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CAMIL MUSCALU

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Institute of Mathematics of the Romanian Academy, P.O. Box 1-764,
RO-70700 Bucharest, Romania, e-mail: muscalu stoilow.imar.ro

A joint norm control Nehari type theorem for N-tuples of Hardy spaces

Camil Muscalu

Institute of Mathematics
of the Romanian Academy
P.O.Box 1-764
RO-70700 Bucharest, Romania
e-mail: muscalu@stoilow.imar.ro

Abstract

If $N \in \mathbb{N}$, $0 < p \leq 1$ and $(X_k)_{k=1}^N$ are r.i.p-spaces it is shown that there is $C > 0$ such that for every $f \in \bigcap_{k=1}^N X_k$ there exists $\widetilde{f} \in \bigcap_{k=1}^N H(X_k)$ with $\|f - \widetilde{f}\|_{X_k} \leq C \cdot dist_{X_k}(f, H(X_k))$ for every $1 \leq k \leq N$. Also, if Π is a convex polygon in $\mathbb{R}^2$ it is proved that the N-tuple $(H(X_1), ..., H(X_N))$ is K_Π-closed with respect to $(X_1, ..., X_N)$ in the sense of G.Pisier.

1 Preliminaries

The results of this note are closely related to previous work in [KLW] and [Pi]. It is very well known that if $f \in L^\infty$ there exists $\widetilde{f} \in H^\infty$ such that $||f-\widetilde{f}||_\infty = dist_\infty(f, H^\infty)$ (cf. for instance [Po]). Also, it is obvious that such a relation holds when L^2 and H^2 are instead of L^∞ and H^∞ respectively. On the other hand, V.Kaftal, D.Larson and G.Weiss [KLW] have recently obtained the following joint norm control Nehari type theorem (we shall give a qualitative version of it):

Theorem 1.1 *There is $C > 0$ such that for every $f \in L^\infty$ there exists $\widetilde{f} \in H^\infty$ with*

$$||f-\widetilde{f}||_\infty \leq C \cdot dist_\infty(f, H^\infty)$$
$$||f-\widetilde{f}||_2 \leq C \cdot dist_2(f, H^2).$$

They have used this result to obtain another proof of Sarason's theorem on the closure of $H^\infty + C$ [Sa]. The best approximant in L^∞- norm (or in L^2- norm) is not a solution for the question in theorem 1.1 (cf. [KLW]). At the same time, Gilles Pisier has pointed out to the authors that this "qualitative version" can be rephrased in therms of the interpolation theory as follows (see also [Pi]):

Theorem 1.2 *The couple (H^2, H^∞) is K- closed with respect to (L^2, L^∞).*

and it can be deduced from some deep interpolation work of P.W.Jones based on Carleson measure techniques (cf. [J]). This means that there exists $C > 0$ such that

$$K_t(f, H^2, H^\infty) \leq C \cdot K_t(f, L^2, L^\infty)$$

for all $f \in H^2$ and $t > 0$, where K_t is the usual Peetre functional cf.[BS]. Our main task in the present paper is to obtain results of the same type but for N-tuples of Hardy p- spaces, $0 < p \leq 1$. To be mare clearly, we shall give now some notations and definitions. Let $\Pi = \overline{P_1 P_2 ... P_N}$ be a convex polygon in the plane $\mathbb{R}^2$ with vertices $P_k = (v_k, w_k)$. By a N- tuple of Banach (or quasi-Banach) spaces we mean a family $(A_1, ..., A_N)$ of N Banach (or quasi-Banach) spaces A_k $(1 \leq k \leq N)$ which are continuously embedded in a common Hausdorff topological space. We shall denote the inner product of $\mathbb{R}^2$ by $< \cdot, \cdot >$. Given $\bar{t} = (t_1, t_2) \in \mathbb{R}^2$ we define the K_Π- functional by

$$K_\Pi(\bar{t}, a, (A_k)_{k=1}^N) = \inf\{\sum_{k=1}^N e^{<\bar{t}, P_k>} ||a_k||_{A_k} \, ; \, a = \sum_{k=1}^N a_k \, , a_k \in A_k\}$$

