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Completely J-Positive Linear Systems of Finite Order*

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Abstract. Completely J-positive linear systems of finite order are introduced as a generalization of completely symmetric linear spaces. To any completely J-positive linear system of finite order there is associated a defining measure with respect to which the transfer function has a certain integral represention. It is proved that these systems are asymptotically stable. The observability and reachability operators obey a certain duality rule and the number of negative squares of the Hankel operator is estimated. The Hankel operator is bounded if and only if a certain measure associated with the defining measure is of Carleson type.

We prove that a real symmetric operator valued function which is analytic outside the unit disk has a realization with a completely J-symmetric linear space which is reachable, observable and parbalanced. Uniqueness and spectral minimality of the completely J-symmetric realizations are discussed.

1. Introduction

Infinite dimensional systems have proved to be of interest for some time and they have been studied from different points of view, cf. [10], [11], [15], and [13], to quote only a few of the main papers in the field. They proved to be useful not only in the system theory but also in operator theory, e.g. see [31], [33], [36], [37].

Balanced realizations have been formally introduced by Moore [29] to provide a method for the model reduction of finite dimensional systems. They have since also played an important role in areas such as H^∞ control theory (see e.g. [14]), system identification (see e.g. [24]) and the parametrization of linear systems [34]. The parametrization approach to linear systems using balanced realizations provided a powerful method to solve the inverse spectral problem for Hankel operators (see [31], [33], [28]). In the case of infinite dimensional systems, it is the larger class of parbalanced system, introduced by N.J. Young [40], which plays an important role.

A linear system (A, B, C, D) with finite dimensional state space is called *sign-symmetric* if there exists a sign matrix S, i.e. a diagonal matrix whose diagonal entries are ± 1, such that

$$A = SA^*S, \quad B = SC^*, \quad C = B^*S.$$

Clearly, a necessary condition for a system to have a sign-symmetric realization is that the transfer function is symmetric with respect to the real axis. Such realizations play a role in finite dimensional system theory since the trace of S specifies the Cauchy-index of the

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transfer function ([2]). The Cauchy-index itself plays a role in the study of the topology of rational functions of a fixed Mc-Millan degree (see e.g. [3]), since it characterizes the connected components of the manifold of these functions. Kumar and Wilson [23] showed that each stable continuous-time finite dimensional linear system has a sign-symmetric balanced realization. This was generalized to the case of multivariable symmetric systems in [30] and to infinite-dimensional systems whose Hankel operator is compact with discrete spectrum in [32]. The sign-symmetry matrix S has the additional significance that the diagonal entries are the signs of the eigenvalues of the corresponding Hankel operator ([34], [32]).

The aim of this paper is to investigate (discrete time, time invariant, infinite dimensional) linear systems with sign symmetric transfer functions, and hence with the corresponding Hankel operator selfadjoint. We show that under sufficiently general assumptions, cf. Theorem 6.2, these systems are realized in such a way that the main operator is selfadjoint on a Kreĭn space. It is known for some time that the most tractable class of selfadjoint operators on Kreĭn spaces are the definitizable ones, which fortunately covers the class of selfadjoint operators on Pontryagin spaces. The spectral theory of definitizable operators, developed by Heinz Langer [25], [26], shows that some singularities show up, the so-called critical points. In order to study these operators, a useful approach is to "localize" the critical points. Thus, operators for which a certain power becomes positive, and hence 0 is the only possible critical point, represent the first relevant class to be studied. In this paper we will confine to this class of linear systems which we call of completely J-positive systems of finite order.

From yet another point of view, transfer functions associated with completely symmetric linear systems in Pontrayagin spaces were intensively investigated by M.G. Kreĭn and H. Langer in a series of papers [18], [19], [20], [21]. In these papers various problems related to the generalized resolvent formula in Pontryagin spaces are discussed and applications to interpolation problems and Hamburger-Stieltjes type moment problems are treated. In that case, the transfer function is the so-called Q-function and the properties of this function are carefully studied.

Our study is strictly concentrated on system theoretic problems related to completely J-symmetric infinite dimensional linear systems, as generalizations of the results obtained by the second named author in [35]. Thus, the asymptotic stability of these sytems is obtained in Proposition 3.1. The integral representation of the transfer function, in terms of a unbounded measure associated to this kind of systems, is obtained in Theorem 3.2. It turns out that the observability operator and the reachability operator are related by a certain duality property, in a sufficiently general case, see Theorem 4.1. This gives the possibility to characterize the boundedness of the Hankel operator. Yet another characterization of the boundedness of the Hankel operator is obtained in Theorem 5.3, in terms of a Carleson type property of the defining measure, by means of a theorem of H. Widom [39]. Also, in case the input/output space is finite dimensional, the number of negative squares of the Hankel matrix is finite, cf. Theorem 5.1.

The last section is devoted to realization of transfer functions with a certain symmetry property. In Theorem 6.2, a general result of realization of this kind of transfer functions is

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obtained. We follow the approach of M.G. Kreĭn and H. Langer combined with some technicalities, e.g. Kreĭn-Reid-Lax-Dieudonné Lemma and some ideas from a paper of N.J. Young [40], to obtain a realization which is minimal and parbalanced. Spectral minimality and uniqueness of these realizations are also discussed.

We recall in Section 2 the basics of the spectral theory of J-positive operators of finite order that we use here.

2. J-Positive Operators of Finite Order

2.1. The Spectral Function. Let $\mathcal{H}$ be a Hilbert space with the scalar product denoted by $\langle \cdot, \cdot \rangle$ and let J be a fixed symmetry on $\mathcal{H}$, that is $J^* = J = J^{-1}$. Then on $\mathcal{H}$ one can introduce an indefinite inner product denoted $[\cdot, \cdot]$

$$[x, y] = \langle Jx, y \rangle, \quad x, y \in \mathcal{H}.$$

The Hilbert space $\mathcal{H}$ endowed with such an indefinite inner product $[\cdot, \cdot]$ is called a *Kreĭn space*. Most often one does not fix the positive definite inner product (there are infinity many and all of them produce the same strong topology) of a Kreĭn space, but even though this point of view is the most natural, we will not follow this way since it needs to introduce too much Kreĭn space terminology.

A bounded operator $A \in \mathcal{L}(\mathcal{H})$ is called *J-selfadjoint* if $JA = A^*J$. It is clear that the operator A is J-selfadjoint if and only if the operator JA is selfadjoint in the Hilbert space $\mathcal{H}$. A J-selfadjoint operator A on $\mathcal{H}$ is called *J-positive of order n* if $JA^n \geq 0$. Similarly one defines J-negative operators of order n. A J-positive operator of order 1 is called simply a *J-positive operator*.

REMARK 2.1 A J-positive operator of order 0 is simply a selfadjoint operator on the Hilbert space $\mathcal{H}$. Indeed, by definition, if A is a J-positive operator of order 0 then $JA = A^*J$ and $J \geq 0$. But the only positive symmetry is the identity operator and hence $J = I$. Thus, the notion of J-positive operator of finite order is a generalization of selfadjoint operator on a Hilbert space. ∎

In the following we denote by $\mathcal{R}_0$ the Boole algebra generated by intervals Δ in $\mathbb{R}$ such that its boundary $\partial\Delta$ does not contain the point 0. We recall now a particular case of a celebrated theorem of H. Langer and some of its consequences, cf. [25], [26].

THEOREM 2.2 *Let $A \in \mathcal{L}(\mathcal{H})$ be a J-positive operator of order n. Then $\sigma(A) \subset \mathbb{R}$ and there exists a mapping $E \colon \mathcal{R}_0 \to \mathcal{L}(\mathcal{H})$, uniquely determined with the following properties:*

(1) $E(\Delta)$ is J-selfadjoint for all $\Delta \in \mathcal{R}_0$.

(2) E is a Boole algebra morphism, that is, it is additive and multiplicative.

(3) $E(\mathbb{R}) = I$.

(4) For all $\Delta \in \mathcal{R}_0$ such that the polynomial t^n is positive (negative) on Δ, the operator $E(\Delta)$ is J-positive (J-negative).

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(5) *For all $\Delta \in \mathcal{R}_0$ the operator $E(\Delta)$ is in $\{A\}''$ (the bicommutant of the algebra generated by the operator A).*

(6) *For all $\Delta \in \mathcal{R}_0$ we have $\sigma(A|E(\Delta)\mathcal{H}) \subseteq \overline{\Delta}$.*

The mapping E uniquely associated to the J-positive operator A of some finite order n is called *the spectral function* of A. As a consequence of Theorem 2.2, the spectral function has also the following properties.

COROLLARY 2.3 *With the notation as in Theorem 2.2 let $\Delta \in \mathcal{R}_0$ be closed and such that $0 \notin \Delta$. Then:*

(a) *The function E_Δ defined by*

$$E_\Delta(\Lambda) = E(\Delta \cap \Lambda), \quad \Lambda \in \mathcal{R}_0,$$

can be extended uniquely to a bounded measure with supp $E_\Delta \subseteq \Delta$.

(b) *The operator $AE(\Delta)$ is similar with a selfadjoint operator on a Hilbert space, in particular it has spectral measure.*

(c) *E_Δ is the spectral measure of the operator $AE(\Delta)$, in particular*

$$AE(\Delta) = \int_\Delta t\,\mathrm{d}\, E(t).$$

Corollary 2.3 shows that the spectral function E of a J-positive operator of some finite order n can be regarded as a spectral measure, in general unbounded, on $\mathbb{R} \setminus \{0\}$. We now recall the integral representations associated with J-positive operators of finite order.

COROLLARY 2.4 *With the notation as in Theorem 2.2, there exists an operator $N \in \mathcal{L}(\mathcal{H})$ with the following properties:*

(1) *$JN \geq 0$;*
(2) *$NE(\Delta) = 0$, for all $\Delta \in \mathcal{R}_0$ such that $0 \notin \Delta$;*
(3) *$AN = 0$;*
and such that the following integral representations hold:

$$A^n(\lambda I - A)^{-1} = \int_{\mathbb{R}\setminus\{0\}} \frac{t^n}{\lambda - t}\mathrm{d}\, E(t) + \frac{1}{\lambda}N, \quad \lambda \in \rho(A), \tag{2.1}$$

$$A^n = \int_{\mathbb{R}\setminus\{0\}} t^n\mathrm{d}\, E(t) + N, \tag{2.2}$$

where the integrals are improper at 0 and converge in the strong operator topology.

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Let us also record that, as a consequence of Corollary 2.4, apart of the integral representation (2.2), for all $k > n$ we also have

$$A^k = \int_{\mathbb{R} \setminus \{0\}} t^k \mathrm{d}\, E(t), \qquad (2.3)$$

where the integral converges in the strong operator topology.

2.2 Functional Calculus. Let A be a J-positive operator of order n and let E denote its spectral function. In the following, for σ a compact subset of $\mathbb{R}$ we denote by $\mathcal{C}(\sigma)$ the C^*-algebra of continuous complex valued functions onto σ and let $\| \cdot \|_u$ denote its uniform norm.

