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Abstract

In view of possible applications in Control Theory, several types of derived cones to reachable sets of general control systems and differential inclusions are identified.

Though not among the intrinsic tangent cones, Hestenes's derived cones to arbitrary subsets of normed spaces [11] proved to have remarcable properties allowing conceptually simple proofs and significant generalizations of the Minimum Principle in Optimal Control [13].

The main results for differential inclusions rely on a continuous version of Filippov's theorem while in the case of nonsmooth control systems a certain generalization of the Bendixson-Picard theorem on differentiability of solutions of differential equations with respect to initial data is needed.

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1. Introduction

The aim of this paper is to characterize derived cones to reachable sets of differential inclusions and, in particular, of general control systems, in view of possible applications in Control Theory.

The concept of derived cone to an arbitrary subset of a normed space has been introduced by M.Hestenes in [11] and successfully used to obtain necesarry optimality conditions in Control Theory.

However, in the last 25-30 years, this concept has been largely ignored in favor of other concepts of tangents cones , that may intrinsically be associated to a point of a given set : the cone of interior directions, the contingent, the quasitangent and, above all, Clarke's tangent cone ([2], [7] , [9] , [10] , [19] , [20] , etc.).

Recently ([13]) one of the authors obtained "an intersection property" of derived cones that allowed a conceptually simple proof and significant extensions of the minimum principle in optimal control; moreover, other properties of derived cones may be used to obtain controllability and other results in the qualitative theory of control systems.

In the case of control systems defined by smooth parametrized vector fields a derived cone to the reachable set along a reference trajectory is explicitely described in Lemma 3.1. in [13], using the derivative of the flow of the vector field ; in fact, this result may be considered as a suitable interpretation of the computations leading to "cones of variations" to a reference trajectory (e.g. Cesari, [5],Lemmas 7.4.iii, 7.4.iv, etc)

In order to extend the above mentioned results to more general systems defined either by differential inclusions, or by nonsmooth parametrized vector fields, one needs characterizations of derived cones to reachable sets of these types of control systems.

We obtain first an "intrinsic" derived cone which is closely related to set of tangent directions to the trajectories of the system ([14]), then we enlarge this cone by means of certain variational inclusion that generalize the variational equations in the classical theory of Ordinary Differential Equations.

In order to obtain the continuity property in the definition of a derived cone we shall essentially use a continuous version of Filippov's theorem on lipschitzian differential inclusions.

We note that Filippov's theorem has been used by Frankowska ([8], [9], [10]) and Polovinkin and Smirnov ([20], [21]) to obtain estimations of the contingent and quasitangent cones to reachable sets.

Since a derived cone is a very special type of convex subcone of the quasitangent cone, our result may be considered as refinements of the existing results of Frankovska ([8], [9], [10]) and Polovinkin and Smirnov ([20], [21]) concerning the contingent and quasitangent cones to reachable sets.

The paper is organized as follows : in Section 2 we present the notations and the preliminary results from Nonsmooth Analysis to be used in the sequel. In Section 3 we prove the existence of certain continuous imbeddings of a solution of a differential inclusion, in Section 4 we identify an intrinsic derived cone defined by a set of tangent directions to the trajectories of the system, while in the Section 5 we prove the main results providing larger derived cones obtained by the "transport" of the intrinsic cones by certain variational inclusions. In the last section we obtain sharper results of the same type for smooth and nonsmooth control systems that may be considered parametrized differential inclusions.

2. Notations, definitions and preliminary results

In this paper we shall be concerned mainly with the absolutely continuous solutions, $x(\cdot) : [0,T] \to R^n$ of a differential inclusion:

$$x' \in F(t,x),\ x(0) \in X_0 \tag{2.1}$$

which is defined by a given "orientor field", $F(\cdot,\cdot) : D \subset R \times R^n \to \mathcal{P}(R^n)$ (where$\mathcal{P}(R^n)$ denotes the family of all subsets of R^n) and by a given set, $X_0 \subset R^n$, of initial data; occasionally we shall denote by $S_F(T,0,X_0)$ the set of all absolutely continuous(i.e. in the space$W^{1,1}([0,T],R^n) = AC([0,T],R^n))$ solutions of (2.1) that are defined on the interval [0,T]. The set of all solutions of (2.1) through a point $(t_0,x_0) \in D$ will be denoted by $S_F(t_0,x_0)$.

In fact our object of study is the *reachable set* of (2.1) defined by:

$$R_F(T,0,X_0) = \{x(T) :\ x(\cdot) \in S_F(T,0,X_0)\} \tag{2.2}$$

In particular, we shall study the reachable set of a standard control system of the form:

$$x' = f(t,x,u(t)), x(0) \in X_0, u(t) \in U\ a.e.in[0,T] \tag{2.3}$$

which under reasonable hypothesis (e.g. Aubin-Cellina([1]), Cesari([5]), and Frankowska([10]), etc) may be equivalent with parametrized differential inclusion:

$$x' \in f(t,x,U),\ x(0) \in X_0 \tag{2.3'}$$

Since the reachable set in (2.2) is, generally, neither a differentiable manifold, nor a convex set, its infinitesimal properties may be characterized only by tangent cones in a generalized sense, extending the classical concepts of tangent cones in Differential Geometry and Convex Analysis, respectively.

From the rather large number of "convex approximations", "tents", "regular tangents cones", etc. in the literature, we choose the concepts of derived cone introduced by M.Hestenes in [11]:

Definition 2.1([11]) A subset $M \subset R^n$ is said to be a *derived set to* $X \subset R^n atx \in X$ if for any finite subset $\{v_1 \dots v_k\} \subset M$, there exist $s_0 > 0$ and a continuous mapping $a(\cdot) : [0, s_0]^k \rightarrow X$ such that $a(0) = x$ and $a(\cdot)$ is (conically) differentiable at $s = 0$ with the derivative $col[v_1, \dots v_k]$ in the sense that:

$$\lim_{R_+^k \ni \theta \rightarrow 0} \frac{|a(\theta) - a(0) - \sum_{i=1}^k \theta_i v_i|}{|\theta|} = 0 \tag{2.4}$$

We shall write in this case that the derivative of $a(\cdot)$ at $s = 0$ is given by:

$$Da(0)\theta = \sum_{i=1}^k \theta_j v_j, \; \forall \theta = (\theta_1, \dots, \theta k) \in R_+^k := [0, \infty)^k$$

A subset $C \subset R^n$ is said to be a *derived cone* of X at x if it is a derived set and also a convex cone.

For the basic properties of derived sets and cones we refer to M.Hestenes [11]; we recall that if M is a derived set then $M \bigcup \{0\}$ as well as the convex cone generated by M, defined by:

$$cco(M) = \{\sum_{i=1}^k \lambda_j v_j; \; \lambda_j \geq 0; v_j \in M, \; j = 1, \dots k\} \tag{2.5}$$

is also a derived set, hence a derived cone.

The fact that the derived cone is a proper generalization of the classical concepts in Differential Geometry and Convex Analysis is illustrated by the following results([11]): if $X \subset R^n$ is a differentiable manifold and $T_x X$ is the tangent space in the sense of Differential Geometry to X at x

$$T_x X = \{v \in R^n : \exists c : (-s, s) \rightarrow X, of class C^1, c(0) = x, c'(0) = v\} \tag{2.6}$$

then $T_x X$ is a derived cone; also, if $X \subset R^n$ is a convex subset then the tangent cone in the sense of Convex Analysis defined by:

$$TC_x^+ X = Cl\{t(y - x); t \geq 0, \; y \in X\} \tag{2.7}$$

is also a derived cone.

Since any convex subcone of a derived cone is also a derived cone, such an object may not be uniquely associated to a point $x \in X$; moreover, simple examples show that even a maximal with respect to set-inclusion derived cone may not be uniquely defined: if the set $X \subset R^2$ is defined by:

$$X = C_1 \bigcup C_2, C_1 = \{(x, x); x \geq 0\}, C_2 = \{(x, -x), x \leq 0\} \tag{2.8}$$

then C_1 and C_2 are both maximal derived cones of X at the point $(0,0) \in X$.

On the other hand, the uptodate experience in Nonsmooth Analysis shows that for some problems, the use of one of the intrinsic tangent cones may be preferable.

From the multitude of the intrinsic tangent cones in the literature(e.g.[2], [18]),the contingent, the quasitangent and Clarke's tangent cones, defined, respectively, by:

$$K_x^+X = \{v \in R^n; \exists s_m \to 0+,\ x_m \in X:\ \frac{x_m - x}{s_m} \to v\}$$

$$Q_x^+X = \{v \in R^n; \exists c(\cdot) : [0, s_0) \to X,\ c(0) = x,\ c'(0) = v\} \qquad (2.9)$$

$$C_x^+X = \{v \in R^n; \forall (x_m, s_m) \to (x, 0+),\ x_m \in X,\ \exists y_m \in X:\ \frac{y_m - x_m}{s_m} \to v\}$$

seem to be among the most oftenly used in the study of different problems involving nonsmooth sets and mappings.

The rather large gap between Clarke's tangent cone and the quasitangent one may be diminished by the "assymptotic" variants of the contingent and the quasitangent cones defined as follows:

$$AQ_x^+X = \{v \in R^n:\ v + Q_x^+ \subset Q_x^+\},\ AK_X^+X = \{v \in R^n:\ v + K_x^+ \subset K_x^+\} \qquad (2.10)$$

for which equivalent definitions of the same type as those in (2.9) may be obtained(e.g.[18], [19]).

We recall that, in contrast with K_x^+X, Q_x^+X, the cones C_x^+X, AK_x^+X, AQ_x^+X are convex and are related as follows:

$$C_x^+X \subset AK_x^+X \subset AQ_x^+X \subset Q_x^+X \subset K_x^+X \qquad (2.11)$$

We note that the use of the *cone of interior directions* defined by:

$$I_x^+X = \{v \in R^n:\ \exists s_0,\ r > 0 : x + B_r(v) \subset X,\ \forall s \in [0, s_0)\},$$

$$B(v, r) = \{w \in R^n:\ |w - v| < r\} \qquad (2.12)$$

as well as of other types of intrinsic tangent cones is severely limited by the fact that it may be an empty set for large classes of sets.

From Definition 2.1 and from (2.9) it follows that if $C \subset R^n$ is a derived cone of X at x then $C \subset Q_x^+X$ and, on the other hand, Example 2.8, for which $C_0^+X = AQ_0^+X = AK_0^+X = \{0\}$, shows that a derived cone may not be contained into any of the cones C_x^+X, AK_x^+X, AQ_x^+X ;an interesting open question seems to be whether any of these cones is a derived cone.

It is easy to see that if $C \subset I_x^+X$ is a convex cone then C is a derived cone and moreover , from Theorem 4.7.4 in [11] it follows that if C is a derived cone with nonempty interior then $Int(C) \subset I_x^+X$.

Using the fact that the classical (Frechet) derivative of a smooth mapping defined on a differentiable manifold may be defined as the linear mapping whose graph is the tangent space to the graph of the mapping (e.g.[17]), corresponding to each type of tangent cone , say $\tau_x X$ one may introduce ([1]) a *set- valued directional derivative* of a multifunction $G(\cdot) : X \subset R^n \to \mathcal{P}(R^n)$ (in particular of a single-valued mapping) at a point $(x, y) \in Graph(G)$ as follows:

$$\tau_y G(x, v) = \{w \in R^n : (v, w) \in \tau_{(x,y)} Graph(G)\},\ v \in \tau_x X \qquad (2.13)$$

Moreover, in the case of a real-valued function , for each type of tangent cone one may introduce also two corresponding *extreme directional derivatives* in terms of the tangent cone to the epigraph and to the subgraph of the function, which, in, turn, may define corresponding generalized gradients.

Thus, the large variety of the types of tangents cones generates a corresponding variety of generalized differentiability concepts from which one may choose the most suitable one for a given problem that involves nonsmooth sets and mappings.

3. Continuous imbeddings of a solution of a differential inclusion

As already stated, the main tool in characterizing derived cones to reachable sets of differential inclusions is a certain version of Filippov's theorem ([1]) , [10], etc.).

We recall first several preliminary results we shall use in this section.

Lemma 3.1. ([23]) *Let $u(\cdot) : I = [0, T] \to R^n$ be measurable and let $G(\cdot) : I \to \mathcal{P}(R^n)$ be a measurable closed-valued multifunction. Then, for every $r(\cdot) : I \to (0, \infty)$ measurable, there exists a measurable selection $g(\cdot) : I \to R^n$ of $G(\cdot)$ (i.e. such that $g(t) \in G(t)$ a.e. in I) such that:*

$$|u(t) - g(t)| < d(u(t), G(t)) + r(t)\ a.e.\ in\ I \qquad (3.1)$$

where the distance between a point $x \in R^n$ and a subset $A \subset R^n$ is defined as usual by: $d(x, A) = inf\{|x - a|; a \in A\}$.

In what follows we denote by $\mathcal{L}(I)$ the familly of all Lebesgue measurable subsets of the interval I and if $A \subset I$ then $\chi_A(\cdot) : I \to \{0,1\}$ denotes the characteristic function of A; as usual, we denote by $L^1(I,R^n)$ the space of Lebesgue integrable mappings endowed with the norm

$$|u(\cdot)|_1 = \int_0^T |u(t)|dt,\ u(\cdot) \in L^1(I,R^n)$$

Definition 3.2. A subset $D \subset L^1(I,R^n)$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I \backslash A$. We denote by $\mathcal{D}(I,R^n)$ the familly of all decomposable closed subsets of $L^1(I,R^n)$.

In this section (S, d) is a separable metric space, $\mathcal{B}(S)$ denotes the family of Borel measurable subsets of S; we recall that a multifunction $G(\cdot) : S \to \mathcal{P}(R^n)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset R^n$, the subset $\{s \in S; G(S) \subset C\}$ is closed.

Lemma 3.3. ([6]) *Let $F^*(\cdot,\cdot) : I \times S \to \mathcal{P}(R^n)$ be a closed-valued $\mathcal{L} \otimes \mathcal{B}(S)$-measurable multifunction such that $F^*(t,\cdot)$ is l.s.c. for any $t \in I$.*

Then the multifunction $G(\cdot) : S \to \mathcal{D}(I,R^n)$ defined by:

$$G(s) = \{v(\cdot) \in L^1(I,R^n) : v(t) \in F^*(t,s)\ a.e.\ in\ I\} \qquad (3.2)$$

is l.s.c. with nonempty closed values if and only if there exists a continuous mapping $p(\cdot) : S \to L^1(I,R^n)$ such that:

$$d(0, F^*(t,s)) \le p(s)(t),\ \ a.e.\ in\ I,\ \forall s \in S \qquad (3.3)$$

Finally, the key tool in what follows is the next formulation of the Bressan-Colombo result in [3] concerning the existence of a continuous selection of a l.s.c. multifunction with closed decomposable values:

Lemma 3.4. ([6]) *Let $G(\cdot) : S \to \mathcal{D}(I,R^n)$ be a l.s.c. multifunction with closed decomposable values and let $\phi : S \to L^1(I,R^n), \psi : S \to L^1(I,R)$ be continuous such that the multifunction $H(\cdot) : S \to \mathcal{D}(I,R^n)$ defined by:*

$$H(s) = cl\{v(\cdot) \in G(s) : |v(t) - \phi(s)(t)| < \psi(s)(t)\ a.e.\ in\ I\} \qquad (3.4)$$

has nonempty values.