for $a \in A_1 + ... + A_N$ (cf. [CP]). Given any interior point $\overline{\vartheta} = (\vartheta_1, \vartheta_2)$ of Π and any $1 \leq q \leq \infty$ the interpolation space $[A_1, ..., A_N]_{\overline{\vartheta}, q; K}$ is defined as the set of all elements $a \in A_1 + ... + A_N$ for which the norm

$$||a||_{\overline{\vartheta}, q; K} = (\int_{\mathbb{R}^2} (e^{-<\bar{t}, \overline{\vartheta}>} K(\bar{t}, a))^q d\bar{t})^{1/q}$$

is finite (see also [CP]). If we consider the closed subspaces $S_k \subseteq A_k$ $1 \leq k \leq N$ we say that the N- tuple $(S_1, ..., S_N)$ is K_{Π}- closed with respect to $(A_1, ..., A_N)$ if and only if there is $C > 0$ such that

$$K_{\Pi}(\bar{t}, a, (S_k)_{k=1}^N) \leq C \cdot K_{\Pi}(\bar{t}, a, (A_k)_{k=1}^N)$$

for all $\bar{t} \in \mathbb{R}^2$ and $a \in \bigcup_{k=1}^N S_k$.

Following [P] we shall denote simply by X an arbitrary rearrangement invariant p- space of functions $0 < p \leq 1$ (in short r.i.p-space) on the real axis $\mathbb{R}$ equipped with the Lebesgue measure. In a few words this means that X is a p-Banach space of functions which is rearrangement invariant and its norm is p-convex (see [LT]). We also consider

$$H(X) = \{\, F : U \to \mathbb{C}, F \,\text{analytic}\,;\ \sup_{y>0} \|F(\cdot + iy)\|_X < \infty \,\}$$

the classical Hardy space of analytic functions on the upper half plane $\mathbf{U}$ generated by X. We have to remark that $\|F\|_{H(X)} \sim \|f\|_X$ where f is the boundary function of F. This means in particular that the map $i : H(X) \to X$, $i(F) = f$ is an embedding and so, we can see $H(X)$ spaces inside X spaces. Let now $(X_k)_{k=1}^N$ r.i.p- spaces.

It is natural from our point of view (via theorems 1.1 and 1.2) to ask himselfs the following questions:

Question1 Is the N- tuple $(H(X_1), ..., H(X_N))$ K_{Π}- closed with respect to $(X_1, ..., X_N)$?

Question2 Is there any $C > 0$ such that for every $f \in \bigcap_{k=1}^N X_k$ there exists $\tilde{f} \in \bigcap_{k=1}^N H(X_k)$ with

$$\|f - \tilde{f}\|_{X_k} \leq C \cdot dist_{X_k}(f, H(X_k))$$

for every $1 \leq k \leq N$?

These are the problems we are mainly concerned. Actually, in this paper we settle both questions in the affirmative. However, the matricial techniques developed in [KLW] does not seem to extend to this more difficult case of N-tuples of Hardy spaces. Our arguments are based on the powerful method due to Peter Jones [J] which provides constructive solutions of $\bar{\partial}$- equation with Carleson measure data, and L^∞ estimates for this solutions at the boundary.

2 The solution of the problems

First, we shall recall the main result of P.W.Jones (cf. [J] or [BS]). A measure μ on the upper half plane $\mathbf{U}$ is said to be a Carleson measure if there exists a constant $C > 0$ such that $|\mu|(Q) \leq C|I|$ for each Q of the form $Q = I \times (0, |I|)$ where I is an open interval in $\mathbb{R}$. The smallest constant $C > 0$ for which the inequality holds , is called the Carleson norm of μ and will be denoted by $||\mu||_C$. For more results on Carleson measures the reader is referred to [GR]. For $\xi \in U$ we shall denote by

$$B(\xi) = \{w \in U\,;\, 0 \leq Im(w) \leq Im(\xi)\}\,.$$

Define kernels $K_1(z,\xi), K_2(z,\xi), K(z,\xi)$ in $U \times U$ by

$$K_1(z,\xi) = \frac{2iIm(\xi)}{z-\overline{\xi}}$$

$$K_2(z,\xi) = \exp\left((i-1)\sqrt{\frac{z-Re(\xi)}{Im(\xi)}} + \sqrt{2}\right)$$

$$K(z,\xi) = \exp\{-i\iint_{B(\xi)}\left(\frac{1}{z-\overline{w}} - \frac{1}{\xi-\overline{w}}\right)d\frac{|\mu|(w)}{||\mu||_C}\}. \tag{2.1}$$

The partial differential operator $\overline{\partial}$ is defined by the formula

$$\overline{\partial} f = \frac{1}{2}(f_x + if_y)$$

where f_x is the derivate of f with respect to variable x (and similarly , f_y). So, we have the following theorem (cf. [J], [BS]).

