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by

MARIAN APRODU

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MARIAN APRODU*

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*) Institute of Mathematics of the Romanian Academy
P.O. Box 1-764, RO-70700, Bucharest, Romania

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Marian Aprodu

Institute of Mathematics of the Romanian Academy, P.O.BOX 1–764, RO-70700 Bucharest, Romania

0 Introduction

Let $S \to B$ be a hyperelliptic surface over a smooth elliptic curve B defined over the field of complex numbers. The aim of this paper is to give a description of the Picard group of S in terms of hermitian forms and multiplicators, similar to Appell-Humbert for complex tori. The main tool used here is the cohomology of the groups and the ideas are similar to those used in [3], [9].

In the first section we recall some fundamental facts on hyperelliptic surfaces, such as the classification theorem and their fundamental groups.

In section 2, we get a description of the group of line bundles whose first Chern classes are torsion elements in the Neron-Severi group, which is usually denoted by $\mathrm{Pic}^{\tau}(S)$ and in the third section, which plays an important role for our purpose, we obtain a description of $\mathrm{Num}(S)$ in terms of hermitian forms.

The fourth section is devoted to the Appell-Humbert theorem and the final section present some direct applications of it such as computing tors $H^2(S,\mathbb{Z})$, finding a basis in $\mathrm{Num}(S)$ (see, also [10]) and computing the space of global sections for the line bundles over S numerically equivalent to a multiple of the fiber of $S \to B$.

1 Preliminaries and notations

There are many approaches concerning the theory of hyperelliptic surfaces ([1], [2], [6], [10], [12], [15]). Firstly, we recall the definition used by Suwa (cf. [12]):

Definition 1.1. A *hyperelliptic surface* is an elliptic bundle S over an elliptic curve B with $b_1(S) = 2$.

Theorem 1.2.(cf. [12]) *Any hyperelliptic surface can be expressed as a quotient of an abelian variety A by the group generated by an automorphism g_5 of A. The period matrix of A and the automorphism g_5 are given as follows:*

$$(a1)\ \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \beta \end{pmatrix} \qquad (a2)\ \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & \frac{1}{2} & \beta \end{pmatrix}$$

$g_5(u,z) = (u+\frac{1}{2}, -z)$

$$(b1)\ \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \rho \end{pmatrix} \qquad (b2)\ \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & \frac{1-\rho}{3} & \rho \end{pmatrix}$$

$g_5(u,z) = (u+\frac{1}{3}, \rho z)$, *where* $\rho = e^{\frac{2\pi i}{3}}$

$$(c1)\ \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & i \end{pmatrix} \qquad (c2)\ \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & \frac{1+i}{2} & i \end{pmatrix}$$

$g_5(u,z) = (u+\frac{1}{4}, iz)$

$$(d1)\ \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \rho \end{pmatrix}$$

$g_5(u,z) = (u+\frac{1}{6}, -\rho z)$.

We say that S is of *first type* if S is of type $(a1)$, $(b1)$, $(c1)$ or $(d1)$ and S is of *second type* otherwise.

For the sake of simplicity, we shall use the following notations:

$$\beta = \begin{cases} \text{arbitrarily} & (a1),(a2) \\ \rho & (b1),(b2),(d1) \\ i & (c1),(c2) \end{cases} \qquad d = \begin{cases} 1/2 & (a2) \\ (1-\rho)/3 & (b2) \\ (1+i)/2 & (c2) \\ 0 & \text{for the other cases} \end{cases}$$

$$\xi = \begin{cases} -1 & (a1),(a2) \\ \rho & (b1),(b2) \\ i & (c1),(c2) \\ -\rho & (d1) \end{cases} \qquad c = \begin{cases} 1/2 & (a1),(a2) \\ 1/3 & (b1),(b2) \\ 1/4 & (c1),(c2) \\ 1/6 & (d1) \end{cases}$$

$\ell = 1/c$.

So, S is the quotient of $\mathbb{C}^2$ by a group G of holomorphic automorphisms of $\mathbb{C}^2$ generated by g_i, $i = \overline{1,5}$, where $g_1(u,z) = (u+1,z)$, $g_2(u,z) = (u, z+1)$, $g_3(u,z) = (u+\alpha, z+d)$, $g_4(u,z) = (u, z+\beta)$ and $g_5(u,z) = (u+c, \xi z)$.

For the next elementary result, see [14].

Lemma 1.3. *The relations between generators are:* g_1, g_2, g_3 *and* g_4 *commute to each other,* $g_5^\ell = g_1$ *and*

$$(a1)\ \begin{aligned} g_2g_5 &= g_5g_2^{-1} \\ g_3g_5 &= g_5g_3 \\ g_4g_5 &= g_5g_4^{-1} \end{aligned} \qquad (a2)\ \begin{aligned} g_2g_5 &= g_5g_2^{-1} \\ g_3g_5 &= g_5g_3g_2^{-1} \\ g_4g_5 &= g_5g_4^{-1} \end{aligned}$$

$$(b1)\quad \begin{aligned} g_2g_5 &= g_5g_2^{-1}g_4^{-1} \\ g_3g_5 &= g_5g_3 \\ g_4g_5 &= g_5g_2 \end{aligned} \qquad (b2)\quad \begin{aligned} g_2g_5 &= g_5g_2^{-1}g_4^{-1} \\ g_3g_5 &= g_5g_3g_2^{-1} \\ g_4g_5 &= g_5g_2 \end{aligned}$$

$$(c1)\quad \begin{aligned} g_2g_5 &= g_5g_4^{-1} \\ g_3g_5 &= g_5g_3 \\ g_4g_5 &= g_5g_2 \end{aligned} \qquad (c2)\quad \begin{aligned} g_2g_5 &= g_5g_4^{-1} \\ g_3g_5 &= g_5g_3g_4^{-1} \\ g_4g_5 &= g_5g_2 \end{aligned}$$

$$(d1)\quad \begin{aligned} g_2g_5 &= g_5g_2g_4 \\ g_3g_5 &= g_5g_3 \\ g_4g_5 &= g_5g_2^{-1}. \end{aligned}$$

From the lemma above, one may see that any element $g \in G$ has an unique expresion as a product $g = g_2^{l_2}g_4^{l_4}g_3^{l_3}g_5^{l_5}$. The action of a such g on $\mathbb{C}^2$ is given by:

$$g(u,z) = (u + l_3\alpha + l_5c, \xi^{l_5}z + l_2 + l_4\beta + l_3d).$$

Another way of representing the hyperelliptic surface S is as follows. Let $\Gamma = \mathbb{Z} + \mathbb{Z}\beta$, $\Lambda = \mathbb{Z}\alpha + \mathbb{Z}c$, $\Lambda_1 = \mathbb{Z}\alpha + \mathbb{Z}$ and

$$\Lambda_2 = \begin{cases} \mathbb{Z}\alpha + \mathbb{Z} = \Lambda_1 & S \text{ of first type} \\ 2\mathbb{Z}\alpha + \mathbb{Z} & S \text{ of type } (a2) \text{ or } (c2) \\ 3\mathbb{Z}\alpha + \mathbb{Z} & S \text{ of type } (b2) \end{cases}$$

Let $\Delta = \mathbb{C}/\Lambda_2$ and $E = \mathbb{C}/\Gamma$. Then S can be expressed as $S = (\Delta \times E)/\mathcal{G}$ where $\mathcal{G}$ is a finite translations group of Δ, acting on E not by translations only, given by the Bagnera-deFranchis table (see for example [1], [2], [10]).

Moreover, $\Delta/\mathcal{G} \cong B$, $E/\mathcal{G} \cong \mathbb{P}^1$ and S has two fibrations: first of them is $S \to B$ from the definition 1.1., with fiber E, and the other one is $S \to \mathbb{P}^1$ with generic fiber Δ. Since Λ is the lattice of B, the short exact sequence of homotopy groups of the first fibration leads us to the following extension

$$0 \longrightarrow \Gamma \xrightarrow{j} G \xrightarrow{\pi} \Lambda \longrightarrow 0$$

where $j(\gamma) = g_2^{l_2}g_4^{l_4}$ and $\pi(g) = l_3\alpha + l_5c$.

Choosing as a cross-section of π the map $s : \Lambda \to G$, $s(\lambda) = g_3^{l_3}g_5^{l_5}$ for $\lambda = \alpha l_3 + cl_5 \in \Lambda$, we see that s is a groups morphism if S is of first type.

Next, we identify an element $\gamma \in \Gamma$ by $j(\gamma) \in G$ and $\lambda \in \Lambda$ by $s(\lambda) \in G$. In other words, we make no distinctions between $\gamma = l_2 + l_4\beta$ and $g_2^{l_2}g_4^{l_4}$ or between $\lambda = l_3\alpha + l_5c$ and $g_3^{l_3}g_5^{l_5}$. So $\lambda\lambda'$ will mean $s(\lambda)s(\lambda')$ and $\lambda + \lambda'$ is the same as $s(\lambda + \lambda')$. This convention simplifies our formulae and produces no ambiguity.

The natural action of an element $\lambda \in \Lambda$ on Γ is given by $\lambda\gamma\lambda^{-1} = \xi^{l_5}\gamma$. If we write $\lambda\lambda' = h(\lambda,\lambda')(\lambda + \lambda')$, then $h(\lambda,\lambda') = (\xi^{l_5} - 1)l_3'd$.