Then $H(\cdot)$ has a continuous selection i.e. there exists a continuous mapping $h(\cdot) : S \to L^1(I,R^n)$ such that:

$$h(s) \in H(s)\ \forall s \in S \qquad (3.5)$$

In what follows, the orientor field in (2.1) is assumed to satisfy the following hypothesis:

Hypothesis 3.5. (i) $F(\cdot,\cdot) : D \subset R \times R^n \to \mathcal{P}(R^n)$ *has nonempty closed values and is* $\mathcal{L}(I) \otimes \mathcal{B}(S)$ *measurable.*

(ii) $z(\cdot) \in AC(I, R^n)$ *is a solution of (2.1) and there exists* $\epsilon_0 > 0, L(\cdot) \in L^1(I, R_+)$ *such that , for any* $t \in I, F(t,\cdot)$ *is L(t)-Lipschitz on* $B(z(t), \epsilon_0)$ *in the sense that:*

$$d(F(t,x), F(t,y)) \leq L(t)|x-y| \, \forall x, y \in B(z(t), \epsilon_0), \; t \in I \tag{3.6}$$

where d(A,B) is the Hausdorff distance

$$d(A,B) = max\{d^*(A,B), d^*(B,A)\}, d^*(A,B) = sup\{d(a,B); a \in A\}$$

In the theorem to follow, S is a separable metric space and $a(\cdot) : S \to X_0, y(\cdot) : S \to AC(I, R^n)$ are given continuous mappings for which there exists $s_0 \in S$ and a continuous function $p(\cdot) : S \to L^1(I, R^n)$ such that:

$$a(s_0) = z(0), y(0)(t) = z(t), p(s_0)(t) = 0, \; \forall t \in I \tag{3.7}$$

$$a(s) \in B(z(0), \frac{\epsilon_0}{2}), y(s)(t) \in B(z(t), \frac{\epsilon_0}{2}) \, \forall t \in I \tag{3.8}$$

$$d(y'(s)(t), F(t, y(s)(t)) \leq p(s)(t) \; a.e.inI, \; \forall s \in S \tag{3.9}$$

We shall use the following notations:

$$m(t) = \int_0^t L(u)du$$

$$\xi(s,t) = T(d(s,s_0))^2 \exp(m(t)) + T|a(s) - y(s)(0)|(\exp(m(t)) + 1) +$$
$$+ \int_0^t p(s)(u) \exp(m(t) - m(u))du \tag{3.10}$$

and assume that the following condition is satisfied:

$$\xi(s,T) < \frac{\epsilon_0}{2} \, \forall s \in S \tag{3.11}$$

Theorem 3.6 *Let* $z(\cdot) \in AC(I, R^n)$ *be asolution of (2.1) and assume that* $F(\cdot,\cdot)$ *satisfies Hypothesis 3.5*

Further on, (S, d) be a separable metric space, let $a : S \to X_0, y : S \to AC(I, R^n), p : S \to L^1(I, R_+)$ *be continuous mappings and* $s_0 \in S$ *such that (3.7)-(3.9), (3.11) are satisfied.*

Then there exists a continuous mapping $x(\cdot) : S \rightarrow AC(I, R^n)$ such that for any $s \in S$ the mapping $x(s)(\cdot)$ is a solution of (2.1) satisfying the following conditions:

$$x(s)(0) = a(s) \, \forall s \in S \tag{3.12}$$

$$x(s)(s_0) = z(t) \, \forall t \in I \tag{3.13}$$

$$|x(s)(t) - y(s)(t)| \leq \xi(s,t) \, \forall (t,s) \in I \times S \tag{3.14}$$

$$|x'(s)(t) - y'(s)(t)| \leq L(t)\xi(s,t) + p(s)(t) + (d(s,s_0))^2 \ : a.e.in I, \forall s \in S \tag{3.15}$$

Proof We denote $\epsilon_i = (d(s,s_0))^2 \frac{i+1}{i+2}$, $b(s) = |a(s) - y(s)(0)|, i = 0, 1, \ldots,$ $S_0 = S - \{s_0\}, p_0(s)(t) = p(s)(t) + \epsilon_0(s)$

$$p_i(s)(t) = \int_0^t p(s)(t) \frac{(m(t) - m(u))^{i-1}}{(i-1)!} du +$$

$$+ \frac{(m(t))^{i-1}}{(i-1)!} (\epsilon_i(s)T + b(s)), i = 1, 2, \ldots \tag{3.16}$$

We note that $\sum_{i \geq 0} p_i(s)(t) \leq \xi(s,t)$. Integrating by parts one has ([1] formula (14) page 122):

$$\int_0^t L(u) p_i(u) du = \int_0^t p(s)(t) \frac{(m(t) - m(u))^{i-1}}{(i-1)!} du +$$

$$+ \frac{(m(t))^i}{i!} (\epsilon_i(s)T + b(s)) < p_{i+1}(s)(t) \; a.e.in I \tag{3.17}$$

Using the same construction as in [4], we shall constuct a Cauchy sequence of succesive approximations $x_i(s)(\cdot) \in AC(I, R^n)$ such that for all $i \geq 1$ the mappings $s \rightarrow x_i(s)(\cdot)$ are continuous and have the following properties:

$$(i) x_i(s)(t) \in B(z(t), \epsilon_0) \, \forall t \in I, x_i(s)(0) = a(s)$$

$$(ii) x'_{i+1}(s)(t) \in F(t, x_i(s)(t)) \; a.e.in I \tag{3.18}$$

$$(iii) |x'_{i+1}(s)(t) - x'_i(s)(t)| \leq L(t) p_i(s)(t) \; a.e.in I$$

From (i) and (ii) we have

$$|x_{i+1}(s)(t) - x_i(s)(t)| \leq \int_0^t L(u) p_i(s)(u) du \; a.e.in I$$

and therefore from (3.17) it follows :

$$|x_{i+1}(s)(t) - x_i(s)| < p_{i+1}(s)(t) \, a.e.in I \tag{3.19}$$

For $s \in S_0$ we put $x_0(s)(t) = y(s)(t)$ and note that according to (3.8) we have $x_0(s)(t) \in B(z(t), \epsilon_0)$, $\forall \in I$.

At the first step we consider the multifunctions $G_0(\cdot), H_0(\cdot)$ defined, respectively, by:

$$G_0(s) = \{v \in L^1(I, R^n) : v(t) \in F(t, y(s)(t)) \, a.e. \, in \, I\}$$

$$H_0(s) = cl\{v \in G(s) : |v(t) - y'(s)(t)| < p(s)(t) + \epsilon_0(s)\}$$

Since $d(y'(s)(t), F(t, y(s)(t)) < p(s)(t) + \epsilon_0(s)$, according with Lemma 3.1, the set $H_0(s)$ is not empty.

Set $F_0^*(t,s) = F(t, y(s)(t))$ and note that:

$$d(0, F_0^*(t,s)) \leq |y'(s)(t)| + p(s)(t) = p^*(s)(t)$$

and $p^* : S \to L^1(I, R^n)$ is continuous.

Applying now Lemmas 2.3 and Lemmas 2.4 we obtain the existence of a continuous selection h_0 of H_0 i.e. such that:

$$h_0(s)(t) \in F(t, y(s)(t)) \, a.e.in I$$

$$|h_0(s)(t) - y'(s)(t)| \leq p_0(s)(t) = p(s)(t) + \epsilon_0(s)$$

We define $x_1(s)(t) = a(s) + \int_0^t h_0(s)(u)du$ and note that x_1 verifies (3.18). Indeed, one has:

$$|x_1'(s)(t) - x_0'(s)(t)| = |h_0(s)(t) - y'(s)(t)| \leq p_0(s)(t)$$

$$|x_1(s)(t) - x_0(s)(t)| = |a(s) - y(s)(0)| + \int_0^t |x_1'(s)(u) - x_0'(s)(u)|du \leq$$

$$\leq b(s) + \int_0^t p_0(s)(t)(u)du \leq p_1(s)(t) - T\epsilon_1(s) < p_1(s)(t)$$

On the other hand :

$$|x_1(s)(t) - z(t)| \leq |x_1(s)(t) - x_0(s)(t)| + |x_0(s)(t) - z(t)| \leq \xi(s,T) + \frac{\epsilon_0}{2} \leq \epsilon_0$$

Hence (i) is also satisfied since obviously $s \to x_1(s)(\cdot)$ is continuous

Suppose we have defined the functions $x_0, \ldots, x_i$ satisfying (3.18) . Observe that, since $F(t,\cdot)$ is L(t)-Lipschitz on $B(z(t), \epsilon_0)$ from (i), (ii) in (3.18) and (3.19) it follows:

$$d(x_i'(s)(t), F(t, x_i(s)(t)) \leq L(t)|x_i(s)(t) - x_{i-1}(s)(t)| < L(t)p_i(s)(t) \, a.e.inI \tag{3.20}$$

Denote $G_i(s) = \{v \in L^1(I, R^n) : v(t) \in F(t, x_i(s)(t)) \, a.e.inI\}$ and consider the map :

$$H_i(s) = cl\{v \in G_i(s) : |v(t) - x_i'(s)(t)| < L(t)p_i(s)(t) \, a.e.inI\} \tag{3.21}$$

To prove that $H_i(s)$ is nonempty we note first that the real function $t \rightarrow r_i(s)(t) = (d(s, s_0))^2 \frac{TL(t)(m(t))^{i-1}}{(i+1)(i+2)(i-1)!}$ is measurable and strictly positive for any s. Using (3.19) we get:

$$d(x_i'(s)(t), F(t, x_i(s)(t)) \leq L(t)|x_i(s)(t) - x_{i-1}(s)(t)| \leq$$

$$L(t) \int_0^t |x_i'(s)(u) - x_{i-1}'(s)(u)| du \leq L(t) \int_0^t L(u)p_{i-1}(s)(u) du =$$

$$= L(t)(\int_0^t p(s)(u) \frac{(m(t) - m(u))^{i-1}}{(i-1)!} du + \frac{(m(t))^{i-1}}{(i-1)!}(T\epsilon_{i-1}(s) + b(s)))$$

From the last inequality and (3.16) we infer that

$$d(x_i'(s)(t), F(t, x_i(s)(t)) \leq L(t)p_i(s)(t) - r_i(s)(t) < L(t)p_i(s)(t)$$

and therefore according to Lemma 3.1 there exists $v \in L^1(I, R^n)$ such that $v(t) \in F(t, x_i(s)(t))$ a.e. in I and

$$|v(t) - x_i'(s)(t)| < d(x'i(s)(t), F(t, x_i(s)(t)) + r_i(s)(t)$$

and hence $H_i(s)$ is not empty.

Set $F_i^*(t,s) = F(t, x_i(s)(t))$ and note that we may write

$$d(0, F_i^*(t,s)) \leq |x_i'(s)(t)| + L(t)p_i(s)(t) = p_i^*(s)(t) \, a.e.inI$$

and $p_i^* : S \rightarrow L^1(I, R^n)$ is continuous.

By Lemmas 3.3 and 3.4 there exists a continuous map $h_i : S \rightarrow L^1(I, R^n)$ such that

$$h_i(s)(t) \in F(t, x_i(s)(t)) \, a.e.inI$$

$$|h_i(s)(t) - x_i'(s)(t)| \leq L(t)p_i(s)(t)\, a.e.inI$$

Define now $x_{i+1}(s)(t) = a(s) + \int_0^t h_i(s)(u)du$ and note that one has:

$$|x_{i+1}(s)(t) - z(t)| \leq |x_{i+1}(s)(t) - x_0(s)(t)| + |x_0(s)(t) - z(t)| \leq$$

$$\leq \sum_{k=1}^{i+1} |x_k(s)(t) - x_{k-1}(s)(t)| + \frac{\epsilon_0}{2} \leq \xi(s,T) + \frac{\epsilon_0}{2} < \epsilon_0$$

Thus $x_{i+1} \in B(z(t), \epsilon_0)$. We infer that $x_{i+1}(\cdot)$ verifies (i),(ii) and (iii) in (3.18).

From (3.19) and (iii) we obtain

$$|x_{i+1}(s)(\cdot) - x_i(s)(\cdot)|_{AC} \leq p_{i+1}(s)(T) \tag{3.22}$$

On the other hand one has:

$$p_{i+1}(s)(T) \leq \frac{(m(T))^i}{i!}(|p(s)| + T(d(s,s_0))^2 + |a(s) - y(s)(0)|) \tag{3.23}$$

where by definition $|p(s)| = \int_0^T |p(s)(u)|du$. From (3.21) and (3.22) we get:

$$|x_{i+1}(s)(\cdot) - x_i(s)(\cdot)|_{AC} \leq \frac{(m(T))^i}{i!}(|p(s) + T(d(s,s_0))^2 + |a(s) - y(s)(0)| \tag{3.24}$$

The function $s \to |p(s)|_{AC}$ is continuous. Therefore (3.24) implies that the sequence $x_i(s)(\cdot)$ is Cauchy in the Banach space $AC(I, R^n)$ and it converges to some function $x(s)(\cdot) \in AC(I, R^n)$. Moreover (3.24) implies that for every $s \in S_0$ the sequence $\{x_i(s')(\cdot)\}$ satisfies the Cauchy conditions uniformly in s' on some neighbourhood.Hence $s \to x(s)(\cdot)$, is continuous from S_0 into $AC(I, R^n)$.

To verify that $x(s)(\cdot)$ is solution of (2.1) it is enough to see that

$$d(x_i'(s)(t), F(t, x(s)(t)) \leq L(t)|x_i(s)(t) - x(s)(t)|\, a.e.\, in\, I$$

By adding the inequalities (iii) in (3.18), we obtain that:

$$|x_{i+1}'(s)(t) - y'(s)(t)| \leq p(s)(t) + L(t)\int_0^t p(s)(u)(\sum_{k=1}^{i} \frac{(m(t) - m(u))^{k-1}}{(k-1)!})du +$$

$$+(T(d(s,s_0)^2 + |y(s)(0) - a(s)|)L(t)\sum_{k=0}^{i} \frac{(m(t))^k}{k!} + (d(s,s_0))^2$$

Similary by adding (3.19) we get:

$$|x_{i+1}(s)(t) - y(s)(t) - a(s) + y(s)(0)| \leq \int_0^t (\sum_{k=0}^{i} \frac{(m(t) - m(u))^k}{k!})du +$$

$$+(\epsilon_0(s)T + b(s)) \sum_{k=0}^{i} \frac{(m(t))^k}{k!}$$

By passing to the limit we obtain (3.14) and (3.15).

For $s = s_0$ we define $x(s_0)(t) = z(t)$ for all $t \in I$. It remains to verify that $x : S \to AC(I, R^n)$ is continuous in s_0.

One has:

$$|x(s)(\cdot) - x(s)(\cdot)|_{AC} = |a(s) - a(s_0)| + |x'(s)(\cdot) - x'(s_0)(\cdot)|_1 \leq$$

$$|a(s) - a(s_0)| + |y'(s)(\cdot) - y'(s_0)(\cdot)|_1 + |x'(s)(\cdot) - y'(s)(\cdot)|_1$$

Using the continuity of a, y, (3.15) and the fact that $p(s_0)(t) = 0 \forall t \in I$, we infer that x is continuous in s_0.

Remark 3.6. Theorem 3.5 above may be interpreted as a continuous version of the Filippov's theorem on differential inclusions that is related in some way to Theorem 3.1in [6]. We note that according to Theorem 3.1 in [4] under similar hypothesis, for any $\epsilon > 0$ there exists a continuous mapping $x_\epsilon : S \to AC(I, R^n)$ satisfying (3.12)-(3.15) in which $\xi(\cdot,\cdot)$ is replaced by :

$$\xi_\epsilon(s,t) = T\epsilon \exp(m(t)) + |a(s) - y(s)(0)|(\exp(m(t)) + 1) +$$

$$+ \int_0^t p(s)(u) \exp(m(t) - m(u))du$$

If we take $\epsilon = d(s, s_0)$ for any s we obtain from this result a family of continuous mapping $s \to x_s(\cdot,\cdot)$ satisfying (3.12)-(3.14), but we cannot infer that $s \to x_s(\cdot,\cdot)$ is continuous.

On the other hand , in the proof of Theorem 3.1 in [6], instead of Lemma 3.1. it is used a similar result replacing the measurable function $r(t) > 0$ by a constant $\epsilon > 0$;however such a result cannot be used to prove that the set $H_i(s)$ in (3.21) is nonempty since the difference $p_{i+1}(s)(t) - \int_o^t L(u)p_i(s)(u)du$ may not be minorized by an $\epsilon > 0$.