Theorem 2.1 *Let μ be a Carleson measure on U. Then , the functions defined for $j = 1,2$ by*

$$f_j(z) = \frac{1}{\pi}\iint_U \frac{1}{z-\xi}K_j(z,\xi)K(z,\xi)d\mu(\xi)$$

satisfy the distributional equation $\overline{\partial} f_j = \mu$ and the estimate $||f_j||_{L^\infty(\mathbb{R})} \leq ||\mu||_C$. Moreover , if I is an interval centred at a point x_0 and if $Q = I \times (0,|I|)$,then the function defined by

$$f(z;Q) = \frac{1}{\pi}\iint_Q \frac{1}{z-\xi}K_2(z,\xi)K(z,\xi)d\,\mu(\xi)$$

satisfies $\overline{\partial}\, f(\cdot;Q) = \mu_Q$ (the restriction of μ to Q) and also the inequalyties

$$|f(x;Q)| \leq C||\mu||_C \cdot \exp\left(-\sqrt{\frac{|x-x_0|}{|I|}}\right), x \in \mathbb{R}\,.$$

The main step in the proof of our problems will be solved with the help of the following divisibility result which may be of independent interest. The reader is also referred to the work of Jean Bourgain [Bo] for other analytic decomposition theorems.

Lemma 2.2 *Let $0 < p \leq 1$ and $N \in \mathbb{N}$. There is $C(= C(p,N)) > 0$ such that if $a \in H^p + H^\infty$ can be written as $a = m_1 + ... + m_N$ where $(m_k)_{k=1}^N \subseteq L^p + L^\infty$ then, there exists $(a_k)_{k=1}^N \subseteq H^p + H^\infty$ with $a = a_1 + ... + a_N$ and*

$$\| \sum_{k \in A} |a_k| \|_X \leq C \cdot \| \sum_{k \in A} |m_k| \|_X$$

for every r.i.p- space X and for all $A \subseteq \{1, ..., N\}$.

Proof: Let $0 < p \leq 1$, $N \in \mathbb{N}$ and $a \in H^p + H^\infty$. We also consider a sequence $(m_k)_{k=1}^N \subseteq L^p + L^\infty$ with $a = m_1 + ... + m_N$. For F an analytic function on $\mathbf{U}$ we recall the definition of the nontangential maximal operator (cf. [BS])

$$NF(x) := \sup_{t+iy \in \Gamma_x} |F(t+iy)|, \; x \in \mathbb{R}$$

where $\Gamma_x := \{t + iy\,;\, |x - t| < y\} \subseteq U$. Also, we denote by g^* the decreasing rearrangement of $|g|$ (see [LT]). The present proof bears the same ideas as in the proofs of [BS Theorem 5.10.6] and [M Lemma 2.4]. We have two cases.

Case 1: $\lim_{t \to \infty} (Na)^*(t) = 0$

For r a negative integer we denote by $\mathcal{A}_r$ the following subset of $\mathbb{R}$:

$$\mathcal{A}_r = \{\, x \in \mathbb{R}\,;\, Na(x) > 2^r \,\}$$

As there, we obtain an infinite colection $\mathcal{C}(= \mathcal{C}_r)$ of dyadic intervals and an integer valued function m with the following properties:

a) each $I \in \mathcal{C}$ is a Whitney interval for some $\mathcal{A}_n$.

b) If $I, J \in \mathcal{C}$ and their interiors have non - empty intersection , then one of the intervals is contained in the other.

c) If $I, J \in \mathcal{C}$, $J \subseteq I$, $J \neq I$ then

$$m(I) < m(J).$$

d) If $\mathcal{C}(I) := \{J \in \mathcal{C}\,;\, J \subseteq I, J \neq I\}$ then

$$\sum_{J \in \mathcal{C}(I)} |J| \leq |I|.$$

e) Na is bounded by $2^{m(I)}$ on the set $E(I)$ defined by

$$E(I) = I \setminus \bigcup_{J \in \mathcal{C}(I)} J = I \setminus \mathcal{A}_{m(I)}.$$

f) If $I \in \mathcal{C}$, then

$$|\, E(I)\, | \geq |\, I\, | /2.$$

g) the colection $\{E(I)\}_{I \in \mathcal{C}}$ is disjoint with union equal to $\mathcal{A}_r$.