Next, let us point out the following useful lemma:

Lemma 1.4. *Let $v \in \mathrm{Hom}(G, \mathbb{C}^*)$. Then*

$$
\begin{array}{llll}
(a1) & v(g_2)=\pm 1 & , v(g_4)=\pm 1 & ; \quad (a2) \quad v(g_2)=1 \quad , v(g_4)=\pm 1 \ ; \\
(b1) & v(g_2)=v(g_4) & , v(g_2)^3=1 & ; \quad (b2) \quad v(g_2)=1 \quad , v(g_4)=1 \ ; \\
(c1) & v(g_2)=v(g_4) & , v(g_2)=\pm 1 & ; \quad (c2) \quad v(g_2)=1 \quad , v(g_4)=1 \ ; \\
(d1) & v(g_2)=1 & , v(g_4)=1 &
\end{array}
$$

2 The group $\mathrm{Pic}^\tau(S)$

The vanishing of the cohomology groups $H^i(\mathbb{C}^2,\mathbb{Z})$, $H^i(\mathbb{C}^2,\mathbb{C})$, $H^i(\mathbb{C}^2,\mathcal{O}_{\mathbb{C}^2})$, $H^i(\mathbb{C}^2,\mathcal{O}^*_{\mathbb{C}^2})$, $H^i(\mathbb{C}^2,\mathbb{C}^*)$ for all $i\geq 1$ yields to the natural isomorphisms (see [9]): $H^i(S,\mathbb{Z})\cong H^i(G,\mathbb{Z})$, $H^i(S,\mathbb{C})\cong H^i(G,\mathbb{C})$, $H^i(S,\mathbb{C}^*)\cong H^i(G,\mathbb{C}^*)$, $H^i(S,\mathcal{O}_S)\cong H^i(G,H)$, $H^i(S,\mathcal{O}^*_S)\cong H^i(G,H^*)$, where $H=H^0(\mathbb{C}^2,\mathcal{O}_{\mathbb{C}^2})$, $H^*=H^0(\mathbb{C}^2,\mathcal{O}^*_{\mathbb{C}^2})$.

The exponential sequence

$$0\longrightarrow \mathbb{Z}\longrightarrow \mathcal{O}_S \xrightarrow{exp} \mathcal{O}^*_S\longrightarrow 0$$

gives rise to the cohomology sequence

$$\ldots\to H^1(S,\mathcal{O}_S)\to \mathrm{Pic}(S)\xrightarrow{c_1} H^2(S,\mathbb{Z})\to 0.$$

Recall that the universal coefficients theorem leads us to:

Lemma 2.1. tors $H^2(S,\mathbb{Z})\cong \mathrm{Ker}\,(i: H^2(S,\mathbb{Z})\to H^2(S,\mathbb{C}))$.

For any $L\in \mathrm{Pic}(S)$, $c_1(L)$ is the Chern class of L and $\mathrm{Pic}^0(S)=\mathrm{Ker}(c_1)$. The subgroup $\mathrm{Pic}^\tau(S)\subset \mathrm{Pic}(S)$ (see [3]) is defined as $\mathrm{Ker}(ic_1)$ (where $i: H^2(S,\mathbb{Z})\to H^2(S,\mathbb{C})$ is canonical) and this is the group of the elements $L\in\mathrm{Pic}(S)$ so that $c_1(L)$ is a torsion element in $H^2(S,\mathbb{Z})$ (as we saw in Lemma 2.1.).

Then $\mathrm{Pic}^\tau(S)=\zeta(H^1(S,\mathbb{C}^*))$ where ζ is the natural morphism $H^1(S,\mathbb{C}^*)\to H^1(S,\mathcal{O}^*_S)$ (see [3]).

Let us compute next $\mathrm{Ker}(\zeta)$, by using the isomorphisms from the beginning of this section. So, $v\in\mathrm{Ker}(\zeta)$ if and only if there is $h\in H^*$ such that

$$(1)\qquad h(g(u,z))=v(g)h(u,z),\ \forall g\in G,\ (u,z)\in\mathbb{C}^2.$$

By taking the logarithmic derivatives $\omega_1=h'_u/h$ and $\omega_2=h'_z/h$ (in order to eliminate v from (1)), these functions verify the following relations:

$$
\begin{array}{lrcl}
(2) & \omega_i(u,z) & = & \omega_i(u+1,z),\\
 & \omega_i(u,z) & = & \omega_i(u,z+1),\\
 & \omega_i(u,z) & = & \omega_i(u,z+\beta),\\
 & \omega_i(u,z) & = & \omega_i(u+\alpha,z+d),\ i=1,2\\
(3) & \omega_1(u,z) & = & \omega_1(u+c,\xi z)\\
(4) & \omega_2(u,z) & = & \xi\omega_2(u+c,\xi z)
\end{array}
$$

for all $(u,z)\in\mathbb{C}^2$.

From (2), if we take $K \subset \mathbb{C}^2$ a compact set with $K + (\Gamma \times \Lambda) = \mathbb{C}^2$ and apply the maximum principle, we deduce that ω_i are constants.

From (4) it follows that $\omega_2 = 0$, so h doesn't depend on z. This means that there is a holomorphic function $\tilde{h}$ on $\mathbb{C}$ so that $h(u,z) = \tilde{h}(u)$, $\forall u, z \in \mathbb{C}$. Moreover, since $\tilde{h}'/\tilde{h}$ is constant, we get $h(u,z) = e^{2\pi i(au+b)}$ with $(a,b) \in \mathbb{C}^2$. Then, by denoting $v_i = v(g_i)$, we have: $v_2 = 1$, $v_4 = 1$, $v_3 = e^{2\pi i a\alpha}$, $v_5 = e^{2\pi i ac}$, where $a \in \mathbb{C}$.

Then we proved the following:

Lemma 2.2. $\mathrm{Ker}(\zeta) = \{v \in \mathrm{Hom}(G, \mathbb{C}^*) : \ v(g) = e^{2\pi i a\lambda}, g = \gamma\lambda \in G, a \in \mathbb{C}\}$.

Next, we try to describe $\mathrm{Pic}^{\tau}(S) \cong \mathrm{Hom}(G, \mathbb{C}^*)/\mathrm{Ker}(\zeta)$.

Let $v \in \mathrm{Hom}(G, \mathbb{C}^*)$. If S is of first type, s is a morphism, so $v(\lambda\lambda') = v(\lambda + \lambda')$.

Otherwise, we know that $\lambda\lambda' = h(\lambda, \lambda')(\lambda + \lambda')$ where $h(\lambda, \lambda') = (\xi^{l_5} - 1)l_3'd \in \Gamma$. But, if S is of type $(a2)$, then $h(\lambda, \lambda')$ depends only on g_2 and, by taking into account Lemma 1.4.,it follows that $v(h(\lambda, \lambda')) = 1$. If S is of type $(b2)$ or $(c2)$, then again from Lemma 1.4. we have $v(h(\lambda, \lambda')) = 1$.

In any case we obtained $v(\lambda\lambda') = v(\lambda + \lambda')$.

Now, we write $v(\lambda) = e^{2\pi i r(\lambda)}$. Since $r(\lambda) + r(\lambda') - r(\lambda + \lambda') \in \mathbb{Z}$, $\forall \lambda, \lambda' \in \Lambda$, $\varphi := \mathrm{Im}\, r$ must be $\mathbb{Z}$–linear. Then φ has an unique $\mathbb{R}$–linear extension $\tilde{\varphi} : \mathbb{C} \to \mathbb{R}$. We define $k : \mathbb{C} \to \mathbb{C}$, $k(u) = \tilde{\varphi}(iz) + i\tilde{\varphi}(z)$ which is $\mathbb{C}$–linear and $\tilde{r} := i\tilde{\varphi} - k$ is real–valued.

The function k being $\mathbb{C}$–linear, there exists $a \in \mathbb{C}$ so that $k(u) = au$, $\forall u \in \mathbb{C}$ and we take $v_0 \in \mathrm{Ker}(\zeta)$, $v_0(g) = e^{2\pi i a\lambda}$. Then $\alpha_G := v/v_0$ has the property that $\alpha_G(\lambda) \in U(1)$, $\forall \lambda \in \Lambda$ and it is uniquely determined by this property in the class of v in $\mathrm{Hom}(G, \mathbb{C}^*)/\mathrm{Ker}(\zeta)$.

Then we have:

$$\mathrm{Pic}^{\tau}(S) \cong \{\alpha_G \in \mathrm{Hom}(G, \mathbb{C}^*)\,, \alpha_G(\lambda) \in U(1), \ \forall \lambda \in \Lambda\}.$$

Moreover, $\alpha_G(\gamma) \in U(1)$, $\forall \alpha_G \in \mathrm{Hom}(G, \mathbb{C}^*)$, so we got:

Proposition 2.3. *There is a canonical isomorphism:*

$$\Psi' : \mathrm{Hom}(G, U(1)) \xrightarrow{\sim} \mathrm{Pic}^{\tau}(S).$$

3 The group $\mathrm{Num}(S)$

In this section we shall give a description of $\mathrm{Num}(S)$ in terms of hermitian forms related to Λ_1 and Γ. It is well-known (see, for example [10]) that $\mathrm{Num}(S) \cong H^2(S, \mathbb{Z})/\mathrm{tors}\, H^2(S, \mathbb{Z})$ and, as we saw in section 2, the cohomology of S is computed by cohomology of groups.

The inclusion $j : \Gamma \to G$ induces a morphism of restriction $\mathrm{res}_{\Gamma} : H^2(G, \mathbb{Z}) \to H^2(\Gamma, \mathbb{Z})$.

