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by

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An explicit study of l-sequences

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Introduction

From the cohomological viewpoint, maximal Cohen–Macaulay modules over a local ring have many similarities with finitely generated modules over an Artinian ring. Therefore, the representation theory of maximal Cohen–Macaulay modules has been developed by pursuing a fruitful analogy with the representation theory of modules of finite type over an Artinian ring. Besides giving motivation, hints and inspiration, the work done in the artinian case can be useful in a more direct manner. Specifically, one may try to reduce the questions about maximal Cohen–Macaulay modules to similar problems stated for modules over suitable Artinian rings. This simply conceived strategy cannot be carried out in a straightforward way. Such an approach is feasible as soon as one has a vehicle to descend the significant properties of the objects we are interested in. Moreover, the transfer technique must allow to recover in the original framework the information about the transformed objects.

By some work done by E.Dieterich [3] , Y.Yoshino [6] , D. Popescu [5] we know a large class of Cohen–Macaulay isolated singularities (R, M, K) for which there exists an M–primary ideal I such that the assignment $L \mapsto L/IL$ defines an embedding of the set of isomorphism classes of indecomposable maximal Cohen–Macaulay R–modules into the set of isomorphism classes of indecomposable finitely generated R/I–modules. This embedding could be more appropiately mastered if one could describe its image in more concrete terms. Recently Popescu [4] found out an answer to this problem in the case I is generated by a system of parameters for the given ring.

Theorem. ([4]) *Let I be an ideal generated by a system of parameters $x_1, \ldots, x_n$ of an excellent,Henselian Cohen-Macaulay local ring (R, M, K) and let N be a finitely generated R/IR - module . Suppose that R is an isolated singularity containing a field and $[K : K^p] < \infty$, where p:= charK . Then there exists a maximal Cohen-Macaulay R - module L such that $L/IL \simeq N$ if and only if there exists an $R/(x_1^2, \ldots, x_n^2)R$ - module E such that $E/IE \simeq N$ and*

$$(*) \qquad ((x_1, \ldots, x_t)E : x_{t+1})_E = (x_1, \ldots, x_{t+1})E \quad \textit{for all } t = 0, \ldots, n-1.$$

To characterize the R/I–modules of finite type which have the form L/IL for a certain maximal Cohen–Macaulay R–module L amounts to find out the R/I–modules which are liftable to R in the sense of M.Auslander, S.Ding and Ø.Solberg [1] . This remark explains the name dubbed for elements fulfilling condition (*) - *lifting sequences* shortly l–sequences.

It is natural to study l–sequences not only in their original context but also in much more general circumstancies than would be required by the representation theory. We believe that such an explicit study of l–sequences illuminates their nature, significance and limits. This study appears to us to be justified also by the expectation that l–sequences would have applications beyond the situation that motivated to consider condition (*) .

The aim of this paper is to report some of the consequences that condition (*) has. Most results are valid in a quite general framework–arbitrary modules over a commutative ring. *A posteriori*, it turns out that we do not gain any simplicity by commiting ourselves to local or Noetherian rings.

In order to point out some of the results proved in the sequel, we shall introduce now some notations. For some elements $\mathbf{x} := x_1, \ldots, x_n$ of a commutative ring R with unity we shall denote by I the ideal $\mathbf{x}R$. For any positive integer t , $t \leq n$, I_t will denote the ideal generated by $x_1, \ldots, x_t$. For the sake of uniformity, we denote by I_0 the null ideal.

The first section is entirely devoted to clarify the definition of l–sequences. On the one hand, we obtain several conditions that are equivalent to (*). On the other hand, there are given some examples showing that these characterizations can not be improved. The main result of this section is stated below:

Theorem 1.11 *The following statements are equivalent* :
(i) $\mathbf{x}$ *is an* l*-sequence on* E
(ii) $(0 : x_1 \cdots x_t)_E = I_t E$ *for all* $t, 1 \leq t \leq n$
(iii) $x_1 \cdots x_{t-1} E \cap (0 : x_t)_E = x_1 \cdots x_t E$ *for all* $t, 1 \leq t \leq n$
(iv) $(0 : I_t)_E = x_1 \cdots x_t E$ *for all* $t, 1 \leq t \leq n$.

In the second section we point out some properties of l–sequences. Our main concern here is to answer the question: if given elements form l–sequence on a module, under what circumstancies do they form l–sequence on other modules appearing in a short exact sequence along with the given module ? The statements of the principal results refer to a short exact sequence of modules

$$0 \longrightarrow E' \xrightarrow{u} E \xrightarrow{v} E'' \longrightarrow 0.$$

Theorem 2.5 *Suppose* $\mathbf{x}$ *is* l*–sequence on* E'' . *Then:*
1) *If* $\mathbf{x}$ *is* l*–sequence on* E, *then* $\mathbf{x}$ *is* l*–sequence on* E' .
2) *If* $\mathbf{x}$ *is* l*–sequence on* E' *and also on* R, *then* $\mathbf{x}$ *is* l*–sequence on* E .

Theorem 2.8 *If* $\mathbf{x}$ *is* l*–sequence on* E, *then the following statements are equivalent:*
(i) $\mathbf{x}$ *is* l*–sequence on* E'
(ii) $\mathbf{x}$ *is* l*–sequence on* E''
(iii) $I_t E \cap u(E') = I_t u(E')$ *for all* $t, 1 \leq t \leq n$.

Apart from their intrinsic interest, such results are useful in examining l–sequences on the graded module associated to a decreasing filtration (cf. (2.11)) .

In the theory of various kind of sequences (e.g. regular, relative regular, proper, filter, standard) one encounters a result to the effect that under certain assumptions, the powers of elements form a sequence of the same kind. One of the basic properties of an l–sequence on E is

$$x_t^{2^t} E = 0 \text{ , for all } t, 1 \leq t \leq n.$$

From this relation it is obvious that one can never obtain an l–sequence on E of the type $x_1^2, \ldots, x_n^2$, if $x_1, \ldots, x_n$ is already an l–sequence on E .

Another theme of common interest is to see to what extent a given property of sequences is preserved by all permutations of the given elements. As it turns out, the permutability of an l–sequence is easily recognized.

Theorem 3.3 *For an l-sequence* x *on E, the following statements are equivalent:*
(i) $x_t^2 E = 0$ *for all* $t, 1 \leq t \leq n$
(ii) *any permutation of* x *is an l-sequence on E* .

Section 3 is devoted to the study of *strong l–sequences* , i.e. those l–sequences satisfying condition (i) of the theorem quoted above. Besides several results - for whose statements the reader is reffered to (3.5), (3.7), (3.8) - the outcome of the study is the feeling that strong l–sequences have many similarities with regular sequences. This is admitedly an intriguing point - nilpotent elements behave much alike regular ones ! We draw the reader's attention to Popescu's theorem: in their native context, l–sequences are strong and moreover they are obtained by a natural construction from regular sequences. Should we look for another explanation?