For each $I \in C$, let $R(I) = I \times (0, 5|I|)$ and let

$$\mathcal{U}_r = \bigcup_I R(I)$$

$$\Gamma(I) = R(I) \setminus \bigcup_{J \in C(I)} R(J) \quad (I \in C).$$

It follows that

$$| a(z) | \leq 2^{m(I)} \quad (z \in \Gamma(I))$$

and so , the function $a_I := a \cdot 1_{\Gamma(I)}$ then satisfies

$$| a_I(z) | \leq 2^{m(I)} \cdot 1_{\Gamma(I)}.$$

Moreover , $\overline{\partial}\, a_I$ is absolutely continuous with respect to arclength measure on the boundary (relative to $\mathcal{U}_r$) of $\Gamma(I)$ and

$$\|\overline{\partial}\, a_I\|_C \leq 25 \cdot 2^{m(I)}.$$

If we denote by $\widetilde{a} := \sum_{I \in C} 2^{m(I)} 1_{E(I)}$ it follows that

$$(\widetilde{a})^*(t) \leq 2(Na)^*(t/8)\, (t > 0) \tag{2.2}$$

(For all this estimates see the detailes in [BS] p.421). Define now μ to be the arclength measure on the union of the boundaries (relative to $\mathcal{U}_r$) of the sets $\Gamma(I)(I \in C)$. Then , μ is a positive Carleson measure with $\|\mu\|_C \leq 225$ (see [BS] p.421). We define by

$$A_I(z) := a_I(z) - \frac{1}{\pi} \iint_{\mathcal{U}_r} \frac{1}{z - \xi} K_2(z, \xi) K(z, \xi)\, d(\overline{\partial}\, a_I)(\xi)$$

where K is given by (2.1) using the measure $\frac{\mu}{\|\mu\|_C}$. It follows that A_I is analytic and $\sum_{I \in C} A_I$ converges uniformly on compact subsets to analytic function A_r. It also follows that

$$\sum_{I \in C} 2^{-m(I)} \mid A_I(z) \mid \leq C \tag{2.3}$$

cf. [BS] p.422 . Now , using the relation (2.2) it follows that

$$\int_0^t (\widetilde{a}^p)^*(u) du \leq C \cdot \int_0^t (Na^p)^*(u) du$$

and so, by [M Remark 2.3] we obtain

$$\int_0^t (\widetilde{a}^p)^*(u) du \leq C \cdot \int_0^t (|a|^p)^*(u) du \quad (t > 0)$$

With the help of the [BS Corollary 5.10.5] since $\lim_{t\to\infty}(Na)^*(t) = 0$, there exists disjoint measurable sets $e(I)$ such that $| e(I) |=| E(I) |$ and

$$C \cdot \int_{e(I)} | a(x) |^p \, dx \geq 2^{m(I)p} \cdot | E(I) | .$$

So , there exists weights w_I such that $\sup_{I\in C} w_I \leq C$ and

$$w_I \cdot \frac{(\int_{e(I)} | a(x) |^p \, dx)^{1/p}}{2^{m(I)} \cdot | E(I) |^{1/p}} = 1$$

There exists also weights λ_I such that $\sup_{I\in C} \lambda_I \leq C(= C(p,N))$ and

$$\lambda_I \cdot w_I \sum_{j=1}^{N} \frac{(\int_{e(I)} | m_j(x) |^p \, dx)^{1/p}}{2^{m(I)} | E(I) |^{1/p}} = 1. \tag{2.4}$$

We consider now the functions

$$(H_j^r(x))_{r=-1}^{-\infty} := (\sum_{I\in C_r} \lambda_I w_I \frac{(\int_{e(I)} | m_j |^p)^{1/p}}{2^{m(I)} | E(I) |^{1/p}} | A_I(x) |)_{r=-1}^{-\infty} \; 1 \leq j \leq N.$$

