INSTITUTUL DE MATEMATICA
AL ACADEMIEI ROMANE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

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Preprint No. 5/95

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Open sets with Stein hypersurface sections in Stein spaces

M. Coltoiu and K. Diederich

1 Introduction

Let $D \subset \mathbb{C}^n$, $n \geq 3$, be an open set such that for any linear hyperplane $H \subset \mathbb{C}^n$ the intersection $H \cap D$ is Stein. According to a theorem of Lelong [13] it follows in this case, that D is itself Stein. Using the solution of the Levi problem by Docquier-Grauert [7], this result can easily be generalized to open subsets in Stein manifolds. Therefore, it is natural to raise the following question (see [4]):

Problem of hypersurface sections: Let X be a Stein space of dimension $n \geq 3$ and $D \subset X$ an open subset such that $H \cap D$ is Stein for every hypersurface $H \subset X$. Does it follow that D is Stein?

This question is closely related to the following classical form of the Levi problem (for a survey see [18]):

Local Steiness problem: Is a locally Stein open subset D of a Stein space X necessarily Stein?

A complete answer to this problem is not known. There are only partial results (see [4] for a discussion of this subject). In particular, for $\dim X = 2$ the answer is known to be positive [1].

It is immediate, that, therefore, a positive answer to the "Problem of hypersurface sections" would, by induction over the dimension, also provide a positive answer to the "Local Steiness problem".

Concerning the "Problem of hypersurface sections" it is known (see [2]), that D as above is indeed Stein, if one knows in addition, that $H^1(D, \mathcal{O}) = 0$ (in this case the hypothesis $\dim X \geq 3$ is not necessary; a weaker statement was already proved in [9]).

In this note we, now, produce a counter-example to the "Problem of hypersurface sections". This shows, that, in order to get the Steiness of D some additional hypothesis (like $H^1(D, \mathcal{O}) = 0$ as above) is necessary, if X is singular. More precisely, our main result can be stated as follows:

Theorem 1.1 *There is a normal Stein space X of pure dimension 3 with only one singular point, and a closed connected analytic subset $A \subset X$ of pure dimension 2, such that $D := X \setminus A$ has the following properties:*
a) D is not Stein;
b) For every hypersurface $H \subset X$ (i. e. closed analytic subset of X of pure codimension 1) the intersection $H \cap D$ is Stein.

This shows again, that in the case of singular Stein spaces the situation may be drastically different from the smooth case (see also [8], [9], [11], [12], [19] for other examples concerning the Levi problem on singular Stein spaces).

2 Tools and Lemmas

For the construction of an example proving Theorem 1.1 some properties of weakly 1-complete manifolds will play an essential role. In order to fix the terminology we state

Definition 2.1 *A complex manifold X is called weakly 1-complete if there exists a $\mathcal{C}^\infty$ plurisubharmonic exhaustion function $\varphi : X \to \mathbb{R}$.*

Notice, that even for complex manifolds with smooth boundary the requirement of being weakly 1-complete is much stronger than Levi-pseudoconvexity of the boundary (see [6]).

Nakano [15] proved the following generalization of the Kodaira vanishing theorem to weakly 1-complete manifolds:

Theorem 2.2 *(Nakano) Let X be a weakly 1-complete manifold, K_X the canonical line bundle of X and E a positive line bundle on X. Then*

$$\begin{aligned} H^i(X, K_X \otimes E) &= 0 \text{ if } i \geq 1 \\ H^i_c(X, E^*) &= 0 \text{ if } 0 \leq i < \dim X \end{aligned} \tag{2.1}$$

where E^* denotes the dual bundle of E.

Next we want to consider a special class of weakly 1-complete manifolds. For this we start from an arbitrary complex manifold S and a holomorphic line bundle $\pi : L \to S$. Then we denote by $\overline{\pi} : \overline{L} \to S$ the bundle obtained from $\pi : L \to S$ by adding the point at infinity to each fiber (i.e. $\overline{L} = \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L})$ is the projective bundle associated to $\mathcal{O}_S \oplus \mathcal{L}$ where $\mathcal{L}$ is the invertible sheaf corresponding to L). We write L_0 for the zero section of L and put $L_\infty := \overline{L} \setminus L$.