REMARK 2.5 As a consequence of Corollary 2.4 one can prove (e.g. see [12]) that for all $f \in \mathcal{C}(\sigma(A))$, the integral

$$\int_{\sigma(A) \setminus \{0\}} f(t)\, t^n \mathrm{d}\, E(t) \qquad (2.4)$$

converges in the strong operator topology as an improper integral and, in addition, the mapping

$$\mathcal{C}(\sigma(A)) \ni f \mapsto \int_{\sigma(A) \setminus \{0\}} f(t) t^n \mathrm{d}\, E(t) \in \mathcal{L}(\mathcal{H}) \qquad (2.5)$$

is uniformly continuous. In particular, this shows that the operator valued measure $t^n \mathrm{d}\, E(t)$ can be extended to a finite measure onto the whole real line $\mathbb{R}$ such that $t^n \mathrm{d}\, E(t)(\{0\}) = 0$ and the improper integral in (2.4) can be equivalently considered as an integral of the function f with respect to this finite measure. ∎

In the following we define a certain Banach algebra of continuous functions and using this and the previous results we will recall a natural functional calculus associated to the operator A. Let $\sigma \subset \mathbb{R}$ be a compact subset such that $\{0\}$ is an accumulation point of σ. By $\mathcal{C}^n(\sigma; 0)$ we denote the class of functions $f \in \mathcal{C}(\sigma)$ with the property that there exists $h_f \in \mathcal{C}(\sigma)$ and a complex polynomial p_f of degree at most $n - 1$ such that

$$f(t) = t^n h_f(t) + p_f(t), \quad t \in \sigma. \qquad (2.6)$$

It is easy to verify that the representation in (2.6) is unique and that $\mathcal{C}^n(\sigma; 0)$ is an algebra. On $\mathcal{C}^n(\sigma; 0)$ we consider the following norm: if $f \in \mathcal{C}^n(\sigma; 0)$ is represented as in (2.6) then

$$\|f\| = \max\{\|h_f\|_u, \|p_f\|_u\}. \qquad (2.7)$$

Then $(\mathcal{C}^n(\sigma; 0), \| \cdot \|)$ is a complete normed algebra, with continuous product and isometric involution. Multiplication of the norm in (2.7) with a certain constant turns $\mathcal{C}^n(\sigma; 0)$ into an involutive Banach algebra.

We now recall the functional calculus with continuous functions obtained by P. Jonas [16] (here we follow the form as in [12]) which is a refinement of the functional calculus

obtained by H. Langer [25]. In the next theorem, the algebra $\mathcal{L}(\mathcal{H})$ is considered with the isometric involution ♯

$$T^\sharp = JT^*J, \quad T \in \mathcal{L}(\mathcal{H}).$$

THEOREM 2.6 *With the notation as before, there exists a unique uniformly continuous mapping*

$$\mathcal{C}^n(\sigma; 0) \ni f \mapsto E(f) \in \mathcal{L}(\mathcal{H}),$$

such that for any polynomial q we have $E(q) = q(A)$. In addition, the mapping E is a homomorphism of Banach algebras with involution and it is given by the formula

$$E(f) = \int_\sigma h_f(t)\, t^n \mathrm{d}\, E(t) + h_f(0)N + p_f(A), \quad f \in \mathcal{C}^n(\sigma; 0).$$

As a consequence of Theorem 2.6, the spectral function E can be considered as a spectral distribution, in the sense of C. Foiaş [5], which is of measure type everywhere on $\mathbb{R}$ except at 0 where it is the derivative of order n of some measure, in the sense of Radon-Nikodym.

For a compact subset σ of the real line we denote by $\mathcal{B}(\sigma)$ the C^*-algebra of bounded borelian functions on σ, endowed with the essential supremum norm $\|\cdot\|_\infty$. In the following we will assume that σ is a compact subset of $\mathbb{R}$ such that $\sigma(A) \subseteq \sigma$ and 0 is a point of accumulation of σ. Also, we will denote by $\mathcal{B}^n(\sigma)$ the involutive Banach algebra consisting of those functions $f \in \mathcal{B}(\sigma)$ such that the represenation (2.6) holds with p_f polynomial of order at most $n-1$ and $h_f \in \mathcal{B}(\sigma)$ continuous in 0. The strong toplogy on $\mathcal{B}^n(\sigma)$ is defined by the norm

$$\|f\| = \max\{\|h_f\|_\infty, \|p_f\|_\infty\}.$$

Clearly, $\mathcal{C}^n(\sigma)$ is a closed subalgebra of $\mathcal{B}^n(\sigma)$.

THEOREM 2.7 *Assume that σ is a compact subset of $\mathbb{R}$ such that $\sigma \supseteq \sigma(A)$ and 0 is an accumulation point of σ. Then there exists a mapping*

$$\mathcal{B}^n(\sigma) \ni f \mapsto E(f) \in \mathcal{L}(\mathcal{H})$$

uniquely determined by the following properties:

(i) *For all polynomials p we have $E(p) = p(A)$.*

(ii) *If $f_k, f \in \mathcal{B}^n(\sigma)$ are such that $\sup_{k \geq 1} \|f_k\| < \infty$ and f_k converges pointwise to f componentwise (i.e. for all $t \in \sigma$ $h_{f_k}(t) \to h_f(t)$ and $p_{f_k}(t) \to p_f(t)$ for $k \to \infty$) then $E(f_k) \to E(f)$ for $k \to \infty$.*

In addition, the mapping E is a homomorphism of involutive Banach algebras and for all $f \in \mathcal{B}^n(\sigma)$ we have

$$E(f) = p_f(A) + \int_{\sigma \setminus \{0\}} h_f(t) t^n \mathrm{d}\, E(t) + h_f(0)N,$$

in particular, the mapping E is an extension of the functional calculus in Theorem 2.6.

As a consequence of Theorem 2.7 and of approximation Theorems of Baire and, respectively, of Weierstraß we have:

COROLLARY 2.8 *Let $\Delta \in \mathcal{R}_0(A)$. Then, if $0 \in \Delta$ ($0 \notin \Delta$) then $E(\Delta)$ can be approximated strongly with operators of type $p(A)A^n + I$ (respectively, with operators of type $p(A)A^n$) where p is polynomial.*

3. The Definining Measure

In this section we consider *linear systems* regarded as quadruples (A, B, C, D) where $A \in \mathcal{L}(\mathcal{H})$ is a contraction, $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$, $C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ and $\mathcal{H}$, $\mathcal{U}$, and $\mathcal{Y}$ are Hilbert spaces. Usually the spaces $\mathcal{U}$, $\mathcal{H}$, and $\mathcal{Y}$ are called, respectively, the *input space*, the *state space* and the *output space*. Also, the operators A, B, C, and D are called, respectively, the *main operator*, the *input operator*, the *output operator*, and the *external operator*.

With every linear system (A, B, C, D) there is associated its *transfer function* $G \colon \rho(A) \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ as follows

$$G(\lambda) = D + C(\lambda I - A)^{-1}B, \quad \lambda \in \rho(A). \tag{3.1}$$

Since the main operator A is assumed contractive, the transfer function is defined and analytic for all $|\lambda| > 1$.

Let us assume that $\mathcal{U} = \mathcal{Y}$ and that on $\mathcal{H}$ there is fixed a symmetry J (and hence the associated Kreĭn space $(\mathcal{H}, [\cdot, \cdot])$). A linear system (A, B, C, D) is called *completely J-symmetric* if the operator A is J-selfadjoint, $C = JB^*$, and $D = D^*$. The completely J-symmetric system is called *completely J-positive of order n* if the operator A is J-positive of order n.

As a consequence of Remark 2.1, the notion of completely J-positive linear system of finite order is a generalization of the notion of completely symmetric linear system as in [35].

We consider first the question of asymptotic stability of completely J-positive liner systems of finite order.

PROPOSITION 3.1 *A completely J-positive linear system (A, B, C, D) of finite order such that $\pm 1 \notin \sigma_p(A)$ is asymptotically stable, that is, $A^k \to 0$ ($k \to \infty$) in the strong operator topology.*

Proof. Let n be the order of the J-positive linear system (A, B, C, D) and E the spectral function of the operator A. As a consequence of Corollary 2.4 for all $h \in \mathcal{H}$ and all $k > n$ we have

$$A^k h = \int\limits_{[-1,1]\setminus\{0\}} t^k \mathrm{d}\, E(t)h = \lim_{\varepsilon \to 0} \int\limits_{[-1,1]\setminus(-\varepsilon,\varepsilon)} t^k \mathrm{d}\, E(t)h.$$

We consider $0 < \delta < 1$ and taking into account of Corollary 2.3 we get

$$A^k h = \int\limits_{[-\delta,\delta]\setminus\{0\}} t^{k-n} t^n \mathrm{d}\, E(t)h + \int\limits_{[-1,1]\setminus(-\varepsilon,\varepsilon)} t^k \mathrm{d}\, E(t)h. \tag{3.2}$$

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Since $t^n \mathrm{d}\, E(t)$ is a finite operator valued measure and $\delta^k \to 0$ $(k \to \infty)$ we have

$$\left\| \int_{[-\delta,\delta]\setminus\{0\}} t^{k-n} t^n \mathrm{d}\, E(t)h \right\| \leq \int_{[-\delta,\delta]\setminus\{0\}} \|t^n \mathrm{d}\, E(t)h\|\, \delta^{k-n} \to 0\, (k \to \infty). \qquad (3.3)$$

On the other hand, since by assumption $\pm 1 \notin \sigma_p(A)$, using Corollary 2.3 we get $\mathrm{d}\, E(\{-1,1\})$ $= 0$ and hence

$$\int_{[-1,1]\setminus(-\varepsilon,\varepsilon)} t^k \mathrm{d}\, E(t)h = \int_{(-1,1)\setminus(-\varepsilon,\varepsilon)} t^k \mathrm{d}\, E(t)h. \qquad (3.4)$$

Further, on the ground of Corollary 2.3, modulo a similarity we can assume that the spectral measure $E|(-1,1)\setminus(-\varepsilon,\varepsilon)$ is selfadjoint with respect to the positive definite inner product $\langle\cdot,\cdot\rangle$. Therefore, using the theorem of dominated convergence of Lebesgue we obtain that

$$\left\| \int_{(-1,1)\setminus(-\varepsilon,\varepsilon)} t^k \mathrm{d}\, E(t)h \right\|^2 = \int_{(-1,1)\setminus(-\varepsilon,\varepsilon)} t^{2k}\langle \mathrm{d}\, E(t)h, h\rangle \to 0\, (k \to \infty). \qquad (3.5)$$

From (3.3), (3.4), and (3.5) we obtain that $\|A^k h\| \to 0$ $(k \to \infty)$. ∎

In the next theorem we consider only linear systems (A, B, C, D) whose input and output spaces are finite dimensional, that is, $\mathcal{U} = \mathbb{C}^m$ and $\mathcal{Y} = \mathbb{C}^p$ for some nonnegative integers m and p. This is sufficient for applications and avoids the complications which are usually encountered when dealing with infinite dimensional vector measures.

