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ABSTRACT

An input-output linear time-varying differential system with homogeneous jump Markov parameters and mean square exponential stable evolution is considered.
We define a family $T(t), t \geq 0$ of linear bounded input-output operators. It is proved that if $\sup \|T(t)\| < \gamma$ then a parametrized by γ differential Riccati type system has a unique global bounded and stabilizing solution. An application to the estimate of a stability radius is given.

1. INTRODUCTION

It is well-known [1], [3]-[7] that the input-output operators play a crucial role in the characterization of the stability radii of some linear deterministic and stochastic systems. The relationship between the contractivennes property of the input-output operator and the existence of a global bounded stabilizing solution of a nonstandard parametrized Riccati equation has been explored in [3], [4],[6].

The purpose of this paper is to derive such results for input-output linear time-varying differential systems with jump Markov perturbations.

A family $T(t), t \geq 0$ of linear bounded input-output operators is associated to a time-varying linear control system with jump Markov parameters.

It is proved that if $\sup \|T(t)\| < \gamma$ then a corresponding parametrized by γ differential Riccati type system has a unique bounded and stabilizing solution.

In the last section, an application to the estimate of a stability radius of a differential system with jump Markov parameters is given.

A parametrized nonstandard quadratic problem for input–output linear time–varying differential systems with jump Markov perturbations is also discussed.

1. NOTATIONS AND PRELIMINARIES

The following notations will be used throughout this paper. R^n is the real n–dimensional space. $B(R^n)$ is the family of Borel sets in $R^n \cdot R_+$ is the set of nonnegative real numbers. If X is a matrix or a vector, X^* is the transpose of X. $|A|$ is the operator norm of the matrix A. $H \geq 0$ means that H is symmetric positive semidefinite. I is the identity matrix.

By $\mathcal{S}$ we denote the space of all $n \times n$ symmetric matrices and by $\mathcal{S}^d$ we denote the space of all

$H = (H(1), \ldots, H(d)), H(i) \in \mathcal{S}$. If $H \in \mathcal{S}^d, |H| = \max\{|H(i)|; i \in D\}$. where $D = \{1, 2, \ldots, d\}$.

In this paper $\{\Omega, \mathcal{F}, \mathcal{P}\}$ is a given probability space; the argument $\omega \in \Omega$ will often not be written.

If $S \in \mathcal{F}$ by $\mathcal{X}_S$ we denote the indicator function of the set S.

Ex denotes expectation of the random variable x.

If $\mathcal{G}$ is a σ–algebra of subsets of $\Omega, \mathcal{G} \subset \mathcal{F}$ by $E[x|\mathcal{G}]$ we denote the conditional mean (expectation) of x with respect to $\mathcal{G}$; $E[x|w(t) = i]$ denotes expectation conditional on the event $w(t) = i$.

Throughout this paper, $w(t), t \geq 0$ is a right continuous homogeneous Markov chain with state space the set $D = \{1, 2, \ldots, d\}$ and probability transition matrix $P(t) = [p_{ij}(t)] = e^{Qt}, t > 0$; here $Q = [q_{ij}]$ with $\sum\limits_{j=1}^{d} q_{ij} = 0, i \in D$ and $q_{ij} \geq 0$ if $i \neq j$.

In this paper we assume that $\pi_i = \mathcal{P}\{w(0) = i\} > 0$ for all $i \in D$.

Therefore, since $p_{ii}(t) > 0$ for all $t > 0$ and $i \in D$ (see [2]) from the elementary inequality $\mathcal{P}\{w(t) = i\} \geq \pi_i p_{ii}(t)$ it follows that $\mathcal{P}\{w(t) = i\} > 0$ for all $t \geq 0$ and $i \in D$.

Throughout this paper $\mathcal{F}_{t_0,t}, t \geq t_0$ is the smallest σ–algebra containing all sets $S \in \mathcal{F}$ with $\mathcal{P}(S) = 0$ and with respect to which all functions $w(s), t_0 \leq s \leq t$ are measurable.

By definition $\mathcal{F}_t = \mathcal{F}_{0,t}, t \geq 0$. For every $t_0 \geq 0$ by $\tilde{L}^2([t_0, \infty) \times \Omega, R^m)$ we denote the space of all measurable functions $u : [t_0, \infty) \times \Omega \rightarrow R^m$ with the properties: $u(t)$ is measurable with respect to the σ–algebra $\mathcal{F}_{t_0,t}$ for every $t \geq t_0$ (i.e. $u(t)$ is $\mathcal{F}_{t_0,t}$–adapted) and $\sum\limits_{i=1}^{d} E[\int_{t_0}^{\infty} |u(t)|^2 dt | w(t_0) = i] < \infty$.

As usually, two measurable functions $u, v : [t_0, \infty) \times \Omega \to R^m$ are identified if $u = v$ a. e. (almost everywhere). Thus, since for every $t \geq t_0 \geq 0$ the σ-algebra $\mathcal{F}_{t_0,t}$ contains all sets $S \in \mathcal{F}$ with $\mathcal{P}(S) = 0$ it is not difficult to verify that $\tilde{L}^2([t_0, \infty) \times \Omega, R^m)$ is a Banach space with the norm

$$\|u\| = \left(\sum_{i=1}^{d} E[\int_{t_0}^{\infty} |u(t)|^2 dt | w(t_0) = i]\right)^{\frac{1}{2}}.$$

Moreover $\tilde{L}^2([t_0, \infty) \times \Omega, R^m)$ is a real Hilbert space with the inner product $< u, v >= \sum_{i=1}^{d} E[\int_{t_0}^{\infty} u^*(t) v(t) dt | w(t_0) = i]$. Obviously $\alpha(t_0)\|u\|^2 \leq E \int_{t_0}^{\infty} |u(t)|^2 dt \leq \|u\|^2$ for every $u \in \tilde{L}^2([t_0, \infty) \times \Omega, R^m)$ where $\alpha(t_0) = \min_{1 \leq i \leq d} \mathcal{P}\{w(t_0) = i\}$

By $\tilde{L}^2([t_0, T] \times \Omega, R^m), T > t_0$ we denote the Banach space of all measurable and $\mathcal{F}_{t_0,t}$ adapted, $t \in [t_0, T]$ functions $u : [t_0, T] \times \Omega \to R^m$ with $\sum_{i=1}^{d} E[\int_{t_0}^{T} |u(t)|^2 dt | w(t_0) = i] < \infty$

2. STATEMENT OF THE PROBLEM

Consider the following linear control system

$$\frac{dx(t)}{dt} = A(t, w(t))x(t) + B(t, w(t))u(t), t \geq 0 \tag{1}$$

and the output

$$y(t) = C(t, w(t))x(t)$$

where A, B, C are $n \times n, n \times m, p \times n$, respectively real matrix valued functions and $u(t)$ is a control vector.

The solutions of (1) are random processes which verify with probability one the corresponding integral equation.

Throughout this paper we assume that $A(t,i), B(t,i)$ and $C(t,i)$ are continuous and bounded on R_+ for every $i \in D$.

We consider also the linear system

$$\frac{dx(t)}{dt} = A(t, w(t))x(t), t \geq 0 \tag{2}$$

By $X(t, t_0)$ we denote the fundamental (random) matrix solution associated with system (2).

It is easy to verify that

$$|X(t, t_0)| \leq e^{a(t-t_0)} \text{ for all } t \geq t_0 \geq 0$$

where $a = \sup\{|A(t,i)|;\ t \geq 0, i \in D\}$

If $t_0 \geq 0, x_0 \in R^n$ and $u : [t_0, \infty) \times \Omega \to R^m$ is a measurable and integrable function on every set $[t_0, T] \times \Omega, T > t_0$ by $x_u(t, t_0, x_0)$ we denote the solution of system (1) corresponding to the control u, with $x_u(t_0, t_0, x_0) = x_0; x_u(\cdot, t_0, x_0)$ is a continuous process (with probability one) and by the variation of constants formula we have

$$x_u(t, t_0, x_0) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s)B(s, w(s))u(s)ds, t \geq t_0 \qquad (3)$$

Now, we consider the cost

$$V_\gamma(t_0, x_0, i, u) = E[\int_{t_0}^{\infty} [|y_u(t, t_0, x_0)|^2 - \gamma^2|u(t)|^2]dt|w(t_0) = i], t_0 \geq 0,$$

$$x_0 \in R^n, i \in D, u \in \tilde{L}^2([t_0, \infty) \times \Omega, R^m)$$

where γ is a given positive number and

$$y_u(t, t_0, x_0) = C(t, w(t))x_u(t, t_0, x_0)$$

In the next section we shall see that if the system (2) is exponentially L^2-stable, then $y_u(\cdot, t_0, x_0) \in \tilde{L}^2([t_0, \infty) \times \Omega, R^p)$ and therefore $V_\gamma(t_0, x_0, i, u) < \infty$

By definition $\tilde{V}_\gamma(t_0, x_0, u) = \sum_{i=1}^{d} V_\gamma(t_0, x_0, i, u)$. In this paper we solve the problem:

Given arbitrary, but fixed $t_0 \geq 0$ and $x_0 \in R^n$, find $\hat{u}(t_0, x_0) \in \tilde{L}^2([t_0, \infty) \times \Omega, R^m)$ such that $V_\gamma(t_0, x_0, i, u) \leq V_\gamma(t_0, x_0, i, \hat{u}(t_0, x_0))$ for all $i \in D$ and all $u \in \tilde{L}^2([t_0, \infty) \times \Omega, R^m)$.