Remark 3.7. After the completion of this work we became aware of another related result, Theorem 3.1 in [24], which, however, does not concern the problem of a continuous imdedding of a given solution. On the other hand, studying the more complex problems of viability and relaxation, the proof of Theorem 3.1 in [24] is much more complicated and the evluation in (3.14) is replaced by another one in terms of a nonexplicitely defined monotone function.

4. Intrinsic derived cones defined by tangent directions

In this section we shall identify certain "intrinsic" derived cones to the reachable set in (2.2) in terms of the *set of tangent directions to the trajectories* of (2.1) through a point $(t_0, x_0) \in D$, defined as follows ([14]):

$$T_F^-(t_0, x_0) = \{v \in R^n : \exists x(\cdot) \in S_F(t_0, x_0) : x'_-(t_0) = v, \} \qquad (4.1)$$

where $x'_-(t_0) = \lim_{s \to 0_-} \frac{x(t_0+s)-x(t_0)}{s}$ is the derivative to the left.

The results in [14] provide certain characterizations of the set $T_F^-(t_0, x_0)$ in (4.1) proving, for istance, that one has:

$$F(t_0, x_0) \subset T_F^-(t_0, x_0) \subset \overline{co}F(t_0, x_0) \qquad (4.2)$$

provided either $F(\cdot, \cdot)$ is Hausdorff continuous with closed convex values, or locally-lipschitz or (continuously) parametrized by a continuous mapping f as in (2.3)'

On the other hand, according to Theorem 3.3 in [14] the following upper estimate holds for any multifunction $F(\cdot, \cdot)$:

$$T_F^-(t_0, x_0) \subset \bigcap_{J \in L_0} \bigcap_{\theta, r > 0} \overline{co}F((t_0 - \theta, t_0] \backslash J, B(x_0, r)) \qquad (4.3)$$

where $L_0 \subset \mathcal{L}(R)$ denotes the family of all subsets of R of zero Lebesgue measure.

We shall prove first that if $z(\cdot)$ is a solution of (2.1) and F satisfies the Hypothesis 3.5. then for a.e. $\tau \in (0, T)$, the set of tangent directions in (4.1) coincides with the following set:

$$F_0^-(\tau, z(\tau)) = \{v \in R^n : \exists \overline{v}(\cdot) \in L^1([\tau - \theta_0, \tau], R^n)\, \overline{v}(t) \in F(t, z(t))$$

$$on[\tau - \theta_0, \tau],\ \lim_{\theta \to 0+} \frac{1}{\theta} \int_{\tau-\theta}^{\tau} \overline{v}(t)dt = v\} \qquad (4.4)$$

We recall that $\tau \in (0,T)$ is said to be a Lebesgue point of an integrable mapping, $g(\cdot) \in L^1(I, R^n)$ if one has:

$$\lim_{\theta \to 0} \frac{1}{\theta} \int_{\tau}^{\tau+\theta} g(t)dt = g(\tau) \tag{4.5}$$

and we recall that the set $\mathcal{L}(g)$ of all Lebesgue points of g is of full measure.

Proposition 4.1. *If $z(\cdot)$ is a solution of (2.1) and $F(\cdot,\cdot)$ satisfies Hypothesis 3.5 then for any Lebesgue point τ of $L(\cdot)$ one has:*

$$T_F^-(\tau, z(\tau)) = F_0^-(\tau, z(\tau)) \tag{4.6}$$

Proof. Let $v \in T_F^-(\tau, z(\tau))$ and $x(\cdot) \in S_F(\tau, z(\tau))$ such that $x(\tau) = z(\tau)$ and $x'_-(\tau) = v$. Since $F(\cdot, z(\cdot))$ is measurable with closed values, according to a known result (e.g. [10]) it has an integrable selection $\overline{v}$ such that:

$$|x'(t) - \overline{v}(t)| = d(x'(t), F(t, z(t)) \leq L(t)|x(t) - z(t)| \, a.e.in[\tau - \theta_0, \tau]$$

Since $x(\cdot), z(\cdot)$ are continuous, $x(\tau) = z(\tau)$ and τ is assumed to be a Lebesgue point for $L(\cdot)$ from this inequality it follows that:

$$\lim_{\theta \to 0+} \frac{1}{\theta} \int_{\tau-\theta}^{\tau} (x'(t) - \overline{v}(t))dt = 0$$

and therefore we may write successively

$$\lim_{\theta \to 0+} \frac{1}{\theta} \int_{\tau-\theta}^{\tau} \overline{v}(t)dt = \lim_{\theta \to 0+} [\frac{1}{\theta} \int_{\tau-\theta}^{\tau} (\overline{v}(t) - x'(t))dt + \frac{1}{\theta} \int_{\tau-\theta}^{\tau} x'(t)dt]$$

$$= x'(\tau) = v$$

Hence $v \in F_0^-(\tau, z(\tau))$.

Conversely, if $v \in F_0^-(\tau, z(\tau))$ and $\overline{v} \in L^1(I, R^n)$ is such that

$$\overline{v}(t) \in F(t, z(t)) \, a.e.in[\tau - \theta_0, \tau], \lim_{\theta \to 0+} \frac{1}{\theta} \int_{\tau-\theta}^{\tau} \overline{v}(t)dt = v$$

then we consider the mapping y defined by:

$$y(t) = z(\tau) + \int_{\tau}^{t} \overline{v}(s)ds \; t \in [\tau - \theta_0, \tau]$$

and note that from Hypothesis 3.5. it follows that:

$$p(t) := L(t)|z(t) - y(t)| \geq d(F(t, z(t)), F(t, y(t)) \geq$$

$$\geq d(\overline{v}(t), F(t, y(t)) = d(y'(t), F(t, y(t)) \; a.e.in[\tau - \theta_0, \tau]$$

We use now Filippov's theorem on differential inclusions (e.g. [1], [10]) to obtain the existence of a solution $x(\cdot) : [\tau - \theta_0, \tau] \to R^n$ of (2.1) such that:

$$x(\tau) = y(\tau), |x(t) - y(t)| \leq \int_t^\tau \exp(m(t) - m(s))p(s)ds, \; m(t) = \int_t^\tau L(s)ds$$

In particular, one has:

$$|\frac{x(\tau - \theta) - y(\tau - \theta)}{\theta}| \leq \frac{1}{\theta} \int_{\tau-\theta}^{\tau} \exp^{m(\tau-\theta)-m(s)} p(s)ds \leq$$

$$\leq \frac{1}{\theta} \int_{\tau-\theta}^{\tau} \exp^{m(\tau-\theta)-m(s)} L(s)|x(s) - y(s)|ds$$

which converges to 0 as $\theta \to 0+$ since (x(s)-y(s)) converges to zero as $s \to \tau$ and τ is a Lebesgue point of $L(\cdot)$.

Therefore we may write:

$$\lim_{\theta \to 0+} \frac{x(\tau - \theta) - x(\tau)}{\theta} = \lim_{\theta \to 0+} [\frac{x(\tau - \theta) - y(\tau - \theta)}{\theta} + \frac{y(\tau - \theta) - y(\tau)}{\theta}] =$$

$$= \lim_{\theta \to 0+} \frac{y(\tau - \theta) - y(\tau)}{\theta} = \lim_{\theta \to 0+} \frac{1}{\theta} \int_{\tau-\theta}^{\tau} \overline{v}(t)dt = v$$

and Proposition 4.1. is proved.

The next statement gives another characterizitations of the set $F_0^-(t, z(t)), t \in I$, defined in (4.4.)

Proposition 4.2. *If Hypothesis 3.5. is satisfied and $F_0^-(t, z(t))$ is the set in (4.4.) then there exists a null subset $J \subset I$ such that:*

$$F(t, z(t)) \subset Cl(F_0^-(t, z(t))), \; \forall t \in I \backslash J \tag{4.7}$$

Proof. Since $z(\cdot) \in AC(I, R^n)$ there exists a null subset $J_1 \subset I$ such that $|z'(t)| < \infty \forall t \in I \backslash J_1$

For each $m \in N$ consider the set-valued map:

$$F_m(t, z(t)) = F(t, z(t)) \cap \overline{B}(z'(t), m), \; t \in I \backslash J_1$$

$F_m(\cdot)$ is measurable and by the Castaing-Valadier representation theorem (Theorem 3.7 in [3]) it follows that there exists $f_m^j(t) \in F_m(t), f_m^j(\cdot)$ measurable for all j, m such that:

$$F_m(t) = Cl\{f_m^j(t); j \in N\} \tag{4.8}$$

Moreover $|f_m^j(t)| \leq m + |z'(t)|$. Thus $f_m^j \in L^1(I, R^n)$, $\forall j, m \in N$. If we denote $L(f_m^j)$ the set of Lebesgue points of the integrable mapping $f_m^j(\cdot)$, then $\mu(L(f_m^j) = T$; also if $J_m = I \setminus \bigcap_{j \geq 1} L(f_m^j)$, $J_0 = \bigcup_{m \in N} J_m$ then $\mu(J_m) = \mu(J_0) = 0$.Furter on, if we denote $F_m^0(t) = \{f_m^j(t); j \in N\}$ then one has: $F_m(t) = Cl(F_m^0(t))$and $F_m^0(t) \subset F_0^-(t, z(t))\, \forall t \in I \setminus J_m$

Let us put $G(t) = \{f_m^j(t)\, j, m \in N\}$. Hence if $t \in I \setminus J_0$, then

$$G(t) \subset F_0^-(t, z(t)) \tag{4.9}$$

It remains to prove that $F(t, z(t)) \subset Cl(G(t)) \forall t \in I \setminus J$, $J = J_0 \cup J_1$ If $t \in I \setminus J$ and $v \in F(t, z(t))$ then there exists $m \in N$ such that $m \geq |v| + |z'(t)| \geq |v - z'(t)|$ thus $v \in F_m(t) \subset Cl(G(t))$.

The main result of this section is the following:

Theorem 4.3. *Let $z(\cdot) : I \to R^n$ be a solution of (2.1), let F be satisfying Hypothesis 3.5. and for any common Lebesgue point $\tau \in J(z(\cdot)) = \mathcal{L}(L(\cdot)) \cap \mathcal{L}(z'(\cdot))$ let $F_0^-(\tau, z(\tau))$ be the set of tangent directions in (4.4) and let $M_0(\tau), C_0(\tau)$ be the sets defined by:*

$$M_0(\tau) := F_0^-(\tau, z(\tau)) - z'(\tau),\ C_0(\tau) := cco(M_0),\ \tau \in J(z(\cdot)) \tag{4.10}$$

Then for any $\tau \in J(z)$, $M_0(\tau)$ is a derived set, hence $C_0(\tau)$ is a derived cone, to the reachable set $R_F(\tau, 0, X_0)$ at $z(\tau)$.

Proof. Let us consider

$$\tau \in J(z(\cdot)),\ \{w_1, \ldots, w_m\} \subset M_0(\tau) \text{and} \{v_1, \ldots, v_m\} \subset F_0^-(\tau, z(\tau))$$

such that $w_i = v_i - z'(\tau)$, $i = 1, 2, \ldots, m$ From Proposition 4.1 it follows that there exist$\theta_0 \in (0, \frac{\tau}{m})and\overline{v}_i \in L^1([\tau - m\theta_0, \tau], R^n)$, $i = 1, 2, \ldots, m$ such that

$$\overline{v}_i(t) \in F(t, z(t))\ a.e.in[\tau - m\theta_0, \tau]$$

$$\lim_{\theta \to 0+} \frac{1}{\theta} \int_{\tau-\theta}^{\tau} \overline{v}_i(t) dt = v,\ i = 1, 2 \ldots, m \tag{4.11}$$

We consider now the separable metric space $S = [0, \theta_0]^m$ endowed with the distance induced by the norm $|s| = \sum_{i=1}^m s_i$; if $s = (s_1, \ldots, s_m) \in S$ and we define the mappings $t_i(\cdot), y_i(\cdot), y(\cdot)$ as follows:

$$t_0(s) = 0,\ t_1(s) = \tau - \sum_{i=1}^m s_i,\ t_i(s) = t_{i-1}(s) + s_i,\ t_{m+1}(s) = \tau,$$

$$y_0(s)(t) = z(t)$$

$$y_i(s)(t) = \begin{cases} y_{i-1}(s)(t) & \text{if } t \in [0, t_i(s)] \\ y_{i-1}(s)(t_i(s)) + \int_{t_i(s)}^t \overline{v}_i(u)du & \text{if } t \in [t_i(s), \tau] \end{cases} \tag{4.12}$$

$$y(s)(t) = y_m(s)(t),\ s \in S,\ t \in [0, \tau]$$

Since the mapping $s \to y(s)(\tau)$ may alternatively defined by:

$$y(s)(\tau) = z(t_1(s)) + \sum_{i=1}^m \int_{t_i(s)}^{t_{i+1}(s)} \overline{v}(u)du,\ s \in S \tag{4.13}$$

from (4.11) it follows that

$$\frac{\partial}{\partial s_i} y(0)(\tau) = w_i,\ i = 1, 2, \ldots, m \tag{4.14}$$

and moreover, the mapping $s \to y(s)(\tau)$ is conically differentiable at $s = 0 \in S$ as one may write successively:

$$\frac{|y(s)(\tau) - y(0)(\tau) - \sum_{i=1}^m s_i w_i|}{|s|} \le \frac{|z(t_1(s)) - z(\tau) - |s| z'(\tau)|}{|s|} +$$

$$+ \sum_{i=1}^m \frac{1}{|s|} \left| \int_{t_i(s)}^{t_{i+1}(s)} \overline{v}_i(u)du - s_i v_i \right| \le \left| \frac{z(\tau - |s|) - z(\tau)}{|s|} - z'(\tau) \right| +$$

$$+ \sum_{i=1}^m \frac{\tau - t_i(s)}{|s|} \left| \frac{1}{\tau - t_i(s)} \int_{t_i(s)}^\tau (\overline{v}_i(u) - v_i)du \right| +$$

$$+ \sum_{i=1}^{m-1} \frac{\tau - t_{i+1}(s)}{|s|} \left| \frac{1}{\tau - t_{i+1}(s)} \int_{t_{i+1}(s)}^\tau (\overline{v}_i(u) - v_i)du \right|$$

which converges to 0 as $|s| \to 0$.