THEOREM 3.2 *Let (A, B, C, D) be a linear system which is completely J-positive of order n, such that $\mathcal{U} = \mathcal{Y} = \mathbb{C}^m$, and consider its transfer function G as in (3.1). Then, there exist a J-positive operator $N \in \mathcal{L}(\mathcal{H})$, such that $N^2 = AN = 0$, and a symmetric matrix valued Borel measure $\mathrm{d}\, \nu$ on $[-1,1] \setminus \{0\}$ such that*

$$G(\lambda) = D + \sum_{k=1}^{n} \frac{1}{\lambda^k} B^* J A^{k-1} B + \frac{1}{\lambda^{n+1}} B^* J N B + \frac{1}{\lambda^n} \int_{[-1,1]\setminus\{0\}} \frac{t^n}{(\lambda - t)} \mathrm{d}\, \nu(t). \qquad (3.6)$$

The measure $\mathrm{d}\, \nu$ has also the following two properties:

(a) *$t^n \mathrm{d}\, \nu(t)$ is a positive matrix valued finite Borel measure on $[-1,1]$;*
(b) *The function*

$$G^\perp(z) = \frac{1}{z}\left(G(\frac{1}{z}) - D\right) = \sum_{k \geq 0} a_k z^k, \qquad (3.7)$$

which is analytic for $|z| < 1$, has its Taylor coefficients

$$a_k = \begin{cases} B^* J A^k B, & 1 \leq k \leq n-1; \\ B^* J N B + \int_{[-1,1]\setminus\{0\}} t^n \mathrm{d}\, \nu(t), & k = n; \\ \int_{[-1,1]\setminus\{0\}} t^k \mathrm{d}\, \nu(t), & k \geq n+1. \end{cases} \qquad (3.8)$$

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The measure $\mathrm{d}\,\nu$ is uniquely determined by these two properties, more precisely, if E denotes the spectral function of A we have $\mathrm{d}\,\nu(t) = \mathrm{d}\,B^*JE(t)B$, and the operator N can be chosen

$$N = A^n - \int\limits_{\mathbb{R}\backslash\{0\}} t^n \mathrm{d}\,E(t). \tag{3.9}$$

If, in addition, $\pm 1 \notin \sigma_p(A)$ then $\mathrm{d}\,\nu(\{-1,1\}) = 0$ and $\lim\limits_{k\to\infty} \|a_k\| = 0$.

Proof. Let E be the spectral function of A and the operator N defined as in (3.9). As a consequence of Corollary 2.4 the principal value of the integral in (3.9) exists with a singularity in 0, the operator N is J-positive and $N^2 = AN = 0$. We consider the transfer function G as in (3.1) and then the function $G^\perp$ defined as in (3.7) is analytic in the open unit disc $\mathbb{D}$. Let z be a complex number in $\mathbb{D}$. Taking into account that $C = JB^*$ and that $\|A\| \le 1$ we have

$$G^\perp(z) = B^*J(zI - A)^{-1}B = \sum_{k\ge 0} z^k B^*JA^kB.$$

We now take into account the definition of the operator N and from (2.3) it follows

$$G^\perp(z) = \sum_{k=0}^{n-1} z^k B^*JA^kB + z^n B^*JNB + \sum_{k\ge n} z^n \int\limits_{[-1,1]\backslash\{0\}} t^k \mathrm{d}\,B^*JE(t)B.$$

Further

$$\sum_{k\ge n} z^k \int\limits_{[-1,1]\backslash\{0\}} t^k \mathrm{d}\,B^*JE(t)B = \int\limits_{[-1,1]\backslash\{0\}} \sum_{k\ge n} z^k t^k \mathrm{d}\,B^*JE(t)B$$

$$= \int\limits_{[-1,1]\backslash\{0\}} \frac{z^n t^n}{1 - zt} \mathrm{d}\,B^*JE(t)B.$$

Letting $\mathrm{d}\,\nu(t) = \mathrm{d}\,B^*JE(t)B$ we obtain that

$$G^\perp(z) = \sum_{k=0}^{n-1} z^k B^*JA^kB + z^n B^*JNB + \int\limits_{[-1,1]\backslash\{0\}} \frac{z^n t^n}{1 - zt} \mathrm{d}\,\nu(t). \tag{3.10}$$

From (3.10) and $\lambda = 1/z$ we obtain the representation (3.6) of the transfer function G. Clearly, $\mathrm{d}\,\nu$ is a symmetric M_m-valued Borel measure and taking into account of the properties of the spectral function E as in Theorem 2.2 and Remark 2.5 it follows that $t^n\mathrm{d}\,\nu(t)$ is a finite M_m-valued positive Borel measure on $[-1, 1]$.

We now prove that the measure $\mathrm{d}\,\nu$ is uniquely determined by the properties (a) and (b). To see this, let $\mathrm{d}\,\mu$ be another M_m-valued symmetric Borel measure on $[-1, 1] \backslash \{0\}$ satisfying the properties (a) and (b). According to the property (a), both measures $\mathrm{d}\,\mu$ and $\mathrm{d}\,\nu$ can be considered as matrix valued distributions on $[-1, 1]$ of measure type everywhere except at 0 where they are of finite order $\le n$. By means of the polarization formula it

follows that $d\mu = d\nu$ if and only if the scalar measures $d\mu_x = d\nu_x$ for all $x \in \mathcal{U} = \mathbb{C}^m$, where $d\mu_x(t) = \langle d\mu(t)x, x\rangle$ and similarly for the measure $d\nu$. Note also that the scalar measures $d\mu_x$ and $d\nu_x$ share all the properties that $d\mu$ and, respectively, $d\nu$ have. In particular, the scalar measures $d\mu_x$ and $d\nu_x$ can be equivalently characterized as bounded linear functionals on the Banach algebra $\mathcal{C}^n([-1,1])$ (see subsection 2.2). Thus, in order to prove that they coincide, it is sufficient to prove that for all functions $f \in \mathcal{C}([-1,1])$ such that $f(0) = 0$ and $f(t)/t^k \in \mathcal{C}([-1,1])$ we have

$$\int_{-1}^{1} f(t)d\mu_x(t) = \int_{-1}^{1} f(t)d\nu_x(t), \quad x \in \mathbb{C}^m. \tag{3.11}$$

To this end, for arbitrary fixed $x \in \mathbb{C}^m$ let f be such a function, that is $f(t) = t^n h(t)$ for some $h \in \mathcal{C}([-1,1])$. By Weierstraß theorem there exists a sequence of polynomials $\{p_k\}_{k\geq 1}$ such that $p_k \to h$ uniformly on $[-1,1]$. Since both measures $d\mu$ and $d\nu$ have the property (b) it follows that

$$\int_{-1}^{1} p_k(t)t^n d\mu_x(t) = \int_{-1}^{1} p_k(t)t^n d\nu(t), \quad k \geq 1. \tag{3.12}$$

Therefore, taking into account of (3.12) we have

$$\left\| \int_{-1}^{1} f(t)d\mu_x(t) - \int_{-1}^{1} f(t)d\nu_x(t) \right\|$$

$$\leq \left\| \int_{-1}^{1} f(t)d\mu_x(t) - \int_{-1}^{1} p_k(t)d\mu_x(t) \right\| + \left\| \int_{-1}^{1} f(t)d\nu_x(t) - \int_{-1}^{1} p_k(t)d\nu_x(t) \right\|$$

$$\leq \int_{-1}^{1} |h(t) - p_k(t)|t^n d\mu_x(t) + \int_{-1}^{1} |h(t) - p_k(t)|t^n d\mu_x(t)$$

$$\left(\int_{-1}^{1} t^n d\mu_x(t) + \int_{-1}^{1} t^n d\nu_x(t) \right) \|h - p_k\|_u \to 0 \ (k \to \infty).$$

This proves that equation (3.11) holds for all $x \in \mathbb{C}^m$ and all $f \in \mathcal{C}([-1,1])$ such that $f(0) = 0$ and $f(t)/t^k \in \mathcal{C}([-1,1])$, and hence $d\mu = d\nu$.

Assume now that $\pm 1 \notin \sigma_p(A)$. Then as in 2.3 we have $dE(\{-1,1\}) = 0$ and hence $d\nu(\{-1,1\}) = 0$. Taking into account of the theorem of dominated convergence of Lebesgue applied to the positive finite measures $t^n d\nu_x(t)$ we have that for all $k \geq n$

$$\langle a_k x, x \rangle = \int_{[-1,1]} t^{k-n}t^n d\nu_x = \lim_{\varepsilon \to 0} \int_{-1+\varepsilon}^{1-\varepsilon} t^{k-n}t^n d\nu_x \to 0, \ (k \to \infty).$$

We now use again the polarization formula and the fact that a_k are matrices of order m and conclude from here that $\|a_k\| \to 0 \ (k \to \infty).$ ∎

The matrix valued measure $d\nu$ as in Theorem 3.2 is called the *defining measure* of the system (A, B, C, D). Under the assumptions of Theorem 3.2 and as a consequence of the

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representation (3.6) it follows that the transfer function G has analytic continuation onto $\mathbb{C} \setminus \mathrm{supp}(\mathrm{d}\,\nu)$.

Following [35], a linear system (A, B, C, D) is called *admissible* if $\lim_{\lambda \searrow 1} C(\lambda I + A)^{-1}B$ exists in the strong operator topology.

PROPOSITION 3.3 *A completely J-positive system (A, B, B^*J, D) of order n, with finite dimensional input/output space $\mathcal{U}$ and with defining measure ν, is admissible if and only if the integral* $\int_{[-1,1]\setminus\{0\}} \frac{t^n \mathrm{d}\,\nu(t)}{1+t}$ *is convergent.*

Moreover, in this case we have

$$\lim_{\lambda \searrow 1} B^*J(\lambda I + A)^{-1}B = \sum_{k=0}^{n-1}(-1)^k B^*JA^kB + (-1)^n B^*JNB + \int_{[-1,1]\setminus\{0\}} \frac{t^n \mathrm{d}\,\nu(t)}{1+t}. \quad (3.13)$$

Proof.   Let $\lambda > 1$ be arbitrary. From the integral representations (2.2), (2.3) and the definition of the defining measure as in the proof of Theorem 3.2 we obtain

$$B^*J(\lambda I + A)^{-1}B = \sum_{k=0}^{n-1}\frac{(-1)^k}{\lambda^{k+1}} B^*JA^kB + \frac{(-1)^n}{\lambda^{n+1}}B^*JNB + \frac{(-1)^n}{\lambda^n}\int_{[-1,1]\setminus\{0\}} \frac{t^n \mathrm{d}\,\nu(t)}{\lambda + t}. \quad (3.14)$$

In particular, this shows that the linear system (A, B, B^*J, D) is admissible if and only $\lim_{\lambda \searrow 1} \int_{[-1,1]\setminus\{0\}} \frac{t^n \mathrm{d}\,\nu(t)}{\lambda + t}$ exists (strongly is the same with unformly since the input/output space $\mathcal{U}$ is supposed of finite dimension). Consequenly, if the system is admissible we use the Fatou's Lemma for the monotone sequence of functions $\{\frac{1}{\lambda+t}\}_{\lambda>1}$ and conclude that the integral $\int_{[-1,1]\setminus\{0\}} \frac{t^n \mathrm{d}\,\nu(t)}{1+t}$ is convergent.