We claim that there exists a subsequence $(r_k)_{k=1}^{\infty}$ of negative integers such that $H_j^{r_k}(x)$ converges for a.e $x \in \mathbb{R}$, when $k \to \infty$, $1 \leq j \leq N$. Indeed, since every r.i.p-space X is included in $L^p + L^\infty$ (cf. [P]) it follows that there exists $m_j^{'} \in L^p$ and $m_j^{''} \in L^\infty$ such that $m_j = m_j^{'} + m_j^{''}$, $1 \leq j \leq N$. We can write the following inequalities for a fixed $1 \leq j \leq N$:

$$\| \sum_{I\in C_r} \lambda_I w_I \frac{(\int_{e(I)} | m_j^{'} |^p)^{1/p}}{2^{m(I)} | E(I) |^{1/p}} | A_I | \|_{L^p} \leq C \cdot \| \sum_{I\in C_r} \frac{\int_{e(I)} | m_j^{'} |^p}{2^{m(I)p} | E(I) |} | A_I |^p \|_{L^1}^{1/p} \leq$$

$$\leq C \cdot (\sum_{I\in C_r} \frac{\int_{e(I)} | m_j^{'} |^p}{2^{m(I)p} | E(I) |} \|A_I^p\|_{L^1})^{1/p}.$$

Using the definition of A_I and Jones's theorem 2.2 we deduce the following relations:

$$\| |A_I|^p \|_{L^1} \leq C \cdot 2^{m(I)p} | E(I) | . \tag{2.5}$$

So , we obtain that

$$\| \sum_{I\in C_r} \lambda_I w_I \frac{(\int_{e(I)} | m_j^{'} |^p)^{1/p}}{2^{m(I)} | E(I) |^{1/p}} | A_I | \|_{L^p} \leq C \cdot \|m_j^{'}\|_{L^p} . \tag{2.6}$$

Similarly ,

$$(2.7) \qquad \|\sum_{I\in C_r} \lambda_I w_I \frac{(\int_{e(I)} | m_j^{''} |^p)^{1/p}}{2^{m(I)} | E(I) |^{1/p}} \cdot | A_I | \|_{L^\infty} \leq$$

$$\leq C \cdot \|m_j^{''}\|_{L^\infty} \cdot \| \sum_{I\in C_r} \frac{1}{2^{m(I)}} | A_I | \| \leq$$

$$\leq C \cdot \|m_j^{''}\|_{L^\infty}.$$

by (2.3) . Now , if M is an arbitrary interval in $\mathbb{R}$, using inequalities 2.6 and 2.7 it follows that

$$\|(H_j^r 1_M)^{p/2}\|_{L^2} \leq C$$

with $C > 0$ independent of r and dependent of $m_j^{'}$,$m_j^{''}$ and M. Using standard measure theory arguments together with Alaoglu's theorem it follows, since M is arbitrary, and j belongs to the finite set $\{1, 2, ..., N\}$ that our claim holds. So, we may assume from the begining that $H_j^r(x)$ converges a.e $x \in \mathbb{R}$ when $r \to \infty$, $1 \leq j \leq N$. We take now λ a Banach limit and we shall define the wanted functions $(a_j)_{j=1}^N$ as follows:

$$a_j(z) := \lambda((\sum_{I\in C_r} \lambda_I w_I \frac{(\int_{e(I)} | m_j |^p)^{1/p}}{2^{m(I)} | e(I) |^{1/p}} A_I(z))_{r=-1}^{=\infty}) \ , z \in U \ , 1 \leq j \leq N.$$

We have to remark that $(a_j)_{j=1}^N$ are analytic functions and moreover,

$$a_1 + ... + a_N = \lambda((\sum_{I\in C_r} A_I)_{r=-1}^{-\infty}) = \lambda((A_r)_{r=-1}^{-\infty}) = a,$$

by 2.4 , since $\|a \cdot 1_{U_r^c}\|_{L^\infty(\mathbb{R})} \leq C \cdot 2^r$ and $\|\overline{\partial}(a \cdot 1_{U_r})\|_C \leq C \cdot 2^r$. We define now the operator T on $(L^1 + L^\infty)(\mathbb{R})$ as follows :

$$T(H)(\cdot) := \lambda(((\sum_{I\in C_r} \lambda_I w_I \frac{(\int_{e(I)} | H |)^{1/p}}{2^{m(I)} | E(I) |^{1/p}} \cdot | A_I | (\cdot))^p)_{r=-1}^{=\infty}).$$