We shall prove next that $\theta_0 > 0$ may be chosen sufficiently small such that the mapping y defined in (4.12) satisfies the hypothesis of Theorem 3.6. We note first that from (4.12) it follows y(0)(t)=z(t) and

$$|y(s)(t) - z(t)| \leq \sum_{i=1}^{m} \int_{t_i(s)}^{t_{i+1}(s)} |\overline{v}_i(t) - z'(t)|dt =$$

$$= |y(s)(\cdot) - z(\cdot)|_{AC}$$

hence from the continuity of the functions $t_i(\cdot)$ and the absolute continuity of the Lebesgue integral it follows that if $\epsilon_0 > 0$ is the positive number in Hypothesis 3.5 then there exists $\theta_0 > 0$ such that $y(s)(t) \in B(z(t), \epsilon_0/2), \forall s \in S$, $(|s| \leq m\theta_0)$

To prove the fact that $y(\cdot) : S \to AC(I, R^n)$ is continuous we note that if for $s, s' \in S$, $i = 1, 2, \ldots, m$ we denote:

$$a_i(s, s') = min\{t_i(s), t_i(s')\},\ b_i(s, s') = max\{t_i(s), t_i(s')\}$$

$$\alpha(t) = |z'(t)| + \sum_{i=1}^{m} |\overline{v}_i(t)|,\ t \in [\tau - m\theta_0, \tau] \tag{4.14}$$

then from (4.12) it follows:

$$|y(s)(\cdot) - y(s)(\cdot)|_{AC} \leq \sum_{i=1}^{m} \int_{a_i(s,s')}^{b_i(s,s')} \alpha(t)dt$$

and, now, the continuity of the mapping $y(\cdot)$ follows from the absolute continuity of the Lebesgue integral and from the fact that the functions a_i, b_i in (4.14) satisfy:

$$b_i(s, s') - a_i(s, s') \leq |s'_i - s_i| \leq |s - s'|,\ \forall s, s' \in S, i = 1, 2, \ldots, m$$

Further on, we define the mapping $p(\cdot) : [0, \tau] \to L^1([0, \tau], R_+)$ as follows:

$$p(s)(t) = \begin{cases} 0 & \text{if } t \in [0, t_1(s)] \\ L(t)|y(s)(t) - z(t)| & \text{if } t \in [t_1(s), \tau] \end{cases} \tag{4.15}$$

and note that $p(\cdot)$ is continuous since $y(\cdot)$ is continuous, and moreover, since the derivative is given by:

$$y'(s)(t) = \begin{cases} z'(t) & \text{if } t \in [0, t_1(s)) \\ \overline{v}_i & \text{if } t \in (t_i(s), t_{i+1}(s)) \end{cases} \tag{4.16}$$

from (4.16) it follows that if $t \in [0, t_1(s)]$ then d(y'(s)(t),F(t,y(s)(t)) =0, and if $t \in [t_i(s), t_{i+1}(s)]$ then from (4.16) and from the lipschitzianity of $F(t, \cdot)$ it follows:

$$d(y'(s)(t), F(t, y(s)(t))) = d(\overline{v}_i(t), F(t, y(s)(t))) \leq d(F(t, z(t)), F(t, y(s)(t))) \leq$$

$$\leq L(t)|z(t) - y(s)(t)|$$

Hence the mapping p in (4.15) satisfies the condition:

$$d(y'(s)(t), F(t, y(s)(t))) \leq p(s)(t),\ a.e.in\,[0, \tau]\,\forall s \in S \tag{4.17}$$

and therefore if we define a(s)=z(0) for all $s \in S$, $s_0 = 0 \in S$ then the mapping $y(\cdot)$ in (4.12) satisfies the hypothesis of Theorem 3.6.

It follows that there exists a continuous mapping $s \to x(s)(\cdot) \in AC([0, \tau], R^n)$ as in Theorem 3.6. In particular such that $s \to c(s) = x(s)(\tau) \in R_F(\tau, 0, X_0)$ is continuous and $c(0) = z(\tau)$.

To end the proof of Theorem 4.3. we need to show that $c(\cdot)$ is differentiable at $s = 0 \in S$ and the derivative is given by $Dc(0)s = \sum_{i=1}^{m} s_i w_i$ In view of the inequality (3.14), the last equality is implied by the following property:

$$\lim_{|s| \to 0} \frac{1}{|s|} \int_0^t p(s)(t)dt = 0 \tag{4.18}$$

which follows from the definition in (4.15) of the mapping $p(\cdot)$, from the continuity of $y(\cdot)$ and from the fact that τ is Lebesgue point for $L(\cdot)$: for any $\epsilon > 0$ there exists δ_ϵ such that $|y(s)(t) - z(t)| < \epsilon \forall t \in [0, \tau]$, $|\frac{1}{|s|}\int_{\tau-|s|}^{\tau} L(t)dt - L(\tau)| < \epsilon\,\forall |s| < \delta_\epsilon$ and therefore from (4.15) it follows that for $|s| < \delta_\epsilon$ we have:

$$\frac{1}{|s|}\int_0^\tau p(s)(t)dt = \frac{1}{|s|}\int_{\tau-|s|}^{\tau} L(t)|y(s)(t) - z(t)|dt <$$

$$< \epsilon\frac{1}{|s|}\int_{\tau-|s|}^{\tau} L(t)dt < \epsilon(L(\tau) + \epsilon)$$

and the theorem is proved.

Remark 4.4. According to Proposition 4.2. the derived cones $C_0(\tau)$ in (4.10) satisfy the condition:

$$Cl(F(\tau, z(\tau)) \subset C_0(\tau)\,a.e.\,in\,I \tag{4.19}$$

Moreover if F is ,in addition, continuously parametrized in the sense that:

$$F(t,x) = f(t,x,U)\,\forall(t,x) \in D \tag{4.20}$$

then the relation in (4.19) may be sharpened as follows ([13]):

$$C_0(\tau) = cco(F(\tau, z(\tau)) - z'(\tau))\; a.e.\; in\; I \tag{4.21}$$

5 Derived cones generated by variational inclusions

In this section we shall prove the main result of this paper which provide larger derived cones obtained by the "transport" of the intrinsic ones in the previous section, by certain "variational inclusions" that generalize the variational equations in the classical theory of Ordinary Differential Equations

We recall that a set-valued map, $A(\cdot) : R^n \to \mathcal{P}(R^n)$ is said to be a *convex*(respectively, closed convex) *process* if $GraphA(\cdot) \subset R^n \times R^n$ is a convex(respectively, closed convex)cone.

For the basic properties of convex processes we refer to [10], but we shall use here only the above definition.

As main examples, we have in view the set-valued directional derivatives

$$AQ_y^+G(x;\cdot), AK_y^+G(x;\cdot), C_y^+G(x;\cdot)$$

in (2.13)of a set-valued mapping, $G(\cdot) : X \subset R^n \to \mathcal{P}(R^n)$, that correspond to the assymptotic quasitangent cone, to the assymptotic contingent cone and to Clarke's tangent cone in (2.9)-(2.10).

Everywhere in this section we assume the following:

Hypothesis 5.1. *$z(\cdot)$ is solution of (2.1), the multifunction $F(\cdot,\cdot)$: $D \subset R \times R^n \to \mathcal{P}(R^n)$ is assumed to satisfy Hypothesis 3.5., and a family $A(t,\cdot) : R^n \to \mathcal{P}(R^n)\; t \in I$ of convex processes satisfying the condition*

$$A(t,v) \subset Q^+_{z'(t)}F(t,\cdot)(z(t),v)\,\forall v \in domA(t,\cdot) \subset R^n\; t \in I \tag{5.1}$$

is assumed to be given and defines the variational inclusion:

$$v' \in A(t,v) \tag{5.2}$$

Remark 5.2. We note that for any orientor field $F(\cdot,\cdot)$ one may find an infinite number of families of convex process $A(t,\cdot), t \in I$, satisfying condition

(5.1): in fact any family, $\overline{A}(t) \subset Q^+_{(z(t),z'(t))} graph F(t,\cdot)$
$t \in I$, of closed convex subcones of the quasitangent cones $Q^+_{(z(t),z'(t))} graph F(t,\cdot)$ defines the family of closed convex process:

$$A(t,v) = \{v' \in R^n : (v,v') \in \overline{A}(t)\} \; v \in R^n \; t \in I$$

that satisfy condition (5.1)

One is tempted, of course, to take an "intrinsic" family of such closed convex process such as one of the convex-valued directional derivatives

$$C^+_{z'(t)} F(t,\cdot)(z(t);\cdot) \subset AK^+_{z'(t)} F(t,\cdot)(z(t);\cdot) \subset AQ^+_{z'(t)} F(t,\cdot)(z(t);\cdot) \quad (5.3)$$

but simple examples show that we may choose families of closed convex process satisfying (5.1) and strictly containing the intrinsic ones in (5.3):$F(t,x) = [-|x|,|x|]$ for $(t,x) \in R \times R$ then z(t)=0 $\forall t \in I$ is a solution of (2.1) and one has:

$$Q^+_{(z(t),z'(t))} graph F(t,\cdot) = Q^+_{(0,0)} graph F(t,\cdot) == \{(x,y) : x \in R, \; y \in [-|x|,|x|]\}$$

$$AQ^+_{(z(t),z'(t))} graph F(t,\cdot) = AK^+_{(z(t),z'(t))} == C^+_{(z(t),z'(t))} graph F(t,\cdot) = \{(0,0)\}$$

On the other hand, for any $\gamma \in [-1,1]$ the family of closed convex process $A_\gamma(t,\cdot)$ defined by

$$A_\gamma(t,v) = \{\gamma v; \; v \in R, \; t \in I\}$$

satisfies (5.1) and strictly containes any of the intrinsic ones in (5.3)

It is important also the fact that if $f(\cdot,\cdot)$ is a local selection of $F(\cdot,\cdot)$ in a tubular neighbourhood of $\{(t,z(t)) \, t \in I\}$ in the sense that there exists $\epsilon_0 > 0$ such that:

$$f(t,x) \in F(t,x) \; \forall x \in B(z(t),\epsilon_0) \; t \in I$$

then any family of closed convex process, $A(t,\cdot)$, satisfying:

$$A(t,v) \subset Q^+ f(t,\cdot)(z(t),v) \; \forall v \in dom A(t,\cdot) \; t \in I \quad (5.1)'$$

will satisfy also (5.1)

In particular, if $F(\cdot,\cdot)$ is continuously parametrized by $f(\cdot,\cdot)$ in the sense of (2.3) and if $f(t,\cdot,u)$ is differentiable for any t,u, then we may take:

$$A(t,v) = \frac{\partial f}{\partial x}(t,z(t),\tilde{u}(t))v; \; v \in R^n, \; z'(t) = f(t,z(t),\tilde{u}(t)) \quad (5.4)$$

and therefore the variational inclusion (5.2) becames the usual variational equation:

$$v' = \frac{\partial f}{\partial x}(t, z(t), \tilde{u}(t))v \tag{5.5}$$

We recall ([10],[20]) that since $F(t,\cdot)$ is assumed to be locally- lipschitz at z(t) a.e. on I, the quasitangent directional derivative in (2.13) is given by:

$$Q_{z'(t)}F(t,\cdot)(z(t);v) = \{v' \in R^n; \lim_{\theta\to 0+} \frac{1}{\theta} d(z'(t) + \theta v', F(t, z(t) + \theta v)) = 0\} \tag{5.6}$$

Lemma 4.3. *Let $z(\cdot), F(\cdot,\cdot)$ and $A(\cdot,\cdot)$ satisfy Hypothesis 5.1.,let $0 \leq t_1 < t_2 \leq T$ and let $C_1 \subset R^n$ be a derived cone of $R_F(t_1, 0, X_0)$ at $z(t_1)$.Then the reachable set of (5.2), $R_A(t_2, t_1, C_1)$, is a derived cone of $R_F(t_2, 0, X_0)$ at $z(t_2)$.*

Proof. In view of Definition 2.1., we consider $\{v_1, \ldots, v_m\} \subset R_A(t_2, t_1, C_1)$ hence such that there exist the solutions $u_1(\cdot), \ldots, u_m(\cdot)$ of the variational inclusion (5.2) such that:

$$u_j(t_2) = v_j,\ u_j(t_1) \in C_1,\ j = 1, 2, \ldots, m \tag{5.7}$$

Since $C_1 \subset R^n$ is a derived cone of $R_F(t_1, 0, X_0)$ at $z(t_1)$, there exists a continuous mapping, $a_0 : S = [0, \theta_0]^m \to R_F(t_1, 0, X_0)$ such that:

$$a_0(0) = z(t_1),\ Da_0(0)s = \sum_{j=1}^{m} s_j u_j(t_1) \forall s \in R^m_+ \tag{5.8}$$

Further on, for any $s = (s_1, \ldots, s_m) \in S$ we denote:

$$y(s)(t) = z(t) + u(t,s),\ u(t,s) = \sum_{j=1}^{m} s_j u_j(t)$$

$$p(s)(t) = d(y'(s)(t), F(t, y(s)(t)),\ t \in [t_1, t_2] \tag{5.9}$$

and prove that $y(\cdot), p(\cdot)$ satisfy the hypothesis of Theorem 3.6. at $s_0 = (0, \ldots 0) \in S$

We take $\theta_0 > 0$ small enough such that $y(s)(t) \in B(z(t), \epsilon_0/2)$ *a.e. in I* and use the local-lipschitzianity of $F(t,\cdot)$ to prove that for any $s \in S$, the measurable function $p(s)(\cdot)$ in (5.9) it is also integrable:

$$p(s)(t) = d(z'(t) + u'(t,s)), F(t, z(t) + u(t,s)) \leq |u'(t,s)| +$$

$$+d(F(t,z(t)),F(t,z(t)+u(t,s))\leq |u'(t,s)|+L(t)|u(t,s)|\leq$$

$$\leq |s|\sum_{j=1}^{m}[|u_j'(t)|+L(t)|u_j(t)|]$$

Moreover, the mapping $s\to p(s)(\cdot)\in L^1([t_1,t_2],R_+)$ is continuous (in fact Lipschitzean) since for any $s,s'\in S$ one may write succesively:

$$|p(s)(\cdot)-p(s')(\cdot)|_1\leq \int_{t_1}^{t_2}|p(s)(t)-p(s')(t)|dt\leq$$

$$\leq \int_{t_1}^{t_2}[|u'(t,s)-u'(t,s')|+d(F(t,y(s)(t),F(t,y(s')(t))dt]\leq$$

$$\leq |s-s'|\sum_{j=1}^{m}\int_{t_1}^{t_2}[|u_j'(t)|+L(t)|u_j(t)|]dt$$

Taking $\theta_0>0$ such that $\xi(s,t)$ defined in (3.10) satisfies condition (3.11) , from Theorem 3.6. it follows the existence of a continuous mapping $x(\cdot):S\to AC([t_1,t_2],R^n)$ with the properties (3.12)-(3.15).

Since $x(s)(t_1)=a_0(s)\in R_F(t_1,0,X_0)$ and $R_F(t_2,t_1,R_F(t_1,0,X_0))=R_F(t_2,0,X_0)$ the mapping $a(\cdot)$ defined by

$$a(s)=x(s)(t_2)\in R_F(t_2,0,X_0),\ s\in S \tag{5.10}$$

is continuous and satisfies the condition $a(0)=z(t_2)$.

To end the proof we need to show that $a(\cdot)$ is differentiable at $s_0=0\in S$ and its derivative is given by:

$$Da(0)(s)=\sum_{j=1}^{m}s_jv_j\forall s\in R_+^m$$

which is equivalent with the fact that:

$$\lim_{s\to 0}\frac{1}{|s|}(|a(s)-a(0)-\sum_{j=1}^{m}s_jv_j|)=0 \tag{5.11}$$

Using (5.10) and the property in (3.14) of $x(\cdot)$ we obtain:

$$\frac{1}{|s|}|a(s)-a(0)-\sum_{j=1}^{m}s_jv_j|\leq \frac{1}{|s|}|x(s)(t_2)-y(s)(t_2)|\leq$$

$$\leq \frac{1}{|s|}\xi(s,t) = T|s| + \frac{1}{|s|}|a_0(s) - z(t_1) - \sum_{j=1}^{m} s_j u_j(t_1)|(\exp(m(t_2)) + 1) +$$

$$+ \int_{t_1}^{t_2} \frac{p(s)(u)}{|s|} \exp(m(t_2) - m(u))du$$

and therefore in view of (5.8), relation (5.11) is implied by the following property of the mapping $p(\cdot)$ in (5.9)

$$\lim_{s\to 0} \frac{p(s)(t)}{|s|} = 0 \; a.e. \, in \, [t_1, t_2] \tag{5.12}$$

In order to prove the last property we note since $A(t,\cdot)$ is a convex process and $u(\cdot)$ is the mapping defined in (5.9), for any $s \in S\backslash\{0\}$ one has:

$$u'(t,s) \in A(t, u(t, \frac{s}{|s|})) \subset Q_{z'(t)}F(t,\cdot)(z(t); u(t, \frac{s}{|s|})) \; a.e \, in \, [t_1, t_2]$$

hence from (5.6) it follows that:

$$\lim_{h\to 0+} \frac{1}{h} d(z'(t) + hu'(t, \frac{s}{|s|}), F(t, z(t) + hu(t, \frac{s}{|s|}))) = 0 \tag{5.13}$$

In order to prove that (5.13) implies (5.12) we consider the compact metric space $S_+^{m-1} = \{\sigma \in R_+^m : |\sigma| = 1\}$ and the real function $\phi_t(\cdot,\cdot) : (0, \theta_0] \times S_+^{m-1} \to R_+$ defined by

$$\phi_t(\theta, \sigma) = \frac{1}{\theta} d(z'(t) + \theta u'(t,\sigma), F(t, z(t) + \theta u(t,\sigma))) \tag{5.14}$$

which according to (5.13) has the property

$$\lim_{\theta\to 0+} \phi_t(\theta,\sigma) = 0 \; \forall \sigma \in S_+^{m-1} \; a.e. \, in \, I \tag{5.15}$$

Using the fact that $\phi_t(\theta,\cdot)$ is lipschitzean and the fact that S_+^{m-1} is a compact metric space from (5.15) and from Proposition 5.4 bellow it follows easily that

$$\lim_{\theta\to 0+} \max_{\sigma \in S_+^{m-1}} \phi_t(\theta,\sigma) = 0$$

which implies the fact that

$$\lim_{s\to 0+} \phi_t(|s|, \frac{s}{|s|}) = 0 \; a.e. \, in [t_1, t_2]$$

and our statement is proved.