The map $s|_{\Lambda_1} : \Lambda_1 \to G$ is a groups morphism, so it induces another morphism of restriction $\mathrm{res}_{\Lambda_1} : H^2(G, \mathbb{Z}) \to H^2(\Lambda_1, \mathbb{Z})$.

According to [9], Chapter I, Appendix, we have classical isomorphisms

$$(5) \qquad H^2(\Gamma,\mathbb{Z}) \cong \{H_\Gamma : \mathbb{C}^2 \to \mathbb{C} \text{ hermitian, Im } H_\Gamma(\Gamma\times\Gamma) \subset \mathbb{Z}\},$$

$$(6) \qquad H^2(\Lambda_1,\mathbb{Z}) \cong \{H_\Lambda : \mathbb{C}^2 \to \mathbb{C} \text{ hermitian , Im } H_\Lambda(\Lambda_1\times\Lambda_1) \subset \mathbb{Z}\}.$$

Let us explain the morphisms res_Γ and res_{Λ_1} (cf. [9], Chapter I) passing through the above isomorphisms.

Starting with $F \in H^2(G,\mathbb{Z})$, we construct $A_\Gamma F : \Gamma\times\Gamma \to \mathbb{C}$, $A_\Gamma F(\gamma,\gamma') = F(\gamma',\gamma) - F(\gamma,\gamma')$, bilinear and antisymmetric which can be extended to $E_\Gamma : \mathbb{C}^2 \to \mathbb{C}$, $\mathbb{R}$-bilinear and antisymmetric verifying $E_\Gamma(ix,iy) = E_\Gamma(x,y)$, $\forall\, x,y \in \mathbb{C}$. Then $H_\Gamma : \mathbb{C}^2 \to \mathbb{C}$ defined by $H_\Gamma(x,y) := E_\Gamma(ix,y) + iE_\Gamma(x,y)$ is a hermitian form on $\mathbb{C}^2$ with $\text{Im } H_\Gamma = E_\Gamma$ and H_Γ will be $\text{res}_\Gamma F$ modulo canonical isomorphism (5).

By applying the same argument for Λ_1, res_Γ and res_{Λ_1} will induce a morphism

$$\chi : H^2(G,\mathbb{Z}) \to \mathcal{N}_1$$

where

$$\mathcal{N}_1 := \{\, (H_\Gamma, H_\Lambda)\ ,\ \ H_\Gamma, H_\Lambda \text{ hermitian forms on } \mathbb{C}^2 \\ \text{with Im } H_\Gamma(\Gamma\times\Gamma) \subset \mathbb{Z}\ ,\ \text{Im } H_\Lambda(\Lambda_1\times\Lambda_1) \subset \mathbb{Z}\,\}.$$

We denote by

$$\mathcal{NS} := \begin{cases} \{\, (H_\Gamma,H_\Lambda) \in \mathcal{N}_1\ ,\ \text{Im } H_\Lambda(\Lambda\times\Lambda) \subset \mathbb{Z}\,\} & S \text{ of first type} \\ \{\, (H_\Gamma,H_\Lambda) \in \mathcal{N}_1\ ,\ H_\Gamma(1,1)\text{Im } \beta \in 2\mathbb{Z}\,\} & S \text{ of type } (a2) \\ \{\, (H_\Gamma,H_\Lambda) \in \mathcal{N}_1\ ,\ H_\Gamma(1,1)\text{Im } \rho \in 3\mathbb{Z}\,\} & S \text{ of type } (b2) \\ \{\, (H_\Gamma,H_\Lambda) \in \mathcal{N}_1\ ,\ H_\Gamma(1,1) \in 2\mathbb{Z}\ ,\ 2\text{Im } H_\Lambda(\Lambda\times\Lambda) \subset \mathbb{Z}\,\} & S \text{ of type } (c2) \end{cases}$$

Now, we can state the main theorem of this section:

Theorem 3.1. *χ induces an isomorphism $\tilde{\chi} : \text{Num}(S) \tilde{\to} \mathcal{NS}$.*

Proof. Because $\mathcal{N}_1$ has no torsion it follows that tors $H^2(G,\mathbb{Z}) \subset \text{Ker}(\chi)$. So it remains to prove that $\text{Ker}(\chi) \subset$ tors $H^2(G,\mathbb{Z})$ and $\chi(H^2(G,\mathbb{Z})) = \mathcal{NS}$.

Let F be a normalized cocycle in $H^2(G,\mathbb{Z})$. Then F is the Chern class of a line bundle. If we represent this line bundle as a cocycle $\{e_g\}_g \in H^1(G,H^*)$ then, by standard diagram chasing, we get:

$$(7) \qquad F(g,g') = f_g(g'(u,z)) - f_{gg'}(u,z) + f_{g'}(u,z) \in \mathbb{Z},\ \forall u,z \in \mathbb{C},\ g,g' \in G,$$

where $f_g : \mathbb{C}^2 \to \mathbb{C}$ is a holomorphic function with $e^{2\pi i f_g} = e_g$, $\forall g \in G$ (see, for example [3], [9]).

Now, we divide the proof into two cases corresponding to the two different kinds of hyperelliptic surfaces.

Case 1. S is of first type.

Let us notice that, in this case, s is a morphism and, by denoting res_Λ the corresponding map from $H^2(G,\mathbb{Z})$ to $H^2(\Lambda,\mathbb{Z})$ we have the following commutative diagram, coming from the inclusion $\Lambda_1 \subset \Lambda$.

$$\begin{array}{ccc} H^2(\Lambda,\mathbb{Z}) & \hookrightarrow & H^2(\Lambda_1,\mathbb{Z}) \\ {\scriptstyle \mathrm{res}_\Lambda}\nwarrow & & \nearrow{\scriptstyle \mathrm{res}_{\Lambda_1}} \\ & H^2(G,\mathbb{Z}) & \end{array}$$

Then it is obvious that $\chi(H^2(G,\mathbb{Z})) \subset \mathcal{NS}$.

Step 1. Our next goal is to find f_g and thus to get a nice form of (7).

Since the restriction of F to Γ and Λ are 2–cocycles, it follows (see [9], Chapter I) that

$$f_\gamma(u,z) = \frac{1}{2i}H_\Gamma(z,\gamma) + \beta_\Gamma(u,\gamma),\ \forall\gamma \in \Gamma, \tag{8}$$

$$f_\lambda(u,z) = \frac{1}{2i}H_\Lambda(u,\lambda) + \beta_\Lambda(z,\lambda),\ \forall\lambda \in \Lambda, \tag{9}$$

where $\beta_\Gamma(\ .\ ,\gamma)$, $\beta_\Lambda(\ .\ ,\lambda)$ are holomorphic functions on $\mathbb{C}$.

Next, we write $\equiv$ for congruence modulo $\mathbb{Z}$. From (7) it follows that, if $g = \gamma\lambda$, then

$$f_\gamma(\lambda(u,z)) - f_g(u,z) + f_\lambda(u,z) \equiv 0 \tag{10}$$

so

$$f_g(u,z) \equiv \frac{1}{2i}H_\Gamma(\xi^{l_5}z,\gamma) + \frac{1}{2i}H_\Lambda(u,\lambda) + \beta_\Gamma(u+\lambda,\gamma) + \beta_\Lambda(z,\lambda),\ \forall g \in G. \tag{11}$$

The relation (7) can be read as

$$f_{gg'}(u,z) \equiv f_g(g'(u,z)) + f_{g'}(u,z),\ g,g' \in G.$$

By replacing f_g from (7) in the above formula, we have:

$$\begin{aligned} &\beta_\Gamma(u+\lambda+\lambda',\gamma+\xi^{l_5}\gamma') + \beta_\Lambda(z,\lambda+\lambda') \equiv \\ &\frac{1}{2i}H_\Gamma(\xi^{l_5}\gamma',\gamma) + \frac{1}{2i}H_\Lambda(\lambda',\lambda) + \beta_\Gamma(u+\lambda+\lambda',\gamma) \\ &+\beta_\Gamma(u+\lambda',\gamma') + \beta_\Lambda(\xi^{l'_5}z+\gamma',\lambda) + \beta_\Lambda(z,\lambda'). \end{aligned} \tag{12}$$

Let us denote by $\varepsilon_\Gamma(\ .\ ,\gamma)$ and $\varepsilon_\Lambda(\ .\ ,\lambda)$ the derivatives of $\beta_\Gamma(\ .\ ,\gamma)$ and $\beta_\Lambda(\ .\ ,\lambda)$ respectively. Then, from (12) we obtain:

$$\varepsilon_\Gamma(u+\lambda+\lambda',\gamma+\xi^{l_5}\gamma') = \varepsilon_\Gamma(u+\lambda+\lambda',\gamma) + \varepsilon_\Gamma(u+\lambda',\gamma') \tag{13}$$

$$\varepsilon_\Lambda(z,\lambda+\lambda') = \xi^{l'_5}\varepsilon_\Lambda(\xi^{l'_5}z+\gamma',\lambda) + \varepsilon_\Lambda(z,\lambda') \tag{14}$$

and from these relations we can describe β_Γ and β_Λ.

Firstly, we determine β_Γ.

In (13), we choose $\lambda = \lambda' = 0$ and we get:

$$\varepsilon_\Gamma(u, \gamma + \gamma') = \varepsilon_\Gamma(u, \gamma) + \varepsilon_\Gamma(u, \gamma') \ \forall \gamma, \gamma' \in \Gamma, \tag{15}$$

which means that $\varepsilon_\Gamma(u, \ .\ ) : \Gamma \to \mathbb{C}$ is a morphism of groups.