Obviously, if $\hat{u}$ has the above property then

$$\tilde{V}_\gamma(t_0, x_0, \hat{u}(t_0, x_0)) = \max\{\tilde{V}_\gamma(t_0, x_0, u); u \in \tilde{L}^2([t_0, \infty) \times \Omega, R^m)\}$$

In the deterministic case the above nonstandard parametrized quadratic problem has been studied in [4].

The standard quadratic problem for linear differential control systems with jump Markov perturbations has been discussed in [8], [9].

3. INPUT-OUTPUT LINEAR BOUNDED OPERATORS

Definition 1. We say that the system (2) is exponentially L^2-stable on $[t_0, \infty)$ if there exist $\beta \geq 1$ and $\alpha > 0$ which depend on t_0 such that $E[|X(t, s)|^2|w(s) = i] \leq \beta e^{-\alpha(t-s)}$ for all $s \geq t_0, t \geq s$ and $i \in D$.

Definition 2. We say that the system (2) is exponentially L^2-stable if it is exponentially L^2-stable on $[0, \infty)$

From Proposition 1 in [8] it follows that the system (2) is exponentially L^2-stable iff there exist $\beta_1 \geq 1$ and $\alpha > 0$ such that

$$E[|X(t,s)|^2|w(s)] \leq \beta_1 e^{-\alpha(t-s)} \text{ a.e. for all } s \geq 0, t \geq s \tag{4}$$

Proposition 1. Suppose that the system (2) is exponentially L^2-stable. Then:

a) There exists $c \geq 1$ such that

$$E[\int_{t_0}^{\infty} |x_u(t,t_0,x_0)|^2 dt|w(t_0)=i] \leq c(|x_0|^2 + E[\int_{t_0}^{\infty} |u(t)|^2 dt|w(t_0)=i]),$$

for all $x_0 \in R^n, t_0 \geq 0, u \in \tilde{L}^2([t_0,\infty) \times \Omega, R^m)$ and $i \in D$

b) $\lim_{t\to\infty} E[|x_u(t,t_0,x_0)|^2|w(t_0)=i] = 0$ for all $t_0 \geq 0, x_0 \in R^n$, $u \in \tilde{L}^2([t_0,\infty) \times R^m)$ and $i \in D$.

Proof. From the proof of Lemma 3 in [8] it follows that

$$\begin{aligned} &E[|\textstyle\int_{t_1}^{t} X(t,s)B(s,w(s))u(s)ds|^2|w(t_0)=i] \leq \\ &\leq \frac{2\beta_1\beta_2}{\alpha} E[\int_{t_1}^{t} e^{-\frac{\alpha}{2}(t-s)}|u(s)|^2 ds|w(t_0)=i], \end{aligned} \tag{5}$$

if $t > t_1 \geq t_0, u \in \tilde{L}^2([t_0,t] \times \Omega, R^m)$ and $i \in D$ where $\beta_2 = \sup\{|B(t,i)|^2; t \geq 0, i \in D\}$ and β_1 and α are the constants in the inequality (4). Now, let $t_0 \geq 0, x_0 \in R^n, i \in D$ and $u \in \tilde{L}^2([t_0,\infty) \times \Omega, R^m)$. By using (3), (5) and the Fubini theorem we have

$$\begin{aligned} &E[\textstyle\int_{t_0}^{\infty} |x_u(t,t_0,x_0)|^2 dt|w(t_0)=i] \leq 2E[\int_{t_0}^{\infty} |X(t,t_0)|^2|x_0|^2 dt|w(t_0)=i] + \\ &+2\textstyle\int_{t_0}^{\infty} E[|\int_{t_0}^{t} X(t,s)B(s,w(s))u(s)ds|^2|w(t_0)=i]dt \leq \\ &\frac{2\beta_1}{\alpha}|x_0|^2 + 8\frac{\beta_1\beta_2}{\alpha^2}\int_{t_0}^{\infty} E[|u(t)|^2|w(t_0)=i]dt \end{aligned}$$

and thus the assertion a) is proved.

To prove b), for every $\varepsilon > 0$ let $t_\varepsilon > t_0$ be such that

$$\sum_{i=1}^{d} E[\int_{t_\varepsilon}^{\infty} |u(t)|^2 dt|w(t_0)=i] < \varepsilon$$

Since

$$x_u(t,t_0,x_0) = X(t,t_\varepsilon)x_u(t_\varepsilon,t_0,x_0) + \int_{t_\varepsilon}^{t} X(t,s)B(s,w(s))u(s)ds, \; t \geq t_\varepsilon$$

by making use of (5) we have for $t \geq t_\varepsilon$ and $i \in D$

$$E[|x_u(t,t_0,x_0)|^2|w(t_0)=i] \leq 2E[|X(t,t_\varepsilon)|^2|x_u(t_\varepsilon,t_0,x_0)|^2|w(t_0)=i] + \frac{4\beta_1\beta_2}{\alpha}\varepsilon$$

On the other hand for $t \geq t_\varepsilon, X(t,t_\varepsilon)$ is measurable with respect to the σ-algebra generated by $\{w(s), s \geq t_\varepsilon\}$. Hence, by using (4) and the Markov property of the process $w(t)$, (see [2]) we can write

$$\begin{aligned}
&E[|X(t,t_\varepsilon)|^2|x_u(t_\varepsilon,t_0,x_0)|^2|w(t_0)=i] = \\
&= E[|x_u(t_\varepsilon,t_0,x_0)|^2 E[|X(t,t_\varepsilon)|^2|\mathcal{F}_{t_\varepsilon}]|w(t_0)=i] \\
&= E[|x_u(t_\varepsilon,t_0,x_0)|^2 E[|X(t,t_\varepsilon)|^2|w(t_\varepsilon)]|w(t_0)=i] \\
&\leq \beta_1 e^{-\alpha(t-t_\varepsilon)} E[|x_u(t_\varepsilon,t_0,x_0)|^2|w(t_0)=i]
\end{aligned}$$

Hence for $t \geq t_\varepsilon$ and $i \in D$ we have

$$E[|x_u(t,t_0,x_0)|^2|w(t_0)=i] \leq 2\beta_1 e^{-\alpha(t-t_\varepsilon)} E[|x_u(t_\varepsilon,t_0,x_0)|^2|w(t_0)=i] + \frac{4\beta_1\beta_2}{\alpha}\varepsilon,$$

Taking $t \to \infty$ one gets $\lim_{t\to\infty} E[|x_u(t,t_0,x_0)|^2|w(t_0)=i] = 0$ and thus the proof is complete.

Under the assumption of Proposition 1 it follows that $x_u(\cdot,t_0,x_0) \in \tilde{L}^2([t_0,\infty)\times\Omega,R^n)$ if $u \in \tilde{L}^2([t_0,\infty)\times R^m)$.

Thus, under the assumption of Proposition 1, we can define the following linear bounded input–output operators:

$$T(t_0) : \tilde{L}^2([t_0,\infty)\times\Omega,R^m) \to \tilde{L}^2([t_0,\infty)\times\Omega,R^p), t_0 \geq 0,$$

by

$$(T(t_0)u)(t) = y_u(t,t_0,0) = C(t,w(t))\int_{t_0}^t X(t,s)B(s,w(s))u(s)ds, t \geq t_0$$

Remark 1. If the system (2) is exponentially L^2-stable, from Proposition 1 it follows that $\sup_{t\geq 0}\|T(t)\| < \infty$

Proposition 2. Assume that the system (2) is exponentially L^2–stable and suppose that for every $t > 0$ the transition matrix $P(t)$ is a double stochastic matrix. Then the function $t \to \|T(t)\|$ is monotonically decreasing on R_+.

Proof. Let $t_1 > t_0 \geq 0$ and $u \in \tilde{L}^2([t_1,\infty)\times\Omega,R^m)$. Define $\hat{u} : [t_0,\infty)\times\Omega \to R^m$ by $\hat{u}(t) = u(t)$ if $t \geq t_1$ and $\hat{u}(t) = 0$ if $t \in [t_0,t_1)$. Obviously $\hat{u} \in \tilde{L}^2([t_0,\infty)\times\Omega,R^m)$ and $(T(t_0)\hat{u})(t) = 0$ if $t \in [t_0,t_1)$ and $(T(t_0)\hat{u})(t) = (T(t_1)u)(t)$ if $t \geq t_1$.