Proposition 5.4. *Let (M,d) be a compact metric space and let* $\phi(\cdot,\cdot) : (0,\theta_0] \times M \to R_+$ *be a real function such that* $\phi(\theta,\cdot)$ *is L-lipschitzean and has the property:*

$$\lim_{\theta\to 0+} \phi(\theta,\sigma) = 0 \;\forall \sigma \in M \tag{5.16}$$

Then ϕ *has also the following property:*

$$\lim_{\theta\to 0+} \max_{\sigma\in M} \phi(\theta,\sigma) = 0 \tag{5.17}$$

Proof. Let us assume by contrary that (5.17) is not satisfied; hence there exists $\epsilon > 0, t_m \to 0+, \sigma_m \in M$ such that $\phi(t_m,\sigma_m) \geq \epsilon \forall m \in N$. Since (M,d) is a compact metric space, without loss of generality, we may assume that $\sigma_m \to \sigma_0 \in S$; hence using the lipschitzianity of the function $\phi(\theta,\cdot)$ we may write succesively

$$0 < \epsilon \leq \phi(t_m,\sigma_m) \leq |\phi(t_m,\sigma_m) - \phi(t_m,\sigma_0)| +$$

$$+\phi(t_m,\sigma_0) \leq Ld(\sigma_m,\sigma_0) + \phi(t_m,\sigma_0) \to 0 \; as\, m \to \infty$$

a contradiction.

Remark 5.5. Since $\{0\} \subset R^n$ is a derived cone to any set $X \subset R^n$ at any point $x \in X$, from Lemma 5.3. it follows in particular that for any $0 \leq t_1 < t_2 \leq T$, $R_A(t_2,t_1,\{0\})$ is a derived cone to $R_F(t_2,0,X_0)$ at $z(t_2)$.

The next preliminary result shows that the derived cone $R_A(t_2,t_1,C_1)$ in Lemma 5.3. may be enlarged by adding the intrinsic one, $C_0(t_2)$ in (4.10), if t_2 is a common Lebesgue point of $z(\cdot)'$ and $L(\cdot)$ in Hypothesis 3.5

In what follows we shall use the notation:

$$J = (\mathcal{L}(z'(\cdot)) \cap \mathcal{L}(L(\cdot))) \cup \{0\} \tag{5.18}$$

for the set of all common Lebesgue points in (0,T) of $z'(\cdot)$ and $L(\cdot)$ to which we add the left-end point 0.

Lemma 5.6. *Let* $z(\cdot), F(\cdot,\cdot)$ *and* $A(\cdot,\cdot)$ *satisfy Hypothesis 5.1., let* $0 \leq t_1 < t_2 \leq T$ *be such that* $t_2 \in J$*, let* $C_1 \subset R^n$ *be a derived cone of* $R_F(t_1,0,X_0)$ *at* $z(t_1)$*, let* $C_0(t_2)$ *be the intrinsic derived cone in (4.10) and let* $M_2, C_2 \subset R^n$ *be defined by:*

$$M_2 = C_0(t_2) \cup R_A(t_2,t_1,C_1),\; C_2 = cco(M_2) \tag{5.19}$$

Then M_2 is a derived set, hence C_2 is a derived cone, to $R_F(t_2, 0, X_0)$ at $z(t_2)$.

Proof. We note first that if $M_0(t_2) = F_0^-(t_2, z(t_2)) - z'(t_2), C_0(t_2) = cco(M_0(t_2))$ and since $C_2 = cco(M_0(t_2) \cup R_A(t_2, t_1, C_1))$ it is enough to prove that $M_0(t_2) \cup R_A(t_2, t_1, C_1)$ is a derived set of $R_F(t_2, 0, X_0)$ at $z(t_2)$

We consider

$$\{v_1, \ldots, v_l\} \subset R_A(t_2, t_1, C_1),\ \{v_{l+1}, \ldots v_m\} \subset M_0(t_2)$$

the solutions $u_1, \ldots, u_l$ of (5.2) and the integrable selections $\overline{v}_j(t) \in F(t, z(t)), t \in (t_2 - \theta_0, t_2]$ such that:

$$u_j(t_2) = v_j,\ u_j(t_1) \in C_1\ j = 1, \ldots, l$$

$$\lim_{\theta \to 0+} \int_{t_2-\theta}^{t_2} \overline{v}_j(t)dt = v_j + z'(t_2)\ j = l+1, \ldots, m$$

Combining the proofs of Lemma 5.3. and Theorem 4.3. we define reccurrently the mappings $y_j(\cdot), y(\cdot) : S = [0, \theta_0]^m \to AC([t_1, t_2], R^n), p(s)(\cdot) : S \to L^1([t_1, t_2], R^n)$
as follows:

$$y_l(s)(t) = z(t) + \sum_{j=1}^{l} s_j u_j(t)$$

$$y_j(s)(t) = \begin{cases} y_{j-1}(s)(t) & \text{if } t \in [t_1, t_j(s)) \\ y_{j-1}(s)(t) + \int_{t_j}^{t} \overline{v}(u)du & \text{if } t \in [t_j(s), t_2] \end{cases} \quad j = l+1, \ldots, m$$

$$t_{l+1}(s) = t_2 - \sum_{j=l+1}^{m} s_j, t_{j+1}(s) = t_j(s) + s_j, j = l+1, \ldots, m$$

$$t_m(s) = t_2 - s_m,\ t_{m+1}(s) = t_2 \tag{5.20}$$

$$y(s)(t) = y_m(s)(t)$$

$$p(s)(t) = \begin{cases} p_1(s)(t) = d(y_l'(s)(t), F(t, y_l(s)(t))) & \text{if } t \in [t_1, t_{l+1}(s)] \\ L(t)|y(s)(t) - z(t)| & \text{if } t \in [t_{l+1}(s), t_2] \end{cases}$$

Using the same arguments as in the proofs of Theorem 4.3. and Lemma 5.3. it follows that we may choose $\theta_0 > 0$ sufficiently small that the mappings $s \to y(s)(\cdot), s \to p(s)(\cdot)$ are continuous and satisfy the hypothesis of Theorem 3.6, hence there exists a continuous mapping $x(\cdot) : S \to AC([t_1, t_2], R^n)$ with the properties in (3.12)-(3.15).

Moreover the mapping $s \to p(s)(\cdot)$ has the property in (5.12) and since $s \to y(s)(t_2)$ is differentiable at $s = s_0 = 0$, its partial derivatives being given by :

$$\frac{\partial}{\partial s_j} y(0)(t_2) = v_j \ \forall j = 1, 2, \ldots, m$$

From (3.14) it follows that the mapping $s \to a(s) = x(s)(t_2) \in R_F(t_2, 0, X_0)$ is continuous and has the properties:

$$a(0) = z(t_2), \ Da(0)s = \sum_{j=1}^{m} s_j v_j$$

which proves Lemma 5.6.

Remark 5.7. In what follows we extend the family of the intrinsic derived cones $C_0(t), t \in J$ in (4.10) to all the points $t \in [0, T]$ as follows

$$C_0(t) = \begin{cases} C_0 & \text{if t=0} \\ cco(F_0^-(t, z(t)) - z'(t)) & \text{if } t \in J \setminus \{0\} \\ \{0\} & \text{if } t \in [0, T] \setminus J \end{cases} \tag{5.21}$$

and we shall use the convention $R_A(t, t, C) = C$ for any $t \in I, C \subset R^n$.

Using Lemmas 5.3, 5.6 and Theorem 4.3 we obtain larger derived cones as follows:

Proposition 5.8. *Let $z(\cdot), F(\cdot, \cdot)$ and $A(\cdot, \cdot)$ satisfy Hypothesis 5.1. and for any $t \in I$ let $C_0(t)$ be the intrinsic derived cone in (5.21).*

Then for any $\tau \in (0, T]$ the set $C_1(\tau) \subset R^n$ defined by:

$$C_1(\tau) = cco(\bigcup_{t \in [0, \tau]} R_A(\tau, t, C_0(t))) \tag{5.22}$$

is a derived cones to $R_F(\tau, 0, X_0)$ at $z(\tau)$.

Proof. In view of a basic property of a derived set it is enough to prove that $M_1(\tau)$, defined by:

$$M_1(\tau) = \bigcup_{t \in [0, \tau]} R_A(\tau, t, C_0(t)) \tag{5.23}$$

is a derived set of $R_F(\tau, 0, X_0)$ at $z(\tau)$

To prove this statement we consider $\{v_1, \ldots, v_m\} \subset M_1(\tau)$ and we note that from the definition in (5.23) it follows that there exist $0 \leq t_1 < t_2 < \ldots < t_{q+1} = \tau$ such that $m_i \geq 1, \sum_{i=1}^{q+1} m_i = m$ and:

$$\{v_1, \ldots, v_m\} = \bigcup_{i=1}^{q+1} \{v_i^j, j = 1, \ldots, m_i\}$$

$$v_i^j \in R_A(\tau, t_i, C_0(t_i))\; i = 1, \ldots, q,\; j = 1, \ldots m_i$$

$$v_{q+1}^j \in C_0(\tau),\; j = 1, \ldots, m_{q+1}$$

We consider now the mappings $u_i^j(\cdot) : [t_i, \tau] \to R^n$ that are solutions of the variational inclusion in (5.2) such that:

$$u_i^j(\tau) = v_i^j,\; u_i^j(t_i) \in C_0(t_i), i = 1, \ldots, q, j = 1, \ldots, m_i \tag{5.24}$$

We note first that from Theorem 4.3 and Lemmas 5.3,5.6 it follows that the set $K(t_i, t_{i-1}, \ldots, t_1) \subset R^n$, $i = 2, 3, \ldots q$, defined reccurently as follows:

$$K(t_2, t_1) = C_0(t_2) \cup R_A(t_2, t_1, C_0(t_1)$$

$$K(t_i, t_{i-1}, \ldots, t_1) = C_0(t_i) \cup R_A(t_i, t_{i-1}, K(t_{i-1}, \ldots, t_1)),\; i = 3, \ldots, q+1 \tag{5.25}$$

is a derived set of $R_F(t_i, 0, X_0)$ at $z(t_i)$ and in particular $K(t_q, \ldots, t_1)$ is a derived set of $R_F(\tau, 0, X_0)$ at $z(\tau)$

On the other hand, from the basic properties of the solution u_i^j follow inductively the following inclusions:

$$\{u_p^j(t_2) : p = 1, 2,\; j = 1, 2, \ldots m_p\} \subset K(t_2, t_1)$$

$$\{u_p^j(t_i) : p = 1, \ldots, i;\; j = 1, \ldots m_p\} \subset K(t_i, \ldots, t_1),\; i = 2, 3, \ldots, q+1$$

which proves that the set $\{v_1, \ldots, v_m\}$ is contained into the derived set $K(t_{q+1}, \ldots, t_1) \subset M_1(\tau)$.

Remark 5.9. From Lemma 5.6. and Proposition 5.8. it follows that if $0 \leq t_1 < t_2 \leq T$ and $C_1(t_1), C_1(t_2)$ are the derived cones in (5.22) then $cco[C_0(t_2) \cup R_A(t_2, t_1, C_1(t_1))]$ is also a derived cone to $R_F(t_2, 0, X_0)$ at $z(t_2)$ that may not be contained into $C_1(t_2)$; this fact suggests the possibility of obtaining derived cones that are larger than those in (5.22).Using an ideea similar to that in the proof of Lemma 7.5.5. in [10] we prove now the main result of this paper:

Theorem 5.10 *Let $z(\cdot), F(\cdot,\cdot)$ and $A(\cdot,\cdot)$ satisfy Hypothesis 5.1., let $C_0(t)$ be the cones in (5.21) and for any $j \geq 1$ and any finite set of points $0 \leq t_1 < \ldots < t_j \leq T$ let the cones $K_j(t_j,\ldots,t_1), D_j(t_j,\ldots,t_1)$ be defined recurrently as follows:*

$$K_1(t_1) = R_A(t_1, 0, C_0),\ D_1(t_1) = cco[C_0(t_1) \cup K_1(t_1)]$$

$$K_j(t_j,\ldots,t_1) = R_A(t_j, t_{j-1}, D_{j-1}(t_{j-1},\ldots t_1))\ j \geq 2 \tag{5.26}$$

$$D_j(t_j,\ldots t_1) = cco[C_0(t_j) \cup K_j(t_j,\ldots t_1)]$$

Then for any $\tau \in I$, the convex cone $\tilde{C}(\tau) \subset R^n$ defined by:

$$\tilde{C}(\tau) = cco[\bigcup_{j\geq 1} \bigcup_{0<t_1<\ldots t_j=\tau} K_j(t_j,\ldots,t_1) \cup C_0(\tau)] \tag{5.27}$$

is a derived cone to $R_F(\tau, 0, X_0)$ at $z(\tau)$ and moreover, it has the following properties:

$$R_A(\tau, t, \tilde{C}(t)) \subset \tilde{C}(\tau)\ \forall 0 \leq t < \tau \leq T \tag{5.28}$$

$$\tilde{C}(\tau) = cco[\bigcup_{t\in[0,\tau)} R_A(\tau, t, \tilde{C}(t)) \cup C_0(\tau)]\ \forall \tau \in I \tag{5.29}$$

Proof. We note first that from Lemmas 5.3 and 5.6 it follows by induction that for any sequence $0 \leq t_1 < \ldots < t_i < T$ and for any $j \geq 1$, the sets $K_j(t_j,\ldots,t_1), D_j(t_j,\ldots,t_1)$ in (5.26) are derived cones to $R_F(t_j, O, X_0)$ at $z(t_j)$.