One of the properties shared by regular and strong l–sequences is their independence in Lech's sense. In other words, if I is an ideal generated by a strong l–sequence on R, then I/I^2 is a free R/I - module (cf. Theorem 3.9) . This result is a key ingredient needed to obtain a free resolution for the ideal generated by a strong l-sequence. The last section of this paper is devoted to the proof of the following theorem:

Theorem 4.1 *The R–module R/I has a free resolution* $(F_\bullet(\mathrm{x}), \varphi_\bullet(\mathrm{x}))$ *such that for all $t \geq 0$, $F_t(\mathrm{x})$ is the free R-module of rank* $b_t(n) := \binom{n+t-1}{t}$, $\varphi_1(\mathrm{x}) = (\, x_1 \quad x_2 \quad \ldots \quad x_n\,)$ *and for* $t \geq 2$

$$\varphi_t(\mathrm{x}) = \left(\begin{array}{c|c} \varphi_{t-1}(x_1, -x_2, \ldots, -x_n) & 0 \\ \hline 0 \qquad x_1 U_s & \varphi_t(x_2, x_3, \ldots, x_n) \end{array} \right),$$

where U_s is the unit matrix of order $s := \binom{n+t-3}{t-1} = s(n,t)$.

We emphasize that the matrices of the homomorphisms $\varphi_t(\mathbf{x})$ are highly structured: in each row, the only non-zero entries are the terms of the given strong l–sequence, perhaps with a changed sign. Actually, the block structure is essentially used in the proof of the theorem.

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1 Characterizations

Fix a commutative ring R with unity, an R–module E and some elements of the ring $\mathbf{x} = x_1, \ldots, x_n$. In the sequel we shall denote by I_0 the null ideal, and, for each positive integer t , $t \leq n$, I_t will denote the ideal generated by $x_1, \ldots, x_t$.

Definition 1.1 We say that $\mathbf{x}$ is an *l–sequence on E* if

$$(I_{t-1}E : x_t)_E = I_tE \text{ for all } t, 1 \leq t \leq n.$$

We call $\mathbf{x}$ an *l–sequence* if it is l–sequence on R.

Basic Example 1.2 Let $a_1, \ldots, a_n$ be a regular sequence in a ring A and x_t be the image of a_t in the ring $R := A/(a_1^2, a_2^2, \ldots, a_n^2)$. Then $\mathbf{x}$ is l–sequence on R .

Though obvious, the next remarks are very useful, as we shall convince ourselves later on.

Remark 1.3 If $x_1, \ldots, x_n$ is l–sequence on E, then $x_1, \ldots, x_t$ is l–sequence on E for all $t, 1 \leq t \leq n$.

Remark 1.4 Let t be a positive integer less than n. Some elements $x_1, \ldots, x_n$ form l–sequence on E if and only if $x_1, \ldots, x_t$ is l–sequence on E and $x_{t+1}, \ldots, x_n$ is l–sequence on E/I_tE .

Our first aim is to rephrase the condition stated in the Definition 1.1 .

Lemma 1.5 $\mathbf{x}$ *is l–sequence on E if and only if*

$$(1.5.1)\ (0 : x_1 \cdots x_t)_E = I_tE \textit{ for all } t, 1 \leq t \leq n.$$

Proof. Suppose that the given elements form an l–sequence on E. We show the thesis by induction.

Since (1.5.1) holds for $t = 1$, we assume that $t > 1$ and that (1.5.1) is true for any positive integer less than t. Then we have

$$(0 : x_1 \cdots x_t)_E = ((0 : x_1 \cdots x_{t-1})_E : x_t)_E = (I_{t-1}E : x_t)_E$$

and, since $\mathbf{x}$ is l–sequence on E, this last submodule of E coincides with I_tE, as asserted.

Conversely, let us assume that condition (1.5.1) is fulfilled. Then, for all $t, 1 \leq t \leq n$, we get

$$(I_{t-1}E : x_t)_E = ((0 : x_1 \cdots x_{t-1})_E : x_t)_E = (0 : x_1 \cdots x_t)_E = I_tE.$$

∎

Remark 1.6 One cannot characterize the property of being l–sequence by asking (1.5.1) to hold only for $t = n$, as an instance of example (1.2) shows :

Example 1.7 Let X be an indeterminate over a field K and x be its image in the ring $R := K[X]/(X^4)$. Take $x_1 := x, x_2 := x^2$. Then $\mathbf{x} := x_1, x_2$ is not l–sequence since $(0 : x_1) = x^3R \neq xR = x_1R$, but $(0 : x_1x_2) = (0 : x^3) = xR = (x_1, x_2)R$.

Now we examine the condition obtained by changing the rôles of $x_1 \cdots x_n$ and $(x_1, \ldots, x_n)$ in (1.5.1) .

Lemma 1.8 *The following statements are equivalent :*
(1.8.1) $(0 : I_t)_E = x_1 \cdots x_tE$ *for all* $t, 1 \leq t \leq n$
(1.8.2) $x_1 \cdots x_{t-1}E \cap (0 : x_t)_E = x_1 \cdots x_tE$ *for all* $t, 1 \leq t \leq n$.

Remark 1.9 In the case $t = 1$, the statement (1.8.2) should be read up $(0 : x_1)_E = x_1E$. This interpretation is consistent with the usual convention that a product indexed over an empty set is one.

Proof of (1.8) . If (1.8.1) holds, then

$$x_1 \cdots x_tE = (0 : I_t)_E = (0 : I_{t-1})_E \cap (0 : x_t)_E = x_1 \cdots x_{t-1}E \cap (0 : x_t)_E.$$

The converse implication is proved by induction. For $t = 1$, (1.8.1) and (1.8.2) coincide. So let us consider the case $t > 1$.

$$
\begin{aligned}
(0:I_t)_E &= (0:I_{t-1})_E \cap (0:x_t)_E \\
&= x_1\cdots x_{t-1}E \cap (0:x_t)_E \quad \text{by the inductive hypothesis} \\
&= x_1\cdots x_t E \quad \text{according to (1.8.2)}.
\end{aligned}
$$

■

As it turns out, $\mathbf{x}$ is an l–sequence on E as soon as it fulfils one of the equivalent conditions stated in the above lemma.

Lemma 1.10 *Condition* (1.8.1) *implies* (1.5.1) .

Proof. For all integers j and t such that $1 \leq j \leq t \leq n$, we get by (1.8.1) $x_j x_1 \cdots x_t E = 0$, whence it follows $x_j E \subseteq (0 : x_1 \cdots x_t)_E$. Thus $I_t E \subseteq (0 : x_1 \cdots x_t)_E$ for all $t, 1 \leq t \leq n$.