T is a quasilinear operator and

$$\|TH\|_{L^1} \leq \lambda((\|(\sum_{I\in C_r} \lambda_I w_I \frac{(\int_{e(I)} | H |)^{1/p}}{2^{m(I)} | E(I) |^{1/p}} \cdot | A_I | (\cdot))^p\|_{L^1})_{r=-1}^{=\infty}) \leq$$

$$\leq C \cdot \lambda((\sum_{I\in C_r} \frac{(\int_{e(I)} | H |)}{2^{m(I)p} | E(I) |} \cdot \|A_I^p\|_{L^1})_{r=-1}^{-\infty}) \leq$$

$$\leq \ (by\ (2.5)) \ \leq C \cdot \|H\|_{L^1}.$$

Similarly , for $H \in L^\infty$ we can write:

$$||TH||_{L^\infty} \leq C \cdot \lambda((||(\sum_{I \in \mathcal{C}_r} \frac{\int_{e(I)} | H |)^{1/p}}{2^{m(I)} | E(I) |^{1/p}} \cdot | A_I |)^p||_{L^\infty})_{r=-1}^{-\infty}) \leq$$

$$\leq C \cdot ||H||_{L^\infty} \cdot \lambda((||(\sum_{I \in \mathcal{C}_r} \frac{1}{2^{m(I)}} \cdot | A_I | ||^p_{L^\infty})_{r=-1}^{-\infty}) \leq$$

$$\leq \ (by\ (2.3)) \ \leq C \cdot ||H||_{L^\infty}$$

It follows that T maps Y into itself for every Y r.i.space. Let now consider $\Lambda \subseteq \{1, ..., N\}$ and X be a r.i.p- space. We can thus write:

$$|| \sum_{k \in \Lambda} |a_k| ||_X = ||(\sum_{k \in \Lambda} |a_k|)^p||_{X^p}^{1/p} \leq C \cdot ||T(\sum_{k \in \Lambda} |m_k|^p)||_{X^p}^{1/p}$$

by the claim proved above. Since X is a r.i.p- space then it follows that $X^p = \{f; |f|^{1/p} \in X\}$ is a r.i.space (cf. [P]) and we obtain that

$$|| \sum_{k \in \Lambda} |a_k| ||_X \leq C \cdot || \sum_{k \in \Lambda} |m_k|^p||_{X^p}^{1/p} \leq C(= C(p, N)) \cdot || \sum_{k \in \Lambda} |m_k| ||_X$$

by Jensen's inequality, and the proof is complete in this case.

Case 2: $\lim_{t \to \infty}(Na)^*(t) = \alpha > 0$

If we take a look at the [M Remark 2.3], it follows using [BS Corollary 5.10.5] that there exists an increasing sequence $(E_n)_n$ of sets of finite measure with $| E_n | \uparrow \infty$ such that $| a(x) | > \frac{\alpha}{2}$ whenever $x \in \bigcup_{n=1}^\infty E_n$. This implies in particular that if λ is a Banach limit

$$\gamma(a) := \lambda(((\frac{1}{| E_n |} \int_{E_n} | a |^p)^{1/p})_{n=1}^\infty) \geq \frac{\alpha}{2} \tag{2.8}$$

Let t_0 such that $(Na)^*(t) \leq 2\alpha$ for $t \geq t_0$. Using [M Lemma 2.5] in the case $X = L^p$, there exists a function a_0 in H^p such that

$$||a_0||_{H^p} \leq C \cdot \int_0^{t_0} ((Na)^p)^*(u) du$$

$$||a - a_0||_{H^\infty} \leq C \cdot (Na)^*(t_0) \leq 2C\alpha \tag{2.9}$$

It follows using the above relations, that

$$\int_0^t (Na_0^p)^*(u) du \leq \int_0^t (Na^p)^*(u) du + \int_0^t (N(a - a_0)^p)^*(u) du \leq$$

