 we investigated what is the influence of an inaccurate determination of γ_0 upon the attenuation property of (14).Related to this problem we obtained the following result:

Theorem 4 *Let $\gamma = \gamma_0 + O(\varepsilon)$ and denote by G_r the system (14) obtained when replacing γ_0 with γ;then G_r is antistable and:*

$$\left\| \begin{matrix} G_1(s) - G_r(s) \\ G_2(s) \end{matrix} \right\|_\infty = \gamma_0 + O(\varepsilon) \tag{29}$$

Proof It is known from [14] that if a self-adjoint operator $G(\gamma)$ is smooth then its eigenvalues and its orthonormal eigenvectors $u_i, i=1,..,n$ are smooth functions of γ;therefore when performing the balancing procedure described at the beginning of this section,one obtains a smooth dependence of T and T^{-1} with respect to ε.Then G_r defined in the statement of the theorem will have the following realization:

$$A_r = A_o + O(\varepsilon)\,;\, B_r = B_o + O(\varepsilon)\,;\, C_r = C_o + O(\varepsilon)\,;\, D_r = D_o + O(\varepsilon) \tag{30}$$

therefore G_r is antistable for ε sufficiently small.

We shall prove now that $\|G_r(s)-G_o(s)\|_\infty \le O(\varepsilon)$;indeed,we have:

$$(G_r u)(t) = C_r \int_{-\infty}^{t} e^{A_r(t-s)} B_r u(s)ds + D_r u(t) \tag{31}$$

where:

$$e^{A_r t} = e^{A_o t} + \int_0^t e^{A_r(t-\tau)}(A_r - A_o)e^{A_r \tau} d\tau \tag{32}$$

Since A_r and A_o are stable,there exists $\alpha,\beta>0$ such that:

$$\left\| \int_0^t e^{A_r(t-\tau)}(A_r-A_o)e^{A_r\tau}d\tau \right\|_\infty \le \beta \|A_r-A_o\| \int_0^t e^{-\alpha(t-\tau)}e^{-\alpha\tau}d\tau = \beta \|A_r-A_o\| t e^{-\alpha t}$$

Taking into account that A_r-A_o=$O(\varepsilon)$ we deduce from the last inequality above and from (32) that:

$$e^{A_r t} = e^{A_o t} + \Psi(t,\varepsilon)$$

with $\Psi(t,\varepsilon)$ bounded and $\lim_{\varepsilon\to 0}\Psi(t,\varepsilon)=0$;therefore,from (30) and (31) it follows that $\|G_r(s)-G_o(s)\|_\infty \le O(\varepsilon)$.We also have:

$$\left\| \begin{matrix} G_1(s)-G_r(s) \\ G_2(s) \end{matrix} \right\|_\infty = \left\| \begin{matrix} G_1(s)-G_o(s)+G_o(s)-G_r(s) \\ G_2(s) \end{matrix} \right\|_\infty \le$$

$$\left\| \begin{matrix} G_1(s)-G_o(s) \\ G_2(s) \end{matrix} \right\|_\infty + \|G_r(s)-G_o(s)\|_\infty = \gamma_0 + O(\varepsilon)$$

therefore (29) is proved.

5. H^∞ approximation for a $H_\perp^\infty$ system

In the preceding section a solution has been described for the $H_\perp^\infty$ (antistable) approximation for a H^∞ system (stable).Motivated by applications to the two-block H^∞ approximation problem,we shall describe now the solution to the problem of approximating an $H_\perp^\infty$ system by a H^∞ one.

In state space formulation(time domain) this new problem is obtained in the simpliest way from the former one just by changing the sense of time which amounts to changing $A\leftarrow -A;B\leftarrow -B$.

The time change represents an izometric transformation in $L^2(\mathbb{R})$ spaces and hence if the Nehari problem is stated in terms of input-output operators the distance is not affected.

In frecquency domain approach the transformation amounts in changing s to $-s$ and since the norm is calculated for $s=j\omega$ it is seen again the optimal value γ_0 is the same as for the problem considered in Section 2.