The other inclusion is proved by induction on t. It is clear that (1.5.1) stated for $t = 1$ is $(0 : x_1)_E = x_1 E$, and the same thing is stated by (1.8.1) for $t = 1$. So we may suppose that $n \geq t > 1$ and (1.5.1) holds for all positive integers less than t . For any $e \in (0 : x_1 \cdots x_t)_E$ we have by (1.8)

$$e x_1 \cdots x_{t-1} \in x_1 \cdots x_{t-1} E \cap (0 : x_t)_E = x_1 \cdots x_t E.$$

Therefore , $e x_1 \cdots x_{t-1} = f x_1 \cdots x_t$ for a suitable $f \in E$ and so $e - f x_t \in (0 : x_1 \cdots x_{t-1})_E = I_{t-1}E$ by the inductive hypothesis. Thus $e \in I_{t-1}E + x_t E = I_t E$ and hence it results $I_t E \supseteq (0 : x_1 \cdots x_t)_E$. ■

Now we are ready to prove our main characterization for l–sequences.

Theorem 1.11 *Let* $\mathbf{x}=x_1, \ldots, x_n$ *be some elements of a commutative ring* R *and* E *be an* R*–module. Then the following statements are equivalent:*

(i) $\mathbf{x}$ *is* l*–sequence on* E

(ii) $(0 : x_1 \cdots x_t)_E = I_t E$ *for all* $t, 1 \leq t \leq n$

(iii) $x_1 \cdots x_{t-1} E \cap (0 : x_t)_E = x_1 \cdots x_t E$ *for all* $t, 1 \leq t \leq n$

(iv) $(0 : I_t)_E = x_1 \cdots x_t E$ *for all* $t, 1 \leq t \leq n.$

Proof. The equivalence of the first two conditions is given by Lemma 1.5, while Lemma 1.8 gives that (iii) and (iv) are equivalent. Having in view (1.10) , we are through as soon as we show the implication (ii)$\Rightarrow$ (iii) .

As already remarked, (ii) and (iii) stated for $t = 1$ do coincide. We may consider $t > 1$. Since (ii) gives $x_t E \subseteq (0 : x_1 \cdots x_t)_E$, we have

$x_1 \cdots x_t E \subseteq (0 : x_t)_E \cap x_1 \cdots x_{t-1}E$. Take $f := ex_1 \cdots x_{t-1} \in (0 : x_t)_E$, so that $e \in (0 : x_1 \cdots x_t)_E = I_t E$ acoording to (ii) . Then there exist suitable $e_1, \ldots, e_t \in E$ such that $e = e_1 x_1 + \cdots + e_t x_t$. Therefore

$$f = e_1 x_1^2 x_2 \ldots x_{t-1} + e_2 x_1 x_2^2 \cdots x_{t-1} + \cdots + e_{t-1} x_1 x_2 \cdots x_{t-1}^2 + e_t x_1 x_2 \cdots x_t.$$

In this sum,all but the last term are zero because by (ii) $(0 : x_1 x_2 \cdots x_{t-1})_E = I_{t-1}E$. Thus $f = e_t x_1 x_2 \cdots x_t$ and

$$x_1 \cdots x_{t-1} E \cap (0 : x_t)_E \subseteq x_1 x_2 \cdots x_t E \quad .$$

■

Remark 1.12 As Example 1.7 shows, in general condition $(0 : I_n)_E = x_1 \cdots x_n E$ alone does not imply $x_1 \cdots x_{n-1} E \cap (0 : x_n)_E = x_1 \cdots x_n E$, nor the fact that $\mathbf{x}$ is an l–sequence on E .

2 Properties

The basic property is obvious and apparently innocuous, so much surprising are its consequences.

Lemma 2.1 *If* $\mathbf{x}$ *is an l-sequence on E, then* $x_t^{2^t} E = 0$ *for all* $t, 1 \leq t \leq n$.

Corollary 2.2 *Every element of an l–sequence is nilpotent, therefore the ideal they generate is proper.*

Proposition 2.3 *Any two elements of an l–sequence generate distinct ideals.*

Proof. Suppose, by way of contradiction, that the thesis does not hold. Then, by (1.4), we may assume that $x_1 R = x_t R$ for some $1 < t \leq n$. Since $x_1 = ax_t$ and $x_t = bx_1$ for suitable $a, b \in R$, one gets $1 - ab \in (0 : x_1) = x_1 R$. So $1 - ab$ is nilpotent, whence it follows that $ab = 1 - (1 - ab)$ is invertible. This in turn implies $(I_{t-1} : x_t) = R$. On the other hand, $(I_{t-1} : x_t) = I_t$, and we get a contradiction with (2.2) . ■

Consider a short exact sequence of R–modules

$$(*) \qquad 0 \longrightarrow E' \xrightarrow{u} E \xrightarrow{v} E'' \longrightarrow 0.$$

We shall examine how are related the properties of being l–sequence on the modules appearing in (*) .

Lemma 2.4 *If* x *is an* l*–sequence on* E'' *and* $x_t^2E \subseteq I_{t-1}E$ *for all* $t, 1 \leq t \leq n$*, then the induced sequence*

$$0 \longrightarrow E'/\mathrm{x}E' \longrightarrow E/\mathrm{x}E \longrightarrow E''/\mathrm{x}E'' \longrightarrow 0$$

is exact.

Proof. It is sufficient to obtain the property in the case $n = 1, x = x_1$.

For any $m' \in E', m \in E$ such that $u(m') = xm$ one has $v(m) \in (0 : x)_{E''} = xE''$. Since v is onto, there exists $n \in E$ such that $v(m) = xv(n)$, whence it follows $m - xn = u(n')$ for a certain $n' \in E'$. Therefore $u(m') = xm = x(m - xn) = u(xn')$, whence $m' = xn'$. Thus the homomorphism $E'/\mathrm{x}E' \longrightarrow E/\mathrm{x}E$ induced by u is one-to-one . ∎

Proposition 2.5 *If* x *is* l*–sequence on* E'' *, then the following statements hold:*

1) *If* x *is* l*–sequence on* E *, then* x *is* l*–sequence on* E' *.*

2) *If* x *is* l*–sequence on* E' *and also on* R *, then* x *is* l*–sequence on* E *.*

Proof. From (1.4) and (2.4) it results that we may assume w.l.o.g. $n = 1, x = x_1$.

If x is l–sequence on E and $m' \in (0 : x)_{E'}$, then $u(m') \in (0 : x)_E = xE$, say , $u(m') = xm$. The same reasoning as in the proof of the previous lemma gives $m' \in xE'$. Thus $(0 : x)_{E'} \subseteq xE'$. Since $x^2u(m') = 0$ for all $m' \in E'$ and since u is injective, one has $x^2E' = 0$, whence $xE' \subseteq (0 : x)_{E'}$.