Next we prove that if $0 < t_1 < \ldots < t_j = \tau \leq T$ and $0 < s_1 < \ldots s_m = \tau$ are such that $\{t_1,\ldots t_j\} \subset \{s_1,\ldots,s_m\}, m > j$ then one has:

$$K_j(t_j,\ldots t_1) \subset K_m(s_m,\ldots,s_1), D_j(t_j,\ldots t_1) \subset D_m(s_m,\ldots,s_1) \tag{5.30}$$

In turn, (5.30) follows from the inclusions;

$$K_j(t_j,\ldots,t_i,t_{i-1},\ldots t_1) \subset K_{j+1}(t_j,\ldots,t_i,s,t_{i-1},\ldots t_1)$$

$$D_j(t_j,\ldots,t_1) \subset D_{j+1}(t_j,\ldots,t_i,s,t_{i-1},\ldots t_1) \tag{5.31}$$

Using the obvious properties:

$$R_A(\tau, s, R_A(s,t,C)) = R_A(\tau,t,C)\ \forall C \subset R^n$$

$$K_j(t_j,\ldots,t_1) \subset D_j(t_j,\ldots,t_1) \tag{5.32}$$

we obtain the following inclusions:

$$K_1(t_1) = R_A(t_1, 0, C_0) = R_A(t_1, s, R_A(s, 0, C_0)) =$$

$$= R_A(t_1, s, K_1(s)) \subset K_2(t_1, s)$$

$$D_1(t_1) = cco[C_0(t_1) \cup K_1(t_1)] \subset cco[C_0(t_1) \cup K_1(t_1, s)] = D_2(t_1, s)$$

for any $0 \leq s < t_1 \leq T$ and (5.31) is verified for j=1

Assuming by induction, that (5.31) is true for $j > 1$ we consider $0 < t_1 < \dots t_j < t_{j+1}$ and $s \in (t_j, t_{j+1})$;using (5.32) we may write succesively:

$$K_{j+1}(t_{j+1}, \dots, t_1) = R_A(t_{j+1}, s, R_A(s, t_j, D_j(t_j, \dots t_1))) \subset$$

$$\subset R_A(t_{j+1}, s, D_{j+1}(s, t_j, \dots t_1)) = K_{j+2}(t_{j+1}, s, t_j, \dots t_1),$$

$$D_{j+1}(t_{j+1}, \dots t_1) = cco[C_0(t_{j+1} \cup K_{j+1}(t_{j+1}, \dots t_1)] \subset$$

$$\subset cco[C_0(t_{j+1}) \cup K_{j+2}(t_{j+1}, s, t_j \dots t_1)] = D_{j+2}(t_{j+1}, s, t_j, \dots t_1)$$

hence (5.31) holds for j+1 and $s \in (t_j, t_{j+1})$; moreover if $i \in \{1, 2 \dots j\}$ and $s \in (t_j, t_{j+1})$ then from the induction hypothesis it follows:

$$K_{j+1}(t_{j+1}, \dots t_1) = R_A(t_{j+1}, t_j, D_j(t_j \dots t_1)) \subset$$

$$\subset R_A(t_{j+1}, t_j, D_{j+1}(t_j, \dots t_i, s, t_{i-1} \dots t_1)) = K_{j+2}(t_{j+1}, \dots t_i, s, t_{i-1} \dots t_1)$$

$$D_{j+1}(t_{j+1}, \dots t_1) = cco[C_0(t_{j+1}) \cup K_{j+1}(t_{j+1}, \dots t_1)] \subset$$

$$\subset cco[C_0(t_{j+1}) \cup K_{j+2}(t_{j+1}, \dots t_i, s, t_{i-1}, \dots t_1)] = D_{j+2}(t_{j+1}, \dots t_i, s, t_{i-1}, \dots t_1)$$

and (5.31) hence (5.30) are completely proved.

In order to prove that the set $\widetilde{M}(\tau)$ defined by:

$$\widetilde{M}(\tau) = \bigcup_{j \geq 1} \bigcup_{0 < t_1 < \dots < t_j = \tau} K_j(t_j, \dots t_1) \cup C_0(\tau) \tag{5.33}$$

is a derived set to $R_F(\tau, 0, X_0)$ at $z(\tau)$, we consider $\{v_1, \dots v_m\} \subset \widetilde{M}(\tau)$ and note that it follows that there exist $j_1, \dots j_{q+1} \geq 1, 0 \leq t^1_{j_i} <\leq t^{j_i}_{j_i} = \tau, i = 1, \dots q+1$ such that:

$$\{v_1, \dots v_m)\} = \{v^l_{j_i}; i = 1, \dots q+1, l = 1, \dots j_i\}$$

$$\{v_{j_i}; l = 1, \dots j_i\} \subset K_{j_i}(t^{j_i}_{j_i}, \dots t^1_{j_i}), i = 1, \dots q \tag{5.34}$$

$$\{v^l_{j_{q+1}}; l = 1, \dots j_{q+1}\} \subset C_0(\tau)$$

We consider now the increasing sequence $o \le s_1 < \dots < s_m = \tau$ such that $\{s_1, \dots s_m\} = \{t^l_{j_i} : j = 1, \dots q,\ l = 1, \dots j_i\}$ and note that according to (5.30) we have:

$$K_{j_i}(t^{j_i}_{j_i}, \dots t^1_{j_i}) \subset K_m(s_m, \dots s_1)$$

and therefore from (5.31) and (5.33) we infer that:

$$\{v_1, \dots v_m\} \subset D_m(s_m, \dots s_1)$$

Since $D_m(s_m, \dots s_1)$ is a derived cone to $R_F(\tau, 0, X_0)$ at $z(\tau)$ there exists a continuous mapping $a(\cdot) : [0, \theta_0] \to R_F(\tau, 0, X_0)$ that has the properties in (5.10)-(5.11) and therefore $\widetilde{M}(\tau)$ is a derived set hence the first statement of Theorem 5.10 is proved.

In view of the definition in (5.10) of the cones $\tilde{C}(t)$ the equalities in (5.28)-(5.29) follow from the inclusions:

$$R_(\tau, \theta, \tilde{C}(\theta)) \subset \bigcup_{m > \ge 1} \bigcup_{0 \le t_1 < \dots < t_m = \tau} K_m(t_m, \dots t_1) \forall \theta \in [0, \tau) \tag{5.35}$$

If $v \in R_A(\tau, \theta, \tilde{C}(\theta)$ then there exists $w(\cdot) \in S_A(\tau, \theta, \tilde{C}(\theta))$ such that $w(\tau) = v, w(\theta) \in \tilde{C}(\theta)$ hence there exist

$$c_j > 0,\ v_0 \in C_0(\theta),\ v_j \in K_{m_j}(\theta, \theta^{m_j - 1}_{m_j}, \dots \theta^1_{m_j})$$

such that

$$w(\theta) = \sum_{j=0}^{q} c_j v_j$$

further on from the properties in (5.30) it follows that if $0 \le s_1 < \dots s_m = \theta$ is such that:

$$\{s_1, \dots s_m\} = \{\theta^p_{m_j} : p = 1, \dots m_j, j = 1, \dots q\}$$

then

$$w(\theta) = c_0 v_0 + \sum_{j=1}^{q} c_j v_j \in cco[C_0(\theta) \cup K_m(\theta, s_{m-1}, \dots s_1)] = D_m(\theta, s_{m-1}, \dots s_1)$$

and therefore

$$w(\tau) \in R_A(\tau, \theta, D_m(\theta, s_{m-1}, \dots s_1)) = K_{m+1}(\tau, \theta, s_{m-1}, \dots s_1)$$

and Theorem 5.10 is completely proved.

Taking into account Proposition 4.2 the result in Theorem 5.10 may be reformulated as follows:

Corollary 5.11. *If $z(\cdot), F(\cdot,\cdot)$ and $A(\cdot,\cdot)$ satisfy Hypothesis 5.1. and $C_0 \subset R^n$ is a derived cone to X_0 at $z(0) \in X_0$ then for any $\tau \in (0,T]$ there exists a derived cone $C(\tau)$ to $R_F(\tau,0,X_0)$ at $z(\tau)$ such the relations (5.28) and the following ones are satisfied:*

$$Cl[F_0^-(\tau,z(\tau)) - z'(\tau)] \subset \tilde{C}(\tau)\, a.e.\, in\, I \tag{5.36}$$

6 Derived cones to reachable sets of standard control systems

In this section we consider shortly the case of standard control systems of the form (2.3) to show that the Hypothesis 3.5 may be considerably weakened and the proofs of Theorems 4.3 and 5.10 may be simplified.

We shall use the well known Peano's existence theorem and the following result proved in Mirica [15].

Lemma 6.1.([15] Lemma 3) *Let $D \subset R \times R^n$ be open and let $\tilde{f}(\cdot,\cdot): D \to R^n$ be a continuous vector field defining the differential equations:*

$$x' = \tilde{f}(t,x) \tag{6.1}$$

Then, for any $(t_0,x_0) \in D$ there exists $a, r > 0$ such that for any $s \in I_0 = [t_0 - a, t_0 + a]$ and $y \in B_r(x_0)$ there exists a solution $\widehat{x}(\cdot,s,y): I_0 \to R^n$ of (6.1) such that:

$$\widehat{x}(s,s,y) = y, \quad \lim_{(\theta,s,y)\to(0,t_0,x_0)} \frac{\widehat{x}(s+\theta,s,y) - y}{\theta} = \tilde{f}(t_0,x_0) \tag{6.2}$$

We obtain first a sharpened version of Theorem 4.3 for control systems of the form (2.3) under the following hypothesis:

Hypothesis 6.2. *The subset $D \subset R \times R^n$ is open, U is a Hausdorff topological space, $f_u(t,\cdot) = f(t,\cdot,u)$ is locally Lipschitz at each point in D (with a Lipschitz constant deppending possibly on u and on t); $\widetilde{u}(\cdot): I = [0,T] \to U$ is measurable and $z(\cdot): I \to R^n$ is an absolutely continuous solution of the problem:*

$$x' = \tilde{f}(t,x),\ x(0) \in X_0 \subset R^n,\ \tilde{f}(t,x) = f(t,x,\widetilde{u}(t)) \tag{6.3}$$

Moreover we suppose $\tilde{f}$ is locally integrably bounded and locally-Lipschitz at each point $(t_0, z(t_0)) \in I$ in the sense that there exist

$$\epsilon > 0,\ m(\cdot), L(\cdot) \in L^1([t_0 - \epsilon, t_0 + \epsilon], R_+)$$

such that:

$$|\tilde{f}(t,x)| \leq m(t)\ \forall (t,x) \in B((t_0, z(t_0)), \epsilon)$$

$$|\tilde{f}(t,x) - \tilde{f}(t,y)| \leq L(t)|x-y|\ \forall (t,x),(t,y) \in B((t_0, z(t_0)), \epsilon)$$

We note that under this hypothesis, the continuously parametrized orientor field $F(\cdot,\cdot)$ defined by:

$$F(t,x) = f(t,x,U)\ \forall (t,x) \in D \tag{6.4}$$

need not be even Hausdorff continuous (if U is not a compact topologocal space).

From Cauchy-Lipschitz existence and uniqueness theorem it follows that the set of tangent directions in (4.1) as well as the set defined in (4.4) are given by:

$$T_F^-(t,z(t)) = F_0^-(t,z(t)) = f(t,z(t),U)\ \forall t \in (0,T] \tag{6.5}$$

since for any $u \in U$, the vector field $f_u(\cdot,\cdot) : D \to R^n$ defined by

$$f_u(t,x) = f(t,x,u)\ \forall (t,x) \in D \tag{6.6}$$

is continuous and locally-Lipschitz with respect to the second variable and therefore has a unique classical (continuously differentiable) solution through each point $(s,y) \in D$.

Lemma 6.3. *Let $z(\cdot)$ be a solution of (6.2) and let f be satisfying Hypothesis 6.2.*

Then for any Lebesgue point $\tau \in \mathcal{L}(z'(\cdot))$ the set $M_0(\tau)$ defined by:

$$M_0(\tau) = f(\tau, z(\tau), U) - z'(\tau)$$

is a derived set at $z(\tau)$ to the reachable set $R_F(\tau, 0, X_0)$ and therefore the convex cone $C_0(\tau)$ defined by:

$$C_0(\tau) = cco(M_0(\tau))\ \tau \in \mathcal{L}(z'(\cdot)) \tag{6.8}$$

is a derived cone at $z(\tau)$ to $R_F(\tau, 0, X_0)$

Proof. We consider

$$\tau \in \mathcal{L}(z'(\cdot)),\ \{v_1, \dots v_m\} \subset M_0(\tau),\ \{u_1, \dots, u_m\} \subset U$$

such that

$$w_j = f(\tau, z(\tau), u_j),\ v_j = w_j - z'(\tau) \in M_0(\tau), j = 1, \dots m \tag{6.9}$$

and use the classical results in the theory of Ordinary Differential Equations for the Peano-Lipschitz vector fields $f_j(\cdot,\cdot) = f(\cdot,\cdot,u_j)$, $j = 1,\dots m$, to obtain the existence and uniqueness of the maximal flows $\widehat{x}_j(\cdot,\cdot,\cdot) : D_j \subset R \times R^n \to R^n$ which are locally-Lipschitz mappings and according to Lemma 5.1. have the property:

$$\widehat{x}_j(s,s,y) = y,\ \lim_{(\theta,s,y)\to(0,\tau,z(\tau))} \frac{\widehat{x}_j(s+\theta,s,y) - y}{\theta} = w_j\, j = 1,\dots m \tag{6.10}$$

Further on, we consider $\theta_0 > 0$ small enough, $S = [0,\theta_0]^m$ and for any $s = (s_1, \dots s_m) \in S$ we define:

$$|s| = -\sum_{i=1}^{m} s_i,\ t_1(s) = \tau - \sum_{i=1}^{m} s_i$$

$$\begin{gathered} t_j(s) = t_{j-1}(s) + s_j,\ t_m(s) = \tau - s_m,\ t_{m+1}(s) = \tau \\ a_0(s) = z(t_1(s)) \\ a_j(s) = \widehat{x}_j(t_{j+1}(s), t_j(s), a_{j-1}(s))\, j = 1,\dots m \\ a(s) = a_m(s) = \widehat{x}_m(\tau, t_m(s), a_{m-1}(s)) \end{gathered} \tag{6.11}$$

We note first that since $\widehat{x}_j$ are locally-Lipschitz the mappings $a_j(\cdot) : S \to R^n$, $j = 1,\dots m$ are continuous, $a_j(0) = z(\tau)$ and moreover $a_m(s) = a(s) \in R_F(\tau, 0, z(0)) \subset R_F(\tau, 0, X_0)$ since for any $s \in S$ may define the admisible pair $(u_s(\cdot), x_s(\cdot)$ as follows:

$$u_s(t) = \begin{cases} \tilde{u}(t) & \text{if } t \in [0, t_1(s)) \\ u_j & \text{if } t \in [t_j(s), t_{j+1}(s)) \end{cases}$$

$$x_s(t) = \begin{cases} z(t) & \text{if } t \in [0, t_1(s)) \\ \widehat{x}_j(t, t_j(s), a_{j-1}(s)) & \text{if } t \in [t_j(s), t_{j+1}(s)] \end{cases}$$

which obviously satisfies $a_m(s) = x_s(\tau)$.

We shall prove by induction that the mappings $a_j(\cdot)$ in (6.11) are differentiable at s=0, their derivatives being given by:

$$Da_j(0)s = \sum_{i=1}^{j} s_i w_i - |s|z'(\tau),\ \forall s \in S \tag{6.12}$$

We note first that since $z(\cdot)$ is differentiable at τ the mapping $a_0(\cdot)$ in (6.11) is differentiable at s=0, its derivatives being given by:

$$Da_0(s) = -|s|z'(\tau) \tag{6.13}$$

To prove (6.12) for j=1, we note that using (6.10) and (6.13) we may write succesively:

$$\lim_{s\to 0} \frac{a_1(s) - a_1(0) + (\sum_{i=1}^{m} s_i)z'(\tau)}{|s|} =$$

$$= \lim_{s\to 0}\Big[\frac{\hat{x}_1(t_1(s) + s_1, t_1(s), a_0(s)) - a_0(s) - s_1 w_1}{s_1}\frac{s_1}{|s|} + \frac{a_0(s) - a_0(0) + |s|z'(\tau)}{|s|}\Big] = 0$$

since $\frac{s_1}{|s|} \in (0,1]\ \forall s \neq 0$

Assuming that (6.12) is verified for $j = 1, \ldots m-1$ we use again (6.10) and (6.13) to obtain:

$$\lim_{s\to 0} \frac{a_{j+1}(s) - a_{j+1}(0) - \sum_{i=1}^{j+1} s_i w_i + |s|z'(\tau)}{|s|} =$$

$$= \lim_{s\to 0}\Big[\frac{\hat{x}_{j+1}(t_{j+1}(s) + s_{j+2}, t_{j+1}(s), a_j(s)) - a_j(s) - s_{j+1}w_{j+1}}{s_{j+2}}\frac{s_{j+2}}{|s|} +$$

$$+ \frac{a_j(s) - a_j(0) - \sum_{i=1}^{j} s_i w_i + |s|z'(\tau)}{|s|}\Big] = 0$$

Since for j=m the relation (6.12) may be written as $Da(0)s = \sum_{i=1}^{m} s_i v_i$, Lemma 6.3. is completely proved.