Corresponding to the modifications indicated above the optimal solution to the new Nehari problem is readily obtained.

Theorem 3' *Let $G_1(s):=(A,B,C_1,D_1)$ and $G_2(s):=(A,B,C_2,D_2)$ be two minimal systems with antistable evolution.Associate the Lyapunov equation:*

$$A^TQ+QA=C_1^TC_1+C_2^TC_2$$

with the solution Q positive definite and the Riccati equation:

$$AR+RA^T-(RC_2^T-BD_2^T)(\gamma^2I-D_2D_2^T)^{-1}(C_2R-D_2B^T)-BB^T=0$$

Let $R(\gamma)>0$ be the solution to this equation such that $-A+[R(\gamma)C_2^T-BD_2^T)(\gamma^2I-D_2D_2^T)^{-1}C_2$ is stable.Assume again that $C_{11}^TC_{11}$ is nonsingular.Let γ_0 be the unique solution to the equation $\gamma^2=\rho(QR(\gamma))$;then γ_0 is the optimal Nehari distance and the system $G_o(s):=(A_o,B_o,C_o,D_o)$ with:

$$\begin{aligned}
A_o &= C_{12}^TC_{11}(C_{11}^TC_{11})^{-1}[A_{21}^T+C_{11}^TC_{12}W_{22}(\gamma_0)]-A_{22}^T-C_{12}^TC_{12}W_{22}(\gamma_0)\\
B_0 &= C_{12}^TC_{11}(C_{11}^TC_{11})^{-1}(\gamma_0B_1-C_{21}^TD_2)-R_{22}(\gamma_0)B_2+C_{22}^TD_2\\
C_o &= -C_{11}(C_{11}^TC_{11})^{-1}[A_{21}^T+C_{11}^TC_{12}W_{22}(\gamma_0)]+C_{12}W_{22}(\gamma_0)\\
D_o &= -C_{11}(C_{11}^TC_{11})^{-1}(\gamma_0B_1-C_{21}^TD_2)+D_1
\end{aligned} \tag{33}$$

where $A_{ij},B_i,C_{ij},D_i,i,j=1,2$ and $W_{22}(\gamma_0),R_{22}(\gamma_0)$ are defined as in Section 4,is an optimal stable approximation to the given antistable system.

6. The two-block H^∞ problem

Consider the system:

$$\begin{aligned}
\dot{x} &= Ax+B_1u_1+B_2u_2\\
y_1 &= C_1x+D_{11}u_1+D_{12}u_2\\
y_2 &= C_2x+u_1
\end{aligned}$$

with $D_{12}^TD_{12}$ invertible.We look for a stabilizing controller:

$$\begin{aligned}
\dot{x}_c &= A_cx_c+B_cu_c\\
y_c &= C_cx_c+D_cu_c
\end{aligned}$$

such that after taking $u_c=y_2$ and $u_2=y_c$ the norm of the input-output operator from u_1 to y_1 is minimal.This problem can be reduced to a two-block Nehari problem.Let X,Y be the stabilizing solutions to the standart Riccati equation:

$$A^TX + XA - (XB_2 + C_1^TD_{12})(D_{12}^TD_{12})^{-1}(B_2^TX + D_{12}^TC_1) + C_1^TC_1 = 0$$

$$AY + YA^T - YC_2^TC_2Y + B_2B_2^T = 0$$

Construct the corresponding double coprime factorization

$(A,B_2,C_2)=NM^{-1}=\tilde{M}^{-1}\tilde{N}$ with:

$$\begin{bmatrix} Y & U \\ -\tilde{N} & \tilde{M} \end{bmatrix}\begin{bmatrix} M & -U \\ N & V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

where:

$$\begin{bmatrix} M & -U \\ N & V \end{bmatrix}(s) := \left[\begin{array}{c|cc} A+B_2F & B_2 & -H \\ \hline F & I & 0 \\ C_2 & 0 & I \end{array}\right]$$

with $F:=-(D_{12}^TD_{12})^{-1}(B_2X+D_{12}^TC_1)$ and $H:=-YC_2^T$.