Let us prove now 2). For any $m \in (0 : x)_E$ one gets $v(m) \in (0 : x)_{E''} = xE''$. Therefore one may find $n \in E$ and $m' \in E'$ such that $m - xn = u(m')$. Then $xu(m') = xm - x^2n = 0$ because $x^2 = 0$ by the hypothesis of 2) . Hence $m' \in (0 : x)_{E'}$ and so $m \in xE$. The other inclusion $xE \subseteq (0 : x)_E$ is a direct consequence of the fact that $x^2 = 0$. ∎

Remark 2.6 In the second part of the previous result , the hypothesis that the given elements form an l–sequence on R is not needed if the exact sequence (*) splits. However, in general it is not superfluous.

Example 2.7 Let X and Y be indeterminates over a field K and denote by x and y their images in the ring $R := K[X, Y]/(X^4, Y^4)$. Consider the canonical injection u of $E' := x^2R$ into $E := R$. Then $E'' := E/E' \simeq$

$K[X,Y]/(X^2,Y^4)$ and it is easily seen that x is l-sequence on E' and also on E'' , though it is not l-sequence on E .

Theorem 2.8 *If* $\mathbf{x}$ *is* l*-sequence on* E *, then the following statements are equivalent:*

(i) $\mathbf{x}$ *is* l*-sequence on* E''

(ii) $\mathbf{x}$ *is* l*-sequence on* E'

(iii) $I_tE \cap u(E') = I_tu(E')$ *for all* $t, 1 \leq t \leq n$

Proof. We already know that (i) implies (ii) .

Suppose now that (ii) holds. For t between 1 and n and $m' \in E'$ such that $u(m') \in I_tE$, we get by (1.5) $x_1 \cdots x_t u(m') = 0$. Hence $m' \in (0 : x_1 \cdots x_t)_{E'}$. Therefore $I_tE \cap u(E') \subseteq I_tu(E')$. As the other inclusion is always true, we have obtained condition (iii) .

Next we show that (ii) is a consequence of (iii) . Fix a positive integer $t, 1 \leq t \leq n$. If $m' \in (I_{t-1}E' : x_t)_{E'}$, then $u(m') \in (I_{t-1}E : x_t)_E = I_tE$. From (iii) it follows $u(m') \in I_tu(E')$. Since u is injective, one gets $m' \in I_tE'$ and thus $(I_{t-1}E' : x_t)_{E'} \subseteq I_tE'$. On the other hand, for $m' \in E'$ one has

$$u(x_t^2m') = x_t^2u(m') \in I_{t-1}E \cap u(E') \qquad \text{since } \mathbf{x} \text{ is } l\text{-sequence on } E$$

$$= I_{t-1}u(E') \qquad \text{by (iii)}$$

Using again the injectivity of u , it results $x_t^2m' \in I_{t-1}E'$. Therefore $(I_{t-1}E' : x_t)_{E'} = I_tE'$.

Finally, let us prove the implication (iii) $\Rightarrow$ (i) . For $m'' \in (0 : x_1)_{E''}$ consider $m \in E$ and $m' \in E'$ for which $m'' = v(m)$, $x_1m = x_1u(m')$. Then $m - u(m') \in (0 : x_1)_E = x_1E$, so that $m = u(m') + x_1n$ with $n \in E$. Passing to E'' we get $m'' = x_1v(n) \in x_1E''$. Therefore $(0 : x_1)_{E''} \subseteq x_1E''$. The opposite inclusion follows from the surjectivity of v and the relation $x_1^2E = 0$.

Actually, for any $t, 1 \leq t \leq n$, the relation $x_t^2E \subseteq I_{t-1}E$ and the fact that v is onto imply $x_t^2E'' \subseteq I_{t-1}E''$, or in other words $I_tE'' \subseteq (I_{t-1}E'' : x_t)_{E''}$. If $t > 1$ and $m_1, \ldots, m_t \in E$ are such that $x_tv(m_t) = x_1v(m_1) + \cdots + x_{t-1}v(m_{t-1})$, then

$$x_tm_t - x_1m_1 - \cdots - x_{t-1}m_{t-1} = u(m')$$

for a suitable $m' \in E'$. Since $\mathbf{x}$ is l–sequence on E, $x_t^2 m_t \in I_{t-1}E$, whence it results $x_t u(m') \in I_{t-1}E \cap u(E') = I_{t-1}u(E')$. As u is injective, one gets $x_t m' \in I_{t-1}E'$. We already know that $\mathbf{x}$ is l–sequence on E' , and therefore $m' \in (I_{t-1}E' : x_t)_{E'} = I_t E'$. Choose some $n'_1, \ldots, n'_t \in E'$ for which $m' = x_1 n'_1 + \cdots + x_t n'_t$. Then it follows $m_t - u(n'_t) \in (I_{t-1}E : x_t)_E = I_t E$. Hence $v(m_t) \in I_t E''$. As m_t was a preimage of an arbitrary element of $(I_{t-1}E'' : x_t)_{E''}$, one has obtained $(I_{t-1}E'' : x_t) \subseteq I_t E''$. ∎

Remark 2.9 If $\mathbf{x}$ is not l–sequence on E, then the statements (i) and (ii) are not anymore equivalent. The example below shows also that in general (i) does not imply condition (iii) .

Example 2.10 Suppose that x is a non-zero divisor in the ring R , so that there exists an exact sequence $0 \longrightarrow R \xrightarrow{u} R \longrightarrow R/x^2R \longrightarrow 0$, where u is multiplication by x^2 . Clearly x is l–sequence on $E'' := R/x^2R$, it is not l–sequence on $E' := R$ and $xR \cap x^2R = x^2R \neq x^3R = xu(E')$.

As an application of these transfer properties we examine l–sequences on the graded module associated to a decreasing filtration of submodules.

Proposition 2.11 *Let $\mathcal{F} := (F^tE)_{t\geq 0}$ be a decreasing filtration of submodules of a module E and $G := \oplus_{t\geq 0}F^tE/F^{t+1}E$ be the associated graded module. An element $x \in R$ such that $x^2F^0E = 0$ is l-sequence on G if and only if x is l-sequence on F^0E/F^tE for all $t \geq 1$.*

Proof. Clearly, x is l–sequence on G if and only if it is l–sequence on $F^tE/F^{t+1}E$ for all $t \geq 0$. Stated in the given module E , this last condition is rewritten as

$$(*) \qquad (F^{t+1}E : x)_{F^tE} = xF^tE + F^{t+1}E \text{ , for all } t \geq 0 .$$

Similarly, x is l–sequence on F^0E/F^tE for all $t \geq 1$ precisely when it satisfies

$$(**) \qquad (F^tE : x)_{F^0E} = xF^0E + F^tE \quad \text{for all } t \geq 1 .$$

We have to show the equivalence of (*) and (**).