Since $u(t)$ is measurable with respect to $\mathcal{F}_{t_1,t}$ by using the Markov property of the process $w(t)$ we have

$$E[|u(t)|^2|\mathcal{F}_{t_1}] = E[|u(t)|^2|w(t_1)] = \sum_{j=1}^d \mathcal{X}_{w(t_1)=j} E[|u(t)|^2|w(t_1)=j]$$

Hence

$$E[|u(t)|^2|w(t_0)=i] = \sum_{j=1}^{d} p_{ij}(t_1-t_0)E[|u(t)|^2|w(t_1)=j], i \in D,$$
$$t \geq t_1$$

Therefore, since $\sum_{i=1}^{d} p_{ij}(t_1-t_0) = 1, j \in D$ by the Fubini theorem, we have

$$\begin{aligned}
\|\tilde{u}\|^2 &= \sum_{i=1}^{d} E[\int_{t_1}^{\infty} |u(t)|^2 dt | w(t_0)=i] = \\
&= \sum_{i=1}^{d} \left(\sum_{j=1}^{d} p_{ij}(t_1-t_0) \int_{t_1}^{\infty} E[|u(t)|^2|w(t_1)=j]dt \right) = \\
&= \sum_{j=1}^{d} \left(\int_{t_1}^{\infty} E[|u(t)|^2|w(t_1)=j]dt \right) \sum_{i=1}^{d} p_{ij}(t_1-t_0) = \\
&= \sum_{j=1}^{d} E[\int_{t_1}^{\infty} |u(t)|^2 dt | w(t_1)=j] = \|u\|^2
\end{aligned}$$

Similarly

$$\|T(t_0)\tilde{u}\|^2 = \|T(t_1)u\|^2$$

Hence

$$\|T(t_1)u\| \leq \|T(t_0)\| \, \|\tilde{u}\|; \ \|T(t_1)u\| \leq \|T(t_0)\| \, \|u\|$$

Therefore

$$\|T(t_1)\| \leq \|T(t_0)\|$$

and the proof is complete.

Corollary 1. Under the assumptions of Proposition 2, we have $\sup_{\tau \geq t_0} \|T(\tau)\| = \|T(t_0)\|$ *for all* $t_0 \geq 0$

Proposition 3. Assume that the system (2) is exponentially L^2*-stable and* $\sup_{t \geq 0} \|T(t)\| < \gamma$. *Then* $\tilde{V}_\gamma(t_0, x_0, \cdot) : \tilde{L}^2([t_0,\infty) \times \Omega, R^m) \to \mathbf{R}$ *is a continuous concave function and for every* $t_0 \geq 0$ *and* $x_0 \in R^n$ *there exists a unique* $u(t_0,x_0) \in \tilde{L}^2([t_0,\infty) \times \Omega, R^m)$ *such that* $\max\{\tilde{V}_\gamma(t_0,x_0,u); u \in \tilde{L}^2([t_0,\infty) \times \Omega, R^m)\} = \tilde{V}_\gamma(t_0,x_0,u(t_0,x_0))$. *Moreover there exists* $q > 0$ *such that* $\|u(t_0,x_0)\| \leq q|x_0|$ *and* $0 \leq \tilde{V}_\gamma(t_0,x_0,u(t_0,x_0)) \leq q|x_0|^2$ *for all* $t_0 \geq 0$ *and* $x_0 \in R^n$.

Proof. The idea of the proof is the one in [6]. Let $\varepsilon \in (0,\gamma^2)$ be such that $\sup_{\tau \geq 0} \|T(\tau)\|^2 < \gamma^2 - \varepsilon$. Let $t_0 \geq 0, x_0 \in R^n$ and $u \in \tilde{L}^2([t_0,\infty) \times \Omega, R^m)$. We have $y_u(t,t_0,x_0) = C(t,w(t))x_u(t,t_0,x_0) = C(t,w(t))X(t,t_0)x_0 + (T(t_0)u)(t), \ t \geq t_0$

Define $z_{t_0,x_0}(t) = C(t,w(t))X(t,t_0)x_0, \ t \geq t_0$

Evidently $z_{t_0,x_0} \in \tilde{L}^2([t_0,\infty) \times \Omega, R^p)$, $\|z_{t_0,x_0}\| \leq \delta_1|x_0|$ and

$$\begin{aligned}\tilde{V}_\gamma(t_0,x_0,u) &= \|z_{t_0,x_0} + T(t_0)u\|^2 - \gamma^2\|u\|^2 = \\ &= < R(t_0)u, u > +2 < u, (T(t_0))^* z_{t_0,x_0} > +\|z_{t_0,x_0}\|^2\end{aligned}$$

where $R(t_0) = (T(t_0))^*T(t_0) - \gamma^2 J(t_0)$, $(T(t_0))^*$ being the adjoint operator and $J(t_0)$ is the identity operator on the Hilbert space $\tilde{L}^2([t_0,\infty) \times \Omega, R^m)$.

Since $\|(T(t_0))^*T(t_0)\| \leq \|T(t_0)\|^2 < \gamma^2 - \varepsilon$, $\|\frac{1}{\gamma^2}(T(t_0))^*T(t_0)\| < 1$ the operator $R(t_0)$ is invertible and $(R(t_0))^{-1}$ is a linear bounded operator defined on whole space $\tilde{L}^2([t_0,\infty) \times \Omega, R^m)$ and also we have

$$< R(t_0)u, u > \leq -\varepsilon\|u\|^2, \|R(t_0)u\| \geq \varepsilon\|u\|, \|(R(t_0))^{-1}\| \leq \frac{1}{\varepsilon}, t_0 \geq 0$$

Thus $\tilde{V}_\gamma(t_0,x_0,\cdot)$ is a continuous concave function and there exists a unique $u(t_0,x_0) \in \tilde{L}^2([t_0,\infty) \times \Omega, R^m)$ namely $u(t_0,x_0) = -(R(t_0))^{-1}(T(t_0))^* z_{t_0,x_0}$ such that the equality in the statement holds.

Obviously $\tilde{V}(t_0,x_0,u(t_0,x_0)) = - < (R(t_0))^{-1}v(t_0,x_0), v(t_0,x_0) > +\|z_{t_0,x_0}\|^2$ where $v(t_0,x_0) = (T(t_0))^* z_{t_0,x_0}$, and

$$\begin{aligned}&\|u(t_0,x_0)\| \leq \frac{1}{\varepsilon}\gamma\delta_1|x_0|, 0 \leq \tilde{V}_\gamma(t_0,x_0,u(t_0,x_0)) \leq \\ &\leq \frac{1}{\varepsilon}\gamma^2\delta_1^2|x_0|^2 + \delta_1^2|x_0|^2.\end{aligned}$$

Thus, the proof is complete.

4. MAIN RESULTS

In order to prove the main results in this paper we need some auxiliary results.

Consider next, the system

$$\frac{dx(t)}{dt} = A(t,w(t))x(t) + f(t), \quad t \geq 0 \tag{6}$$

where the random process $f(t)$ is right continuous, $\mathcal{F}_t$-adapted, $t \geq 0$ and f is also bounded on every set $[0,T] \times \Omega$, $T > 0$. For $t_0 \geq 0$, $x_0 \in R^n$ by $x(t,t_0,x_0)$ we denote the solution of system (6) with $x(t_0,t_0,x_0) = x_0$; $x(\cdot,t_0,x_0)$ is a continuous process (with probability one) and by standard way one can easily show that $x(t,t_0,x_0)$ is bounded on every set $[t_0,T] \times \Omega, T > t_0$. Since $w(t)$ and $f(t)$ are right continuous processes one can obtain easily that with probability one we have

$$\lim_{\substack{h \to 0 \\ h > 0}} \frac{x(t+h,t_0,x_0) - x(t,t_0,x_0)}{h} = A(t,w(t))x(t,t_0,x_0) + f(t), t \geq t_0 \tag{7}$$

Lemma 1. If $v : R_+ \times R^n \times D \to \mathbb{R}$ is a function of C^1 class in $(t,x) \in R_+ \times R^n$ for every $i \in D$ then

$$E[v(t,x(t,t_0,x_0),w(t))|w(t_0)=i] - v(t_0,x_0,i) =$$
$$= E[\int_{t_0}^{t}\{\frac{\partial v}{\partial s}(s,x(s,t_0,x_0),w(s))+$$
$$+[x^*(s,t_0,x_0)A^*(s,w(s)) + f^*(s)]\frac{\partial v}{\partial x}(s,x(s,t_0,x_0),w(s))+$$
$$+\sum_{j=1}^{d} v(s,x(s,t_0,x_0),j)q_{w(s)j}\}ds|w(t_0)=i], i \in D, t \geq t_0$$

Proof. Let $t_0 \geq 0, x_0 \in R^n, x(t) = x(t,t_0,x_0)$, and
$G_i(t) = E[v(t,x(t),w(t))|w(t_0)=i], t \geq t_0, i \in D$
We can write

$$G_i(t+h) - G_i(t) =$$
$$= E[(v(t+h,x(t+h),w(t+h)) - v(t,x(t+h),w(t+h)))|w(t_0)=i]+$$
$$+E[(v(t,x(t+h),w(t+h)) - v(t,x(t),w(t+h)))|w(t_0)=i]+$$
$$+E[(v(t,x(t),w(t+h)) - v(t,x(t),w(t)))|w(t_0)=i] =$$
$$= E[\frac{\partial v}{\partial t}(\xi_{t,h},x(t+h),w(t+h))h|w(t_0)=i]+$$
$$+E[(x(t+h)-x(t))^*\frac{\partial v}{\partial x}(t,x(t)+\theta_{t,h}(x(t+h)-x(t)),w(t+h))|w(t_0)=i]$$
$$+E[(\sum_{j=1}^{d} v(t,x(t),j)\mathcal{X}_{w(t+h)=j} - v(t,x(t),w(t)))|w(t_0)=i]$$

with $t < \xi_{t,h} < t+h, \theta_{t,h} \in (0,1)$. On the other hand by the Markov property of the process $w(t)$ we have

$$E[\sum_{j=1}^{d} v(t,x(t),j)\mathcal{X}_{w(t+h)=j}|w(t_0)=i] =$$
$$= E[\sum_{j=1}^{d} v(t,x(t),j)E[\mathcal{X}_{w(t+h)=j}|\mathcal{F}_t]|w(t_0)=i] =$$
$$= E[\sum_{j=1}^{d} v(t,x(t),j)E[\mathcal{X}_{w(t+h)=j}|w(t)]|w(t_0)=i] =$$
$$= \sum_{j=1}^{d} E[v(t,x(t),j)p_{w(t)j}(h)|w(t_0)=i]$$