Conversely, if the integral $\int_{[-1,1]\setminus\{0\}} \frac{t^n \mathrm{d}\,\nu(t)}{1+t}$ is convergent then we use the Theorem of Lebesgue of dominated convergence applied to the sequence of functions $\{\frac{1}{\lambda+t}\}_{\lambda>1}$ and conclude that the system is admissible. In addition, in this case we can pass to the limit in (3.14) following $\lambda \searrow 1$ and obtain the equation (3.13) ∎

REMARK 3.4 Under the assumptions of Proposition 3.3 assume in addition that the integral $\int_{[-1,1]\setminus\{0\}} \frac{t^n \mathrm{d}\,\nu(t)}{1+t}$ is convergent and let $0 < \varepsilon < 1$ be arbitrary. Then

$$\int_{[-1,1]} \frac{t^n \mathrm{d}\,\nu(t)}{1+t} \geq \int_{[-1,-1+\varepsilon]} \frac{t^n \mathrm{d}\,\nu(t)}{1+t} \geq \frac{\nu([-1,-1+\varepsilon])}{\varepsilon}.$$

Taking into account of the behaviour of the function t^n in the neigbourhood of -1 we obtain from here that

$$\sup_{0<\varepsilon<\delta} \frac{\nu([-1,-1+\varepsilon])}{\varepsilon} < \infty,$$

for some (equivalently for all) $0 < \delta < 1$. ∎

4. Observability and Reachability

We consider again a completely J-positive system (A, B, JB^*, D) of order n, where J is a fixed symmetry on the state space $\mathcal{H}$, and let $\mathcal{U}$ denote the input/output space. Following the general theory we consider $O\colon \mathcal{D}(O)(\subseteq \mathcal{H}) \to \ell^2_{\mathcal{U}}$, the *observability operator* defined by

$$\mathcal{D}(O) = \{h \in \mathcal{H} \mid \sum_{k \geq 0} \|B^* J A^k h\|^2 < \infty\},$$

$$Oh = (B^* J A^k h)_{k \geq 0}, \quad h \in \mathcal{D}(O).$$

By duality one introduces the *reachability operator* $R\colon \mathcal{D}(R)(\subseteq \ell^2_{\mathcal{U}}) \to \mathcal{H}$

$$\mathcal{D}(R) = \{(x_k)_{k \geq 0} \in \ell^2_{\mathcal{U}} \mid \sum_{k \geq 0} \|A^k B x_k\|^2 < \infty\},$$

$$R((x_k)_{k \geq 0}) = \sum_{k \geq 0} A^k B x_k, \quad (x_k)_{k \geq 0} \in \mathcal{D}(R).$$

Note that the domain of R is dense in $\ell^2_{\mathcal{U}}$ since it contains all the sequences with finite support.

Theorem 4.1 *If $\pm 1 \notin \sigma_p(A)$ then $\mathcal{D}(O)$ is dense in $\mathcal{H}$, $O = R^*J$ and $R = JO^*$, in particular both operators O and R are closed.*

Proof. Let $h \in \mathcal{D}(O)$ and $x = (x_k)_{k \geq 0} \in \mathcal{D}(R)$. Taking into account that $A^{*k}J = JA^k$ for all $k \geq 0$ we obtain

$$\langle Oh, x \rangle = \sum_{k \geq 0} \langle B^* J A^k h, x \rangle = \sum_{k \geq 0} \langle h, J A^k B x_k \rangle = \langle h, J \sum_{k \geq 0} A^k B x_k \rangle = \langle h, J R x \rangle. \quad (4.1)$$

This proves that $O \subseteq (JR)^* = R^*J$.

To prove the converse inclusion, let $h \in \mathcal{D}(R^*)$, that is, there exists $z \in \ell^2_{\mathcal{U}}$ such that

$$\langle h, J R x \rangle = \langle z, x \rangle, \quad x \in \mathcal{D}(R). \quad (4.2)$$

On the other hand

$$\langle h, J R x \rangle = \sum_{k \geq 0} \langle h, J A^k B x_k \rangle = \sum_{k \geq 0} \langle B^* J A^k h, x \rangle, \quad x \in \mathcal{D}(R),$$

and from here, (4.2), and the remark that the unit ball of $\mathcal{D}(R)$ is dense in the unit ball of $\ell^2_{\mathcal{U}}$, we get

$$\Big(\sum_{k \geq 0} \|B^* J A^k h\|^2\Big)^{1/2} \leq \sup_{x \in \mathcal{D}(R), \, \|x\| \leq 1} \Big| \sum_{k \geq 0} \langle B^* J A^k h, k \rangle \Big| = \|z\| < \infty,$$

and hence $h \in \mathcal{D}(O)$. Thus we proved $O = R^*J$.

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We now prove that $\mathcal{D}(O)$ is dense in $\mathcal{H}$. To see this, consider the linear manifold $\mathcal{D} = \bigcup_{m \geq 1} E([-1 + 1/m, 1 - 1/m])\mathcal{H}$. By Theorem 2.2, its Corollary 2.3, and $\pm 1 \notin \sigma_p(A)$ we have that $\bigvee_{m \geq 2} E([-1 + 1/m, 1 - 1/m])\mathcal{H} = J(\ker(A - 1) \vee \ker(A + 1))^\perp = \mathcal{H}$, therefore $\mathcal{D}$ is dense in $\mathcal{H}$. In the following we prove that $\mathcal{D} \subseteq \mathcal{D}(O)$. To this end, let $m \geq 2$ and $h \in E([-1 + 1/m, 1 - 1/m])\mathcal{H}$ be fixed. Then,

$$\sum_{k \geq n+1} \|JA^k h\|^2 = \sum_{k \geq n+1} \left\| \int_{-1+1/m}^{1-1/m} t^k \, \mathrm{d} E(t) h \right\|^2 \leq \sum_{k \geq n+1} \left(\int_{-1+1/m}^{1-1/m} |t|^{k-n} \|t^n \mathrm{d} E(t) h\| \right)^2. \quad (4.3)$$

Since $t^n \mathrm{d} E(t)$ is a finite measure, by means of the theorem of dominated convergence of Lebesgue it follows that

$$\lim_{k \to \infty} \int_{-1+1/m}^{1-1/m} |t|^{k-n} \|t^n \mathrm{d} E(t) h\| = 0$$

and hence there exits $N \in \mathbb{N}$ such that $N \geq n$ and for all $k \geq N$

$$\int_{-1+1/m}^{1-1/m} |t|^{k-n} \|t^n \mathrm{d} E(t) h\| < 1.$$

Then

$$\sum_{k \geq N} \left(\int_{-1+1/m}^{1-1/m} |t|^{k-n} \|t^n \mathrm{d} E(t) h\| \right)^2 \leq \sum_{k \geq N} \int_{-1+1/m}^{1-1/m} |t|^{k-n} \|t^n \mathrm{d} E(t) h\|$$

$$= \int_{-1+1/m}^{1-1/m} \frac{|t|^N}{1 - |t|} \|\mathrm{d} \, t^n E(t) h\| < \infty. \quad (4.4)$$

From (4.3) and (4.4) it follows that $\mathcal{D} \subseteq \mathcal{D}(O)$ and hence $\mathcal{D}(O)$ is dense in $\mathcal{H}$.

From (4.1) we also have that $R \subseteq JO^*$. To prove the converse inclusion we proceed as in the proof of the inclusion $R^* J \subseteq O$, provided we first prove that the unit ball of $\mathcal{D}(O)$ is dense in the unit ball of $\mathcal{H}$. For the proof of the latter, it is sufficient to note that from Corollary 2.3 it follows that, without loss of generality, we can assume that for all $m \geq 2$ the projections $E([-1 + 1/m, 1 - 1/m])$ are selfadjoint with respect to the positive definite inner product $\langle \cdot, \cdot \rangle$, too. Then the proof follows as in the Hilbert space case. ∎

COROLLARY 4.2 *Assume that* $\pm 1 \notin \sigma_p(A)$. *Then the following assertions are equivalent:*
   (i) *the observability operator O is bounded;*
   (ii) *the reachability operator R is bounded.*

Recall that a system (A, B, C, D) is called *observable* if the observability operator O is bounded and injective. The system is called *reachable* if the reachability operator R is bounded and has dense range. Note that, as a consequence of Theorem 4.1, the completely J-positive operator system (A, B, B^*J, D) of order n, such that $\pm 1 \notin \sigma_p(A)$, is observable if and only if it is reachable. This also makes interesting the characterization of the kernel of the observability operator.

PROPOSITION 4.3 *Let E be the spectral function and let N be the nilpotent operator associated with the main operator A of the completely J-positive system (A, B, B^*J, D) of order n. Then*

$$\ker(O) = \bigcap_{k=0}^{n} \ker(B^*JA^k) \cap \bigcap_{\Delta \in \mathcal{R}_0} (B^*JE(\Delta) \cap \ker(B^*JN)).$$

Proof. As a consequence of the definition of the observability operator O we have $\ker(O) = \bigcap_{k \geq 0} \ker(B^*JA^k)$. Therefore, if $x \in \bigcap_{k \geq 0} \ker(B^*JA^k)$ then $x \in \ker(B^*Jp(A))$ for all polynomials p. Using Corollary 2.8 it follows that $x \in \ker(B^*JE(\Delta))$ for all $\Delta \in \mathcal{R}_0$. Taking into account the representation (2.2) we obtain also $x \in \ker(B^*JN)$.

Conversely, assume

$$x \in \bigcap_{k=0}^{n} \ker(B^*JA^k) \cap \bigcap_{\Delta \in \mathcal{R}_0} (B^*JE(\Delta)) \cap \ker(B^*JN).$$

From (2.2) we get $x \in \ker(B^*JA^n)$ and then from (2.3) we get $x \in \ker(B^*JA^k)$ for all $k \geq n + 1$. ∎

5. The Hankel Operator

We introduce now the operator $H = OR$. Note that in this definition, the domain of H is $\mathcal{D}(H) = \{x \in \mathcal{D}(R) \mid Rx \in \mathcal{D}(O)\}$. The operator H is called the *Hankel operator* associated to the system (A, B, B^*J, D). From the definition of the operators O and R, the operator H has the following Hankel block matrix representation

$$H \sim \left(B^*JA^{i+j}B\right)_{i,j \geq 0}, \tag{5.1}$$

more precisely, for all $x = \sum_{i \geq 0} \in \ell_{\mathcal{U}}^2$, $y = \sum_{j \geq 0} \in \ell_{\mathcal{U}}^2$ with finite supports we have

$$\langle Hx, y \rangle = \sum_{j \geq 0} \sum_{i \geq 0} \langle B^*JA^{i+j}Bx_j, y_i \rangle. \tag{5.2}$$

This formulation allows us to use some terminology from the theory of hermitian kernels. Thus, we can speak about the *number of negative squares* of the kernel H, denoted by $\kappa^-(H)$. To be more precise, we consider the vector space $\mathcal{F}_0(\mathcal{U})$ consisting of sequences $x \in \ell^2(\mathcal{U})$ of finite support and on $\mathcal{F}_0(\mathcal{U})$ we consider the inner product $[\cdot, \cdot]_H$ given as in (5.2). Then $\kappa^-(H)$ coincides with the negative signature of the inner product space $(\mathcal{F}_0(\mathcal{U}), [\cdot, \cdot]_H)$, more precisely, $\kappa^-(H)$ is the maximal algebraic dimension of subspaces $\mathcal{L} \subseteq \mathcal{F}_0(\mathcal{U})$ with the propery $[x, x]_H < 0$ for all $x \in \mathcal{L} \setminus \{0\}$. Note that in case the Hankel operator H is bounded, $\kappa^-(H)$ coincides either with the number of negative eigenvalues of H, counted with their multiplicities, or is the symbol ∞.