In (13) we choose $\lambda' = 0$ and it follows:

$$\varepsilon_\Gamma(u + \lambda, \gamma + \xi^{l_5}\gamma') = \varepsilon_\Gamma(u + \lambda, \gamma) + \varepsilon_\Gamma(u, \gamma'). \tag{16}$$

From (15) and (16) we deduce that:

$$\varepsilon_\Gamma(u + \lambda, \xi^{l_5}\gamma') = \varepsilon_\Gamma(u, \gamma'). \tag{17}$$

We choose $\lambda \in \Lambda_1$ in (17), so $\varepsilon_\Gamma(u + \lambda, \gamma') = \varepsilon_\Gamma(u, \gamma')$, $\forall \lambda \in \Lambda_1$, $\gamma' \in \Gamma$, $u \in \mathbb{C}$.

By standard arguments, $\varepsilon_\Gamma(\ .\ , \gamma')$ must be a constant, so we may write $\varepsilon_\Gamma(\gamma)$ instead of $\varepsilon_\Gamma(u, \gamma)$. On the other side, if we apply (15) and (17) again, ε_Γ must be identically equal to zero and β_Γ doesn't depend on u. Then we write $\beta_\Gamma(\gamma)$ instead of $\beta_\Gamma(u, \gamma)$.

Next, we determine β_Λ. We choose $\lambda = \lambda' = 0$ and $\gamma' = 0$ in (14) so $\varepsilon_\Lambda(z, 0) = 0$, $\forall z \in \mathbb{C}$. We apply these relation to (14) for $\lambda' = 0$ and we obtain

$$\varepsilon_\Lambda(z, \lambda) = \varepsilon_\Lambda(z + \gamma', \lambda), \ \forall \lambda \in \Lambda, \ \gamma' \in \Gamma.$$

For the same reason as above, ε_Λ doesn't depend on u and we write $\varepsilon_\Lambda(\lambda)$ instead of $\varepsilon_\Lambda(z, \lambda)$. With this notation, we turn back to (14) which becomes:

$$\varepsilon_\Lambda(\lambda + \lambda') = \xi^{l'_5}\varepsilon_\Lambda(\lambda) + \varepsilon_\Lambda(\lambda'). \tag{18}$$

An easy computation in (18) will show that $\varepsilon_\Lambda(\lambda) = \frac{1-\xi^{l_5}}{1-\xi}\varepsilon_\Lambda(c)$ and $\beta_\Lambda(z, \lambda) = \frac{1-\xi^{l_5}}{1-\xi}\varepsilon_\Lambda(c)z + \beta_\Lambda(\lambda)$.

Then we get:

$$\begin{aligned} f_g(u, z) &= \frac{1}{2i}H_\Gamma(\xi^{l_5}z, \gamma) + \frac{1}{2i}H_\Lambda(u, \lambda) + \beta_\Gamma(\gamma) \\ &\quad + \frac{1 - \xi^{l_5}}{1 - \xi}\varepsilon_\Lambda(c)z + \beta_\Lambda(\lambda) + \mathrm{const}(g), \ \forall g \in G, \end{aligned} \tag{19}$$

where $\mathrm{const}(g) \in \mathbb{Z}$, $\forall g \in G$ and (7) becomes:

$$\begin{aligned} F(g, g') &= \frac{1}{2i}H_\Lambda(\lambda', \lambda) + \frac{\xi^{l_5}}{2i}H_\Gamma(\gamma', \gamma) + \beta_\Lambda(\lambda) + \beta_\Lambda(\lambda') - \beta_\Lambda(\lambda + \lambda') \\ &\quad + \beta_\Gamma(\gamma) + \beta_\Gamma(\gamma') - \beta_\Gamma(\gamma + \xi^{l_5}\gamma') + \frac{1 - \xi^{l_5}}{1 - \xi}\varepsilon_\Lambda(c)\gamma' \\ &\quad + \mathrm{const}(g) + \mathrm{const}(g') - \mathrm{const}(gg'), \ \forall g, g' \in G \end{aligned} \tag{20}$$

Since $\mathrm{const}(g) + \mathrm{const}(g') - \mathrm{const}(gg')$ is a coboundary in $C^2(G, \mathbb{Z})$, we can ignore this term, without changing the cohomology class of F in $H^2(G, \mathbb{Z})$.

Let $r(g) := \beta_\Lambda(\lambda) + \beta_\Gamma(\gamma) + \frac{1}{1-\xi}\varepsilon_\Lambda(c)\gamma$, $r_\Gamma(\gamma) := r(\gamma) = \beta_\Gamma(\gamma) + \frac{1}{1-\xi}\varepsilon_\Lambda(c)\gamma$ and $r_\Lambda(\lambda) := r(\lambda) = \beta_\Lambda(\lambda)$.

With this notations, (20) gives rise to the final formula for F:

$$(21) \qquad F(g,g') = \frac{1}{2i}H_\Lambda(\lambda',\lambda) + \frac{\xi^{l_5}}{2i}H_\Gamma(\gamma',\gamma) + r(g) + r(g') - r(gg') \in \mathbb{Z}.$$

and thus, if we replace β_Γ by r_Γ, we may always suppose that $\varepsilon_\Lambda(c) = 0$.

From (21), one may see that if $H_\Gamma = 0$ and $H_\Lambda = 0$, then $F(g,g') = r(g) + r(g') - r(gg')$, which means that the cohomology class of F in $H^2(G,\mathbb{C})$ equals to zero. Then, by means of Lemma 2.1., F represents a torsion class in $H^2(G,\mathbb{Z})$. Thus we proved that $\mathrm{Ker}(\chi) \subset \mathrm{tors}H^2(G,\mathbb{Z})$.

Step 2. It remains to prove that $\mathcal{NS} \subset \chi(H^2(G,\mathbb{Z}))$.

We check that for given $(H_\Gamma, H_\Lambda) \in \mathcal{NS}$, there exists $r_\Gamma : \Gamma \to \mathbb{C}$ and $r_\Lambda : \Lambda \to \mathbb{C}$ so that, by defining $r(g) = r_\Gamma(\gamma) + r_\Lambda(\lambda)$, for any $g = \gamma\lambda$ then:

$$(22) \qquad \frac{1}{2i}H_\Lambda(\lambda',\lambda) + \frac{\xi^{l_5}}{2i}H_\Gamma(\gamma',\gamma) + r(g) + r(g') - r(gg') \in \mathbb{Z}.$$

Let us denote by:

$$b_\Gamma(\gamma) = ir_\Gamma(\gamma) - \frac{1}{4}H_\Gamma(\gamma,\gamma),\ \forall\gamma \in \Gamma,$$
$$b_\Lambda(\lambda) = ir_\Lambda(\lambda) - \frac{1}{4}H_\Lambda(\lambda,\lambda),\ \forall\lambda \in \Lambda.$$

One may see that (22) is equivalent to the following three relations:

$$(23) \qquad b_\Gamma(\xi\gamma) - b_\Gamma(\gamma) \in i\mathbb{Z},$$
$$(24) \qquad b_\Gamma(\gamma) + b_\Gamma(\gamma') - b_\Gamma(\gamma+\gamma') + \frac{1}{2}iE_\Gamma(\gamma,\gamma') \in i\mathbb{Z},\ \forall\gamma,\gamma' \in \Gamma,$$
$$(25) \qquad b_\Lambda(\lambda) + b_\Lambda(\lambda') - b_\Lambda(\lambda+\lambda') + \frac{1}{2}iE_\Lambda(\lambda,\lambda') \in i\mathbb{Z},\ \forall\lambda,\lambda' \in \Lambda.$$

Then, the problem of finding r_Γ and r_Λ so that (22) is true reduces to searching for b_Γ and b_Λ which satisfy (23), (24) and (25).

By using (24), a straightforward computation shows that (23) is equivalent to:

$$(26) \qquad \begin{array}{ll} S \text{ of type } (a1) & 2b_\Gamma(1),\ 2b_\Gamma(\beta) \in i\mathbb{Z}, \\ S \text{ of type } (b1) & b_\Gamma(1) - b_\Gamma(\rho) \in i\mathbb{Z},\ 3b_\Gamma(1) - \frac{i\sqrt{3}}{4}H_\Gamma(1,1) \in i\mathbb{Z}, \\ S \text{ of type } (c1) & 2b_\Gamma(1) \in i\mathbb{Z},\ b_\Gamma(1) - b_\Gamma(i) \in i\mathbb{Z}, \\ S \text{ of type } (d1) & b_\Gamma(1) + b_\Gamma(\rho) \in i\mathbb{Z},\ b_\Gamma(1) + \frac{i\sqrt{3}}{4}H_\Gamma(1,1) \in i\mathbb{Z}. \end{array}$$

If we fix $b_\Lambda(c)$, $b_\Lambda(\alpha)$, $b_\Gamma(1)$ and $b_\Gamma(\beta) \in \mathbb{C}$ so that (26) is verified and we set:

$$b_\Gamma(\gamma) := l_2 b_\Gamma(1) + l_4 b_\Gamma(\beta) + \frac{1}{2}il_2l_4E_\Gamma(1,\beta),\ \forall\gamma = l_2 + l_4\beta,$$

$$b_\Lambda(\lambda) := l_3 b_\Lambda(\alpha) + l_5 b_\Lambda(c) + \frac{1}{2} i l_3 l_5 E_\Lambda(c,\alpha),\ \forall\lambda = l_3\alpha + l_5 c,$$

then it is obvious that b_Γ and b_Λ are the functions we were looking for.