A parametrized family of stabilizing controllers is written as:

$$K = K_1K_2^{-1}$$

where:

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} M & -U \\ N & V \end{bmatrix}\begin{bmatrix} L \\ I \end{bmatrix}$$

After coupling this family of controllers to the system one gets the input-output operator:

$$T_{y_1u_1} = T_{11} + T_{12}LT_{21}$$

where:

$$T_{11}(s) := \left(\begin{bmatrix} A+B_2F & -B_2F \\ 0 & A+HC_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_1+H \end{bmatrix}, [C_1+D_{12}F \;\; -D_{12}F], D_{11} \right)$$

$$T_{12}(s) := (A+B_2F, B_2, C_1+D_{12}F, D_{12})$$

$$T_{21}(s) := (A+HC_2, B_1+H, C_2, I)$$

In order to have $T_{21}, T_{21}^{-1} \in RH^{\infty}$ we assume that $A\text{-}B_1C_2$ is hurwitz; such an assumption is usually made in the literature related to the so-called DF problem[2].

Under such assumption,by denoting $\tilde{L} = -LT_{21}$ we may write:

$$T_{y_1u_1} = T_{11} - T_{12}\tilde{L}$$

Taking into account the choice for F and H we get that $T_{12}(D_{12}^TD_{12})^{-1/2}$ is inner.Let $T_{12}^{\perp}$ be a completation such that $[T_{12}\ T_{12}^{\perp}]$ is inner.A realization for $[T_{12}\ T_{12}^{\perp}]$ is:

$$[\,T_{12}\quad T_{12}^{\perp}\,](s)=\left[\begin{array}{c|cc} A+B_2F & B_2(D_{12}^TD_{12})^{-\frac{1}{2}} & -X^{-1}C_1^TD_{12}^{\perp} \\ \hline C_1+D_{12}F & D_{12}(D_{12}^TD_{12})^{-\frac{1}{2}} & D_{12}^{\perp} \end{array}\right]$$

where $D_{12}^{\perp}$ is such that $[D_{12}(D_{12}^TD_{12})^{-\frac{1}{2}}\ D_{12}^{\perp}]$ is unitary.

Write:

$$T_{y_1u_1}=T_{11}-\left[\,T_{12}\quad T_{12}^{\perp}\right]\begin{bmatrix}L\\0\end{bmatrix}$$

Since $[T_{12}\ T_{12}^{\perp}]$ is inner,we have:

$$\|T_{y_1u_1}\|_{\infty}=\left\|[\,T_{12}\quad T_{12}^{\perp}]^{*}\,T_{11}-\begin{bmatrix}L\\0\end{bmatrix}\right\|_{\infty} \tag{34}$$

A realization for $[T_{12}\ T_{12}^{\perp}]^{*}T_{11}$ is:

$$\begin{aligned}
\dot{\xi}&=-(A+B_2F)^T\xi-(C_1+D_{12}F)^T(C_1+D_{12}F)x_1+\\
&\qquad (C_1+D_{12}F)^TD_{12}Fx_2-(C_1+D_{12}F)^TD_{11}u\\
\dot{x}_1&=(A+B_2F)x_1-B_2Fx_2+B_1u\\
\dot{x}_2&=(A+HC_2)x_2+(B_1+H)u\\
\hat{y}_1&=B_2^T\xi+D_{12}^T(C_1+D_{12}F)x_1-D_{12}^TD_{12}Fx_2+D_{12}^TD_{11}u\\
\hat{y}_2&=-(D_{12}^{\perp})^TC_1X^{-1}\xi+(D_{12}^{\perp})^T(C_1+D_{12}F)x_1+(D_{12}^{\perp})^TD_{11}u
\end{aligned}$$