$$\leq \int_0^t (Na^p)^*(u) du + Ct\alpha^p \leq C \cdot \int_0^t (Na^p)^*(u) du$$

and so, by [M Remark 2.3]

$$\int_0^t (|a_0|^p)^*(u)du \leq C \cdot \int_0^t (|a|^p)^*(u).$$

Since (L^1, L^∞) is a Calderón couple (cf. [BS]) we deduce that there exists a linear operator Ψ bounded on L^1 and on L^∞ with $\Psi(f) \geq 0$ for $f \geq 0$ and $\Psi(|a|^p) = |a_0|^p$. Also, there is a function $0 \leq \varphi_1 \leq 1$ such that $|a|^p = \varphi_1|m_1|^p + ... + \varphi_1|m_N|^p$ and thus $f_j = \Psi(\varphi_1|m_j|^p) \geq 0$ for $1 \leq j \leq N$. We deduce that there exists $0 \leq |\varphi_2| \leq C(p, N)$ with the property $a_0 = \varphi_2 f_1^{1/p} + ... + \varphi_2 f_N^{1/p}$. Since $a_0 \in H^p$ we deduce that $\lim_{t\to\infty}(Na_0)^*(t) = 0$ and so, we can apply the first case of our proof and obtain a sequence $(a_k^{'})_{k=1}^N$ of analytic functions with sum a_0. On the other hand , there exists weights $(\beta_n)_n$ such that $0 \leq \beta_n \leq C(= C(p, N))$ and

$$(2.10) \qquad \beta_n \sum_{j=1}^N (\int_{E_n} | m_j |^p)^{1/p} = (\int_{E_n} | a |^p)^{1/p}.$$

We shall define now the quasilinear operator on $L^p + L^\infty$ by the formulae

$$V(H) := \frac{1}{\gamma(a)} \cdot \lambda(((\beta_n \frac{1}{|E_n|} \int_{E_n} | H |^p)^{1/p})_{n=1}^\infty) \mid a - a_0 \mid .$$

For $H \in L^p$ then $T(H) = 0$ and if $H \in L^\infty$ then

$$||T(H)||_{L^\infty} \leq ||H||_{L^\infty} \cdot \frac{2C\alpha}{\frac{\alpha}{2}} \leq C \cdot ||H||_{L^\infty}$$

by inequalyties 2.8 and 2.9. Using a result from [P] it follows that V maps X on X boundedly, for every r.i.p-space X. We consider the functions :

$$a_j^{''} = \frac{1}{\gamma(a)} \lambda(((\beta_n \frac{1}{|E_n|} \int_{E_n} | m_j |^p)^{1/p})_{n=1}^\infty)(a - a_0) \ , 1 \leq j \leq N.$$

We have $a_1^{''} + ... + a_N^{''} = a - a_0$, by 2.10. We put now $a_j = a_j^{'} + a_j^{''}$, $j \in \{1, ..., N\}$ and we remark that $a_1 + ... + a_N = a_0 + a - a_0 = a$. We can write

$$||\sum_{j\in A} |a_j^{'}|||_X \leq C \cdot ||\sum_{j\in A} |\varphi_2| f_j^{1/p}||_X \leq C \cdot ||\sum_{j\in A} f_j||_{X^p}^{1/p} =$$

$$= C \cdot ||\Psi(\sum_{j\in A} \varphi_1|m_j|^p)||_{X^p}^{1/p} \leq C(p, N)||\sum_{j\in A} |m_j|||_X$$

by Jensen's inequality. Similar,

$$||\sum_{j\in A} |a_j^{''}|||_X \leq C(p, N) \cdot ||V((\sum_{j\in A} |m_j|^p)^{1/p})||_X \leq C \cdot ||\sum_{j\in A} |m_j|||_X$$

as above. These two relations completes the proof Δ.

We can give now the complete solution of our problems. The following two results are easy consequences of the above lemma (applied for $\Lambda_k = \{k\}$, $1 \leq k \leq N$) and of definition of K_Π- functional. Let Π a convex polygon in $\mathbb{R}^2$ and $(X_k)_{k=1}^N$ r.i.p- spaces, $0 < p \leq 1$.