Case 2. S is of second type.

The proof is similar to the proof of *Case 1.*, but it needs more computations. As in the previous case, we try to find a decent form of f_g.

Since the restriction of F to Γ and Λ_1 are cocycles, then we must have, as in the first case:

$$f_\gamma(u,z) = \frac{1}{2i} H_\Gamma(z,\gamma) + \beta_\Gamma(u,\gamma),\ \forall\gamma\in\Gamma, \tag{27}$$

$$f_{\lambda_1}(u,z) = \frac{1}{2i} H_\Lambda(u,\lambda_1) + \beta_\Lambda(z,\lambda_1),\ \forall\lambda_1\in\Lambda_1, \tag{28}$$

where $\beta_\Gamma(\ .\ ,\gamma)$, $\beta_\Lambda(\ .\ ,\lambda_1)$ are holomorphic functions on $\mathbb{C}$. Let us denote by $\varepsilon_\Gamma(\ .\ ,\gamma)$, $\varepsilon_\Lambda(\ .\ ,\lambda_1)$ the derivatives of $\beta_\Gamma(\ .\ ,\gamma)$ and $\beta_\Lambda(\ .\ ,\lambda_1)$ respectively.

Step 1. We show that $\varepsilon_\Gamma(\ .\ ,\ .\ )$, and $\varepsilon_\Lambda(\ .\ ,\ .\ )$ are constants in their first variable and groups morphism to $\mathbb{C}$ in their second variable.

For $g = \gamma\lambda \in G$ with $\lambda\in\Lambda_1$, then g is also equal to $\lambda\gamma$ and we apply (7) two times:

$$f_g(u,z) \equiv f_\gamma(\lambda(u,z)) + f_\lambda(u,z) \equiv f_\lambda(\gamma(u,z)) + f_\gamma(u,z)$$

to get the following:

$$\frac{1}{2i} H_\Gamma(l_3 d,\gamma) + \beta_\Gamma(u+\lambda,\gamma) + \beta_\Lambda(z,\lambda) \equiv \beta_\Gamma(u,\gamma) + \beta_\Lambda(z+\gamma,\lambda),\ \lambda\in\Lambda_1. \tag{29}$$

By taking the derivatives with respect to u and z respectively in (29) it will follow that $\varepsilon_\Gamma(u+\lambda,\gamma) = \varepsilon_\Gamma(u,,\gamma)$ and $\varepsilon_\Lambda(z+\gamma,\lambda) = \varepsilon_\Lambda(z,\lambda)$, $\forall\gamma\in\Gamma,\ \lambda\in\Lambda_1,\ u,z\in\mathbb{C}$ and thus ε_Γ and ε_Λ are constant in their first variable.

Then we write $\varepsilon_\Gamma(\gamma)$ instead of $\varepsilon_\Gamma(u,,\gamma)$ and $\varepsilon_\Lambda(\lambda)$ instead of $\varepsilon_\Lambda(z,\lambda)$ and,by denoting $\beta_\Gamma(\gamma) = \beta_\Gamma(0,\gamma)$ and $\beta_\Lambda(\lambda) = \beta_\Lambda(0,\lambda)$, we deduce that:

$$\beta_\Gamma(u,\gamma) = \varepsilon_\Gamma(\gamma)u + \beta_\Gamma(\gamma) \tag{30}$$

$$\beta_\Lambda(z,\lambda) = \varepsilon_\Lambda(\lambda)z + \beta_\Lambda(\lambda). \tag{31}$$

Next, we turn back to (7) and we choose $g, g' \in G$, $g = \gamma\lambda$, $g' = \gamma'\lambda'$ with $\lambda,\lambda'\in\Lambda_1$. Then we obtain:

$$\begin{aligned}&\frac{1}{2i} H_\Gamma(l_3 d,\gamma') + \varepsilon_\Gamma(\gamma+\gamma')(u+\lambda+\lambda') + \varepsilon_\Lambda(\lambda+\lambda')z \\ &+\beta_\Gamma(\gamma+\gamma') + \beta_\Lambda(\lambda+\lambda') \equiv \frac{1}{2i} H_\Gamma(\gamma',\gamma) + \frac{1}{2i} H_\Lambda(\lambda',\lambda) \\ &+\varepsilon_\Gamma(\gamma)(u+\lambda+\lambda') + \varepsilon_\Gamma(\gamma')(u+\lambda') + \varepsilon_\Lambda(\lambda)(z+\gamma'+l'_3 d) \\ &+\varepsilon_\Lambda(\lambda')z + \beta_\Gamma(\gamma) + \beta_\Gamma(\gamma') + \beta_\Lambda(\lambda) + \beta_\Lambda(\lambda').\end{aligned} \tag{32}$$

Now, we take the derivatives with respect to u and z respectively in (32) and it will follow that $\varepsilon_\Gamma \in \mathrm{Hom}(\Gamma, \mathbb{C})$ and $\varepsilon_\Lambda \in \mathrm{Hom}(\Lambda_1, \mathbb{C})$.

If we apply (30) and (31) in (29) we will obtain the following relation:

$$\frac{1}{2i} H_\Gamma(l_3 d, \gamma) - \varepsilon_\Lambda(\lambda)\gamma + \varepsilon_\Gamma(\gamma)\lambda \equiv 0, \ \forall \lambda \in \Lambda_1, \ \gamma \in \Gamma. \tag{33}$$

Step 2. We prove that β_Λ can be extended to $\beta_\Lambda : \mathbb{C} \times \Lambda \to \mathbb{C}$, also holomorphic in the first variable so that:

$$f_\lambda(u, z) = \frac{1}{2i} H_\Lambda(u, \lambda) + \beta_\Lambda(z, \lambda), \ \forall \lambda \in \Lambda.$$

In fact, by taking into account (7) and (28), it is sufficient to prove this only for $\lambda = c$.

Let $\eta_\lambda = \frac{\partial f_\lambda}{\partial u}$, $\mu_\lambda = \frac{\partial^2 f_\lambda}{\partial u^2}$ and $\nu_\lambda = \frac{\partial^2 f_\lambda}{\partial u \partial z}$, $\forall \lambda \in \Lambda$.

By using induction on m, one may apply (7) several times to prove that:

$$f_{mc} \equiv \sum_{k=0}^{m-1} f_c(u + kc, \xi^k z), \ \forall m \in \mathbb{N}, \tag{34}$$

which implies

$$\eta_{mc} = \sum_{k=0}^{m-1} \eta_c(u + kc, \xi^k z), \tag{35}$$

$$\mu_{mc} = \sum_{k=0}^{m-1} \mu_c(u + kc, \xi^k z), \ \forall m \in \mathbb{N}. \tag{36}$$

In particular, for $mc = n \in \mathbb{N}$, we get

$$\sum_{k=0}^{m-1} \eta_c(u + kc, \xi^k z) = \frac{1}{2i} H_\Lambda(1, n), \tag{37}$$

$$\sum_{k=0}^{m-1} \mu_c(u + kc, \xi^k z) = 0. \tag{38}$$

Our next goal is to prove that η_c is a constant and then, from (37), we deduce that this constant must be equal to $\frac{1}{2i} H_\Gamma(1, c)$ and this step will be finished.

We apply (7) for $l_3\alpha$, $l_5 c$ and then, for $\lambda = l_3\alpha + l_5 c$, we have:

$$\begin{aligned} f_\lambda(u, z) &\equiv f_{l_3\alpha}(u + l_5 c, \xi^{l_5} z) + f_{l_5 c}(u, z) \\ &\equiv f_{l_5 c}(u + l_3\alpha, z + l_3 d) + f_{l_3\alpha}(u, z). \end{aligned} \tag{39}$$

But $l_3\alpha \in \Lambda_1$ and, by meaning of (28) and (39) the following two formulae holds:

$$\eta_{l_5 c}(u, z) = \eta_{l_5 c}(u + l_3\alpha, z + l_3 d), \tag{40}$$

$$\mu_{l_5 c}(u, z) = \mu_{l_5 c}(u + l_3\alpha, z + l_3 d), \ \forall l_3, l_5 \in \mathbb{Z}. \tag{41}$$

We apply again (7) for l_5c and mc, where we choose m such that $mc = n \in \mathbb{Z} \subset \Lambda_1$. A similar argument as in (39) leads us to:

$$\eta_{l_5c}(u,z) = \eta_{l_5c}(u+n,z), \tag{42}$$

$$\mu_{l_5c}(u,z) = \mu_{l_5c}(u+n,z),\ \forall l_5, n \in \mathbb{Z}. \tag{43}$$

From (7), applied for γ, λ and $g = \gamma\lambda$, we obtain:

$$f_g(u,z) \equiv \frac{1}{2i} H_\Gamma(\xi^{l_5}z + l_3 d, \gamma) + \varepsilon_\Gamma(\gamma)(u+\lambda) + \beta_\Gamma(\gamma) + f_\lambda(u,z). \tag{44}$$