We argue by induction on t that x is l–sequence on F^0E/F^tE , provided x is l–sequence on G. By (*), this is true for $t = 1$. So let us consider

$e \in (F^tE : x)_{F^0E} \subseteq (F^{t-1}E : x)_{F^0E}$ for some value of t greater than one. By (**) written for $t-1$ we get $e = xf_0 + f_{t-1}$, for some $f_0 \in F^0E$, $f_{t-1} \in F^{t-1}E$. From $F^tE \ni xe = x^2 f_0 + xf_{t-1} = xf_{t-1}$ it follows $f_{t-1} \in (F^tE : x)_{F^{t-1}E}$. Then (*) yields $f_{t-1} = xg + f_t$ for some $f_t \in F^tE$, $g \in F^{t-1}E$. Therefore $e = xf_0 + xg + f_t \in xF^0E + F^tE$. Since always one has $xF^0E + F^tE \subseteq (F^tE : x)_{F^0E}$, we have checked condition (**) for t.

Conversely, suppose that (**) is fulfilled for all $t \geq 1$. Then (2.5) applied to the canonical exact sequence of modules

$$0 \longrightarrow F^{t-1}E/F^tE \longrightarrow F^0E/F^tE \longrightarrow F^0E/F^{t-1}E \longrightarrow 0$$

gives that x is l–sequence on $F^{t-1}E/F^tE$ for all $t \geq 1$. ∎

3 Strong l–sequences

We shall keep the same notations as in the previous section.

Definition 3.1 The given elements $\mathbf{x} := x_1, \ldots, x_n$ form a *strong l-sequence on E* if $\mathbf{x}$ is an l–sequence on E and $x_t^2 E = 0$ for all $t, 1 \leq t \leq n$.

Examples 3.2 1. Obviously an l–sequence of length one is strong.

2. The construction given in Example 1.2 gives strong l–sequences which may have arbitrary length.

3. To see that not every l–sequence is strong we invoke once again the ring $R := K[X]/(X^4)$ introduced in Example 1.7 . In this ring $x_1 := x^2$, $x_2 := x$ form an l–sequence having the property $x_1^2 = 0 \neq x_2^2$.

4. Let $\mathbf{x}$ be a strong l–sequence on R and let E be a module such that $\mathbf{x}$ is l–sequence on E. Then $\mathbf{x}$ is a strong l–sequence on E .

5.The first element of an l–sequence satisfies $x_1^2 E = 0$. Hence it follows that $\mathbf{x}$ is strong, provided that any permutation of it is an l–sequence. Actually, this sufficient condition is also necessary in order that a given l–sequence be strong.

Theorem 3.3 *Let $\mathbf{x}=x_1, \ldots, x_n$ be an l-sequence on E. Then the following statements are equivalent:*

(i) $\mathbf{x}$ *is a strong l-sequence on E*

(ii) *any permutation of* $\mathbf{x}$ *is an l-sequence on E .*

Proof. Since the symmetric group is generated by the transpositions (1,2) , (2,3) , ... , (n-1,n) , it is sufficient to show that, for all $t = 1, \dots, n-1$, $x_1, \dots, x_{t-1}, x_{t+1}, x_t, x_{t+2}, \dots, x_n$ is an l–sequence along with $x_1, \dots, x_n$. Having in view (1.3) and (1.4) , everything boils down to the case where $t = 1$, $n = 2$.

As $x_2^2 E = 0$, certainly $x_2 E \subseteq (0 : x_2)_E$.

Take $e \in E$ such that $ex_2 = 0$. Then $e \in (0 : x_2)_E \subseteq (x_1 E : x_2)_E = (x_1, x_2)E$, say , $e = e_1 x_1 + e_2 x_2$ for some $e_1, e_2 \in E$. Since $0 = ex_2 = e_1 x_1 x_2 + e_2 x_2^2 = e_1 x_1 x_2$ (where the last equality holds because $\mathbf{x}$ is a strong l–sequence on E) , we get by (1.5) $e_1 \in (0 : x_1 x_2)_E = (x_1, x_2)E$. Thus $e_1 = f_1 x_1 + f_2 x_2$, with $f_1, f_2 \in E$ and therefore

$$e = e_1 x_1 + e_2 x_2 = (f_1 x_1 + f_2 x_2) x_1 + e_2 x_2 = (f_2 x_1 + e_2) x_2 \in x_2 E.$$

As e was an arbitrary element of $(0 : x_2)_E$, we have obtained altogether $(0 : x_2)_E = x_2 E$.

Now consider $e \in (x_2 E : x_1)_E$, i.e. $ex_1 = fx_2$ for some $f \in E$. Then it follows $f \in (x_1 E : x_2)_E = (x_1, x_2)E$, let us say $f = f_1 x_1 + f_2 x_2$. The relation $ex_1 = fx_2$ is rewritten in the form $ex_1 = f_1 x_1 x_2$, whence one gets $e - f_1 x_2 \in (0 : x_1)_E = x_1 E$, that is $e \in (x_1, x_2)E$. Therefore $(x_2 E : x_1)_E \subseteq (x_1, x_2)E$.

For the other inclusion we use again the granted condition $x_1^2 E = 0$. ∎

Remark 3.4 It may happen that some permutations of an l–sequence give again an l–sequence, while other permutations of the original l–sequence do not have anymore this property.

For instance, let us examine in the Example 2.7 the l–sequence $x_1 := x^2, x_2 := x, x_3 := y^2, x_4 := y$. As is easily seen, x_3, x_1, x_4, x_2 is an l–sequence too, while x_4, x_1, x_2, x_3 is not l–sequence.

One way to phrase the difference between an l–sequence and a strong one is the following: for a strong l–sequence we know exactly the order of nilpotency for each of its elements. Correspondingly, we can get the order of nilpotency of the ideal generated by a strong l–sequence.

Proposition 3.5 *Let $x_1, \dots, x_n$ be a strong l-sequence on E and denote by I the ideal generated by these elements. Then $I^n E \neq 0 = I^{n+1} E$.*

Proof. Each monomial in $x_1, \dots, x_n$ generating I^{n+1} is a multiple of the second power of some x_t , for a suitable $t, 1 \leq t \leq n$ (by the pigeonhole principle). Therefore $I^{n+1}E = 0$, because $x_t^2 E = 0$ by hypothesis. For the same reason one has $I^n E = x_1 x_2 \dots x_n E$. Were $I^n E = 0$, then would follow from (1.5) $E = (0 : x_1 x_2 \dots x_n)_E = IE$,whence $E = IE = \dots = I^n E = 0$, contradiction. ∎

Remark 3.6 One may prove that the ideal I generated by an arbitrary l-sequence on E verifies relations $I^n E \neq 0 = I^u E$ for $u := 2^n$, where n is the length of the given l-sequence on E. As Example (3.2.3) shows, the value of u can not be diminished in general.

The result we are going to prove now shows another property specific to strong l–sequences and which is not shared by all l–sequences. It is reminiscent of the theory of regular sequences.