Hence

$$E[(\sum_{j=1}^{d} v(t,x(t),j)\mathcal{X}_{w(t+h)=j} - v(t,x(t),w(t)))|w(t_0)=i] =$$
$$E[\sum_{j\neq w(t)} \{v(t,x(t),j) - v(t,x(t),w(t))\}p_{w(t)j}(h)|w(t_0)=i]$$

Since $P(h) = e^{Qh}$ and $\sum_{j=1}^{d} q_{ij} = 0, i \in D$, by using (7) and the Lebesgue bounded convergence theorem one gets

$$\lim_{\substack{h\to 0\\ h>0}} \frac{1}{h}[G_i(t+h) - G_i(t)] = E[\{\frac{\partial v}{\partial t}(t, x(t), w(t)) + \\ +(x^*(t)A^*(t, w(t)) + f^*(t))\frac{\partial v}{\partial x}(t, x(t), w(t)) + \\ + \sum_{j=1}^{d} v(t, x(t), j) q_{w(t)j}\} | w(t_0) = i] \tag{8}$$

Further, since the process $w(t)$ is continuous in probability (see [2]), by virtue of the Lebesgue bounded convergence theorem one concludes that for every $i \in D$ the function $G_i(t)$ is continuous.

Consequently, according to (8), the equalities in the statement hold and thus the proof is complete.

Now, let us consider the following differential system of Riccati type

$$\frac{d}{dt}K(t,i) + A^*(t,i)K(t,i) + K(t,i)A(t,i) + \sum_{j=1}^{d} K(t,j)q_{ij} + \\ C^*(t,i)C(t,i) + \gamma^{-2}K(t,i)B(t,i)B^*(t,i)K(t,i) = 0 \\ t \geq 0, i \in D \tag{9}$$

Proposition 4. Assume that the system (2) is exponentially L^2-stable. If $K : [0,\infty) \times D \to S$ is a bounded solution of (9) then

$$V_\gamma(t_0, x_0, i, u) = x_0^* K(t_0, i)x_0 - E[\textstyle\int_{t_0}^{\infty} |\gamma u(t) - \\ - \frac{1}{\gamma}B^*(t, w(t))K(t, w(t))x_u(t, t_0, x_0)|^2 dt | w(t_0) = i]$$

for all $x_0 \in R^n, i \in D, t_0 \geq 0$, and $u \in \tilde{L}^2([t_0,\infty) \times \Omega, R^m)$

Proof. Let $x_0 \in R^n, t_0 \geq 0, T > t_0, u \in \tilde{L}^2([t_0,\infty) \times \Omega, R^m)$. Let $K(t,i), t \geq 0, i \in D$ be a symmetric bounded solution of (9). Employing (3) we deduce easily that there exists $\beta_1(t_0,T) > 0$ such that $|x_u(t, t_0, x_0)|^2 \leq \beta_1(t_0,T)(|x_0|^2 + \int_{t_0}^{t} |u(t)|^2 dt)$ for all $t_0 \leq t \leq T$. Firstly, we assume that the function u is bounded on $[t_0,\infty) \times \Omega$. Define $u_k : [t_0,\infty) \times \Omega \to R^m, k \geq 1$ by $u_k(t) = k\int_{\max\{t-\frac{1}{k}, t_0\}}^{t} u(s)ds$. Evidently that $u_k(t)$ are continuous, bounded and $\mathcal{F}_t$-adapted processes and by the Lebesgue theorem we have $\lim_{k\to\infty} \int_{t_0}^{T} |u_k(t) - u(t)|^2 dt = 0$ and therefore by using the Lebesgue bounded convergence theorem we get

$$\lim_{k\to\infty} E[\int_{t_0}^{T} |u_k(t) - u(t)|^2 dt | w(t_0) = i] = 0 \text{ for all } i \in D \tag{10}$$

On the other hand it is easy to verify that

$$\sup_{t_0 \leq t \leq T} |x_k(t) - x_u(t)|^2 \leq \beta_2(t_0, T) \int_{t_0}^{T} |u_k(t) - u(t)|^2 dt \tag{11}$$

where $x_u(t) = x_u(t, t_0, x_0)$ and $x_k(t) = x_{u_k}(t, t_0, x_0)$

Now, applying Lemma 1 for $f(t) = B(t, w(t))u_k(t), t \geq t_0, v(t, x, i) = x^* K(t, i)x$, $t \in R_+, x \in R^n, i \in D$, and taking into account the equations (9) we obtain

$$\begin{aligned}
&E[x_k^*(T)K(T, w(T))x_k(T)|w(t_0) = i] - x_0^* K(t_0, i)x_0 = \\
&= -E[\textstyle\int_{t_0}^{T}(|C(t, w(t))x_k(t)|^2 - \gamma^2 |u_k(t)|^2)dt|w(t_0) = i] \\
&-E[\textstyle\int_{t_0}^{T} |\gamma u_k(t) - \frac{1}{\gamma} B^*(t, w(t))K(t, w(t))x_k(t)|^2 dt|w(t_0) = i],\ i \in D
\end{aligned}$$

By using (10), (11) and taking $k \to \infty$ in the above relations one gets

$$\begin{aligned}
&E[x_u^*(T)K(T, w(T))x_u(T)|w(t_0) = i] - x_0^* K(t_0, i)x_0 = \\
&= -E[\textstyle\int_{t_0}^{T}(|C(t, w(t))x_u(t)|^2 - \gamma^2 |u(t)|^2)dt|w(t_0) = i)] \\
&-E[\textstyle\int_{t_0}^{T} |\gamma u(t) - \frac{1}{\gamma} B^*(t, w(t))K(t, w(t))x_u(t)|^2 dt|w(t_0) = i]
\end{aligned} \tag{12}$$

for every $i \in D$ and $u \in \widetilde{L}([t_0, \infty) \times \Omega, R^n)$ with the property that u is bounded on $[t_0, \infty) \times \Omega$

Further, in the general case, let us define $\hat{u}_k : [t_0, \infty) \times \Omega \to R^m$ by $\hat{u}_k(t) = u(t)$ if $|u(t)| \leq k$ and $\hat{u}_k(t) = 0$ if $|u(t)| > k$. Obviously $|\hat{u}_k(t)| \leq |u(t)|, k \geq 1, t \geq t_0$

Therefore by the Lebesgue convergence theorem we have

$$\lim_{k \to \infty} E[\int_{t_0}^{T} |\hat{u}_k(t) - u(t)|^2 dt|w(t_0) = i] = 0, \quad i \in D$$

Let $\hat{x}_k(t) = x_{\hat{u}_k}(t, t_0, x_0)$. Since for every $k \geq 1, \hat{u}_k$ and $\hat{x}_k$ verify the corresponding equality (12) and inequality (11) we can conclude that (12) holds if $i \in D$ and $u \in \widetilde{L}^2([t_0, \infty) \times \Omega, R^m)$. From Proposition 1 it follows that $x_u \in \widetilde{L}^2([t_0, \infty) \times \Omega, R^m)$ and $\lim_{T \to \infty} E[|x_u(T)|^2 |w(t_0) = i] = 0, i \in D$.

Therefore taking $T \to \infty$ in (12) we obtain the equalities in the statement and thus the proof is complete.

In the following we shall discuss a parametrized quadratic control problem on a finite horizon

We associate the performances

$$\begin{aligned}
&H_\gamma(T, t_0, x_0, i, u) = E[\int_{t_0}^{T} (|y_u(t, t_0, x_0)|^2 - \gamma^2 |u(t)|^2)dt|w(t_0) = i], \\
&t_0 \geq 0, T > t_0, x_0 \in R^n, i \in D \text{ and } u \in \widetilde{L}^2([t_0, T] \times \Omega, \mathbf{R}^m)
\end{aligned}$$

Proposition 5. Under the assumptions of Proposition (3), for every $T > 0$, the symmetric solution $K_T(t,i)$ of system (9) with $K_T(T,i) = 0, i \in D$ is defined for all $t \in [0,T]$ and $i \in D$ and has the properties:

a) $H_\gamma(T,t_0,x_0,i,\tilde{u}_T) = x_0^* K_T(t_0,i)x_0$, $H_\gamma(T,t_0,x_0,i,u) \leq x_0^* K_T(t_0,i)x_0$ for all $i \in D$ and $u \in \tilde{L}^2([t_0,T]\times\Omega, R^m)$ where $\tilde{u}_T(t) = \gamma^{-2}B^*(t,w(t))K_T(t,w(t))\tilde{x}(t)$, $t \in [t_0,T]$ and $\tilde{x}(t), t \in [t_0,T]$ is the solution of the system

$$\frac{d\tilde{x}}{dt} = [A(t,w(t)) + \gamma^{-2}B(t,w(t))B^*(t,w(t))K_T(t,w(t))]\tilde{x}(t)$$

and $\tilde{x}(t_0) = x_0$($\tilde{u}_T$ depends also on t_0 and x_0)

b) $0 \leq K_T(t,i) \leq q\ I$ for all $0 \leq t \leq T, T > 0$ with some $q > 0$

c) $K_{T_1}(t,i) \leq K_{T_2}(t,i)$ for all $0 \leq t \leq T_1 < T_2,\ i \in D$

Proof. Let $t_0 \geq 0$ be such that the solution K_T of (9) is defined on $[t_0,T] \times D$. Let $x_0 \in R^n, u \in \tilde{L}^2([t_0,T] \times \Omega, R^m)$ and $x_u(t) = x_u(t,t_0,x_0), t \in [t_0,T]$. Since $K_T(t,i), t \in [t_0,T], i \in D$ verify equations (9), by using the same reasoning as in the proof of Proposition 4 we have.