Again in (7), we take $g = \gamma\lambda$, $g' = \gamma'\lambda'$ with $l'_3 = 0$ (and this implies that $h(\lambda,\lambda') = 0$) and $(l_5 + l'_5)c \in \mathbb{Z} \subset \Lambda_1$ and use (44) and (28):

$$\begin{aligned} &\frac{1}{2i} H_\Gamma(z + l_3 d, \gamma + \xi^{l_5}\gamma') + \frac{1}{2i} H_\Lambda(u, \lambda+\lambda') + \varepsilon_\Gamma(\gamma + \xi^{l_5}\gamma')(u+\lambda+\lambda') \\ &+\beta_\Gamma(\gamma + \xi^{l_5}\gamma') + \beta_\Lambda(z, \lambda+\lambda') \equiv \frac{1}{2i} H_\Gamma(z + \xi^{l_5}\gamma', \gamma) + \frac{1}{2i} H_\Gamma(\xi^{l'_5}z, \gamma') \\ &+\varepsilon_\Gamma(\gamma)(u+\lambda+\lambda') + \varepsilon_\Gamma(\gamma')(u+\lambda') + \beta_\Gamma(\gamma) + \beta_\Gamma(\gamma') + f_{\lambda'}(u,z) \\ &+f_\lambda(u+\lambda', \xi^{l'_5}z + \gamma'). \end{aligned} \tag{45}$$

Then,

$$\begin{aligned} \varepsilon_\Gamma(\gamma + \xi^{l_5}\gamma') + \frac{1}{2i} H_\Lambda(1, \lambda+\lambda') &= \varepsilon_\Gamma(\gamma) \\ +\varepsilon_\Gamma(\gamma') + \eta_\lambda(u+\lambda', \xi^{l'_5}z + \gamma') &+ \eta_{\lambda'}(u,z) \end{aligned} \tag{46}$$

and

$$\mu_\lambda(u+\lambda', \xi^{l'_5}z + \gamma') = -\mu_{\lambda'}(u,z). \tag{47}$$

In particular, $\forall u, z \in \mathbb{C}$, $\forall \gamma' \in \Gamma$, $\forall l_5, l'_5 \in \mathbb{Z}$ so that $(l_5 + l'_5)c \in \mathbb{Z}$ we have:

$$\mu_{l'_5c}(u,z) = -\mu_{l_5c}(u + l'_5c, \xi^{l'_5}z + \gamma'). \tag{48}$$

From this relation, one may imediatelly obtain that:

$$\mu_{l'_5c}(u,z) = \mu_{l'_5c}(u+n, z+\gamma),\ \forall \gamma \in \Gamma,\ n \in \mathbb{Z}. \tag{49}$$

We apply (43) and (49) for $l'_5 = 1$ to deduce that $\mu_c(u,z)$ doesn't depend on z and we write $\mu_c(u) = \mu_c(u,z)$. Now, we take into account (41) and (43) which show us that $\mu_c(u+\lambda) = \mu_c(u)$, $\forall \lambda \in \Lambda_1$. But this means nothing else than μ_c is a constant. From (38), this constant must be zero, so η_c depends only on z, say $\eta_c(z) = \eta_c(u,z)$. In fact, it is easy to see that η_λ depends only on z, $\forall \lambda \in \Lambda$.

Then ν_λ will depend only on z for any $\lambda \in \Lambda$ and, from (46), we have:

$$\nu_\lambda(\xi^{l'_5}z + \gamma') = -\nu_{\lambda'}(z),\ \forall z \in \mathbb{C},\ \gamma' \in \Gamma, \tag{50}$$

as soon as $l'_3 = 0$ and $(l_5 + l'_5)c \in \mathbb{Z}$.

In particular, $\forall z \in \mathbb{C}$, $\forall \gamma' \in \Gamma$, $\forall l_5, l_5' \in \mathbb{Z}$ so that $(l_5 + l_5')c \in \mathbb{Z}$ we have:

$$\nu_{l_5'c}(z) = -\nu_{l_5c}(\xi^{l_5'}z + \gamma').$$

As we have already done for μ_c, we get that ν_c must be a constant and, by means of (40), η_c must be a constant too.

Step 3. Next, we try to find β_Λ and thus to get the finest form of F.

If we apply (46) for $l_5 = -l_5' = 1$ and $l_3 = 0$, then we get $\varepsilon_\Gamma(\gamma + \xi\gamma') = \varepsilon_\Gamma(\gamma) + \varepsilon_\Gamma(\gamma')$, $\forall \gamma, \gamma' \in \Gamma$. Since ε_Γ is a morphism, it must be identically zero.

So, we find the following relation for f_g:

$$(51) \qquad f_g(u,z) \equiv \frac{1}{2i}H_\Gamma(\xi^{l_5}z + l_3 d, \gamma) + \frac{1}{2i}H_\Lambda(u,\lambda) + \beta_\Gamma(\gamma) + \beta_\Lambda(z,\lambda).$$

Let $\varepsilon_\Lambda(z,\lambda) = \frac{\partial \beta_\Lambda}{\partial z}(z,\lambda)$. We turn again to (7) to replace f_g obtained in (51) and then, by taking the derivatives with respect to z, we get:

$$(52) \quad \frac{\xi^{l_5+l_5'}}{2i}H_\Gamma(1, h(\lambda,\lambda')) + \varepsilon_\Lambda(z, \lambda+\lambda') = \xi^{l_5'}\varepsilon_\Lambda(\xi^{l_5'}z + \gamma' + l_3'd, \lambda) + \varepsilon_\Lambda(z,\lambda').$$

By using the same computations as before, one may see that ε_Λ doesn't depend on z, so we write $\varepsilon_\Lambda(\lambda) = \varepsilon_\Lambda(z,\lambda)$ and

$$(53) \qquad \varepsilon_\Lambda(\lambda) = \frac{1}{2i}H_\Gamma(1, l_3 d) + \frac{1-\xi^{l_5}}{1-\xi}\varepsilon_\Lambda(c),$$

$$(54) \qquad \beta_\Lambda(z,\lambda) = \frac{\xi^{l_5}}{2i}H_\Gamma(z, l_3 d) + \frac{1-\xi^{l_5}}{1-\xi}\varepsilon_\Lambda(c)z + \beta_\Lambda(\lambda),$$

where $\beta_\Lambda(\lambda) := \beta_\Lambda(0,\lambda)$.

In particular, for $\lambda \in \Lambda_1$, we have $\varepsilon_\Lambda(\lambda) = \frac{1}{2i}H_\Gamma(1, l_3 d)$ and, by applying (33), we get the following extra-condition for H_Γ:

$$(55) \qquad \frac{1}{2i}H_\Gamma(l_3 d, \gamma) - \frac{1}{2i}H_\Gamma(\gamma, l_3 d) \in \mathbb{Z}, \ \forall \gamma \in \Gamma, \ l_3 \in \mathbb{Z},$$

which is equivalent to:

$$(56) \qquad \begin{array}{ll} (a2) & H_\Gamma(1,1)\mathrm{Im}\,\beta \in 2\mathbb{Z}, \\ (b2) & H_\Gamma(1,1)\mathrm{Im}\,\rho \in 3\mathbb{Z}, \\ (c2) & H_\Gamma(1,1) \in 2\mathbb{Z}. \end{array}$$

Next, we turn back to (7).

Firstly, let us notice that (51) is read here:

$$(57) \quad f_g(u,z) = \frac{1}{2i}H_\Gamma(\xi^{l_5}z + l_3 d, \gamma) + \beta_\Gamma(\gamma) + \frac{1}{2i}H_\Lambda(u,\lambda) + \frac{\xi^{l_5}}{2i}H_\Gamma(z, l_3 d) + \frac{1-\xi^{l_5}}{1-\xi}\varepsilon_\Lambda(c)z + \beta_\Lambda(\lambda) + \mathrm{const}(g),$$

where $\mathrm{const}(g) \in \mathbb{Z}$. As in the proof of *Case 1*, we may suppose that $\mathrm{const}(g) = 0$, without changing the cohomology class of F in $H^2(G, \mathbb{Z})$.

Let us denote by $r(g) := \beta_\Lambda(\lambda) + \beta_\Gamma(\gamma) + \frac{1}{1-\xi}\varepsilon_\Lambda(c)(\gamma + l_3 d)$ and $r_\Lambda(\lambda) := r(\lambda) = \beta_\Lambda(\lambda) + \frac{1}{1-\xi}\varepsilon_\Lambda(c) l_3 d$, $r_\Gamma(\gamma) := r(\gamma) = \beta_\Gamma(\gamma) + \frac{1}{1-\xi}\varepsilon_\Lambda(c)\gamma$. Then, we may suppose that $\varepsilon_\Lambda(c) = 0$ and we find the following final formula for F:

$$
\begin{aligned}
(58) \qquad F(g, g') \;=\; & \frac{1}{2i} H_\Lambda(\lambda', \lambda) + \frac{\xi^{l_5}}{2i} H_\Gamma(\gamma' + l_3' d, \gamma) + \frac{1}{2i} H_\Gamma(l_3 d, \gamma) \\
& + \frac{1}{2i} H_\Gamma(l_3' d, \gamma') - \frac{1}{2i} H_\Gamma((l_3 + l_3')d, \gamma + \xi^{l_5}\gamma' + h(\lambda, \lambda')) \\
& + \frac{\xi^{l_5}}{2i} H_\Gamma(\gamma' + l_3' d, l_3 d) + r(g) + r(g') - r(gg') \in \mathbb{Z}.
\end{aligned}
$$

From (58), one may see that if $H_\Lambda = 0$ and $H_\Gamma = 0$, then F has the cohomology class in $H^2(G, \mathbb{C})$ equal to zero, so the cohomology class of F in $H^2(G, \mathbb{Z})$ is a torsion element. This fact shows that $\mathrm{Ker}(\chi) \subset \mathrm{tors}\, H^2(G, \mathbb{Z})$.

Step 4. We show next that $\mathcal{NS} = \chi(H^2(G, \mathbb{Z}))$.