In the case where S is compact 1-dimensional, which is of particular interest to us, $\overline{L}$ is a ruled surface and hence algebraic [3]. We show

Lemma 2.3 *Assume that the complex manifold S is compact, 1-dimensional and let L be a topologically trivial line bundle on S. Then the surfaces L, $\overline{L} \setminus L_0$ and $\overline{L} \setminus (L_0 \cup L_\infty)$ are weakly 1-complete.*

Proof: Since $\dim S = 1$, S is Kähler and, therefore, the map $H^1(S, \mathbb{R}) \to H^1(S, \mathcal{O}_S)$ is surjective. Together with the fact, that L is topologically trivial, it follows that with respect to a suitable open covering of S the bundle L can be given by constant transition functions $\{g_{ij}\}$ with $|g_{ij}| = 1$ for all i, j. Hence, the function $\varphi := |w|^2$, where w is the coordinate along the fibers, is well-defined on L. Clearly, φ is a plurisubharmonic exhaustion function on L. (Notice, that this function φ may depend on the chosen local trivializations of L.) Similarly, the functions $\frac{1}{|w|^2}$, $|w|^2 + \frac{1}{|w|^2}$ are well-defined, exhaustive and plurisubharmonic on $\overline{L} \setminus L_0$, $\overline{L} \setminus (L_0 \cup L_\infty)$ respectively. $\square$

This Lemma has as immediate consequence

Corollary 2.4 *Under the assumptions of Lemma 2.3 we have: If F is any negative line bundle over $\overline{L}$ and s a section of F defined on an open connected neighbourhood of L_0 (resp. L_∞), then s vanishes identically.*

Proof: Indeed, from the Theorem of Nakano (Theorem 2.2) we get the vanishing $H^1_c(\overline{L} \setminus L_0, F) = 0$ (resp. $H^1_c(L, F) = 0$). Therefore, the given section s can be extended to a global section $\overline{s} \in H^0(\overline{L}, F)$. But $H^0(\overline{L}, F) = 0$ by Kodaira's vanishing theorem, since F is negative on $\overline{L}$. $\square$

The following Lemma will be crucial for the construction of our example:

Lemma 2.5 *Let X be a compact, connected complex manifold of dimension k, $\pi : L \to X$ a holomorphic line bundle, and assume that there exists a compact analytic subset $A \subset L$ of pure dimension k such that $A \cap L_0 = \emptyset$. Then there is a positive integer λ such that the bundle L^λ is analytically trivial.*

Proof: We define λ to be the sheet number of the ramified analytic covering $p := \pi|A : A \to X$. Let $\{U_i\}$ be an open covering of X such that $L|U_i \simeq U_i \times \mathbb{C}$ and let $\{g_{ij}\}$ be the corresponding transition functions for L. For $x \in U_i$ the fiber $p^{-1}(x)$ can be written in the trivialization $L|U_i \simeq U_i \times \mathbb{C}$ as $p^{-1}(x) = \{s_1^{(i)}(x), \ldots, s_\lambda^{(i)}(x)\} \subset \mathbb{C}^\lambda$ if counted with multiplicities. Then the product $h_i(x) := s_1^{(i)}(x) \cdot \ldots \cdot s_\lambda^{(i)}(x)$ defines a holomorphic function on U_i. From the hypothesis $A \cap L_0 = \emptyset$ it follows, that $h_i(x) \neq 0$ for $x \in U_i$. Moreover, $h_i(x) = g_{ij}^\lambda(x) h_j(x)$ when $x \in U_i \cap U_j$. Therefore, the collection $\{h_i\}$ defines a non-vanishing holomorphic section in L^λ showing that L^λ is analytically trivial. $\square$

The following Lemma is a special case of a more general result of Matsushima and Morimoto (see [14], Théorème 5):

Lemma 2.6 *Let X be a complex manifold, $\pi : L \to X$ a holomorphic line bundle and assume that $L \setminus L_0$ is Stein. Then X is itself Stein.*