$$H_\gamma(T,t_0,x_0,i,u) = x_0^* K_T(t_0,i)x_0 - E[\int_{t_0}^T |\gamma u(t) - \frac{1}{\gamma}B^*(t,w(t))K_T(t,w(t))x_u(t)|^2 dt | w(t_0) = i], i \in D$$

Thus, since $\tilde{u}_T \in \tilde{L}^2([t_0,T] \times \Omega, R^m)$ the proof of the assertion a) is complete. Now, from a) it follows that $x_0^* K_T(t_0,i)x_0 \geq H_\gamma(T,t_0,x_0,i,0) \geq 0,\ i \in D, x_0 \in R^n$. Hence $K_T(t_0,i) \geq 0,\ i \in D$.

Further, define $u_T : [t_0,\infty) \times \Omega \to R^m$ as follows

$$u_T(t) = \tilde{u}_T(t) \text{ if } t \in [t_0,T] \text{ and } u_T(t) = 0 \text{ if } t > T \tag{13}$$

Obviously $u_T \in \tilde{L}^2([t_0,\infty) \times \Omega, R^m)$. By using Proposition 3 and a) we have

$$x_0^* K_T(t_0,i)x_0 \leq \sum_{j=1}^d x_0^* K_T(t_0,j)x_0 \leq \tilde{V}_\gamma(t_0,x_0,u_T) \leq q|x_0|^2 \tag{14}$$

Hence $0 \leq K_T(t_0,i) \leq qI, i \in D$ and since q is an absolute constant the solution K_T of (9) is defined on $[0,T] \times D$ and the assertions in a), b) hold. It remains only to prove c). Let $0 < T_1 < T_2$ and $t_0 \in [0,T_1]$. Define $\hat{u} : [t_0,T_2] \times \Omega \to R^m$ as follows: $\hat{u}(t) = \tilde{u}_{T_1}(t)$ if $t \in [t_0,T_1]$ and $\hat{u}(t) = 0$ if $t \in (T_1,T_2]$. Obviously $\hat{u} \in \tilde{L}^2([t_0,T_2] \times \Omega, R^m)$ and by using a) we have

$$x_0^* K_{T_2}(t_0,i)x_0 \geq H_\gamma(T_2,t_0,x_0,i,\hat{u}) \geq H_\gamma(T_1,t_0,x_0,i,\tilde{u}_{T_1}) = x_0^* K_{T_1}(t_0,i)x_0$$

Hence $K_{T_2}(t_0,i) \geq K_{T_1}(t_0,i)$ for all $t_0 \in [0,T_1]$ and $i \in D$ and thus, the proof ends.

Definition 3. A solution $K : [t_0, \infty) \times D \to \mathcal{S}$ of system (9) is said to be stabilizing if the system

$$\frac{dx}{dt} = [A(t, w(t)) + \gamma^{-2} B(t, w(t)) B^*(t, w(t)) K(t, w(t))] x(t) \tag{15}$$

is exponentially L^2-stable on $[t_0, \infty)$

Theorem 1. Suppose that the system (2) is exponentially L^2-stable and $\sup_{t \geq 0} \|T(t)\| < \gamma$. Then the system (9) has a unique bounded and stabilizing solution $\widetilde{K} : [0, \infty) \times D \to \mathcal{S}$.

Moreover

$$\widetilde{K}(t, i) \geq 0, t \in R_+, i \in D, V_\gamma(t_0, x_0, i, \tilde{u}_{t_0, x_0}) = x_0^* \widetilde{K}(t_0, i) x_0, V_\gamma(t_0, x_0, i, u) \leq x_0^* \widetilde{K}(t_0, i) x_0$$

for all $i \in D, x_0 \in R^n, t_0 \geq 0$ and all $u \in \widetilde{L}^2([t_0, \infty) \times \Omega, R^m)$ where $\tilde{u}_{t_0, x_0}(t) = \gamma^{-2} B^*(t, w(t)) \widetilde{K}(t, w(t)) \tilde{x}(t), t \geq t_0$ and $\tilde{x}(t)$ is the solution of system (15) corresponding to the solution $\widetilde{K}$, with $\tilde{x}(t_0) = x_0$

If in addition all functions $A(\cdot, i), B(\cdot, i), C(\cdot, i)$ are θ periodic for every $i \in D$ then $\widetilde{K}(\cdot, i)$ is also a θ periodic function for every $i \in D$.

Proof. From Proposition 5 it follows that $\lim_{T \to \infty} K_T(t, i)$ exists for every $t \geq 0$ and $i \in D$. Let $\widetilde{K}(t, i) = \lim_{T \to \infty} K_T(t, i)$.

From Proposition 5 it follows that $0 \leq \widetilde{K}(t, i) \leq qI, t \in R_+, i \in D$ and $\widetilde{K}$ is a solution of the system (9). We shall verify that if $A(\cdot, i), B(\cdot, i), C(\cdot, i)$ are θ-periodic functions for every $i \in D$ then $\widetilde{K}(t + \theta, i) = \widetilde{K}(t, i)$ for all $t \geq 0$ and $i \in D$. Indeed, let $\widehat{K}_T : [0, T] \times D \to \mathcal{S}$ defined as follows:

$$\widehat{K}_T(t, i) = K_{T+\theta}(t + \theta, i)$$

Obviously $\widehat{K}_T(t, i)$ verify equations (9) for $t \in [0, T]$ and $i \in D$ and since $\widehat{K}_T(T, i) = 0 = K_T(T, i), i \in D$ from uniqueness it follows that $\widehat{K}_T(t, i) = K_T(t, i)$ for all $t \in [0, T]$ and $i \in D$. Taking $T \to \infty$ in the above equality one gets

$$\widetilde{K}(t + \theta, i) = \widetilde{K}(t, i), t \in R_+, i \in D.$$

Further we shall prove that $\widetilde{K}$ has the properties in the statement and $\widetilde{K}$ is a stabilizing solution of (9). Consider the functions u_T defined by (13), $T > t_0$. From Proposition 5 and the proof of Proposition 3 it follows that

$$0 \leq \sum_{i=1}^{d} x_0^* K_T(t_0, i) x_0 \leq \widetilde{V}_\gamma(t_0, x_0, u_T) \leq$$
$$-\varepsilon \|u_T\|^2 + 2\gamma\delta_1 \|u_T\| |x_0| + \delta_1^2 |x_0|^2$$
$$\leq -\frac{\varepsilon}{2} \|u_T\|^2 + (\delta_1 + \frac{2\gamma^2 \delta_1^2}{\varepsilon}) |x_0|^2$$

Hence $\|u_T\|^2 \le \frac{2}{\varepsilon}(\delta_1 + \frac{2\gamma^2\delta_1^2}{\varepsilon})|x_0|^2$ for all $T > t_0$. Therefore a sequence $t_k \to \infty$ exists such that the sequence u_{t_k} converges weakly to some $\hat{u}_{t_0,x_0} \in \tilde{L}^2([t_0,\infty)\times \Omega, R^m)$ and $\|\hat{u}_{t_0,x_0}\| \le \delta|x_0|^2$ for all $t_0 \ge 0, x_0 \in R^n$, with some $\delta > 0$. By using Proposition 4 we obtain

$$\tilde{V}_\gamma(t_0, x_0, u_T) \le x_0^* \sum_{i=1}^{d} \widetilde{K}(t_0, i)x_0, \ k \ge 1$$

Thus, taking into account (14) we get $\lim_{T\to\infty} \tilde{V}_\gamma(t_0, x_0, u_T) = x_0^* \sum_{i=1}^{d} \widetilde{K}(t_0,i)x_0$. Hence $\lim_{k\to\infty} \tilde{V}_\gamma(t_0,x_0,u_{t_k}) = x_0^* \sum_{i=1}^{d} \widetilde{K}(t_0,i)x_0$. Since $\tilde{V}_\gamma(t_0,x_0,\cdot)$ is a continuous concave function we have $\overline{\lim_{k\to\infty}} \tilde{V}_\gamma(t_0,x_0,u_{t_k}) \le \tilde{V}_\gamma(t_0,x_0,\hat{u}_{t_0,x_0})$ Therefore $x_0^* \sum_{i=1}^{d} \widetilde{K}(t_0,i)x_0 \le \tilde{V}_\gamma(t_0,x_0,\hat{u}_{t_0,x_0})$. But, applying again Proposition 4, we deduce

$$\begin{aligned}\tilde{V}_\gamma(t_0, x_0, \hat{u}_{t_0,x_0}) &= x_0^* \sum_{i=1}^{d} \widetilde{K}(t_0,i)x_0 - \sum_{i=1}^{d} E[\int_{t_0}^{\infty} |\gamma \hat{u}_{t_0,x_0}(t) - \\ &-\frac{1}{\gamma}B^*(t,w(t))\widetilde{K}(t,w(t))\hat{x}(t)|^2 dt|w(t_0) = i]\end{aligned}$$