THEOREM 5.1 *The number of negative squares of H is $\leq \left\lceil \frac{n+1}{2} \right\rceil \cdot \dim \mathcal{U}$.*

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Proof.   We recall first briefly the proof corresponding to the case $n = 0$. With the notation as in Theorem 3.2, if $n = 0$ and taking into account Remark 2.1, the block matrix $H = [h_{ij}]_{i,j \geq 0}$ has the entries

$$h_{ij} = B^* J A^{i+j} B = \int_{-1}^{1} t^{i+j} \mathrm{d}\,\nu(t).$$

If $x = (x_i)_{i \geq 0} \in \mathcal{F}_0(\mathcal{U})$ we let $x(t) = \sum_{i \geq 0} t^i x_i$ for $-1 \leq t \leq 1$. Then

$$[x, x]_H = \langle Hx, x \rangle = \int_{-1}^{1} \langle \mathrm{d}\,\nu(t) \cdot x(t), x(t) \rangle \geq 0$$

which proves that $\kappa^-(H) = 0$ in this case.

Assume now that $n \geq 1$. As before we identify the operator H with the matrix $[h_{ij}]_{i,j \geq 0}$ as in (5.1) and consider the infinite hermitian kernel H_0 with the representation

$$H_0 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & [h_{ij}]_{i,j \geq \left[\frac{n+1}{2}\right]} \end{array} \right]. \tag{5.3}$$

Note that for $i, j \geq \left[\frac{n+1}{2}\right]$ we have $i + j \geq n$ and

$$h_{ij} = B^* J N^{i+j-n} B + \int_{-1}^{1} t^n \mathrm{d}\,\nu(t) = \begin{cases} B^* J N B + \int_{-1}^{1} t^n \mathrm{d}\,\nu(t), & i + j = n, \\ \int_{-1}^{1} t^{i+j-n} \cdot t^n \mathrm{d}\,\nu(t), & i + j > n. \end{cases}$$

Taking into account that the operator valued measure $t^n \mathrm{d}\,\nu(t)$ is nonnegative and that $JN \geq 0$ we proceed as in the case $n = 0$ and prove that $\kappa^-(H_0) = 0$.

Further, consider the matrix $H_1 = H - H_0$. Then

$$H_1 = \left[\begin{array}{c|c} [h_{ij}]_{0 \leq i,j \leq \left[\frac{n+1}{2}\right]-1} & * \\ \hline * & 0 \end{array} \right], \tag{5.4}$$

where the block matrices marked with $*$ are of now importance here. Clearly we have

$$\kappa^-\left([h_{ij}]_{0 \leq i,j \leq \left[\frac{n+1}{2}\right]-1}\right) \leq \left[\frac{n+1}{2}\right] \cdot \dim \mathcal{U}.$$

Applying [6], Corollary 3.2 we obtain

$$\kappa^-(H_1) \leq \left[\frac{n+1}{2}\right] \cdot \dim \mathcal{U}.$$

Taking into account that $H = H_0 + H_1$ we obtain

$$\kappa^-(H) \leq \kappa^-(H_0) + \kappa^-(H_1) \leq \left[\frac{n+1}{2}\right] \cdot \dim \mathcal{U}. \blacksquare$$

In the following we are interested in characterizing the boundedness of the Hankel operator. As an immediate consequence of Proposition 4.1 we have

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PROPOSITION 5.2 *Assume that $\pm 1 \notin \sigma_p(A)$. Then the following assertions are equivalent:*

(i) *the observability operator O is bounded, equivalently the reachability operator is bounded;*

(ii) *the Hankel operator is everywhere defined in $\ell_{\mathcal{U}}^2$.*

(iii) *the Hankel operator is bounded and everywhere defined in $\ell_{\mathcal{U}}^2$.*

A more interesting criterion of boundedness of the Hankel operator can be obtained in terms of the defining measure ν. Recall that a planar measure μ on the open unit disk $\mathbb{D}$ is called a Carleson measure if the Poisson integral induces a bounded operator from $L^p(\partial\mathbb{D})$ into $L^p(\mu)$, for some $p \geq 1$. According to a celebrated theorem of L. Carleson [4], a planar measure μ is a Carleson measure if and only if $\sup_I \mu(R(I))/|I| < \infty$, where I runs through the set of all arcs of $\mathbb{T} = \partial\mathbb{D}$ and $R(I)$ denotes the set of all complex numbers $z \in \mathbb{D}$ such that $z/|z| \in I$ and $1 - |z| \leq |I|/2\pi$.

In the following theorem the definig measure ν will be considered as a vector valued planar measure on $\mathbb{D}$ with support in $[-1, 1]$.

THEOREM 5.3 *Assume $\pm 1 \notin \sigma_p(A)$ and that the input-output space $\mathcal{U}$ is finite dimensional. The following assertions are equivalent:*

(1) *the Hankel block matrix H in (5.1) defines a bounded operator in $\ell_{\mathcal{U}}^2$;*

(2) *for all $x \in \mathcal{U}$ the measure $\Delta \mapsto \int_\Delta t^n \langle \mathrm{d}\nu(t)x, x \rangle$, $\Delta \in \mathcal{B}(\mathbb{D})$, is a Carleson measure.*

(3) *For some (equivalently, for all) $0 < \delta < 1$*

$$\sup_{0 < \varepsilon < \delta} \left(\frac{\|\nu([-1, -1 + \varepsilon])\| + \|\nu([1 - \varepsilon, 1])\|}{\varepsilon} \right) < \infty.$$

Proof. $(1) \Leftrightarrow (2)$. The Hankel block matrix in (5.1) can be partitioned in the following way:

$$\left[\begin{array}{c|c} [B^*JA^{i+j}B]_{0 \leq i,j \leq n} & [B^*JA^{i+j}B]_{0 \leq i \leq n,\ j \geq 0} \\ \hline [B^*JA^{i+j}B]_{i \geq 0,\ 0 \leq j \leq n} & [B^*JA^{i+j}B]_{i,j \geq n} \end{array} \right].$$

Note that the north-west matrix is bounded and that the north-east and south-west matrices are contained in the south-east matrix. Thus, the Hankel block matrix H defines a bounded operator in $\ell_{\mathcal{U}}^2$ if and only the Hankel matrix $[B^*JA^{i+j}B]_{i,j \geq n}$ induces a bounded operator in $\ell_{\mathcal{U}}^2$.

Recall now that by Theorem 3.2 the entries of the Hankel block matrix can be described as the moments of the defining measure ν

$$B^*JA^{i+j}B = \int_{[-1,1]\backslash\{0\}} t^{i+j}\mathrm{d}\nu(t) = \int_{-1}^1 t^{i+j-n}t^n\mathrm{d}\nu(t), \quad i,j \geq n. \tag{5.5}$$

Moreover, since $\mathcal{U}$ is finite dimensional, the Hankel block matrix $(B^*JA^{i+j}B)_{i,j \geq n}$ induces a bounded operator in $\ell_{\mathcal{U}}^2$ if and only if for all $x \in \mathcal{U}$ the Hankel block matrix $[\langle B^*JA^{i+j}Bx, x \rangle]_{i,j \geq n}$ induces a bounded operator on ℓ^2. We now use (5.5) and a theorem

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of H. Widom [39] which characterizes those planar measures with support in $[-1,1]$ whose moments define a bounded Hankel operator, to conclude the equivalence of (i) and (2).

$(2) \Leftrightarrow (3)$. We consider the $\mathcal{U}$-valued Borel finite positive measure μ

$$\mu(\Delta) = \int_\Delta t^n \mathrm{d}\,\nu(t), \quad \Delta \in \mathcal{B}(\mathbb{D}).$$

Of course μ has its support contained in $[-1,1]$. For arbitrary $x \in \mathcal{U}$ we consider the scalar Borel finite positive measure $\mu_x = \langle \mu x, x \rangle$. Taking into account that $\operatorname{supp} \mu \subseteq [-1,1]$ and of the above mentioned theorem of L. Carleson, μ_x is a Carleson measure if and only if $\mu_x((-1,-1+\varepsilon]) \leq K\varepsilon$ and $\mu_x([1-\varepsilon,1)) \leq K\varepsilon$ as $\varepsilon \to 0$, which is equivalent with the assertion (3) (we again take into account that $\mathcal{U}$ has finite dimension). ∎

6. Realization Theory

In this section we deal with realizations of functions $G(z)$ of type obtained in Section 3, by linear systems which we wish to be completely J-selfadjoint within some Kreĭn state space and specified fundamental symmetry J. From system theoretic consideration we would also want that these realizations be observable, reachable and parbalanced. It turns out that this kind of realization is possible for a class of functions which in a certain case is larger than the class of functions obtained in Section 3, more precisely these functions satisfy only a symmetry property, but in another sense it is more restrictive, namely the Hankel operator is supposed to be bounded. We first recall the definition of Kreĭn spaces induced by selfadjoint operators and a lifting property.

Let us consider a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and let H be a bounded selfadjoint operator on $\mathcal{H}$. On $\mathcal{H}$ we consider the (in general, indefinite) inner product $[\cdot, \cdot]_H$ defined by

$$[x,y]_H = \langle Hx, y \rangle, \quad x, y \in \mathcal{H}.$$

Let $\widehat{\mathcal{H}} = \mathcal{H} \ominus \ker H$ and note that the restriction of the inner product $[\cdot, \cdot]_H$ to $\widehat{\mathcal{H}}$ is nondegenerate. On $\widehat{\mathcal{H}}$ we consider the norm $\||H|^{1/2} \cdot \|$ and let $\mathcal{K}_H$ be the completion of $(\widehat{\mathcal{H}}, \||H|^{1/2} \cdot \|)$ to a Hilbert space. Now remark that, since the operator H is bounded, we have

$$|[x,y]_H| \leq \||H|^{1/2}x\| \, \||H|^{1/2}y\|, \quad x, y \in \mathcal{H},$$

and hence the inner product $[\cdot, \cdot]_H$ can be uniquely extended to $\mathcal{K}_H$. It is now easy to see that $(\mathcal{K}_H, [\cdot, \cdot]_H)$ is a Kreĭn space, the strong topology on this Kreĭn space is induced by the norm $\||H|^{1/2}\|$. We also note that the corresponding fundamental symmetry is the extension of the operator S_H, where $H = S_H|H|$ is the polar decomposition of H and S_H denotes the corresponding selfadjoint partial isometry.