When performing the coordinate transformation:

$$S:=\begin{bmatrix}I&-X&0\\0&I&0\\0&0&I\end{bmatrix}$$

one obtains the following equivalent realization for (34):

$$\begin{aligned}
\dot{\xi}_1&=-(A+B_2F)^T\xi_1-\hat{B}_1u\\
\dot{x}_1&=(A+B_2F)x_1-B_2Fx_2+B_1u\\
\dot{x}_2&=(A+HC_2)x_2+(B_1+H)u\\
\hat{y}_1&=B_2^T\xi_1-D_{12}^TD_{12}Fx_2+D_{12}^TD_{11}u\\
\hat{y}_2&=-(D_{12}^{\perp})^TC_1X^{-1}\xi_1+(D_{12}^{\perp})^TD_{11}u
\end{aligned}$$

where $\hat{B}_1:=(C_1+D_{12}F)^TD_{11}+XB_1$ .After reducing the unobservable part one obtains the equivalent realization of $[T_{12}\ T_{12}^{\perp}]^{*}T_{11}$:

$$[T_{12} \quad T_{12}^{\perp}]^* T_{11}(s) := \left[\begin{array}{cc|c} -(A+B_2F)^T & 0 & -\hat{B}_1 \\ 0 & A+HC_2 & B_1+H \\ \hline B_2^T & -D_{12}^T D_{12} F & D_{12}^T D_{11} \\ -(D_{12}^{\perp})^T C_1 X^{-1} & 0 & (D_{12}^{\perp})^T D_{11} \end{array}\right]$$

Let consider the partition:

$$[T_{12} \quad T_{12}^{\perp}]^* T_{11} = \begin{bmatrix} \hat{G}_1 \\ \hat{G}_2 \end{bmatrix}$$

where it is obvious that $\hat{G}_1 = \hat{G}_{1s} + \hat{G}_{1a}$ with:

$$\hat{G}_{1s}(s) := (A+HC_2, B_1+H, -D_{12}^T D_{12} F, 0)$$

$$\hat{G}_{1a}(s) := (-(A+B_2F)^T, -\hat{B}_1, B_2^T, D_{12}^T D_{11})$$

$$\hat{G}_2(s) := (-(A+B_2F)^T, -\hat{B}_1, -(D_{12}^{\perp})^T C_1 X^{-1}, (D_{12}^{\perp})^T D_{11})$$

where $\hat{G}_{1a}$ and $\hat{G}_2$ are antistable and $\hat{G}_{1s}$ is stable.

Denoting by $\hat{L} = \hat{L} - \hat{G}_{1s}$,the two-block H^∞ has been transformed in a two-block Nehari problem analysed in the previous section.

The computation for the optimal distance γ_0 can be performed as in Section 3 and the realization of an optimal solution is given by (33).

Remark 7 In the construction of an optimal robust controller with respect to perturbations in the normalized left coprime factorization the robust controller solves a disturbance attenuation problem associated to a fictious plant.This problem is also a DF problem and hence we may use the procedures in this paper to compute the optimal robustness radius and the optimal robust controller;in fact this remark may be considered as providing a test for our computations.

7. An example

In the same way as in the previous section a weighted mixed sensitivity problem is also reduced to a two-block Nehari problem[3],[15].In this section weshall consider an example of such problem taken from [1] and we shall compute by our procedure the optimal γ_0 and the corresponding optimal solution.This problem consist in determining:

$$\inf_{K(s)\in RH^{\infty}} \left\| \begin{bmatrix} N(s)+\phi(s)K(s) \\ S(s) \end{bmatrix} \right\|_{\infty} := \gamma_0$$

where:

$$N(s) = \frac{-2(s+10)(s+0.125)(s-0.12)}{(s+0.1)(s+1)(10s+\sqrt{2})} \ ; \ S(s) = \frac{0.1s+1}{10s+\sqrt{2}}$$

$$\phi(s) = \frac{(10s-\sqrt{2})(s-1)}{(10s+\sqrt{2})(s+1)}$$

We transformed this problem in a two-block Nehari problem and when applying the algorithm described in Section 3 we obtained for the tolerance level $\varepsilon=10^{-12}$, $\gamma_0=1.100437963947$;this optimal distance can be achieved with the optimal solution $K(s):=[A_o,B_o,C_o,D_o]$ determined using formulae (33),where:

$$A_o = \begin{bmatrix} -0.2039 & -0.0104 \\ 1 & 0 \end{bmatrix} \ ; \ B_o = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C_o = [\,-0.0469 \ \ -0.0049\,] \ ; \ D_o = -0.9003$$

Appendix

A1. Proposition A1. Let A_1,A_2 and B be three symmetric, positive-definite matrices having the same dimension and assume that $A_1<A_2$;then $\rho(A_1B)<\rho(A_2B)$ where $\rho(.)$ denotes the spectral radius of (.).