Thus

$$\sum_{i=1}^{d} E[\int_{t_0}^{\infty} |\gamma \hat{u}_{t_0,x_0}(t) - \frac{1}{\gamma}B^*(t,w(t))\widetilde{K}(t,w(t))\hat{x}(t)|^2 dt|w(t_0) = i] = 0$$

Hence $\hat{u}_{t_0,x_0} = \tilde{u}_{t_0,x_0}$ a.e., $\tilde{u}_{t_0,x_0} \in \tilde{L}^2([t_0,\infty)\times\Omega, R^m)$, $\|\tilde{u}_{t_0,x_0}\| \le \delta|x_0|$. Applying Proposition 1 for $u = \tilde{u}_{t_0,x_0}$ we get

$$E[\int_{t_0}^{\infty} |\tilde{x}(t)|^2 dt|w(t_0) = i] \le c(|x_0|^2 + \|\tilde{u}_{t_0,x_0}\|^2) \le c(1+\delta)|x_0|^2$$

for all $t_0 \ge 0, x_0 \in R^n$ and $i \in D$

Thus by virtue of Lemma 3 in the next section (see also Corollary 1 in [8]) we can conclude that $\widetilde{K}$ is a stabilizing solution.

The properties of the solution $\widetilde{K}$ and the uniqueness follow directly from Proposition 4. The proof is complete.

In the deterministic case Theorem 1 has been proved in [4]. The non-standard parametrized Riccati equations for stochastic differential Itô equations have been discussed in [6]. The standard Riccati equations associated to quadratic control problem for differential systems with jump Markov perturbations have been studied in [8], [9].

Further, let us consider $\tilde{\gamma}(t_0) = \sup_{\tau \geq t_0} \|T(\tau)\|$ and

$\Gamma(t_0) = \{\gamma > 0;$ the system (9) has a bounded and stabilizing solution $K_\gamma : [t_0, \infty) \times D \to \mathcal{S}\}$, $t_0 \geq 0$

Theorem 2. Suppose that the system (2) is exponentially L^2-stable.

Then $\Gamma(t_0) = (\tilde{\gamma}(t_0), \infty)$ for all $t_0 \geq 0$

Proof. Let $t_0 \geq 0$ and $\gamma > 0$ be such that $\gamma > \tilde{\gamma}(t_0)$. From the proof of Theorem 1 it follows that the system (9) has a unique bounded and stabilizing solution $\widetilde{K}_\gamma : [t_0, \infty) \times D \to \mathcal{S}$. Hence $(\tilde{\gamma}(t_0), \infty) \subset \Gamma(t_0)$. Let $\gamma \in \Gamma(t_0)$ and $K_\gamma : [t_0, \infty) \times D \to \mathcal{S}$ be a bounded and stabilizing solution of system (9). From Proposition 4 it follows that

$$\widetilde{V}_\gamma(\tau, x_0, u) \leq \sum_{i=1}^{d} x_0^* K_\gamma(\tau, i) x_0$$

for all $\tau \geq t_0, x_0 \in R^n$ and $u \in \widetilde{L}^2([\tau, \infty) \times \Omega, R^m)$. Therefore $\tilde{\gamma}(t_0) \leq \gamma$. We shall prove that $\tilde{\gamma}(t_0) < \gamma$. Indeed suppose on the contrary that $\tilde{\gamma}(t_0) = \gamma$.

Hence for every $\varepsilon \in (0, \gamma^2)$ there exists $\tau_\varepsilon \geq t_0$ and $u_\varepsilon \in \widetilde{L}([\tau_\varepsilon, \infty) \times \Omega, R^m)$ such that $\|u_\varepsilon\| = 1$ and $\|T(\tau_\varepsilon)u_\varepsilon\|^2 > \gamma^2 - \varepsilon$.

Let $x_\varepsilon(t) = x_{u_\varepsilon}(t, \tau_\varepsilon, 0), t \geq \tau_\varepsilon$. From Proposition 4 it follows that

$$\|T(\tau_\varepsilon)u_\varepsilon\|^2 - \gamma^2 = -\gamma^2 \|u_\varepsilon - v_\varepsilon\|^2,$$

where $v_\varepsilon(t) = \gamma^{-2} B^*(t, w(t)) K_\gamma(t, w(t)) x_\varepsilon(t), t \geq \tau_\varepsilon$.

Hence

$$\|u_\varepsilon - v_\varepsilon\|^2 < \gamma^{-2}\varepsilon \tag{16}$$

Since K_γ is a stabilizing solution of system (9) it follows that the system $\frac{dx(t)}{dt} = \widetilde{A}(t, w(t))x(t)$ is exponentially L^2-stable on $[t_0, \infty)$ where $\widetilde{A}(t, i) = A(t, i) + \gamma^{-2} B(t, i) B^*(t, i) K_\gamma(t, i), t \geq t_0, i \in D$. But $\frac{dx_\varepsilon(t)}{dt} = \widetilde{A}(t, w(t))x_\varepsilon(t) + B(t, w(t))[u_\varepsilon(t) - v_\varepsilon(t)]$.

Thus, from Proposition 1 it follows that there exists $\delta_1(t_0) > 0$ such that $\|x_\varepsilon\|^2 \leq \delta_1(t_0)\|u_\varepsilon - v_\varepsilon\|^2$.

Therefore, according to (16) one gets

$$1 = \|u_\varepsilon\| \leq \|u_\varepsilon - v_\varepsilon\| + \|v_\varepsilon\| \leq \delta_2(t_0)\sqrt{\varepsilon}, \varepsilon > 0$$

Thus we get a contradiction. The proof ends.

Corollary 2. Under the assumption of Theorem 2, we have $\tilde{\gamma}(t_0) = \inf \Gamma(t_0)$ for all $t_0 \geq 0$

Corollary 3. Under the assumptions of Proposition 2, we have $\Gamma(t_0) = (\|T(t_0)\|, \infty), t_0 \geq 0$ *and therefore* $\|T(t_0)\| = \inf \Gamma(t_0)$ *for every* $t_0 \geq 0$.

Proposition 6. Under the assumption of Theorem 2, suppose that $A(\cdot, i)$, $B(\cdot, i)$ *and* $C(\cdot, i)$ *are* θ*-periodic functions for every* $i \in D$. *Then* $\Gamma(t_0) = \Gamma(t_0 + \theta)$ *for all* $t_0 \geq 0$.

Proof. Obviously $\Gamma(t_0) \subset \Gamma(t_0 + \theta)$. Let $\gamma \in \Gamma(t_0 + \theta)$ and $K_\gamma : [t_0 + \theta, \infty) \times D \to \mathcal{S}$ be a bounded and stabilizing solution of system (9). Define $\widehat{K}_\gamma : [t_0, \infty) \times D \to \mathcal{S}$ by $\widehat{K}_\gamma(t, i) = K_\gamma(t + \theta, i), t \geq t_0, i \in D$. Evidently $\widehat{K}_\gamma$ is a bounded solution of (9).

We shall prove that $\widehat{K}_\gamma$ is a stabilizing solution.

Let us define the following liniar operators on the space $\mathcal{S}^d$

$$\begin{aligned}(L(t)H)(i) &= [A(t,i) + \gamma^{-2}B(t,i)B^*(t,i)K_\gamma(t,i)]H(i) + \\ &+ H(i)[A^*(t,i) + \gamma^{-2}K_\gamma(t,i)B(t,i)B^*(t,i)] \\ &+ \sum_{j=1}^{d} H(j)q_{ji},\ t \geq t_0 + \theta, i \in D, H \in \mathcal{S}^d\end{aligned}$$

$$\frac{dS(t,s)}{dt} = L(t)S(t,s), t \geq s \geq t_0 + \theta, S(s,s) = J,$$

$$\begin{aligned}(\widehat{L}(t)H)(i) &= [A(t,i) + \gamma^{-2}B(t,i)B^*(t,i)\widehat{K}_\gamma(t,i)]H(i) + \\ &H(i)[A^*(t,i) + \gamma^{-2}\widehat{K}_\gamma(t,i)B(t,i)B^*(t,i)] + \\ &+ \sum_{j=1}^{d} H(j)q_{ji}, t \geq t_0, i \in D, H \in \mathcal{S}^d\end{aligned}$$

$$\frac{d\widehat{S}(t,s)}{dt} = \widehat{L}(t)\widehat{S}(t,s), t \geq s \geq t_0, \widehat{S}(s,s) = J,$$

J being the identity operator on the space $\mathcal{S}^d$.

From Proposition 2 in [8] it follows that K_γ is a stabilizing solution of system (9) iff there exist $\beta \geq 1$ and $\alpha > 0$ which depend on t_0 such that $\|S(t,s)\| \leq \beta e^{-\alpha(t-s)}$ for all $s \geq t_0 + \theta$ and $t \geq s$.

Since $L(t + \theta) = \widehat{L}(t)$ for all $t \geq t_0$ it follows that $\widehat{S}(t,s) = S(t + \theta, s + \theta)$, $s \geq t_0, t \geq s$.