"$\supset$". Let $(H_\Gamma, H_\Lambda) = \chi(F)$ where $F \in H^2(G, \mathbb{Z})$. We have already seen in *Step 3* that (56) must be true. It remains to prove that $2\mathrm{Im}\, H_\Lambda(\Lambda \times \Lambda) \subset \mathbb{Z} \times \mathbb{Z}$ if S is of type $(c2)$. In fact, we have some more relations which lead us to the conclusion and which are also useful for the Appell-Humbert Theorem.

Let $b_\Gamma(\gamma) = i r_\Gamma(\gamma) - \frac{1}{4} H_\Gamma(\gamma, \gamma)$ and $b_\Lambda(\lambda) = i r_\Lambda(\lambda) - \frac{1}{4} H_\Lambda(\lambda, \lambda)$. As in the case when S is of first type, we have the following relations:

$$
\begin{aligned}
(59) \qquad & S \text{ of type } (a2) \quad 2b_\Gamma(1),\; 2b_\Gamma(\beta) \in i\mathbb{Z}, \\
& S \text{ of type } (b2) \quad b_\Gamma(1) - b_\Gamma(\rho) \in i\mathbb{Z},\; 3b_\Gamma(1) - \frac{i\sqrt{3}}{4} H_\Gamma(1,1) \in i\mathbb{Z}, \\
& S \text{ of type } (c2) \quad 2b_\Gamma(1) \in i\mathbb{Z},\; b_\Gamma(1) - b_\Gamma(i) \in i\mathbb{Z}.
\end{aligned}
$$

We start from the relation $F(\lambda', \lambda) - F(\lambda, \lambda') \in \mathbb{Z}$, $\forall \lambda, \lambda' \in \Lambda$, we replace F from the formula (58) for $\gamma = \gamma' = 0$, $l_5' = l_3 = 0$ and we use (55) to get:

$$
(60) \; iE_\Lambda(l_5 c, l_3' \alpha) + b_\Gamma(h(l_5 c, l_3' \alpha)) + \frac{1}{4} H_\Gamma(1,1) l_3'^2 |d|^2 (\bar{\xi}^{l_5} - \xi^{l_5}) \in i\mathbb{Z},\; \forall l_5, l_3' \in \mathbb{Z}.
$$

This condition is equivalent to:

$$
\begin{aligned}
(61) \qquad & S \text{ of type } (a2) \quad b_\Gamma(1) + iE_\Lambda(c, \alpha) \in i\mathbb{Z}, \\
& S \text{ of type } (b2) \quad b_\Gamma(1) + iE_\Lambda(c.\alpha) - \frac{i\sqrt{3}}{12} H_\Gamma(1,1) \in i\mathbb{Z}, \\
& S \text{ of type } (c2) \quad -b_\Gamma(1) + iE_\Lambda(c, \alpha) - \frac{i}{4} H_\Gamma(1,1) \in i\mathbb{Z}
\end{aligned}
$$

and, because of (56) and (59), if S is of type $(c2)$ then $2E_\Lambda(c, \alpha) \in \mathbb{Z}$.

Moreover, from (55), (58) and (60), we have the following relation for b_Λ:

$$(62)\qquad b_\Lambda(\lambda)+b_\Lambda(\lambda')-b_\Lambda(\lambda+\lambda')+\frac{1}{2}iE_\Lambda(l_5'c,l_3\alpha)+iE_\Lambda(l_5c,l_3'\alpha)$$
$$+\frac{1}{2}H_\Gamma(l_3d,l_3'd)\in i\mathbb{Z},\ \forall\lambda,\lambda'\in\Lambda.$$

"$\subset$". To prove this inclusion, we have to prove that if $(H_\Gamma,H_\Lambda)\in\mathcal{NS}$, then there exist r_Γ and r_Λ so that

$$(63)\qquad \frac{1}{2i}H_\Lambda(\lambda',\lambda)+\frac{\xi^{l_5}}{2i}H_\Gamma(\gamma'+l_3'd,\gamma)+\frac{1}{2i}H_\Gamma(l_3d,\gamma)$$
$$+\frac{1}{2i}H_\Gamma(l_3'd,\gamma')-\frac{1}{2i}H_\Gamma((l_3+l_3')d,\gamma+\xi^{l_5}\gamma'+h(\lambda,\lambda'))$$
$$+\frac{\xi^{l_5}}{2i}H_\Gamma(\gamma'+l_3'd,l_3d)+r_\Lambda(\lambda)+r_\Lambda(\lambda')-r_\Lambda(\lambda+\lambda')$$
$$+r_\Gamma(\gamma)+r_\Gamma(\gamma')-r_\Gamma(\gamma+\xi^{l_5}\gamma'+h(\lambda,\lambda'))\in\mathbb{Z}.$$

We start with $b_\Gamma(1)$ and $b_\Gamma(\beta)$ so that (59) and (61) are satisfied. We set, as in the first case,

$$(64)\qquad b_\Gamma(\gamma)=l_2b_\Gamma(1)+l_4b_\Gamma(\beta)+\frac{1}{2}il_2l_4E_\Gamma(1,\beta)$$

and this b_Γ will satisfy:

$$(65)\qquad b_\Gamma(\gamma)+b_\Gamma(\gamma')-b_\Gamma(\gamma+\gamma')+\frac{1}{2}iE_\Gamma(\gamma,\gamma')\in i\mathbb{Z},$$
$$(66)\qquad b_\Gamma(\xi\gamma)-b_\Gamma(\gamma)\in i\mathbb{Z}.$$

We define

$$(67)\qquad r_\Gamma(\gamma)=-ib_\Gamma(\gamma)-\frac{i}{4}H_\Gamma(\gamma,\gamma).$$

Next, we start with $r_\Lambda(\alpha)$ and $r_\Lambda(c)$ in $\mathbb{C}$ and we take:

$$(68)\quad r_\Lambda(\lambda)\ =\ \frac{(l_3-1)l_3}{4i}H_\Lambda(\alpha,\alpha)+\frac{(l_5-1)l_5}{4i}H_\Lambda(c,c)+\frac{(l_3-1)l_3}{4i}H_\Gamma(d,d)$$
$$+\frac{1}{2i}H_\Lambda(l_5c,l_3\alpha)+l_3r_\Lambda(\alpha)+l_5r_\Lambda(c)$$

A straightforward computation, by using the relations (55), (60), (64), (65), (66), (67) and (68) leads us to the conclusion $\Box$.

We denote by $\Psi'':\mathcal{NS}\tilde{\rightarrow}\mathrm{Num}(S)$ the isomorphism obtained in Theorem 2.2.

4 Appell-Humbert theorem

Keeping the notations from the previous sections, we define $\alpha_\Gamma(\gamma):=e^{2\pi b_\Gamma(\gamma)}$ and $\alpha_\Lambda(\lambda):=e^{2\pi b_\Lambda(\lambda)}$. Recall that, since $b_\Gamma(\xi\gamma)-b_\Gamma(\gamma)\in i\mathbb{Z}$, b_Γ must be purely imaginary.

If S is of first type, then α_Γ and α_Λ will verify

$$\alpha_\Lambda(\lambda+\lambda') = \alpha_\Lambda(\lambda)\alpha_\Lambda(\lambda')e^{\pi i E_\Lambda(\lambda,\lambda')} \tag{69}$$

$$\alpha_\Gamma(\gamma+\gamma') = \alpha_\Gamma(\gamma)\alpha_\Gamma(\gamma')e^{\pi i E_\Gamma(\gamma,\gamma')} \tag{70}$$

$$\alpha_\Gamma(\xi\gamma) = \alpha_\Gamma(\gamma), \tag{71}$$

where $(H_\Gamma, H_\Lambda) \in \mathcal{NS}$.

If S is of second type, then α_Γ and α_Λ will verify

$$\alpha_\Lambda(\lambda+\lambda') = \alpha_\Lambda(\lambda)\alpha_\Lambda(\lambda')e^{\pi i E_\Lambda(l_5'c, l_3\alpha)+\pi i E_\Lambda(l_5c, l_3'\alpha)+\pi H_\Gamma(l_3d, l_3'd)} \tag{72}$$

$$\alpha_\Gamma(\gamma+\gamma') = \alpha_\Gamma(\gamma)\alpha_\Gamma(\gamma')e^{\pi i E_\Gamma(\gamma,\gamma')} \tag{73}$$

$$\alpha_\Gamma(\xi\gamma) = \alpha_\Gamma(\gamma) \tag{74}$$

and

$$\alpha_\Gamma(1) = \begin{cases} e^{-2\pi i E_\Lambda(c,\alpha)} & S \text{ of type } (a2) \\ e^{-2\pi i E_\Lambda(c,\alpha)+\pi\frac{i\sqrt{3}}{6}H_\Gamma(1,1)} & S \text{ of type } (b2) \\ e^{-2\pi i E_\Lambda(c,\alpha)-\pi\frac{i}{2}H_\Gamma(1,1)} & S \text{ of type } (c2), \end{cases} \tag{75}$$

where $(H_\Gamma, H_\Lambda) \in \mathcal{NS}$.

Let $\mathcal{P}_1 = \{$ Group of data $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \alpha_\Lambda)\}$ with natural group operation and $\mathcal{P} = \mathcal{P}_1/\sim$ where $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \alpha_\Lambda) \sim (H'_\Gamma, H'_\Lambda, \alpha'_\Gamma, \alpha'_\Lambda)$ if and only if $H_\Gamma = H'_\Gamma$, $H_\Lambda = H'_\Lambda$, $\alpha_\Gamma = \alpha'_\Gamma$ and there exists $a \in \mathbb{C}$ so that $\alpha_\Lambda(\lambda) = \alpha'_\Lambda(\lambda)e^{2\pi i a\lambda}$, $\forall \lambda \in \Lambda$. For simplicity, we shall denote by $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$ instead of $\widehat{(H_\Gamma, H_\Lambda, \alpha_\Gamma, \alpha_\Lambda)}$ and $\alpha_\Lambda \sim \alpha'_\Lambda$ for the equivalence.