Proposition 3.7 *For any strong l-sequence on E one has*

$$\bigcap_{i=1}^{t} x_i E = x_1 \cdots x_t E \quad \text{for all } t,\ 1 \leq t \leq n \ .$$

Proof. Indeed, for t and $\mathbf{x}$ as above we get

$$x_1 \cdots x_t E = (0 : I_t)_E = \cap_{i=1}^{t} (0 : x_i)_E = \cap_{i=1}^{t} x_i E.$$

The first equality holds for any l–sequence on E by (1.11) , while the latest invokes the hypothesis that the given l–sequence is strong. ∎

The transfer of the property of being l–sequence between modules appearing in a short exact sequence is also satisfactory. A reasoning similar to that used to obtain (2.5) and (2.8) gives

Theorem 3.8 *For a short exact sequence of modules (*) , the following statements hold:*

1) *Suppose that* $\mathbf{x}$ *is a strong l–sequence on E''. Then:*

a) *If* $\mathbf{x}$ *is a strong l–sequence on E , then* $\mathbf{x}$ *is a strong l–sequence on E' .*

b) $\mathbf{x}$ *is a strong l–sequence on E , provided that one of the additional conditions is fulfilled:*

α) $\mathbf{x}$ *is a strong l–sequence on E' and an l–sequence on R*

β) $\mathbf{x}$ *is l–sequence on E' and a strong l–sequence on R.*

2) *Assume that* $\mathbf{x}$ *is a strong l–sequence on E. Then* $\mathbf{x}$ *is strong l–sequence on E'' if and only if* $\mathbf{x}$ *is a strong l–sequence on E' .*

Theorem 3.9 *If* x *is a strong* l*–sequence , then the coefficients of any syzygy of* x *belong to the ideal* $I := \mathrm{x}R$.

Proof. Induct on the length n.

For $n = 1$, the thesis is a direct consequence of the defining relation $xR = (0 : x)$.

Suppose that $n \geq 2$ and $a_1, \ldots, a_n \in R$ are such that $a_1x_1 + \cdots + a_nx_n = 0$. Then $a_n \in (I_{n-1} : x_n) = I_n$, which yields $a_n = b_1x_1 + \cdots + b_nx_n$ for some $b_i \in R$. Using this relation in the previous equality one obtains $(a_1 + b_1x_n)x_1 + \cdots + (a_{n-1} + b_{n-1}x_n)x_{n-1} = 0$, because $x_n^2 = 0$. By inductive hypothesis, the coefficients of this syzygy of the l–sequence $x_1, \ldots, x_{n-1}$ are elements of the ideal $(x_1, \ldots, x_{n-1})R = I_{n-1}$, so that $a_i \in I_{n-1} + x_nR = I_n$ for all $i = 1, \ldots, n$. ∎

4 Free resolution for the ideal generated by a strong l–sequence

In this section we shall construct a free resolution for the ideal generated by a strong l-sequence.The main tool to achieve the aim is the freeness of the R/I - module I/I^2 .

Obviously, changing the sign of some elements of a (strong) l–sequence does not destroy the property. Therefore, the description given below for the map $\varphi_t(\mathrm{x})$ is licit.

Theorem 4.1 *The* R*–module* R/I *has a free resolution* $(F_\bullet(\mathrm{x}), \varphi_\bullet(\mathrm{x}))$ *such that for all* $t \geq 0$, $F_t(\mathrm{x})$ *is the free* R*–module of rank* $b_t(n) := \binom{n+t-1}{t}$, $\varphi_1(\mathrm{x}) = (\,x_1 \quad x_2 \quad \ldots \quad x_n\,)$ *and for* $t \geq 2$

$$\varphi_t(\mathrm{x}) = \left(\begin{array}{c|c} \varphi_{t-1}(x_1, -x_2, \ldots, -x_n) & 0 \\ \hline 0 \qquad x_1U_s & \varphi_t(x_2, x_3, \ldots, x_n) \end{array}\right),$$

where U_s *is the unit matrix of order* $s := \binom{n+t-3}{t-1} = s(n,t)$.

For the sake of convenience, we need one more piece of notation. If $\mathbf{x}=x_1,\ldots,x_n$ is the given strong l–sequence, one denotes $x := x_1$ and $\mathbf{y}:=x_2,\ldots,x_n$.

Let us proceed to prove by double induction: on the length n and on t .

In the case where $n = 1$, it is clear that all F_t's coincide with R and all maps $\varphi_t(x_1)$ are given by multiplication by x_1 . This is exactly the conclusion of the theorem , up to usual conventions.

For $n = 2$, let us examine a first syzygy $ax_1+bx_2 = 0$ for the given strong l–sequence x_1, x_2 . Theorem 3.9 yields $b = cx_1+dx_2$ for some $c, d \in R$. Then $(a + cx_2)x_1 = 0$ and therefore there exists $e \in R$ for which $a = ex_1 - cx_2$. In other words the vector ${}^{tr}(a, b)$ belongs to the image of the matrix

$$\varphi_2(\mathbf{x}) := \begin{pmatrix} x_1 & -x_2 & 0 \\ 0 & x_1 & x_2 \end{pmatrix}.$$

Conversely, it is easily checked that the first syzygy module of $\mathbf{x}$ contains the image of the homomorphism $R^3 \longrightarrow R^2$ given by the matrix $\varphi_2(\mathbf{x})$.

So let us suppose that $t > 2$ and $\varphi_{t-1}(x_1, x_2)$ is a $t-1$ by t matrix of the form

$$\varphi_{t-1}(x_1,x_2) = \left(\begin{array}{ccc|c} & \varphi_{t-2}(x_1,-x_2) & & 0 \\ \hline 0 & \ldots & 0 \quad x_1 & x_2 \end{array}\right).$$

For any ${}^{tr}(a_1,\ldots,a_t)$ in the kernel of $\varphi_{t-1}(x_1,x_2)$ one has