Therefore $\|\widehat{S}(t,s)\| \leq \beta e^{-\alpha(t-s)}, s \geq t_0, t \geq s$ and thus applying again Proposition 2 in [8] we conclude that $\widehat{K}_\gamma$ is a stabilizing solution of system (9), hence $\gamma \in \Gamma(t_0)$. The proof is complete.

Since $\tilde{\gamma}(\cdot)$ is a monotonically decreasing function on R_+ the next result holds.

Corollary 4. Under the assumptions of Proposition 6 we have

$$\tilde{\gamma}(t_0) = \tilde{\gamma}(0)$$

for all $t_0 \geq 0$

Corollary 5. Under the assumptions of Proposition 2 if $A(\cdot,i), B(\cdot,i)$ *and* $C(\cdot,i)$ *are* θ*-periodic functions for every* $i \in D$, *then* $\|T(t_0)\| = \|T(0)\|$ *for all* $t_0 \geq 0$.

In what follows we shall discuss the time-invariant case i.e. $A(t,i) = A(i), B(t,i) = B(i), C(t,i) = C(i)$ for all $t \geq 0$ and $i \in D$

Consider the systems

$$\frac{dx(t)}{dt} = A(w(t))x(t), \quad t \geq 0 \tag{17}$$

$$\begin{aligned}\frac{dS(t,i)}{dt} &= A^*(i)S(t,i) + S(t,i)A(i) + \sum_{j=1}^{d} S(t,j)q_{ij} + C^*(i)C(i) \\ &+\gamma^{-2}S(t,i)B(i)B^*(i)S(t,i), \quad t \geq 0, i \in D\end{aligned} \tag{18}$$

In the time-invariant case the differential Riccati type system (9) is replaced by the following algebraic system

$$\begin{aligned}&A^*(i)K(i) + K(i)A(i) + \textstyle\sum_{j=1}^{d} K(j)q_{ij} + C^*(i)C(i) \\ &+\gamma^{-2}K(i)B(i)B^*(i)K(i) = 0, i \in D\end{aligned} \tag{19}$$

Remark 2. From Lemma 1 in [8] it follows that $E[|\widetilde{X}(t,s)x|^2|w(s) = i] = E[|\widetilde{X}(t-s,0)x|^2|w(0) = i]$ for all $t \geq s \geq 0, i \in D$ where $\widetilde{X}(t,s)$ is the fundamental matrix solution associated with system (17).

Remark 3. $S : [t_0, T] \times D \to \mathcal{S}$ is a solution of (18) iff $K : [t_0, T] \times D \to \mathcal{S}$ defined by $K(t,i) = S(t_0 + T - t, i)$ is a solution of system (9), with $A(t,i) = A(i), B(t,i) = B(i)$ and $C(t,i) = C(i), t \geq 0, i \in D$.

From Remark 3 and the proof of Proposition 5 it follows that the next result holds

Proposition 7. Suppose that $A(t,i) = A(i), B(t,i) = B(i), C(t,i) = C(i), t \geq 0, i \in D$ *and assume that the system (17) is exponentially* L^2*-stable. Let* $t_0 \geq 0$ *and* $\gamma > 0$ *be such that* $\|T(t_0)\| < \gamma$. *Then the symmetric solution* S_{t_0} *of system (18) with* $S_{t_0}(t_0, i) = 0, i \in D$ *is defined on* $[t_0, \infty) \times D$ *and has the properties:*

a) $0 \leq S_{t_0}(T_1, i) \leq S_{t_0}(T_2, i) \leq qI$ for all $t_0 \leq T_1 < T_2, i \in D$

b) $H_\gamma(T, t_0, x_0, i, \tilde{u}) = x_0^* S_{t_0}(T,i)x_0$, $H_\gamma(T, t_0, x_0, i, u) \leq x_0^* S_{t_0}(T,i)x_0$ for all $T > t_0 \geq 0, x_0 \in R^n, i \in D$ and $u \in \tilde{L}^2([t_0,T] \times \Omega, R^m)$ where $\tilde{u}(t) = \gamma^{-2}B^*(w(t))S_{t_0}(t_0 + T - t, w(t))\tilde{x}(t), t \in [t_0, T]$ and $\tilde{x}(t)$ verifies $\frac{d\tilde{x}(t)}{dt} = [A(w(t)) + \gamma^{-2}B(w(t))B^*(w(t))S_{t_0}(t_0 + T - t, w(t))]\tilde{x}(t), \tilde{x}(t_0) = x_0$

The next concept follows directly from Definition 3

Definition 4. A solution $K : D \to \mathcal{S}$ of system (19) is said to be stabilizing if the system

$$\frac{dx(t)}{dt} = [A(w(t)) + \gamma^{-2}B(w(t))B^*(w(t))K(w(t))]x(t) \qquad (20)$$

is exponentially L^2-stable.

Theorem 3. Suppose that $A(t,i) = A(i), B(t,i) = B(i), C(t,i) = C(i)$ and assume that the system (17) is exponentially L^2-stable. Then $\|T(t_0)\| = \|T(0)\|$ for all $t_0 \geq 0$

Proof. From Corollary 4 it follows that $\tilde{\gamma}(t_0) = \tilde{\gamma}(0)$ for all $t_0 \geq 0$. Hence $\|T(t_0)\| \leq \tilde{\gamma}(0)$ for all $t_0 \geq 0$. We shall prove that $\|T(t_0)\| = \tilde{\gamma}(0)$ for all $t_0 \geq 0$. Indeed, suppose on the contrary that there exists $\tau \geq 0$ such that $\|T(\tau)\| < \tilde{\gamma}(0)$.

Let $\gamma \in (\|T(\tau)\|, \tilde{\gamma}(0))$. From Proposition 7 it follows that $\lim_{t\to\infty} S_\tau(t,i)$ exists. Let $\widehat{S}(i) = \lim_{t\to\infty} S_\tau(t,i), i \in D$.

Obviously $\widehat{S}(i) \geq 0, i \in D$ and $\widehat{S}$ is a solution of system (19). By using again Proposition 7 and using the reasoning in the proof of Theorem 1 one can prove that there exists $\tilde{c} > 0$ such that $\int_\tau^\infty E[|\widehat{X}(t,\tau)x|^2|w(\tau) = i]dt \leq \tilde{c}|x|^2$ for all $x \in R^n, i \in D$ where $\widehat{X}(t,s)$ is the fundamental matrix solution associated with system (20) corresponding to the solution $\widehat{S}$. Now, by using Remark 2 one gets

$$E[\int_s^\infty |\widehat{X}(t,s)x|^2 dt|w(s) = i] \leq \tilde{c}|x|^2 \text{ for all } s \geq 0, x \in R^n, i \in D$$

Therefore $\widehat{S}$ is a stabilizing solution of system (19). Define $K : [0,\infty) \times D \to \mathcal{S}$ by $K(t,i) = \widehat{S}(i)$ for all $t \geq 0, i \in D$.

It follows that K is a bounded and stabilizing solution of system (9). Hence $\gamma \in \Gamma(0)$ and according to Theorem 2 we obtain that $\gamma > \tilde{\gamma}(0)$. Thus, we get a contradiction and the proof is complete.

Corollary 6. Under the assumptions of Theorem 3, the system (19) has a unique symmetric stabilizing solution iff $\|T(0)\| < \gamma$.

5. STABILITY RADII OF SOME TIME-VARYING DIFFERENTIAL SYSTEMS WITH JUMP MARKOV PERTURBATIONS

In this section we give an estimate for a stability radius of a differential system with jump Markov perturbations.

To prove the main result in this section we need some auxilary results with some interest in themselves.

Lemma 2. Let $\varphi : R^n \times \Omega \to R_+$ be a measurable function with respect to $B(R^n) \otimes \mathcal{G}_t$, where t is a fixed nonnegative real number and $\mathcal{G}_t$ is the σ-algebra generated by $\{w(s), s \geq t\}$. Let $g : \Omega \to R^n$ be a measurable function with respect to the σ-algebra $\mathcal{F}_t$. Then $h(g(\omega), w(t,\omega)) = E[\hat{\varphi}|\mathcal{F}_t](\omega)$ a.e. where $\hat{\varphi}(\omega) = \varphi(g(\omega), \omega)$ and $h(x,i) = E[\varphi(x,\cdot)|w(t) = i], x \in R^n, i \in D$.

Proof. By standard way (see Lemma 3 in [10]) it suffices to verify the equality in the statement for bounded real functions φ of the form $\varphi(x,\omega) = \varphi_1(x)\varphi_2(\omega)$ where φ_1 is a bounded Borel measurable function and φ_2 is a bounded measurable function with respect to the σ-algebra $\mathcal{G}_t$.

By using the Markov property of the process $w(t)$ we can write

$$E[\hat{\varphi}|\mathcal{F}_t] = E[\varphi_1(g)\varphi_2|\mathcal{F}_t] = \varphi_1(g)E[\varphi_2|\mathcal{F}_t] = \varphi_1(g)E[\varphi_2|w(t)]$$

On the other hand

$$h(x, w(t)) = E[\varphi_1(x)\varphi_2|w(t)] = \varphi_1(x)E[\varphi_2|w(t)]$$

Hence $h(g(\omega, w(t,\omega)) = \varphi_1(g(\omega))E[\varphi_2|\omega(t)](\omega)$ and this completes the proof

Consider next, the nonlinear system

$$\frac{dx}{dt} = F(t, x, w(t)), t \geq 0 \tag{21}$$

where $F : R_+ \times R^n \times D \to R^n$ has the following properties: F is continuous in $(t,x) \in R_+ \times R^n, F(t,0,i) = 0, t \in R_+, i \in D, F$ is locally Lipschitz with respect to x, (i.e. for every $T > 0$ and $r > 0$ there exists $L = L(T,r) > 0$ such that $|F(t,x_1,i) - F(t,x_2,i)| \leq L|x_1 - x_2|$ for all $0 \leq t \leq T, |x_1| \leq r, |x_2| \leq r$ and $i \in D$) and there exists $M > 0$ such that $|F(t,x,i)| \leq M|x|$ for all $t \in R_+, x \in R^n$ and $i \in D$.