Theorem 2.3 *The N- tuple $(H(X_1), ..., H(X_N))$ is K_Π- closed with respect to $(X_1, ..., X_N)$.*

Corollary 2.4 *We have the equality*

$$[H(X_1), ..., H(X_N)]_{\overline{\vartheta},q;K} = H([X_1, ..., X_N]_{\overline{\vartheta},q;K})$$

where $\overline{\vartheta} \in Int\,\Pi$ and $1 \leq q \leq \infty$.

Finally, we shall present the proof of the Nehari type theorem.

Theorem 2.5 *There is $C(= C(p, N)) > 0$ such that for every $f \in \bigcap_{k=1}^N X_k$ there exists $\widetilde{f} \in \bigcap_{k=1}^N H(X_k)$ with*

$$\|f - \widetilde{f}\|_{X_k} \leq C \cdot dist_{X_k}(f, H(X_k))$$

for every $1 \leq k \leq N$.

Proof: We will make an induction over N. The case $N = 1$ is obvious. We assume that we know the result for N and we shall prove the case $N + 1$. So, let $f \in \bigcap_{k=1}^{N+1} X_k$. Using the induction hypothesis, there is $f_1 \in \bigcap_{k=1}^N H(X_k)$ such that $\|f - f_1\|_{X_k} \leq C \cdot dist_{X_k}(f, H(X_k))$ for every $1 \leq k \leq N$. Also, there is $f_2 \in H(X_{k+1})$ such that $\|f - f_2\|_{X_{k+1}} \leq 2 \cdot dist_{X_{k+1}}(f, H(X_{k+1}))$. Let now $g_i = f - f_i$, $i = 1, 2$. Then, $f_1 - f_2 = g_2 - g_1$. Since $f_1 - f_2$ is analytic, using lemma 2.2 we find analytic functions h_i, $i = 1, 2$ such that $f_1 - f_2 = h_2 - h_1$ and moreover, $\|h_i\|_{H(X)} \leq C \cdot \|g_i\|_X$, $i = 1, 2$ for every r.i.p- space X. Then, our wanted function is $\widetilde{f} := f_1 + h_1 = f_2 + h_2 \in \bigcap_{k=1}^{N+1} H(X_k)$. Indeed, if $1 \leq k \leq N$ we can write:

$$\|f - \widetilde{f}\|_{X_k} \leq \|f - f_1\|_{X_k} + \|h_1\|_{X_k} \leq C \cdot \|g_1\|_{X_k} \leq C \cdot dist_{X_k}(f, H(X_k)).$$

On the other hand,

$$\|f - \widetilde{f}\|_{X_{k+1}} \leq \|f - f_2\|_{X_{k+1}} + \|h_2\|_{X_{k+1}} \leq C \cdot dist_{X_{k+1}}(f, H(X_{k+1}))$$

which ends the proof Δ.

Corollary 2.6 *Let $0 < p_1 \leq p_2 \leq ... \leq p_N \leq \infty$. There is $C(= C(p_1, N)) > 0$ such that for every $f \in \bigcap_{k=1}^N L^{p_k}$ there exists $\widetilde{f} \in \bigcap_{k=1}^N H^{p_k}$ with*

$$\|f - \widetilde{f}\|_{L^{p_k}} \leq C \cdot dist_{L^{p_k}}(f, H^{p_k}).$$

for all $1 \leq k \leq N$.

Remarks

(1) A moment of reflection on the proof of lemma 2.2 shows us that in the particular case $p = 1$ the constant $C(= C(p, N)) > 0$ becomes an universal constant and so, it does not depend of N.

(2) In the particular case $N = 2$ theorem 2.3 was obtained by the author in [M] (also, the situation $p = 1$ of this case has appeared in [X]). Using it, in [M] it is shown how it is possible to transport the classical "weak interpolation theory" in the analytic context of Hardy "quasi"- spaces.

(3) All the results of this note holds also in the case of Hardy spaces on the unit disk, since it is well known that Jones's $\overline{\partial}$- method can be transported in this situation.

(4) In [Pi] it is also trated the non- commutative case of upper triangular matrix spaces. That paper has motivated (together with [KLW]) the present work. We shall end with an apparently open problem: It is true the non-commutative version of the above theorems 2.3 and 2.5 ? In [Pi] only the cases $N = 2$ and $X = L^{p,q}$ are considered. We conjectured that the answer is yes.

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