Remark 6.4. If, in addition to Hypothesis 6.2., the parametrized multifunction F(.,.) in (6.4) is L(t)-locally Lipschitz on $B(z(t), \epsilon)$ (in particular if f(t,.,u) is L(t)-locally Lipschitz for any u in U) then Theorem 4.3. remains valid using the derived cones $C_0(\tau)$ in (6.8) instead of those in (4.4) that satisfy only the weaker condition (4.6).

However, the lipschitzianity property in Hypothesis 3.5. being quite restrictive (particulary for orientor fields with unbounded values) alternative assumptions avoiding this condition may significantly extend the applicability area of this type of results.

In view of the fact that for any $r > 0$ the quasitangent derivatives at (z,z') of the multifunction $F(t,\cdot)$ and $F_r(t,x) = F(t,x) \cap B(z',r)$ coincide, Hypothesis 3.5. may be replaced by the following weaker one:

Hypothesis 6.5. *The mapping z(.) is a solution of (2.1) and there exist* $\epsilon > 0, L(\cdot) \in L^1(I, R_+), r(t) > 0 \, \forall t \in I$ *such that the truncated multifunction:*

$$F_r(t,x) = F(t,x) \cap B(z'(t), r(t)) \, t \in I \; x \in B(z(t), \epsilon) \tag{6.14}$$

has the following property: $F(t,\cdot)$ *is L(t)-Lipschitz on* $B(z(t),\epsilon)$ *a.e. in I.*

An alternative assumption to Hypothesis 6.5 that may also allow a much easier proof (that may avoid the use of the imbedding Theorem 3.6) is the following:

Hypothesis 6.6. *The mapping* $f(\cdot,\cdot,\cdot) : D \times U \to R^n$ *satisfies Hypothesis 6.2., z(.) is a solution of (6.2), there exists* $\epsilon > 0$ *such that* $f(t,\cdot,\tilde{u}(t))$ *is of class* C^1 *on* $B(z(t),\epsilon)$ *a.e. in I and there exists integrable mappings* $m(\cdot), M(\cdot) \in L^1(I, R_+)$ *such that:*

$$|\tilde{f}(t,x)| \leq m(t), \, |\frac{\partial \tilde{f}}{\partial x}(t,x)| \leq M(t) \, \forall x \in B(z(t),\epsilon) \, a.e. \, in \, I \tag{6.15}$$

We note that, as in the case of Hypothesis 6.2, the multifunction F(.,.) in (6.4) need not be locally Lipschitz as in Hypothesis 3.5.

We recall that according to well known results in the general theory of O.D. E. under Hypothesis 6.6., the vector field $\tilde{f}(\cdot,\cdot)$ in (6.2.) has a continuous maximal flow $\widehat{x}(\cdot,\cdot,\cdot) : \widehat{D} \subset R \times R^n \times R^n \to R^n$ which is differentiable with respect to the last variable, and for any $t_0 \in (0,T), x_0 \in B(z(t),\epsilon)$ the mapping $t \to D_3\widehat{x}(t,t_0,x_0)$ is the unique absolutely continuous matrix-valued solution of the variational equation

$$\frac{dZ}{dt} = \frac{\partial \tilde{f}}{\partial x}(t, \widehat{x}(t,t_0,x_0))Z, \; Z(t_0) = I_n \tag{6.16}$$

On the other hand according to the Scorza-Dragoni property, there exists a subset of full measure $J \subset I$ such that any solution y of the differential equation in (6.1) is differentiable at any $t \in J$ and satisfies the relation:

$$y'(t) = f(t, y(t)) \, \forall t \in J \tag{6.17}$$

Moreover, according to Proposition 3.1 in Mirica [13], the mapping $\widehat{x}(t,\cdot,y)$ is differentiable at $s \in J$, its derivative being given by:

$$D_2\widehat{x}(t,s,y) = -D_3\widehat{x}(t,s,y)\widetilde{f}(s,y) \tag{6.18}$$

In order to unify the notations we note that the family of linear mappings $A(t,\cdot), t \in J$, defined by:

$$A(t,v) = \frac{\partial \widetilde{f}}{\partial x}(t,z(t))v,\ v \in R^n \tag{6.19}$$

is a family of closed convex processes satisfying condition (5.1) for the multifunction F(.,.) in (6.4)

Moreover, according to the above mentioned diferentiability theorem, the reachable set of the inclusion (5.2) may be expressed as follows:

$$R_A(t_2,t_1,C_1) = D_3\widehat{x}(t_2,t_1,z(t_1))C_1\ 0 \le t_1 < t_2 \le T \tag{6.20}$$

In this case the analogues of Lemmas 5.3 and 5.6 allow very simple proofs based on the differentiability of the flow $\widehat{x}(.,.,.)$.

Lemma 6.7. *If Hypothesis 6.6. is verified, $0 \le t_1 < t_2 \le T$ and C_1 is a derived cone to $R_F(t_1,0,X_0)$ at $z(t_1)$ then $R_A(t_2,t_1,C_1)$ in (6.20) is a derived cone to $R_F(t_2,0,X_0)$ at $z(t_2)$.*

Proof We consider

$$\{v_1,\dots v_m\} \subset R_A(t_2,t_1,C_1)\ \{w_1\dots w_m\} \subset C_1$$

such that:

$$v_j = D_3\widehat{x}(t_2,t_1,z(t_1))w_j,\ j = 1,\dots m \tag{6.21}$$

Since $C_1 \subset R^n$ is a derived cone to $R_F(t_1,0,X_0)$ there exists a continuous mapping $a_0(\cdot) : [0,\theta_0]^m \to R_F(t_1,0,X_0)$ such that:

$$a_0(0) = z(t_1),\ Da_0(0)s = \sum_{i=1}^{m} s_i w_i$$

We define $a(\cdot) : S \to R_F(t_2,0,X_0)$ as follows:

$$a(s) = \widehat{x}(t_2,t_1,a_0(s)),\ s \in S \tag{6.22}$$

and note that from the properties of the flow $\widehat{x}(.,.,.)$, a(.) is continuous, $a(0) = z(t_2)$ and

$$Da(0)\sigma = D_3\widehat{x}(t_2, t_1, z(t_1))Da_0(0)\sigma = D_3\widehat{x}(t_2, t_1, z(t_1))(\sum_{i=1}^{m}\sigma_i w_i) = \sum_{i=1}^{m}\sigma_i v_i$$

and Lemma 6.7 is proved.

Lemma 6.8. *Let z(.) and f(.,.,.) satisfy Hypothesis 6.6., let* $J \subset (0, T)$ *the subset of full measure for which (6.17) is satisfied, let* $0 \leq t_1 < t_2 \leq T$ *such that* $t_2 \in J$*, let* $C_1 \subset R^n$ *be a derived cone of* $R_F(t_1, 0, X_0)$ *at* $z(t_1)$*, and let* $C_0(t_2)$ *be the derived cone in (6.8) and let* $A(t, \cdot)$ *be defined in (6.19).*

Then the set $M_2 \subset R^n$ *defined by:*

$$M_2 = C_0(t_2) \cup R_A(t_2, t_1, C_1) \tag{6.23}$$

is a derived set of $R_F(t_2, 0, X_0)$ *at* $z(t_2)$.

Proof. We consider

$$\{v_1, \dots v_m\} \subset R_A(t_2, t_1, C_1)\ \{v_{l+1}, \dots v_m\} \subset C_0(t_2),\ u_j \in U,\ w_j \in C_1$$

such that:

$$v_j = D_3\widehat{x}(t_2, t_1, C_1)w_j,\ j = 1, \dots l$$

$$v_j = w_j - z'(t_2),\ w_j = f(t_2, z(t_2), u_j),\ j = l+1, \dots m \tag{6.24}$$

Since C_1 is a derived cone to $R_F(t_1, 0, X_0)$ at $z(t_1)$ there exists $a_0(\cdot) : [0, \theta_0]^l \to R_F(t_1, 0, X_0)$ continuous and such that:

$$a_0(0) = z(t_1),\ Da_0(0)\sigma = \sum_{i=1}^{l}\sigma_i w_i\ \forall\sigma = (\sigma_1, \dots \sigma_l) \in R^l_+$$

We take $\theta_0 > 0$ sufficiently small, $S = [0, \theta_0]^m$ and for any $s \in S$ we define:

$$|s| = \sum_{i=1}^{m} s_i\ t_{l+1}(s) = t_2 - \sum_{j=l+1}^{m} s_j, t_j(s) = t_{j-1}(s) + s_j,\ t_{m+1}(s) = t_2$$

$$a_l(s) = \widehat{x}(t_{l+1}(s), t_1, a_0(s_1, \dots s_l)) \tag{6.25}$$

$$a_j(s) = \widehat{x}_j(t_{j+1}(s), t_j(s), a_{j-1}(s))\ j = l+1 \dots m$$

$$a(s) = a_m(s) = \widehat{x}(t_2, t_2 - s_m, a_{m-1}(s))$$

where $\widehat{x}(.,.,.)$ is the maximal flow of the vector field $\widetilde{f}(.,.)$ in (6.3) and $\widetilde{x}_j(.,.,.)$ is the maximal flow of $f(\cdot,\cdot,u_j)$, $j = 1,\dots l$

As in the proof of Lemma 6.3 and of Lemma 6.6 it follows that the mapping $a_j(.)$, $j = l,\dots m$ are continuous, $a(s) \in R_F(t_2, 0, X_0), a(0) = z(t_2)$ and a standard computation shows that a(.) is differentiable at 0, its derivative being given by $Da(0)\sigma = \sum_{i=1}^m \sigma_i v_i$.

Using now the cones $C_0(\tau)$ in (6.8) and $R_A(\tau, t, C_0(t))$ in Lemma 6.7, Proposition 5.8 may be formulated as follows:

Theorem 6.9. *If z(.) and f(.,.,.) satisfy Hypothesis 6.2., $J \subset I$ is the set for which (6.17) is satisfied, $C_0 \subset R^n$ is a derived cone of X_0 at z(0), and $\widehat{x}(.,.,.)$ is the maximal flow of the vector field $\widetilde{f}(.,.)$ in (6.3) then for any $\tau \in J$ the convex cone $C(\tau)$ defined by:*

$$C(\tau) = cco \bigcup_{t \in J \cap [0,\tau]} R_A(\tau, t, C_0(t)) =$$

$$cco[\{D_3\widehat{x}(\tau, t, z(t))(f(t, z(t), U) - \widetilde{f}(t, z(t))), t \in J \cap (0, \tau]\} \cup D_3\widehat{x}(\tau, 0, z(0))C_0]$$

is a derived cone to $R_F(\tau, 0, X_0)$ at $z(\tau)$.

Remark 6.10. We note that due to the fact that $A(t,\cdot)$ in (6.19) are linear mappings the cones C(t) in (6.26) satisfy (5.29) and, in fact, coincide with the cones $\overline{C}(t)$ in (5.27).

Theorem 6.9 was proved in a more direct way in [13] Lemmas 3.1,3.2; Lemmas 5.3, 5.7,5.8 may be interpreted as giving another proof of this theorem which, in turn is technically related to Lemmas 7.4.(ii)-(iv) in Cesari ([5]), where certain "cones of variations" are identified.

We note that if in addition to Hypothesis 6.2 the parametrized orientor field F(.,.) in (6.4) verifies also Hypothesis 3.5 then one may use Theorem 5.8 to obtain derived cones generated by the variational inclusion (5.2) with the following choice for the family $A(t,\cdot)$ of closed convex processes:

$$A(t,v) = \frac{\partial f}{\partial x}(t, z(t), \widetilde{u}(t)) + AQ^+_{z'(t)} f(t, z(t), U),\ v \in R^n\ t \in J \qquad (6.27)$$

Particularly interesting are the cases in which one may find explicit description of the assymptotic quasitangent cones

$AQ^+_{z'(t)} f(t, z(t), U)$; such a case is that of the convex-valued multifunction F for which one has([20], etc):

$$AQ^+_{z'} F(t,z) = \overline{cco}(F(t,z) - z'),\ \forall z' \in F(t,z) \qquad (6.28)$$

In the case only Hypothesis 6.2 is satisfied we may choose for any $t \in I$ a closed convex cone

$$\widehat{A}(t) \subset Q^{+}_{(z(t),z'(t))} graph \tilde{f}(t,\cdot) \tag{6.29}$$

that is maximal with respect to set-inclusion among the closed convex cones with this property and note that the corresponding family of closed convex processes defined by

$$A(t,v) = \{v' \in R^n;\ (v,v') \in \widehat{A}(t)\}\ v \in domA(t,\cdot) = pr_1 \widehat{A}(t)\ t \in I \tag{6.30}$$

satisfy condition (5.1); consequently, the results in Section 5 remain valid provided the parametrized vector field f(.,.,.) in (6.3)-(6.4) satisfies the following:

Hypothesis 6.11. *The sets $D \subset R^n$, U and the mappings $f(.,.,.)$,$z(.)$, $\tilde{u}$ satisfy Hypothesis 6.2 and, in addition, one of the following properties hold:*

(i) There exist $\epsilon, r > 0$ and $L(\cdot) \in L^1(I, R_+)$ such that the truncated multifunction $F_r(t,\cdot)$, defined by:

$$F_r(t,x) = f(t,x,U) \cap B(z'(t),r)\ t \in I,\ x \in B(z(t),\epsilon) \tag{6.31}$$

is $L(t)$-Lipschitzean on $B(z(t),\epsilon)$,

(ii) There exist $\epsilon > 0, L(\cdot) \in L^1(I,R_+)$ and $U_0 \subset U$ such that the multifunction $\widetilde{F}_0(t,\cdot)$ defined by:

$$\widetilde{F}_0(t,x) = f(t,x,U_0),\ x \in B(z(t),\epsilon),\ t \in I \tag{6.32}$$

is $L(t)$-Lipschitzean on $B(z(t),\epsilon)$ and $\tilde{u}(t) \in U_0$ for all t in I.

However, as we shall see in what follows, the result in Section 5 for the standard control system (2.3) may be obtained without the additional Hypothesis 5.11 with direct proofs that avoid the use of the imbedding Theorem 3.6.