(1) $${}^{tr}(a_1,\ldots,a_{t-1}) \in \ker \varphi_{t-2}(x_1,-x_2)$$

and

(2) $$a_{t-1}x_1 + a_t x_2 = 0.$$

From (1) and the inductive hypothesis on t it results

$$\begin{pmatrix} a_1 \\ \vdots \\ a_{t-1} \end{pmatrix} = \varphi_{t-1}(x_1,-x_2)\begin{pmatrix} b_1 \\ \vdots \\ b_t \end{pmatrix} = \begin{pmatrix} \varphi_{t-2}(x_1,x_2)\begin{pmatrix} b_1 \\ \vdots \\ b_{t-1} \end{pmatrix} \\ b_{t-1}x_1 + b_t x_2 \end{pmatrix}$$

for suitable $b_1, \ldots, b_t \in R$. This is equivalent to

$$
{}^{tr}(a_1, \ldots, a_{t-2}) = \varphi_{t-2}(x_1, x_2) \begin{pmatrix} b_1 \\ \vdots \\ b_{t-1} \end{pmatrix}, \tag{3}
$$

and

$$
a_{t-1} = b_{t-1}x_1 + b_t x_2. \tag{4}
$$

Using (4) in (2) , one obtains

$$
a_t = -b_t x_1 + b_{t+1} x_2 \tag{5}
$$

for some $b_{t+1} \in R$. Rewrite (3), (4) and (5) in the equivalent forms

$$
\begin{pmatrix} a_1 \\ \vdots \\ a_t \end{pmatrix} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\varphi_{t-2}(x_1, x_2)} & \multicolumn{2}{c}{0} \\ \hline 0 & x_1 & -x_2 & 0 \\ 0 & 0 & x_1 & x_2 \end{array} \right) \begin{pmatrix} b_1 \\ \vdots \\ b_{t-1} \\ -b_t \\ b_{t+1} \end{pmatrix}
$$

$$
= \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\varphi_{t-1}(x_1, -x_2)} & 0 \\ \hline 0 & x_1 & x_2 \end{array} \right) \begin{pmatrix} b_1 \\ \vdots \\ b_{t-1} \\ -b_t \\ b_{t+1} \end{pmatrix}.
$$

At this point we have checked that ker $\varphi_{t-1}(x_1, x_2) \subseteq$ Im $\varphi_t(x_1, x_2)$, where $\varphi_t(x_1, x_2)$ is the t by $t+1$ matrix appearing in the last equality. The converse inclusion follows from the computation of the product $P := \varphi_{t-1}(x_1, x_2)\varphi_t(x_1, x_2)$. In the first $t-2$ rows and t columns one obtains $\varphi_{t-2}(x_1, -x_2)\varphi_{t-1}(x_1, -x_2)$, which is zero by induction on t . The last column of P contains in the first $t-2$ rows only zeroes, while the last row of P is easily seen to be the product of the row vector (x_1, x_2) with the 2 by $t+1$ matrix $(0 \quad \varphi_2(x_1, x_2))$. By the inductive hypothesis, the entries in the last row of P are all zero, and so P is the zero matrix, as desired. This concludes the case $n = 2$.

Now let n be greater than 2 and $t = 2$. Take any $a_1, \ldots, a_n \in R$ such that $a_1x_1 + \cdots + a_nx_n = 0$, and use again Theorem 3.9 to obtain

$$a_1 = b_1x_1 + \cdots + b_nx_n \tag{6}$$

and

$$(a_2 + b_2x_1)x_2 + \cdots + (a_n + b_nx_1)x_n = 0 \quad .$$

By induction on n , there exist $c_1, \ldots, c_m \in R$, $m := \binom{n}{2}$ such that

$$\begin{pmatrix} a_2 + b_2x_1 \\ \vdots \\ a_n + b_nx_1 \end{pmatrix} = \varphi_2(\mathbf{y}) \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$$

or

$$\begin{pmatrix} a_2 \\ \vdots \\ a_n \end{pmatrix} = -x_1 \begin{pmatrix} b_2 \\ \vdots \\ b_n \end{pmatrix} + \varphi_2(\mathbf{y} \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} . \tag{7}$$

With the aid of relations (6) and (7) express the column vector

$${}^{tr}(a_1, \ldots, a_n)$$

as the product of the matrix

$$\varphi_2(\mathbf{x}) := \left(\begin{array}{ccc|c} x_1 & -x_2 & \ldots - x_n & 0 \\ \hline 0 & x_1U_{n-1} & & \varphi_2(\mathbf{y}) \end{array}\right)$$

with the column vector ${}^{tr}(b_1, -b_2, \ldots, -b_n, c_1, \ldots, c_m)$. Note that $\varphi_2(\mathbf{x})$ has n rows and $n + m = \binom{n}{1} + \binom{n}{2} = \binom{n+1}{2} = b_2(n)$ columns, precisely as many as stated in the conclusion of Theorem 4.1 .

Denoting $b_1(n) = n$ and $\varphi_1(\mathbf{x}) : R^n \longrightarrow R$ the homomorphism whose matrix with respect to canonical basis is $(\, x_1 \quad \ldots \quad x_n \,)$, one has obtained $\ker \varphi_1(\mathbf{x}) \subseteq \operatorname{Im} \varphi_2(\mathbf{x})$. On the other hand, invoking once again the inductive hypothesis on n , one gets

$$\varphi_1(\mathbf{x})\varphi_2(\mathbf{x}) = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \left(\begin{array}{ccccc|c} x_1 & -x_2 & -x_3 & \dots & -x_n & 0 \\ & x_1 & 0 & \dots & 0 & \\ 0 & & x_1 & & & \varphi_2(\mathbf{y}) \\ & & & \ddots & & \\ & & & & x_1 & \end{array} \right)$$

$$= \begin{pmatrix} 0 & \dots & 0 & (x_2 & \dots & x_n)\varphi_2(x_2, \dots, x_n)) \end{pmatrix} = 0 \ .$$

Thus we have established the starting step in the induction on t .

Suppose then that the conclusion of the theorem is true for $t-1$. Let us denote

$$p := \binom{n+t-3}{t-2}, q := \binom{n+t-2}{t-1}, v := \binom{n+t-4}{t-2}, r := \binom{n+t-4}{t-3},$$

so that $\varphi_{t-1}(\mathbf{x})$ is a p by q matrix, $\varphi_{t-2}(\mathbf{x})$ is r by p , while $\varphi_{t-1}(\mathbf{y})$ is v by $q-p$. For ${}^{tr}(a_1, \dots, a_q) \in \ker \varphi_{t-1}(\mathbf{x})$ one writes

$$0 = \left(\begin{array}{c|c} \varphi_{t-2}(x, -\mathbf{y}) & 0 \\ \hline 0 \quad x_1 U_v & \varphi_{t-1}(\mathbf{y}) \end{array} \right) \begin{pmatrix} a_1 \\ \vdots \\ a_p \\ \vdots \\ a_q \end{pmatrix} =$$

$$(8) \qquad = \begin{pmatrix} \varphi_{t-2}(x, -\mathbf{y}) \begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix} \\ \\ x_1 \begin{pmatrix} a_{r+1} \\ \vdots \\ a_p \end{pmatrix} + \varphi_{t-1}(\mathbf{y}) \begin{pmatrix} a_{p+1} \\ \vdots \\ a_q \end{pmatrix} \end{pmatrix} .$$