Let now $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $A \in \mathcal{L}(\mathcal{H}_1)$, $A = A^*$, and $B \in \mathcal{L}(\mathcal{H}_2)$. Also, let $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be given and consider the induced Kreĭn spaces $(\mathcal{K}_A, [\cdot, \cdot]_A)$ and $(\mathcal{K}_B, [\cdot, \cdot]_B)$. We say that the operator T induces an operator $\widetilde{T} \in \mathcal{L}(\mathcal{K}_A, \mathcal{K}_B)$ if $T \ker A \subseteq \ker B$ and denoting by $\widehat{T}$ the corresponding factor operator in $\mathcal{L}(\mathcal{H}_1 \ominus \ker A, \mathcal{H}_2 \ominus \ker B)$ the operator $\widehat{T}$ is bounded with respect to the norms $\||A|^{1/2} \cdot \|$ and, respectively, $\||B|^{1/2} \cdot \|$. The operator

$\widetilde{T}$ is the extension by continuity of the operator $\widehat{T}$ and hence it is uniquely determined by T.

We recall now a result originally due to M.G. Kreĭn [17] and obtained independently by W.T. Reid [38], P.D. Lax [27], and J. Dieudonné [8]), whose indefinite variant was obtained by A. Dijksma, H. Langer, and H. de Snoo [9]. For the following formulation, including the norm estimate, we refer to [7].

LEMMA 6.1 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, $H \in \mathcal{L}(\mathcal{H}_1)$ and $G \in \mathcal{L}(\mathcal{H}_2)$ be selfadjoint operators, and $T_1 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ be operators such that*

$$[T_1 x, y]_G = [x, T_2 y]_H, \quad x \in \mathcal{H}_1, \ y \in \mathcal{H}_2,$$

or equivalently, $HT_1 = T^*G$. Then

$$\||G|^{1/2} T_1 x\| \leq \||G|^{1/2} T_1 S_H T_2 S_G T_1\|^{1/2} \||H|^{1/2} x\|, \quad x \in \mathcal{H}_1,$$

and similarly

$$\||H|^{1/2} T_2 y\| \leq \||H|^{1/2} T_2 S_G T_1 S_H T_2\|^{1/2} \||G|^{1/2} y\|, \quad y \in \mathcal{H}_2,$$

and hence the operators T_1 and T_2 induce uniquely determined operators $\widetilde{T}_1 \in \mathcal{L}(\mathcal{K}_H, \mathcal{K}_G)$ and, respectively, $\widetilde{T}_2 \in \mathcal{L}(\mathcal{K}_G, \mathcal{K}_H)$ such that

$$[\widetilde{T}_1 x, y]_G = [x, \widetilde{T}_2 y]_H, \quad x \in \mathcal{K}_H, \ y \in \mathcal{K}_G.$$

Let $\mathcal{U}$ be a Hilbert space and assume that $G \colon \{z \in \mathbb{C} \mid |z| > 1\} \to \mathcal{L}(\mathcal{U})$ is an operator valued function which is analytic everywhere on its domain of definition and at infinity. One can define an operator valued analytic function $g \colon \mathbb{D} \to \mathcal{L}(\mathcal{U})$ by

$$g(z) = \frac{1}{z}\left(G\left(\frac{1}{z}\right) - G(\infty)\right), \quad |z| < 1.$$

Then g has the Taylor expansion

$$g(z) = \sum_{k \geq 0} S_k z^k, \quad |z| < 1.$$

Associated with the function G one can consider the block-operator Hankel matrix

$$H = \begin{bmatrix} S_0 & S_1 & S_2 & \dots & S_k & \dots \\ S_1 & S_2 & S_3 & \dots & S_{k+1} & \dots \\ S_2 & S_3 & \dots & & & \\ \vdots & & & & & \\ S_k & & & & & \\ \vdots & & & & & \end{bmatrix}. \tag{6.1}$$

Following N.J. Young [40], we say that a system (A, B, C, D) is *parbalanced* if the corresponding observability and reachability operators O and, respectively, R are bounded and the observability gramian O^*O coincide with the reachability gramian RR^*.

THEOREM 6.2 *Let $\mathcal{U}$ be a Hilbert space and let $G \colon \{z \in \mathbb{C} \mid |z| > 1\} \to \mathcal{L}(\mathcal{U})$ be an operator valued function which is analytic on its domain and at infinity, such that G is symmetric, that is,*

$$G(\bar{z}) = G(z)^*, \quad |z| > 1.$$

If the Hankel block-operator matrix in (6.1) defines a bounded operator in $\ell_{\mathcal{U}}^2$ then there exists a Kreĭn state space $\mathcal{K}$ with a specified fundamental symmetry J on $\mathcal{K}$ such that G is realized by a completely J-symmetric linear system (A, B, B^*J, D) which is observable, reachable and parbalanced.

Proof. Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the scalar product and the corresponding norm on the Hilbert space $\mathcal{U}$. We consider the Hilbert space $\ell_{\mathcal{U}}^2$, of square summable sequences with entries in $\mathcal{U}$, endowed with scalar product also denoted by $\langle \cdot, \cdot \rangle$

$$\langle f, g \rangle = \sum_{k \geq 0} \langle f_k, g_k \rangle, \quad f = (f_k)_{k \geq 0}, \ g = (g_k)_{k \geq 0} \in \ell_{\mathcal{U}}^2.$$

Consider the Hilbert space $\mathcal{H} = \ell_{\mathcal{U}}^2$. According to our assumption, let H denote the bounded operator defined by the Hankel block-operator H as in (6.1). We consider (A_0, B_0, C_0, D_0) the right shift realization of G, that is,

$$G(z) = D_0 + C_0(zI - A_0)^{-1}B_0, \quad |z| > 1, \tag{6.2}$$

where the operator $A_0 \colon \mathcal{H} \to \mathcal{H}$ is the right shift

$$(A_0 f)_k = \begin{cases} f_{k-1}, & k \geq 1, \\ 0, & k = 0. \end{cases} \tag{6.3}$$

the operator $B_0 \colon \mathcal{U} \to \mathcal{H} = \ell_{\mathcal{U}}^2$ is defined by

$$B_0 = \begin{bmatrix} I & 0 & \dots & 0 & \dots \end{bmatrix}^t, \tag{6.4}$$

where t denotes the matrix transpose, the operator $C_0 \colon \mathcal{H} \to \mathcal{U}$ is defined by

$$C_0 = \begin{bmatrix} S_0 & S_1 & \dots & S_k & \dots \end{bmatrix}, \tag{6.5}$$

and the external operator is $D_0 = Q(\infty)$. Note that the operator C_0 is bounded due to the assumption on boundedness of the Hankel operator H.

Since the function G is assumed symmetric, it follows that the operators $S_k \in \mathcal{L}(\mathcal{U})$ are selfadjoint and hence, the Hankel operator H is selfadjoint on $\mathcal{H}$. Therefore, on the Hilbert space $\mathcal{H}$ we can define the (in general, indefinite) inner product $[\cdot, \cdot]_H$

$$[f, g]_H = \langle Hf, g \rangle = \sum_{j,k \geq 0} \langle S_{j+k} f_j, g_k \rangle, \quad f, g \in \mathcal{H}.$$

We consider the Kreĭn space $(\mathcal{K}_H, [\cdot, \cdot]_H)$ with the positive inner product $\langle |H| \cdot, \cdot \rangle$. The fundamental symmetry J relating the inner products $[\cdot, \cdot]_H$ and $\langle |H| \cdot, \cdot \rangle$ is the operator induced by the partial isometry S_H on $\mathcal{K}_H$.

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We now prove that A_0 is H-selfadjoint. Indeed, if $f = (f_k)_{k \geq 0}$ and $g = (g_k)_{k \geq 0}$ are arbitrary sequences in $\mathcal{H}$ then

$$[A_0 f, g]_H = \sum_{j,k \geq 0} \langle S_{j+k}(Af)_j, g_k \rangle = \sum_{j \geq 1} \langle S_{j+k} f_{j-1}, g_k \rangle \tag{6.6}$$

$$= \sum_{j,k \geq 0} \langle S_{j+k+1} f_j, g_k \rangle = \sum_{j,k \geq 0} \langle f_j, S_{j+k+1} g_k \rangle = [f, A_0]_H.$$

Since A_0 is H-symmetric we can apply Lemma 6.1 and obtain that the operator A_0 induces a uniquely determined operator $A \in \mathcal{L}(\mathcal{K}_H)$ which is selfadjoint with respect to the inner product $[\cdot, \cdot]_H$. In addition,

$$\||H|^{1/2} A_0 x\| \leq \||H|^{1/2} A_0 S_H A_0 S_H A_0\|^{1/2} \||H|^{1/2} x\|, \quad x \in \mathcal{H}. \tag{6.7}$$

Since the shift A_0 is clearly a contraction, as well as the selfadjoint partial isometries S_H, and by assumption H is bounded and hence, by spectral theory, the same is $|H|^{1/2}$, it follows that the operator norm of A, calcutated with respect to the norm $\||H|^{1/2}\|$, is $\leq \||H|^{1/2}\|$.

We prove now that the induced operator A is contractive with respect to the norm $\||H|^{1/2}\|$. To see this, let us first note that an equivalent formulation of this claim is that the opertor $T : \mathcal{R}(|H|^{1/2}) \to \mathcal{H} \ominus \ker H$ defined by,

$$Ty = |H|^{1/2} A_0 |H|^{-1/2} y, \quad y \in \mathcal{R}(|H|^{1/2}),$$

which, by (6.7), is bounded, is contractive. It is easy to see that the bounded adjoint operator B^* is defined by

$$T^* x = |H|^{-1/2} A_0^* |H|^{1/2} x, \quad x \in \mathcal{H} \ominus \ker H. \tag{6.8}$$

Let us note that, an equivalent formulation of (6.6) is $HA_0 = A_0^* H$. Then taking into account that $A_0 A_0^* \leq 1$ it follows

$$H^2 - A_0^* H^2 A_0 = H^2 - A_0^* H A_0^* H = H^2 - H A_0 A_0^* H = H(1 - A_0 A_0^*) H \geq 0.$$

Since $H^2 = |H|^2$, from here we get that there exists a contraction $Z \in \mathcal{L}(\mathcal{H})$ such that

$$|H| A_0 = Z^* |H|. \tag{6.9}$$

From (6.8) and (6.9) we get

$$T^* T = |H|^{1/2} A_0^* |H| A_0 |H|^{1/2} = |H|^{1/2} Z A_0 |H|^{-1/2}. \tag{6.10}$$

Let F denote the spectral measure of the positive selfadjoint operator $|H|$ and for $n \geq 1$ denote $P_n = F(1/n, +\infty)$. Taking into acount that for bounded operators X, Y we have $\sigma(XY) \setminus \{0\} = \sigma(YX) \setminus \{0\}$, from (6.10) we get

$$\sigma(P_n T^* T P_n) \setminus \{0\} = \sigma(P_n |H|^{1/2} Z A_0 |H|^{-1/2} P_n) \setminus \{0\}$$

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$$= \sigma(ZA_0|H|^{-1/2}P_n|H|^{1/2}P_n) \setminus \{0\} = \sigma(ZA_0P_n) \setminus \{0\} \subseteq \overline{\mathbf{D}},$$

where we take into account that all the operators Z, A_0, and P_n are contractions and hence the same is ZA_0P_n. Therefore, the operator TP_n is contractive and hence

$$\|Tx\| \leq \|x\|, \quad x \in \bigcup_{n \geq 1} F(1/n, +\infty)\mathcal{H}.$$

This implies that the operator T is contractive on $\mathcal{H} \ominus \ker H$ and the claim is proved.