For the convenience of the reader we give here a simple proof of this special case:

Proof: If $U \subset X$ is an open set such that $L|U$ is trivial, then any holomorphic function G on $\pi^{-1}(U) \setminus L_0 \simeq U \times \mathbb{C}^*$ has a Laurent expansion along the fibers. The constant term of it is a holomorphic function g on U which does not depend on the choice of the trivialization. In this way we associate to any holomorphic function G on $L \setminus L_0$ a holomorphic function g on X. Let now $\{x_\nu\}$ be an infinite discrete sequence of points in X and $c_\nu \in \mathbb{C}$ arbitrary complex numbers. Since $L \setminus L_0$ is Stein, there is a holomorphic function G on $L \setminus L_0$ such that $G|\pi^{-1}(x_\nu) \setminus \{0\} = c_\nu$ for every $\nu \in \mathbb{N}$. Therefore the corresponding holomorphic function

$g \in \mathcal{O}(X)$ satisfies $g(x_\nu) = c_\nu$ and so X is holomorphically convex. Similarly, for any two points $x_1 \neq x_2$ in X there is a $g \in \mathcal{O}(X)$ with $g(x_1) \neq g(x_2)$. Consequently X is Stein. $\square$

Remark: The converse of this Lemma also holds in the sense, that the Steiness of X implies L and $L \setminus L_0$ being Stein.

For the convenience of the reader we recall the following result of Simha [17]:

Lemma 2.7 *Let X be a normal Stein space of dimension 2 and $A \subset X$ a closed analytic subset without isolated points. Then $X \setminus A$ is Stein.*

Remark: A more general result is proved in [5]: If X is a normal Stein space of dimension n and $A \subset X$ is a closed analytic subset without isolated points, then $X \setminus A$ is a union of $(n-1)$ Stein open subsets. In particular, $X \setminus A$ is $(n-1)$-complete.

An immediate consequence of Lemma 2.7 is:

Corollary 2.8 *Let X be a Stein space of dimension 2 and $A \subset X$ a closed analytic subset. Assume that for any point $x \in A$ and for any local irreducible component $X_{x,i}$ of X at x the point x is not isolated in $A \cap X_{x,i}$. Then $X \setminus A$ is Stein.*

Proof: This follows immediately from Lemma 2.7 and the invariance of the Stein property under normalization (see [16]). $\square$

3 Construction of the example proving Theorem 1.1

In this section we, now, want to construct a normal Stein space of pure dimension 3 with only one singular point and a closed connected analytic subset $A \subset X$ of pure dimension 2 having the properties stated in Theorem 1.1.

We start with a 1-dimensional torus S and choose $\pi : L \to S$ to be a topologically trivial holomorphic line bundle on S such that no power L^k, $k = 1, 2, 3, \ldots$, is analytically trivial. Let $\overline{\pi} : \overline{L} \to S$, L_0, L_∞ be defined as in Section 2 and let $q : F \to \overline{L}$ be a negative line bundle on $\overline{L}$. We shall identify the zero section F_0 of F over $\overline{L}$ with $\overline{L}$. By the results of Grauert [10] we can blow down F_0 to a point x_0 by a contraction map $p : F \to X$ such that $p(F_0) = \{x_0\}$ and X is a normal Stein space (X is, in fact, affine algebraic), p is proper, holomorphic and induces a biholomorphism $F \setminus F_0 \simeq X \setminus \{x_0\}$.

We define $A := p(q^{-1}(L_0 \cup L_\infty)) \subset X$. Then A is a closed connected analytic subset of X of pure dimension 2, and we put $D := X \setminus A$.
We show at first:

Claim 1: D is not Stein.

Clearly D is biholomorphic to $q^{-1}(\overline{L} \setminus (L_0 \cup L_\infty)) \setminus F_0$. If D would be Stein, then, by Lemma 2.6, the manifold $\overline{L} \setminus (L_0 \cup L_\infty)$ also would be Stein. And, therefore, again by Lemma 2.6, also S would be Stein. This is obviously not possible. Consequently D is not Stein.

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