We note first that if $g(\cdot) : X \subset R^n \to R^m$ is locally Lipschitz then at any point $x \in X$ the qusitangent set-valued directional derivative $Q^+g(x,\cdot)$ in (2.9).(2.13) is single-valued and globally Lipschitz on its domain, being given by:

$$Q^+g(x,v) = \{g^+_Q(x,v)\}\ g^+_Q(x,v) = \lim_{(\theta,u)\to(0+,v),x+\theta u \in X} \frac{g(x+\theta u) - g(x)}{\theta}$$

$$v \in dom(g_Q^+ g(x,\cdot)) = pr_1 Q^+_{(x,g(x))} graph\, g(\cdot) \tag{6.33}$$

We note that if $v \in R^n$ is contained in the cone of "feasible direction"

$$F_x^+ X = \{v \in R^n : \exists \theta_0 > 0 : x + \theta v \in X\ \forall \theta \in (0, \theta_0)\}$$

then the quasitangent derivative in (6.33), if it exists, is given by:

$$g_Q^+(x,v) = \lim_{\theta \to 0+} \frac{g(x+\theta v) - g(x)}{\theta}\ v \in F_x^+ X \cap dom g_Q^+(x,\cdot) \tag{6.34}$$

In particular, if g(.) is locally-Lipschitz at $x \in Int(X)$ then (6.34) holds for any $v \in dom g_Q^+(x,\cdot)$. Moreover, for any convex subcone, $\widehat{A}_x \subset Q^+_{(x,g(x))} graph\, g(\cdot)$ (in particular for $\widehat{A}_x = AQ^{+\cdot}_{(x,g(x)} graph\, g(\cdot))$ the restriction to $A_x = pr_1 \widehat{A}_x$ of $g_Q^+(x,\cdot)$ is positively linear in the sense that it satisfies:

$$g_Q^+(x, s_1 v_1 + s_2 v_2) = s_1 g_Q^+(x, v_1) + s_2 g_Q^+(x, v_2)\ \forall v_1, v_2 \in A_x\ s_1, s_2 \geq 0 \tag{6.35}$$

On the other hand, as it is well known, the quasitangent derivative in (6.33)-(6.34) is a proper generalization of the classical derivative in the sense that g is differentiable at $x \in Int(X)$ iff $dom g_Q^+(x,\cdot) = R^n$ and the mapping $g_Q^+(x,\cdot) : R^n \to R^m$ is linear; in this case the two derivatives coincide, i.e. one has:

$$g_Q^+(x,v) = Dg(x)v,\ \lim_{h \to 0} \frac{g(x+h) - g(x) - g_Q^+(x,v)}{|h|} = 0,\ v \in R^n \tag{6.36}$$

The last property may be extended to a type of "conical differentiability" property as follows:

Proposition 6.12. *Let $g(\cdot) : X \subset R^n \to R^m$ be locally- Lipschitz at $x \in Int(X)$ and let $\{v_1, \dots v_m\} \subset dom(g_Q^+)(x,\cdot)$ be such that:*

$$cco\{(v_i, g_Q^+(x, v_i));\ i = 1, \dots m\} \subset Q^+_{(x,g(x))} graph\, g(\cdot) \tag{6.37}$$

Then for any mapping $a_0(\cdot) : S = [0, s_0]^m \to X$ that is differentiable at $s = (0, \dots, 0) \in S$ and has the properties:

$$a_0(0) = x,\ Da_0(0)s = \sum_{i=1}^{m} s_i v_i\ \forall s \in S \tag{6.38}$$

the mapping $a(\cdot) : S \to R^m$ *defined by:*

$$a(s) = g(a_0(s)),\ s \in S \tag{6.39}$$

is differentiable at $s = (0, \ldots 0)$ *and its derivative is given by:*

$$Da(0)s = \sum_{i=1}^{m} s_i g_Q^+(x, v_i)\ \forall s \in S \tag{6.40}$$

Proof. For any $s = (s_1, \ldots, s_m) \in S$ we denote:

$$o_0(s) = a_0(s) - x - \sum_{i=1}^{m} s_i v_i,\ |s| = \sum_{i=1}^{m} s_i$$

$$o(s) = a(s) - a(0) - \sum_{i=1}^{m} s_i g_Q^+(x, v_i) \tag{6.41}$$

$$o_1(s) = g(x + \sum_{i=1}^{m} s_i v_i) - g(x) - \sum_{i=1}^{m} s_i g_Q^+(x, v_i)$$

and note that according to (6.38) we have: $\lim_{s \to 0} \frac{o_0(s)}{|s|} = 0$

On the other hand, from the lipschitzianity of g(.) at x we infer that:

$$|o(s) - o_1(s)| = |g(a_0(s)) - g(x + \sum_{i=1}^{m} s_i v_i)| \leq L|o_0(s)|$$

hence for (6.40) it is enough to prove:

$$\lim_{s \to 0+} \frac{o_1(s)}{|s|} = 0 \tag{6.42}$$

Further on, for any

$$\sigma = (\sigma_1, \ldots, \sigma_m) \in S_+^{m-1} = \{(\sigma_1, \ldots, \sigma_m) \in R_+^m : |\sigma| = 1\}$$

we denote:

$$\phi(\theta, \sigma) = \frac{|g(x + \theta \sum_{i=1}^{m} \sigma_i v_i) - g(x) - \theta \sum_{i=1}^{m} \sigma_i g_Q^+(x, v_i)|}{\theta} \tag{6.43}$$

and note that from the definition in (6.41) of $o_1(\cdot)$ it follows:

$$\frac{|o_1(s)|}{|s|} = \phi(|s|, \frac{s}{|s|}) \, \forall s \in S \backslash \{0\} \tag{6.44}$$

On the other hand, from (6.34) and (6.35) it follows that:

$$\lim_{\theta \to 0+} \phi(\theta, \sigma) = 0 \, \forall \sigma \in S_+^{m-1}$$

and therefore from Proposition 5.4 it follows that:

$$\lim_{s \to 0} \frac{|o_1(s)|}{|s|} = \lim_{s \to 0} \phi(|s|, \frac{s}{|s|}) = 0$$

and (6.40) is proved.

The above statements hold, in particular, for the Caratheodory-Lipschitz vector-field, $\tilde{f}(\cdot,\cdot)$ in (6.3) and its corresponding maximal flow, $\hat{x}(\cdot,\cdot,\cdot)$.

The next auxiliary result, allowing the extension of Lemma 6.7 and Theorem 6.9 to the nonsmooth control system satisfying Hypothesis 6.2, may be interpreted as refinements of the existing generalizations of the classical Bendixson-Picard theorem on differentiability of solutions with respect to initial data (e.g. Mirica [12]).

Theorem 6.13. *Let $\tilde{f}(\cdot,\cdot)$ be the Caratheodory- Lipschitz vector field in (6.3) satisfying (6.3)-(6.4), let $\hat{x}(\cdot,\cdot,\cdot)$ its maximal flow and let $z(t) = \hat{x}(t, 0, x_0)$, $t \in I$ be a reference trajectory.*

(i) An absolutely continuous mapping $w(\cdot) : [t_1, t_2] \subset [0, T] \to R^n$ is a solution of the variational equation:

$$w' = \tilde{f}_Q^+(t,\cdot)(z(t), w) \tag{6.45}$$

if and only if:

$$w(t) \in dom \tilde{f}_Q^+(t,\cdot)(z(t),\cdot) \tag{6.46}$$

$$w(t_1) \in dom \hat{x}_Q^+(t, t_1, \cdot)(z(t_1,\cdot) \, \forall t \; in[t_1, t_2] \tag{6.47}$$

and in this case one has:

$$w(t) = \hat{x}_Q^+(t, t_1, \cdot)(z(t_1), ;, w(t_1)) \tag{6.48}$$

(ii) Let $w_i(\cdot) : [t_1, t_2] \to R^n$, $i = 1, \dots m$, be solutions of the variational equations (6.45) such that for any $c_1, \dots, c_m \geq 0$ the mapping $w(\cdot, c)$ defined by:

$$w(t,c) = \sum_{i=1}^{m} c_i w_i, \; c = (c_1, \dots, c_m) \in R_+^m, \; t \in [t_1, t_2] \tag{6.49}$$

is also a solution of (6.45)

Further on, let $s_0 > 0$, $S = [0, s_0]^m$ and let $a_0 : S \to R^n$ be a mapping that is differentiable at $s = (0, \dots, 0) \in S$ and satisfies the conditions:

$$a_0(0) = z(t_1), \; Da_0(0)s = \sum_{i=1}^{m} s_i w_i(t) \; \forall s \in S \tag{6.50}$$

Then for any $t \in [t_1, t_2]$ the mapping $a_t(\cdot) : S \to R^n$ defined by:

$$a_t(s) = \widehat{x}(t, t_1, a_0(s)), \; s \in S \tag{6.51}$$

is differentiable at $s = (0, \dots, 0)$ and its derivative is given by:

$$Da_t(0)s = w(t,s) \; \forall s \in S \tag{6.52}$$

Proof. We note first that from Hypothesis 6.2 it follows that there exist $\epsilon > 0$ and $m(\cdot), L(\cdot) \in L^1(I, R_+)$ such that (6.3)-(6.4) hold for any $x, y \in B(z(t), \epsilon)$ *a.e.in* I; moreover, from the integral equation:

$$\widehat{x}(t, t_1, x_1) = x_1 + \int_{t_1}^{t} \widetilde{f}(\sigma, \widehat{x}(\sigma, t_1, x_1)) d\sigma \tag{6.53}$$

and using Gronwall's Lemma it follows that if $\widehat{L} = \exp|\int_0^T L(t)dt|$ then $\widehat{x}(t, t_1, \cdot)$ is $\widehat{L}$-Lipschitz on $B(z(t_1), \epsilon_1)$ where $\epsilon_1 = \epsilon \frac{1}{\widehat{L}}$

(i) We assume that $w(\cdot) : [t_1, t_2] \to R^n$ is an absolutely continuous solution of (6.45) and note that, in this case, (6.46) is automatically satisfied ; we shall prove now that (6.48) which implies (6.47).

For $t \in [t_1, t_2]$, $\theta \in (0, \theta_0)$ we denote:

$$\xi(t, \theta) = \frac{\widetilde{x}(t, t_1, z(t_1) + \theta w(t_1)) - z(t) - \theta w(t)}{\theta}$$

$$\phi(t, \theta) = \frac{\widetilde{f}(t, z(t) + \theta w(t_1)) - \widetilde{f}(t, z(t)) - \theta w'(t)}{\theta}, \; \alpha(t, \theta) = \int_{t_1}^{t} |\phi(u, \theta)| du \tag{6.54}$$

$$\psi(t,\theta) = \frac{\tilde{f}(t,\hat{x}(t,t_1,z(t_1)+\theta w(t_1)) - \tilde{f}(t,z(t)+\theta w(t))}{\theta}$$

and note that from (6.4) and (6.53) it follows that:

$$|\psi(t,\theta)| \leq L(t)|\xi(t,\theta)|$$

$$\xi(t,\theta) = \int_{t_1}^{t} \phi(\sigma,\theta)d\sigma + \int_{t_1}^{t_2} \psi(\sigma,\theta)d\sigma$$

and therefore, the functions $|\xi(\cdot,\theta)|$ satisfy the inequality:

$$|\xi(t,\theta)| \leq \alpha(t,\theta) + \int_{t_1}^{t_2} L(u)|\xi(u,\theta)|du$$

From Gronwall's Lemma (e.g. Aubin-Cellina [1], etc) it follows:

$$|\xi(t,\theta)| \leq \alpha(t,\theta)\int_{t_1}^{t} \alpha(u,\theta)\exp(\int_{u}^{t} L(r)dr)du\, \forall\theta \in (0,\theta_0),\ t \in [t_1,t_2] \quad (6.55)$$

On the other hand, from (6.34) and (6.45) it follows that $\lim_{\theta\to 0+}\phi(t,\theta) = 0$ on $[t_1,t_2]$ and since

$$|\phi(t,\theta)| \leq L(t)|w(t)| + |w'(t)|\ on\ [t_1,t_2]$$

from Lebesgue dominated convergence theorem it follows that

$$\lim_{\theta\to 0+}\alpha(t,\theta) = \lim_{\theta\to 0+}\int_{t_1}^{t}|\phi(u,\theta)|du = 0\ \forall t \in [t_1,t_2]$$

therefore, from (6.55) it follows that $\lim_{\theta\to 0+}\xi(t,\theta) = 0$, $\forall t \in [t_1,t_2]$ which in view of (6.54) is equivalent with (6.48).

Conversely, let w(.,.) be given by (6.48) and satisfying (6.46). Since w(.,.) is obviously measurable and satisfies the inequality

$$|w(t)| \leq \hat{L}|w(t_1)|\ t \in [t_1,t_2]$$

it is integrable.

Using the definition in (6.34) of the quasitangent derivative, the integral equation in (6.53) and the notations in (6.54) we obtain:

$$w(t) = w(t_1) + \lim_{\theta\to 0+}[\int_{t_1}^{t}\psi(u,\theta)du + \int_{t_1}^{t}\frac{\tilde{f}(u,z(u)+\theta w(u)) - \tilde{f}(u,z(u))}{\theta}du] \quad (6.56)$$

On the other hand, since according to (6.48)

$$w(t) = \lim_{\theta \to 0+} \frac{\widehat{x}(t, t_1, z(t_1) + \theta w(t_1)) - z(t)}{\theta} \; \forall t \in [t_1, t_2]$$

from (6.54) we obtain:

$$\lim_{\theta \to 0+} |\psi(t, \theta)| \leq L(t) \lim_{\theta \to 0+} \left|\frac{\widehat{x}(t, t_1, z(t_1) + \theta w(t_1)) - z(t) - \theta w(t)}{\theta}\right| = 0$$

and since $|\psi(\cdot, \theta)|$ is integrably-bounded, we get:

$$\lim_{\theta \to 0+} \int_{t_1}^{t} \psi(u, \theta) du = 0 \; \forall t \in [t_1, t_2]$$

Further on, from (6.46) it follows that:

$$\lim_{\theta \to 0+} \frac{\widetilde{f}(\sigma, z(\sigma) + \theta w(\sigma)) - \widetilde{f}(\sigma, z(\sigma))}{\theta} = \widetilde{f}_Q^{+}(\sigma, \cdot)(z(\sigma); w(\sigma)) \; a.e.on[t_1, t_2]$$

Hence using again the Lipschitzianity of $\widetilde{f}(\sigma, \cdot)$ from (6.56) we obtain:

$$w(t) = w(t_1) + \int_{t_1}^{t} \widetilde{f}_Q^{+}(s, \cdot)(z(s); w(s)) ds \; \forall t \in [t_1, t_2]$$

and the fact that w(.,.) is a solution of (6.45) is proved.

(ii) This statement is an immediate consequence of statement (i) and of Proposition 6.12 applied to the locally Lipschitz mappings $\widehat{x}(t, t_1, \cdot)$, $t \in [t_1, t_2]$: if $w_1(\cdot), \ldots, w_m(\cdot)$ are solutions of (6.45) such that for any $c = (c_1, \ldots c_m) \in R_+^m$ $w(\cdot, c) = \sum_{i=1}^m c_i w_i(\cdot)$ is also a solution of (6.45), then from (6.48) it follows that:

$$\sum_{i=1}^{m} c_i \widehat{x}(t, t_1, \cdot)(z(t_1, w_i(t_1)) = \widehat{x}_Q^{+}(t, t_1, \cdot)(z(t_1), \sum_{i=1}^{m} c_i w_i(t_1))$$

hence $\widehat{x}(t, t_1, \cdot)$ satisfies hypothesis (6.37) in Proposition 6.12 and (6.52) follows from (6.40).

Theorem 6.13 allows the straigtforward extensions of Lemmas 6.7, 6.8 and Theorem 6.9 to nonsmooth control systems for which the classical variational equation (6.16) is replaced by a quasitangent variational equation of the form:

$$v' \in A(t, v), \; A(t, v) = \widetilde{f}_Q^{+}(t, \cdot)(z(t), v), \; v \in pr_1 \widehat{A}(t) \qquad (6.57)$$

defined by an arbitrary family of convex cones $\widehat{A}(t) \subset R^n \times R^n$, $t \in J$, that satisfy the condition:

$$\widehat{A}(t) \subset Q^+_{(z(t),z'(t))} graph \widetilde{f}(t,\cdot),\ t \in J \tag{6.58}$$

In fact the family $\widehat{A}(t)$ of convex cones satisfying (6.58) define a family of single-valued convex processes $A(t,\cdot)$ satisfying (5.1).

Practically the same proofs as those of Lemmas 6.7, 6.8 and Proposition 5.8 lead to the following result extending Theorem 6.9:

Theorem 6.14. *Let $z(\cdot)$ and $f(.,.,.)$ satisfy Hypothesis 6.2 ,let $C_0 \subset R^n$ be a derived cone to X_0 at z(0) and for any $t \in [0,T)$ let $C_0(t) \subset R^n$ be the cone defined in (6.8) and (3.10).*

Then for any family of convex cones, $\widehat{A}(t)$, $t \in J$, satisfying (6.58) and for any $\tau \in (0,T]$, the convex cone $C_A(\tau)$ defined by:

$$C_A(\tau) = cco \bigcup_{t \in J \cap [0,\tau]} R_A(\tau,t,C_0(t)) = cco \bigcup_{t \in [0,\tau]} \widetilde{x}^+_Q(\tau,t,\cdot)(z(t),C_0(t)) \tag{6.59}$$

is a derived cone to $R_F(\tau,0,X_0)$ at $z(\tau)$ and moreover, these cones satisfy the conditions:

$$R_A(\tau,s,C(s)) \subset C(\tau)\ \forall 0 \le s < \tau \le T \tag{6.60}$$

$$C_0(t) = cco(f(t,z(t),U) - z'(t)) \subset C(t)\ a.e.\ on\ [0,T] \tag{6.61}$$

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