Proof. In the assumptions of the statement we have:

$$B^{\frac{1}{2}}A_1B^{\frac{1}{2}} - B^{\frac{1}{2}}A_2B^{\frac{1}{2}} = B^{\frac{1}{2}}(A_1-A_2)B^{\frac{1}{2}} < 0$$

therefore $B^{\frac{1}{2}}A_1B^{\frac{1}{2}} < B^{\frac{1}{2}}A_2B^{\frac{1}{2}}$.Since $\sigma(B^{\frac{1}{2}}A_1B^{\frac{1}{2}})=\sigma(A_1B)$ and $\sigma(B^{\frac{1}{2}}A_2B^{\frac{1}{2}})=\sigma(A_2B)$,where $\sigma(.)$ denotes the set of eigenvalues of (.),we deduce that $\rho(A_1B)<\rho(A_2B)$

A2.The necessity part of Theorem 1

We shall sketch the proof of the fact that if the suboptimal two-block Nehari problem has a solution G then $\gamma>\|G_2\|_{\infty}$ and $\gamma^2>\rho(QR(\gamma))$.The first inequality follows immediately since we have:

$$\|G_2(s)\|_\infty \le \left\| \begin{matrix} G_1(s)-G(s) \\ G_2(s) \end{matrix} \right\|_\infty < \gamma$$

In order to prove that $\gamma^2>\rho(QR(\gamma))$ we shall consider the augmented plant $P(s)$ defined as:

$$P(s)=\begin{bmatrix} I & 0 \\ G_1(s) & I \\ G_2(s) & 0 \end{bmatrix} := \left(A, [B \;\; 0], \begin{bmatrix} 0 \\ C_1 \\ C_2 \end{bmatrix}, \begin{bmatrix} I & 0 \\ D_1 & I \\ D_2 & 0 \end{bmatrix} \right)$$

Define now the operator $R{:}L^2(-\infty,\infty,\mathbf{R}^{m_1+p_1})\rightarrow L^2(-\infty,\infty,\mathbf{R}^{m_1+p_1})$ where (m_1,p_1) are the dimensions of G_1 and $R{:=}P^*JP$ with:

$$J{:}=\begin{bmatrix} -\gamma^2 I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Since:

$$\begin{bmatrix} I & G^* \\ 0 & I \end{bmatrix} P^* J P \begin{bmatrix} I & 0 \\ G & I \end{bmatrix} = \begin{bmatrix} -\gamma^2 I+(G_1-G)^*(G_1-G)+G_2^*G_2 & G_1-G \\ (G_1-G)^* & I \end{bmatrix}$$

it follows that R has a signature and the anticausal Toeplitz operator associated to R,denoted by $\boldsymbol{R}$ has the same signature.If we define the operator $\Lambda{:}L^2(-\infty,\infty,\mathbf{R}^{m_1+p_1})\rightarrow L^2(-\infty,\infty,\mathbf{R}^{m_1+p_1});(\Lambda u)(t)=u(-t)$,it follows that $\Lambda^*\boldsymbol{R}\Lambda$ and its causal Toeplitz operator have the same signature.Then using a known result[10] it follows that the Riccati equation associated to $\Lambda^*\boldsymbol{R}\Lambda$ has a stabilizing solution; direct calculations show that this equation is just (7).Let denote by W this stabilizing solution;using the operatorial reprezentation of W (see[10]) and the fact that $\boldsymbol{R}$ has a signature,it follows that W is positive definite,therefore from (6) we deduce that $\gamma^2>\rho(QR(\gamma))$.The same idea can also be found in [17] for the infinite dimensional case for the Pritchard-Salamon class of systems.

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