Further, let us note that

$$[B_0 x, h]_H = \langle x, C_0 h \rangle, \quad x \in \mathcal{U}, \ h \in \mathcal{H}. \tag{6.11}$$

Indeed, taking into account of (6.4) and (6.5), for all $x \in \mathcal{U}$ and $h \in \mathcal{H}$ we have

$$[B_0 x, h]_H = \sum_{k \geq 0} \langle S_k x, h_k \rangle = \langle x, \sum_{k \geq 0} S_k h_k \rangle = \langle x, C_0 h \rangle.$$

Again by Lemma 6.1, it follows that the operators B_0 and C_0 induce uniquely determined operators $B \in \mathcal{L}(\mathcal{U}, \mathcal{K}_H)$ and, respectively, $C \in \mathcal{L}(\mathcal{K}_H, \mathcal{U})$ such that

$$[Bx, h]_H = \langle x, Ch \rangle, \quad x \in \mathcal{U}, \ h \in \mathcal{K}_H. \tag{6.12}$$

Letting $D = D_0 = G(\infty)$, it follows that the linear system (A, B, C, D) is completely J-selfadjoint. In order to show that this is also a realization of the analytic function G, we note that for all complex z with $|z| > 1$, since A is contractive the operator $(zI - A)^{-1} \in \mathcal{L}(\mathcal{K}_H)$ exists and an application of Lemma 6.1 shows that it is the uniquely determined operator induced by $(zI - A_0)^{-1}$. Thus, from (6.2) we obtain that

$$G(z) = D + C(zI - A)^{-1}B, \quad |z| > 1,$$

that is, the completely J-symmetric linear system (A, B, C, D) is a realization of the function G.

Taking into account the definition of the shift realization as in (6.3), (6.4), and (6.5) it follows that the observability operator O_0 of the system (A_0, B_0, C_0, D_0) has the matrix representation

$$O_0 = \begin{bmatrix} C_0 & C_0 A_0 & C_0 A_0^2 & \cdots \end{bmatrix} = \begin{bmatrix} S_0 & S_1 & S_2 & \cdots & S_k & \cdots \\ S_1 & S_2 & S_3 & \cdots & S_{k+1} & \cdots \\ S_2 & S_3 & \cdots & & & \\ \vdots & & & & & \\ S_k & & & & & \\ \vdots & & & & & \end{bmatrix} = H. \tag{6.13}$$

Since clearly the operator H is H-hermitian and taking into account of the uniqueness part in Lemma 6.1 it follows that the observability operator O of the system (A, B, C, D)

coincides with the operator induced by H on $\mathcal{K}_H$ and valued in $\ell^2_{\mathcal{U}}$, in particular the operator O is bounded.

Similarly, let R_0 be the reachability operator of the linear system (A_0, B_0, C_0, D_0). From (6.3), (6.4), and (6.5) it follows that the matrix of R_0 is

$$R_0 = \begin{bmatrix} B_0 & A_0 B_0 & A_0^2 B_0 & \cdots \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & \cdots \\ 0 & I & 0 & \cdots & 0 & \cdots \\ 0 & 0 & I & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & I & \cdots \\ \vdots & & & & & \end{bmatrix}. \tag{6.14}$$

Therefore, the reachability operator coincides with the operator of identification of $\ell^2_{\mathcal{U}}$ with $\mathcal{H}$. We again invoke the uniqueness part of Lemma 6.1 and conclude that the reachability operator R of the linear system (A, B, C, D) coincides with the operator induced by R_0, that is, the operator $P_{\mathcal{H} \ominus \ker H} : \ell^2 \to \mathcal{K}_H$, or even more precisely, the composition of the embedding of $\mathcal{H}$ into $\mathcal{K}_H$ with the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H} \ominus \ker H$. In particular, this implies that the reachability operator $R : \ell^2_{\mathcal{U}} \to \mathcal{K}_H$ is bounded. Since by the construction of the Kreĭn space $\mathcal{K}_H$, the space $\mathcal{H} \ominus \ker H$ is dense in $\mathcal{K}_H$ it follows that the reachability operator R has dense range and hence the system (A, B, C, D) is reachable.

We now remark that

$$\langle O_0 h, f \rangle = \langle H h, f \rangle = [h, f]_H = [h, R_0 f]_H, \quad h \in \mathcal{H}, \ f \in \ell^2_{\mathcal{U}}.$$

This implies that the induced operators $O : \mathcal{H} \to \ell^2_{\mathcal{U}}$ and $R : \ell^2_{\mathcal{U}} \to \mathcal{H}$ satisfy the relation

$$\langle O h, f \rangle = [h, R h]_H, \quad h \in \mathcal{K}_H, \ f \in \ell^2_{\mathcal{U}},$$

equivalently, that the observability operator O is the "adjoint" of the reachability operator R. Since, as proved before, R has dense range, this implies that the observability operator O is injective and hence the system (A, B, C, D) is also observable.

Finally we prove that the system (A, B, C, D) is parbalanced, that is, the observability gramian $O^* O$ coincides with the reachability gramian $R R^*$. To see this, recall (6.13) and note that

$$\langle O h, f \rangle = \langle H, f \rangle = \langle |H| h, S_H f \rangle, \quad h \in \mathcal{H} \ominus \ker H, \ f \in \ell^2_{\mathcal{U}}.$$

Taking into account that the scalar product on $\mathcal{K}_H$ is $\langle |H| \cdot, \cdot \rangle$ it follows that $O^* = S_H$, or more precisely, the operator induced by $S_H : \ell^2 \to \mathcal{K}_H$, and hence $O^* O = S_H H = |H|$. From (6.14) we get

$$\langle |H| R_0 f, h \rangle = \langle |H| f, h \rangle = \langle f, |H| h \rangle, \quad f \in \ell^2_{\mathcal{U}}, \ h \in \mathcal{H} \ominus \ker H.$$

This shows that $R^* = |H| : \mathcal{K}_H \to \ell^2_{\mathcal{U}}$ and hence $R R^* = |H|$. Thus the system (A, B, C, D) is parbalanced. ∎

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COROLLARY 6.3 *Assume, in addition to the assumptions of Theorem 6.2, that for some $n \geq 0$ and all $|\lambda| > 1$ the function G has the representation*

$$G(\lambda) = D + \sum_{k=1}^{n} \frac{1}{\lambda^k} S_{k-1} + \frac{1}{\lambda^{n+1}} \Gamma + \frac{1}{\lambda^n} \int_{[-1,1] \setminus \{0\}} \frac{t^n}{(\lambda - t)} d\nu(t),$$

where $\{S_k\}_{k=0}^{n-1}$ is a family of bounded selfadjoint operators on $\mathcal{U}$, $D \in \mathcal{L}(\mathcal{U})$, $D = D^*$, $\Gamma \in \mathcal{L}(\mathcal{U})$, $\Gamma \geq 0$, and ν is a hermitian $\mathcal{L}(\mathcal{U})$-valued measure on $[-1,1] \setminus \{0\}$ such that $t^n d\nu(t)$ is a finite and positive measure. Then the realization (A, B, C, D) constructed as in Theorem 6.2 is completely J-positive of order n.

Proof. With the notation as in the proof of Theorem 6.2 we first note that as in Theorem 5.1 we have that the Hankel block-matrix $[S_{j+k+n}]_{j,k \geq 0}$ is nonnegative on $\ell_{\mathcal{U}}^2$. On the other hand, from the definition of the operator A_0 as in (6.3) we have

$$[A_0^n f, g]_H = \langle \Gamma f_n, g_n \rangle + \sum_{j,k \geq 0} \langle S_{j+k+n} f_j, g_k \rangle, \quad f, g \in \ell_{\mathcal{U}}^2.$$

Therefore

$$[A_0^n f, f]_H = \langle \Gamma f_n, f_n \rangle + \sum_{j,k \geq 0} \langle S_{j+k+n} f_j, f_k \rangle \geq 0, \quad f \in \ell_{\mathcal{U}}^2.$$

It is easy to see that the operator induced by A_0^n is exactly A^n and hence A is J-positive of order n. ∎

For an analytic $\mathcal{L}(\mathcal{U})$-valued function G, let us denote by $\sigma(G)$ the complement in $\mathbb{C}$ of the largest possible domain of analytic continuation of G. If (A, B, C, D) is a realization of G then clearly $\sigma(G) \subseteq \sigma(A)$. If the converse inclusion holds, $\sigma(A) \subseteq \sigma(G)$, then the realization is called *minimal spectral*. We are interested now in this property for completely J-positive realizations of finite order, as in Corollary 6.3. The following proposition gives a partial answer. The approach we follow is closely related to the positive definite case, see [1].

PROPOSITION 6.4 *Let (A, B, B^*J, D) be a completely J-positive realization of finite order of the transfer function G such that the reachability operator R has dense range. Then*

$$\sigma(A) \setminus \{0\} \subseteq \sigma(G)$$

Proof. Let Δ be a compact real interval which does not contain 0. Then either $\Delta \subset (-\infty, 0)$ or $\Delta \subset (0, +\infty)$. To make a choice, assume that $\Delta \subset (0, +\infty)$.

From the construction of the spectral function A of a J-positive operator of finite order as in [26] we have

$$E(\Delta) = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \frac{1}{2\pi i} \int_{C_{\Delta_\varepsilon}^\delta} (zI - A)^{-1} dz, \tag{6.15}$$

where $\Delta_\varepsilon = [a - \varepsilon, b + \varepsilon]$, assuming that $\Delta = [a, b]$, $C_{\Delta_\varepsilon}^\delta$ is a rectangle symmetric with respect to the real axis constructed around the interval Δ_ε from which we remove two segments of length 2δ around the points of coordinates $(-\varepsilon + a, 0)$ and $(b + \varepsilon, 0)$.

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Let us assume now that G has analytic continuation in a neighbourhood of Δ. Then, from (6.15) and taking into account that the system (A, B, B^*J, D) is a realization of G, applying the Cauchy formula we get

$$B^*JE(\Delta)B = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \frac{1}{2\pi i} \int_{C_{\Delta_\varepsilon}^\delta} (G(z) - D)\mathrm{d}\,z = 0.$$

Taking into account of Theorem 2.2 it follows that $E(\Delta)\mathcal{H}$ is a uniformly positive subspace of $\mathcal{H}$ and hence, for arbitrary u in the input/output space $\mathcal{U}$ we have

$$0 = \langle B^*JE(\Delta)Bu, u \rangle = [E(\Delta)Bu, E(\Delta)Bu] \geq \alpha \|E(\Delta)Bu\|^2,$$

for some $\alpha > 0$. Then $E(\Delta)B = 0$ follows and since the spectral function E commutes with the main operator A we obtain

$$0 = A^k E(\Delta)B = E(\Delta)A^k B, \quad k \geq 0.$$

From here we obtain that $E(\Delta)|\mathcal{R}(R) = 0$ and since it is assumed that the reachability operator R has dense range, it follows that $E(\Delta) = 0$ and hence $\Delta \cap \sigma(A) = \emptyset$. ∎

Finally, we are interested to determine to which extent the realizations of symmetric transfer functions is unique. The appropriate notion to be used is that of unitary operators in Kreĭn spaces. Given two Kreĭn spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with specified fundamental symmetries J_1 and, respectively, J_2, a bounded operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ is called (J_1, J_2)-unitary if $U^{-1} = J_1 U^* J_2$.