Definition 5. We say that the system (21) is exponentially L^2-stable if there exist $\beta \geq 1$ and $\alpha > 0$ such that $E[|x(t,t_0,x_0)|^2|w(t_0) = i] \leq \beta e^{-\alpha(t-t_0)}|x_0|^2$ for all $t_0 \geq 0, t \geq t_0$ and all $x_0 \in R^n$ and $i \in D, x(t,t_0,x_0)$ being the solution of system (21) with $x(t_0,t_0,x_0) = x_0$

Lemma 3. If there exists $c > 0$ such that $E[\int_t^\infty |x(s,t,x)|^2 ds|w(t) = i] \leq c|x|^2$ for all $t \geq 0, x \in R^n$ and $i \in D$, then the system (21) is exponentially L^2-stable.

Proof. Let $v(t,x,i) = \displaystyle\int_t^\infty h(s,t,x,i)ds$, where $h(s,t,x,i) =$ $E[|x(s,t,x)|^2|w(t) = i], s \geq t, x \in R^n, i \in D$. Applying Lemma 2 for $\varphi(x,\omega) =$ $|x(s,t,x,\omega)|^2$ and $g(\omega) = x(t,t_0,x_0,\omega), t \geq t_0, x_0 \in R^n$, we get

$$h(s,t,x(t,t_0,x_0),w(t)) = E[|x(s,t,x(t,t_0,x_0))|^2|\mathcal{F}_t] = E[|x(s,t_0,x_0)|^2|\mathcal{F}_t]$$

Hence

$$\begin{aligned} V_i(t) &= E[v(t,x(t,t_0,x_0),w(t))|w(t_0)=i] = \\ &= \int_t^\infty E[h(s,t,x(t,t_0,x_0),w(t))|w(t_0)=i]ds = \\ &= \int_t^\infty E[E[|x(s,t_0,x_0)|^2|\mathcal{F}_t]|w(t_0)=i]ds = \\ &= \int_t^\infty E[|x(s,t_0,x_0)|^2|w(t_0)=i]ds \end{aligned}$$

Therefore $V_i'(t) = -E[|x(t,t_0,x_0)|^2|w(t_0)=i] \leq -\frac{1}{c}V_i(t)$

But, since $|F(t,x,i)| \leq M|x|$, by standard way one gets $\frac{d}{ds}|x(s,t,x|^2 \geq -2M|x(s,t,x)|^2, |x(s,t,x)|^2 \geq e^{-2M(s-t)}|x|^2, s \geq t, x \in R^n$; hence $v(t,x,i) \geq \frac{1}{2M}|x|^2$,

$$E[|x(t,t_0,x_0|^2|w(t_0)=i] \leq 2MV_i(t) \leq 2Mce^{-\frac{1}{c}(t-t_0)}|x_0|^2$$

The proof is complete.

In the linear case Lemma 3 was proved in [8].

Now for system (21) we define the operator $\mathcal{L}$ as follows

$$(\mathcal{L}(v)(t,x,i) = \frac{\partial x}{\partial t}(t,x,i) + F^*(t,x,i)\frac{\partial v}{\partial x}(t,x,i) + \sum_{j=1}^{d} v(t,x,j)q_{ij},\ t \geq 0, x \in R^n, i \in D$$

where $v(t,x,i)$ are real functions of class C^1 in (t,x) for every $i \in D$.

By using the reasoning in the proof of Lemma 1 one can prove the following known formula (see [9] p. 143).

$$\begin{aligned} &E[v(t,x(t,t_0,x_0),w(t))|w(t_0)=i] - v(t_0,x_0,i) = \\ &= E[\int_{t_0}^t (\mathcal{L}v)(s,x(s,t_0,x_0),w(s))ds|w(t_0)=i], \\ &t \geq t_0 \geq 0, x_0 \in R^n, i \in D \end{aligned} \tag{22}$$

Let us consider the following perturbed differential system

$$\begin{aligned} \frac{dx(t)}{dt} &= A(t,w(t))x(t) + B(t,w(t))\Delta(t,y(t),w(t)), t \geq 0 \\ y(t) &= C(t,w(t))x(t) \end{aligned} \tag{23}$$

By $\mathcal{D}$ we denote the set of all functions $\Delta : R_+ \times R^p \times D \to R^m$ with the following properties: $\Delta(t,0,i) = 0, t \geq 0, i \in D; \Delta$ is continuous in $(t,y) \in$

$R_+ \times R^p$ for every $i \in D$; Δ is locally Lipschitz with respect to y and there exists $\delta > 0$ such that $|\Delta(t,y,i)| \le \delta|y|$ for all $t \ge 0, y \in R^p$ and $i \in D$ (δ depends on the function Δ). If $\Delta \in \mathcal{D}$, we take $|\Delta| = \sup\{\frac{|\Delta(t,y,i)|}{|y|}; t \ge 0, y \ne 0, i \in D\}$ and by $x_\Delta(t,t_0,x_0)$ we denote the solution of system (23) with $x_\Delta(t_0,t_0,x_0) = x_0$.

Definition 6. The stability radius of system (2) with respect to the perturbation structure (B,C) is $\tilde{r} = \inf\{|\Delta|, \Delta \in \mathcal{D}$; (23) is not exponentially L^2-stable $\}$.

We shall give some characterizations for the stability radius $\tilde{r}$. Let $\mathcal{S}_+$ be the set of all $H \in \mathcal{S}$ with $H \ge 0$ and let $\hat{\gamma} = \inf\{\gamma > 0$; the system (9) has a bounded solution $K_\gamma : R_+ \times D \to \mathcal{S}_+\}$

Theorem 4. Assume that the system (2) is exponentially L^2-stable.

Then $\tilde{r} \ge \frac{1}{\hat{\gamma}}$

Proof. Let $\Delta \in \mathcal{D}$ with $|\Delta| < \frac{1}{\hat{\gamma}}$. We shall prove that the system (23) corresponding to this Δ is exponentially L^2-stable. By definition of $\hat{\gamma}$ it follows that the system (9) has a bounded solution $K_\gamma : R_+ \times D \to \mathcal{S}_+$ with $\gamma \in [\hat{\gamma}, \frac{1}{|\Delta|})$. Let $t_0 \ge 0, x_0 \in R^n, x(t) = x_\Delta(t,t_0,x_0), y(t) = C(t,w(t))x(t)$. By using the formula (22) for system (23) and $v(t,x,i) = x^*K_\gamma(t,i)x$ and taking into account the equations (9) for $K_\gamma(t,i)$ we can write for $T > t_0$

$$
\begin{aligned}
&E[x^*(T)K_\gamma(T,w(T))x(T)|w(t_0)=i] - x_0^*K_\gamma(t_0,i)x_0 = \\
&= -E[\int_{t_0}^{T} \{|y(t)|^2 - \gamma^2|\Delta(t,y(t),w(t))|^2\}dt|w(t_0)=i] \\
&-E[\int_{t_0}^{T} |\gamma\Delta(t,y(t),w(t)) - \\
&-\frac{1}{\gamma}B^*(t,w(t))K_\gamma(t,w(t))x(t)|^2 dt|w(t_0)=i]
\end{aligned}
$$

Since $0 \le K_\gamma(t,i) \le qI, t \ge 0, i \in D$, we have

$$E[\int_{t_0}^{\infty} |y(t)|^2 dt|w(t_0)=i] \le \frac{q}{1-\gamma^2|\Delta|^2}|x_0|^2, t_0 \ge 0, x_0 \in R^n, i \in D$$

Taking in Proposition 1, $u(t) = \Delta(t,y(t),w(t))$ one gets

$E[\int_{t_0}^{\infty} |x_\Delta(t,t_0,x_0)|^2 dt|w(t_0) = i] \le M|x_0|^2$ for all $t_0 \ge 0$ and $x_0 \in R^n$.

Hence by Lemma 3 it follows that the system (23) is exponentially L^2-stable for every $\Delta \in \mathcal{D}$, with $|\Delta| < \hat{\gamma}$.

Hence $\tilde{r} \ge \hat{\gamma}$ and the proof is complete.

We remark that if the system (2) is exponentially L^2-stable, from Theorem 1 it follows that $\tilde{\gamma}(0) \geq \tilde{\gamma}$. Thus, the next result holds

Corollary 7. Assume that the system (2) is exponentially L^2-stable. Then $\tilde{r} \geq \frac{1}{\tilde{\gamma}(0)}$ *where* $\tilde{\gamma}(0) = \sup_{t_0 \geq 0} \|T(t_0)\|$.

In the deterministic case Corollary 7 has been proved in [4]. The stability radii for stochastic differential equations have been studied in [1], [5]-[7].

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