From the first r rows it follows (by induction on t) that there exist $b_1, \dots, b_q \in R$ such that

$$\begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix} = \varphi_{t-1}(x, -\mathbf{y}) \begin{pmatrix} b_1 \\ \vdots \\ b_q \end{pmatrix} = \left(\begin{array}{c|c} \varphi_{t-2}(x, \mathbf{y}) & 0 \\ \hline 0 \quad x_1 U_v & \varphi_{t-1}(-\mathbf{y}) \end{array} \right) \begin{pmatrix} b_1 \\ \vdots \\ b_q \end{pmatrix}$$

$$(9) \qquad = \begin{pmatrix} \varphi_{t-2}(x, \mathbf{y}) \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix} \\ x_1 \begin{pmatrix} b_{r+1} \\ \vdots \\ b_p \end{pmatrix} + \varphi_{t-1}(-\mathbf{y}) \begin{pmatrix} b_{p+1} \\ \vdots \\ b_q \end{pmatrix} \end{pmatrix}.$$

Then the last $p - r$ equations of (8) become

$$0 = x_1 \begin{pmatrix} a_{r+1} \\ \vdots \\ a_p \end{pmatrix} + \varphi_{t-1}(\mathbf{y}) \begin{pmatrix} a_{p+1} \\ \vdots \\ a_q \end{pmatrix} = x_1 \varphi_{t-1}(-\mathbf{y}) \begin{pmatrix} b_{p+1} \\ \vdots \\ b_q \end{pmatrix}$$

$$+\varphi_{t-1}(\mathbf{y}) \begin{pmatrix} a_{p+1} \\ \vdots \\ a_q \end{pmatrix} = \varphi_{t-1}(\mathbf{y}) \begin{pmatrix} a_{p+1} - x_1 b_{p+1} \\ \vdots \\ a_q - x_1 b_q \end{pmatrix}.$$

Therefore one has for suitable $w := \binom{n+t-2}{t}$ elements $c_1, \ldots, c_w \in R$

$$(10) \qquad \begin{pmatrix} a_{p+1} \\ \vdots \\ a_q \end{pmatrix} = x_1 \begin{pmatrix} b_{p+1} \\ \vdots \\ b_q \end{pmatrix} + \varphi_t(\mathbf{y}) \begin{pmatrix} c_1 \\ \vdots \\ c_w \end{pmatrix}.$$

Putting (9) and (10) together , one gets

$$
(11)\qquad \begin{pmatrix} a_1 \\ \vdots \\ a_p \\ a_{p+1} \\ \vdots \\ a_q \end{pmatrix} = \begin{pmatrix} \varphi_{t-1}(x,-\mathbf{y}) \begin{pmatrix} b_1 \\ \vdots \\ b_q \end{pmatrix} \\ \\ x_1 \begin{pmatrix} b_{p+1} \\ \vdots \\ b_q \end{pmatrix} + \varphi_t(\mathbf{y}) \begin{pmatrix} c_1 \\ \vdots \\ c_w \end{pmatrix} \end{pmatrix}
$$

$$
= \left(\begin{array}{c|c} \varphi_{t-1}(x,-\mathbf{y}) & 0 \\ \hline 0 \quad x_1 U_{q-p} & \varphi_t(\mathbf{y}) \end{array}\right) \begin{pmatrix} b_1 \\ \vdots \\ b_q \\ c_1 \\ \vdots \\ c_w \end{pmatrix} .
$$

Denoting by $\varphi_t(\mathbf{x})$ the matrix appearing in the last relation, it is clear that it has the structure predicted by the statement of Theorem 4.1 . Moreover,

$$
q - p = \binom{n+t-3}{t-1} = s(n,t),
$$

$$
q + w = \binom{n+t-1}{t} = b_t(n).
$$

It remains to show that the image of the matrix $\varphi_t(\mathbf{x})$ is contained in the kernel of $\varphi_{t-1}(\mathbf{x})$. Keeping the same notations, we get

$$
\varphi_{t-1}(\mathbf{x})\varphi_t(\mathbf{x}) = \left(\begin{array}{c|c} \varphi_{t-2}(x,-\mathbf{y}) & 0 \\ \hline 0 \quad x_1 U_v & \varphi_{t-1}(\mathbf{y}) \end{array}\right) \left(\begin{array}{c|c} \varphi_{t-1}(x,-\mathbf{y}) & 0 \\ \hline 0 \quad x_1 U_{q-p} & \varphi_t(\mathbf{y}) \end{array}\right) .
$$

In the upper left r by q block of this product matrix one gets

$$
\varphi_{t-2}(x,-\mathbf{y})\varphi_{t-1}(x,-\mathbf{y}) \;,
$$

which is zero by induction on t . The other entries in the first r rows are obviously zero. To express in a convenient way the others $p-r$ rows, we use the block form of the matrix $\varphi_{t-1}(x, -\mathrm{y})$. Since $v = p - r$, we get

$$\left(x_1 U_v \mid \varphi_{t-1}(\mathrm{y}) \right) \left(\begin{array}{c|c|c|c} 0 & x_1 U_v & \varphi_{t-1}(-\mathrm{y}) & 0 \\ \hline 0 & 0 & x_1 U_{q-p} & \varphi_t(\mathrm{y}) \end{array} \right).$$

Clearly, the first r columns of the product matrix are all zero, while the others coincide with the following product

$$\left(x_1 U_v \mid \varphi_{t-1}(\mathrm{y}) \right) \left(\begin{array}{c|c|c} x_1 U_v & \varphi_{t-1}(\mathrm{y}) & 0 \\ \hline 0 & x_1 U_{q-p} & \varphi_t(\mathrm{y}) \end{array} \right).$$

Since $x_1^2 = 0$, the first v columns are zero . The others are given by

$$(x_1\varphi_{t-1}(-\mathrm{y}) + x_1\varphi_{t-1}(\mathrm{y}) \mid \quad \varphi_{t-1}(\mathrm{y})\varphi_t(\mathrm{y})).$$

The former block is zero because $\varphi_{t-1}(-\mathrm{y}) = -\varphi_{t-1}(\mathrm{y})$, the latter is zero by induction on n . ■

Corollary 4.2 *If the ring is local, the free resolution given by (4.1) is minimal.*

References

[1] M.Auslander, S.Ding, Ø.Solberg: Liftings and weak liftings of modules, J.Algebra,**156**(1993), 273-317

[2] M.Cipu, J.Herzog, D.Popescu: Indecomposable generalized Cohen-Macaulay modules, Trans.Amer.Math.Soc., **342**(1994), 107-136

[3] E.Dieterich: Reduction of isolated singularities, Comment.Math. Helvetici, **62**(1987), 654-676

[4] D.Popescu: Maximal Cohen-Macaulay modules over isolated singularities, preprint, 1994

[5] D.Popescu: Indecomposable Cohen-Macaulay modules and their multiplicities, Trans.Amer.Math.Soc., **323**(1991), 369-387

[6] Y.Yoshino: Brauer-Thrall type theorem for maximal Cohen-Macaulay modules, J.Math.Soc.Japan,**39**(1987), 719-739