THEOREM 6.5 *Let G be a symmetric $\mathcal{L}(\mathcal{U})$-valued function, for some Hilbert space $\mathcal{U}$, analytic outside the closed unit disk and at infinity, and such that the Hankel operator H is bounded. We assume, in addition, that there exists some $\varepsilon > 0$ such that either $(-\varepsilon, 0) \subset \rho(H)$ or $(0, \varepsilon) \subset \rho(H)$. If (A_i, B_i, C_i, D_i) are completely J_i-selfadjoint systems, $i = 1, 2$, which are observable, equivalently, reachable, realizations of G, then there exists a uniquely determined (J_1, J_2)-unitary operator $U \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that $A_2 = UA_1U^{-1}$, $B_2 = UB_1$, $C_2 = C_1U^{-1}$, (and, of course, $D_1 = D_2 = G(\infty)$).*

Proof. We consider the reachability operators $R_i : \ell_\mathcal{U}^2 \to \mathcal{H}_i$ and the observability operators $O_i : \mathcal{H}_i \to \ell_\mathcal{U}^2$, corresponding to the system (A_i, B_i, C_i, D_i), $i = 1, 2$ Clearly, since the systems are completely J_i-selfadjoint, we have $O_i = R_i^* J_i$ and hence

$$R_i^* J_i R_i = O_i R_i = H. \tag{6.16}$$

To make a choice, assume that for some $\varepsilon > 0$ we have $(-\varepsilon, 0) \subset \rho(H)$. Taking into account of the construction of the induced Kreĭn space $\mathcal{K}_H$ (see the beginning of Section 6) this implies that the spectral subspace corresponding to H and the interval $(-\infty, 0)$ is a maximal uniformly negative subspace in $\mathcal{K}_H$. Taking into account that the reachability operators R_i have dense range, we can apply Lemma 2.3 in [7] and get that R_i induces

uniquely determined (S_H, J_i)-unitary operators $\Phi_i \colon \mathcal{K}_H \to \mathcal{H}_i$. Let then $U = \Phi_2 \Phi_1^{-1} \colon \mathcal{H}_1 \to \mathcal{H}_2$. Then U is (J_1, J_2)-unitary and $R_2 = U R_1$ and hence

$$R_2 = [B_2 \quad A_2 R_2] = U R_1 = U[B_1 \quad A_1 R_1]. \tag{6.17}$$

Identifying the first components in (6.17) we get $B_2 = U B_1$ and hence $C_2 = B_2^* J_2 = B_1^* U^* J_2 = B_1^* J_1 U^{-1} = C_1 U^{-1}$. Finally, identifying the second components in (6.17) we get

$$A_2 R_2 = U A_1 R_1 = U A_1 U^{-1} R_2,$$

whence, since R_2 has dense range, we get that $A_2 = U A_1 U^{-1}$.

In case $(0, \varepsilon) \subset \rho(H)$, a similar argument applies. ∎

REMARK 6.6 If the Hankel operator H associated with some symmetric transfer function G is bounded and $\kappa(H) = \min\{\kappa^-(H), \kappa^+(H)\} < \infty$ then the assumption on the topology of the spectrum of H as in Theorem 6.5 holds. For example, this is true if the function G has the representation as in Corrolary 6.3 and the input/output space $\mathcal{U}$ is finite dimensional (the argument is as in the proof of Theorem 5.1). ∎

References

[1] J.S. BARAS, R.W. BROCKETT, P.A. FUHRMAN: State-space models for infinite-dimensional systems, *IEEE Trans. Autom. Control*, 19(1974), pp. 693–700.

[2] R.R BITMEAD, B.D.O. ANDERSON: The matrix Cauchy-Index: Properties and Applications, *SIAM J. Applied Mathematics*, **33**(1977), 655–672.

[3] R.W. BROCKETT: Some Geometric Questions in the theory of linear systems, *IEEE Transactions on Automatic Control*, **21**(1976), 449–454.

[4] L. CARLESON: Interpolation by bounded analytic functions and the corona problem, *Ann. of Math.*, **76**(1962), 547–559.

[5] I. COLOJOARĂ, C. FOIAŞ: *Theory of Generalized Spectral Operators*, Gordon & Breach, New York 1968.

[6] T. CONSTANTINESCU, A. GHEONDEA: The negative signature of some hermitian matrices, *Lin. Alg. Appl.*, **178**(1993), 17–42.

[7] T. CONSTANTINESCU, A. GHEONDEA: Elementary rotations of operators in Kreĭn spaces, *J. Operator Theory*, **29**(1993), 167–203.

[8] J. DIEUDONNÉ: Quasi-hermitian operators, in *Proceedings of International Symposium on Linear Spaces*, Jerusalem 1961, pp. 115–122.

[9] A. DIJKSMA, H. LANGER, H. DE SNOO: Unitary colligations in Kreĭn spaces and their role in extension theory of isometric and symmetric linear relations in Hilbert spaces, in *Functional Analysis II*, Lecture Notes in Mathematics, no **1242**, Springer Verlag, Berlin - Heidelberg - New York 1987, pp. ?–?.

[10] P. A. FUHRMANN: Realization theory in Hilbert space for a class of transfer functions, *J. of Functional Analysis*, **18**(1975), 338–349.

[11] P.A. FUHRMANN: *Linear Systems and Operators in Hilbert Space*, McGraw-Hill, 1981.

[12] A. GHEONDEA: Spectral theory of selfadjoint operators on Kreĭn spaces [Romanian], *Studii şi Cercetări Matematice*, **45**(1993), 177-265.

[13] K. GLOVER, R.F. CURTAIN, J.R. PARTINGTON: Realisation and approximation of linear infinite dimensional systems with error bounds, *SIAM J. Control and Optimization*, **26**(1988), 863–898.

[14] M. GREEN, D.J.N. LIMEBEER: *Linear Robust Control*, Prentice Hall 1995.

[15] J.W. HELTON: Systems with infinite dimensional state space, *Proc. IEEE*, **64**(1976), 145–160.

[16] P. JONAS: On the functional calculus and the spectral function for definitizable operators, *Beiträge Anal.*, **16**(1981), 121–135.

[17] M.G. KREĬN : On linear completely continuous operators in functional spaces with two norms, (Ukrainian), *Zbirnik Prac. Inst. Mat. Akad. Nauk USSR*, **9**(1947), 104–129.

[18] M.G. KREĬN , H. LANGER: Über die Q-Funktion eines π-hermiteschen Operators im Raume Π_κ, *Acta Sci. Math.*, **34**(1973), 191–230.

[19] M.G. KREĬN, H. LANGER: Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume Π_κ zusammenhängen. I. Einige Funktionenklassen und ihre Darstellungen, *Math. Nachr.* **77**(1977), 187–236.

[20] M.G. KREĬN, H. LANGER: Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume Π_κ zusammenhängen. II. Veralgemeinerte Resolventen, u-Resolventen und ganze Operatoren, *J. Functional Analysis*, **30**(1978), 390–447.

[21] M.G. KREĬN, H. LANGER: On some extension problems which are closely connected with the theory of hermitian operators in a space Π_κ. III. Indefinite analogues of the Hamburger and Stieltjes moment problems, *Beiträge zur Analysis*, part (I) **14**(1979), 25–40; part (II) **15**(1981), 27–45.

[22] M.G. KREĬN , H. LANGER: Some propositions on analytic matrix functions related to the theory of operators in the space Π_κ, *Acta. Sci. Math*, **43**(1981), 181–205.

[23] A. KUMAR, D.A. WILSON: Symmetry properties of balanced systems, IEEE TRANSACTIONS ON AUTOMATIC CONTROL, **28**(1983), 927–929.

[24] S.Y. KUNG: A new indentification and model reduction via a singular value decomposition, in *Proc. 12th Asilomar Conference on Circuits, Systems and Computers*, IEEE 1978, pp. 705–714.

[25] H. Langer: *Spektraltheorie lineare Operatoren in J-Räumen und einige Anwendungen auf die Schar $L(\lambda) = \lambda^2 I + \lambda B + C$*, Habilitationsschrift, Dresden 1965.

[26] H. Langer: Spectral functions of definitizable operators in Kreĭn spaces, in *Functional Analysis*, Lecture Notes in Mathematics, vol. 948, Springer-Verlag, Berlin 1982, pp. 1–46.

[27] P.D. Lax: Symmetrizable linear transformations, *Comm. Pure Appl. Math.* **7**(1954). 633–647.

[28] A.V. Megretskii, V.V. Peller, S.R. Treil: The inverse spectral problem for self-adjoint Hankel operators, *Acta Mathematica*, **174**(1995), 241–309.

[29] B.C. Moore: Principal component analysis in linear systems: controllability, observability and model reduction, *IEEE Transactions on Automatic Control*, **26**(1981), 17–32.

[30] R.J. Ober: *Balanced Parametrizations for Linear Systems*, PhD Dissertation, Cambridge University, 1987.

[31] R.J. Ober: A note on a system theoretic approach to a conjecture by Peller-Krushchev, *Systems and Control Letters*, **8**(1987), 303–306.

[32] R.J. Ober: Stability and structural properties of infinite dimensional balanced realizations, in *Proceedings of 5th IFAC Symposium on Control of Distributed Parameter Systems*, Perpignan, France, 1989.

[33] R.J. Ober: A note on a system theoretic approach to a conjecture by Peller-Krushchev: the general case, *IMA J. Math. Contr. Inform.*, **7**(1990), 35–45.

[34] R.J. Ober: Balanced parametrizations of classes of linear systems, *SIAM J. Control and Optimization*, **29**(1991), 1251–1287.

[35] R. Ober: System theoretic aspects of completely symmetric systems, in *Operator Theory: Advances and Applications* (to appear).

[36] R.J. Ober, S. Montgomery-Smith, Bilinear transformation of infinite-dimensional state-space systems and balanced realizations of non-rational transfer functions, *SIAM J. Control and Optimization*, **28**(1990), 438–465.

[37] R.J. Ober, Y. Wu: Asymptotic stability of infinite dimensional discrete-time balanced realizations, *SIAM J. Control and Optimization*, **31**:5(1993), 1321–1339.

[38] W.T. Reid: Symmetrizable completely continuous linear transformations in Hilbert space, *Duke Math. J.* **18**(1951), 41–56.

[39] H. Widom: Hankel matrices, *Trans. Amer. Math. Soc.*, **121**(1966), 1–35.

[40] N.J. YOUNG: Balanced realizations in infinite dimensions, in *Operator Theory: Advances and Applications*, Vol. 19, Birkhäuser Verlag, Basel 1986, pp. 449–471.

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