Remark 4.1. By using a classical argument that have been already used in section 2 (cf. [9], Chapter I), one may see that if S is of second type and $H_\Gamma = 0$ or if S is of first type, then there exists an unique α'_Λ so that $\alpha_\Lambda \sim \alpha'_\Lambda$ and $\alpha'_\Lambda(\lambda) \in U(1)$, $\forall \lambda \in \Lambda$.

This argument allows us many times to suppose that the multiplicators appearing in theorems of Appell-Humbert kind are $U(1)$–valued (see [9] for tori and [3] for primary Kodaira surfaces).

Lemma 4.2. *We have an exact short sequence*

$$0 \longrightarrow \mathrm{Hom}(G, U(1)) \xrightarrow{\mu} \mathcal{P} \xrightarrow{\eta} \mathcal{NS} \longrightarrow 0$$

where η is the canonical projection and $\mu(\alpha_G) = (0, 0, \alpha_G|_\Gamma, \alpha_G|_\Lambda)$.

Proof. The morphism η is surjective from the proof of the Theorem 3.1. By the above remark, μ is injective. Since $\eta\mu = 0$ it remains to check that $\mathrm{Ker}(\eta) \subset \mu(\mathrm{Hom}(G, U(1))$.

Indeed, let $(0, 0, \alpha_\Gamma, \widehat{\alpha_\Lambda}) \in \mathcal{P}$. Since the corresponding hermitian forms are equal to zero, it follows that $\alpha_\Gamma \in \mathrm{Hom}(\Gamma, U(1))$ and $\alpha_\Lambda \in \mathrm{Hom}(\Lambda, \mathbb{C}^*)$. From Remark 4.1., $\widehat{\alpha_\Lambda}$ has a representative that is $U(1)$–valued, say α'_Λ.

Then we define $\alpha_G(g) := \alpha_\Gamma(\gamma)\alpha'_\Lambda(\lambda) \in U(1)$, $\forall g = \gamma\lambda \in G$, which is an element of $\mathrm{Hom}(G, U(1))$ and verifies $\mu(\alpha_G) = (0, 0, \alpha_\Gamma, \widehat{\alpha_\Lambda})$ □.

Theorem 4.3. *There is the following isomorphism of exact sequences*

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}(G,U(1)) & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{NS} & \longrightarrow & 0 \\ & & \wr\downarrow \Psi' & & \wr\downarrow \Psi & & \wr\downarrow \Psi'' & & \\ 0 & \to & \mathrm{Pic}^\tau(S) & \longrightarrow & \mathrm{Pic}(S) & \longrightarrow & \mathrm{Num}(S) & \longrightarrow & 0 \end{array}$$

where Ψ' is the isomorphism from section 2, Ψ'' is the isomorphism from section 3 and Ψ maps an element $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda}) \in \mathcal{P}$ to the cocycle $\{e_g\}_g \in H^1(G, H^*)$ given by

$$e_g(u,z) = \alpha_\Gamma(\gamma)\alpha_\Lambda(\lambda)e^{\pi H_\Lambda(u,\lambda)+\pi H_\Gamma(\xi^{l_5}z+\gamma,\gamma+l_3d)-\frac{\pi}{2}H_\Gamma(\gamma,\gamma)+\frac{\pi}{2}H_\Lambda(\lambda,\lambda)}.$$

Proof. All we have to check is that Ψ is well-defined, so let us suppose that $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$ maps by Ψ to $\{e_g\}_g \in H^1(G, H^*)$ and we change the representative of α_Λ by α'_Λ. If $e''_g = \frac{\alpha_\Lambda(\lambda)}{\alpha'_\Lambda(\lambda)} \overset{not}{=} \alpha''_\Lambda(\lambda)$, then is is easy to see that $\{e''_g\}_g$ is a coboundary in $C^1(G, H^*)$.

Indeed, there exists $a \in \mathbb{C}$ so that $\alpha''_\Lambda(\lambda) = e^{2\pi i a\lambda}$ and we choose $h(u,z) = e^{2\pi i a u}$. Then, $e''_g = h(g(u,z))h^{-1}(u,z)$, $\forall u, z \in \mathbb{C}$, $g \in G$ □.

Definition 4.4. For any $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda}) \in \mathcal{P}$, the line bundle over S associated to the cocycle $\{e_g\}_g = \Psi(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda}) \in H^1(G, H^*)$ will be denoted by $L(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$.

Remark 4.5. $L(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$ is the quotient of $\mathbb{C}^2 \times \mathbb{C}$ given by the equivalence relation $((u,z),w) \sim (g(u,z), e_g(u,z)w)$, $\forall g \in G$.

5 Applications

The first application of Appell–Humbert theorem is a description of tors $H^2(G,\mathbb{Z})$ and its generators in terms of the groups cohomology (see, also [10], [12] for precised characterisation).

By taking into account that torsion cocycles F are given by the vanishing of their corresponding hermitian forms H_Γ and H_Λ, one may obtain very easy the following table (see, also [5] for a similar result on primary Kodaira surfaces):

Type	tors $H^2(G,\mathbb{Z})$	Action of generators of tors $H^2(G,\mathbb{Z})$ on (g,g')
(a1)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(1-(-1)^{l_5})l'_2/2$ and $(1-(-1)^{l_5})l'_4/2$
(a2)	$\mathbb{Z}_2$	$(1-(-1)^{l_5})l'_4/2$
(b1)	$\mathbb{Z}_3$	$(\mathrm{Re}((1-\rho^{l_5})\gamma') + \sqrt{3}\mathrm{Im}((1-\rho^{l_5})\gamma'))/3$
(b2)	0	0
(c1)	$\mathbb{Z}_2$	$(\mathrm{Re}((1-i^{l_5})\gamma') + \mathrm{Im}((1-i^{l_5})\gamma'))/2$
(c2)	0	0
(d1)	0	0

Next, we may apply Appell–Humbert theorem to compute a basis in Num(S) (see, also [10], Theorem 1.4.).

Let us denote by q the cardinal of $\mathcal{G}$.

If we fix isomorphisms $H^2(\Gamma,\mathbb{Z}) \cong H^2(E,\mathbb{Z}) \overset{deg}{\cong} \mathbb{Z}$ and $H^2(\Lambda_2,\mathbb{Z}) \cong H^2(\Delta,\mathbb{Z}) \overset{deg}{\cong} \mathbb{Z}$, then the inclusions $\mathcal{NS} \subset \mathcal{N}_1 \subset \mathcal{N}_2 = \mathbb{Z} \oplus \mathbb{Z}$ will become:

Type	$\mathcal{N}_1$	$\mathcal{NS}$	q	basis in $\mathcal{NS}$	
				e_1	e_2
$(a1)$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}\oplus 2\mathbb{Z}$	2	$(1,0)$	$(0,2)$
$(a2)$	$\mathbb{Z}\oplus 2\mathbb{Z}$	$2\mathbb{Z}\oplus 2\mathbb{Z}$	4	$(2,0)$	$(0,2)$
$(b1)$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}\oplus 3\mathbb{Z}$	3	$(1,0)$	$(0,3)$
$(b2)$	$\mathbb{Z}\oplus 3\mathbb{Z}$	$3\mathbb{Z}\oplus 3\mathbb{Z}$	9	$(3,0)$	$(0,3)$
$(c1)$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}\oplus 4\mathbb{Z}$	4	$(1,0)$	$(0,4)$
$(c2)$	$\mathbb{Z}\oplus 2\mathbb{Z}$	$2\mathbb{Z}\oplus 4\mathbb{Z}$	8	$(2,0)$	$(0,4)$
$(d1)$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}\oplus 6\mathbb{Z}$	6	$(1,0)$	$(0,6)$

It is easy to determine the numerical classes of $\mathcal{O}_S(E)$ and $\mathcal{O}_S(\Delta)$ in $\mathcal{NS}$. Indeed, according to [10], since the intersection number $E.\Delta$ is equal to q, then via isomorphism $\mathcal{N}_2 \cong \mathbb{Z}\oplus\mathbb{Z}$, we have $c_1(E) = (0,q)$ and $c_1(\Delta) = (q,0)$.

Then, by using the previous table, we get the following (compare also with [10],Theorem 1.4.):

Type	Basis of Num(S)	
$(a1)$	$1/2\Delta$	E
$(a2)$	$1/2\Delta$	$1/2E$
$(b1)$	$1/3\Delta$	E
$(b2)$	$1/3\Delta$	$1/3E$
$(c1)$	$1/4\Delta$	E
$(c2)$	$1/4\Delta$	$1/2E$
$(d1)$	$1/6\Delta$	E

The next application of Appell-Humbert theorem is computing the space of global sections of some line bundles over S.

As we saw, any element $L \in \text{Pic}(S)$ can be written as $L = L(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$, where $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda}) \in \mathcal{P}$.

From [10], Theorem 1.4., the numerical type of L is of form $c_1(L) = a\Delta + bE$, where $a,b \in \mathbb{Q}$, or $c_1(L) = a_1e_1 + b_1e_2$ with $a_1, b_1 \in \mathbb{Z}$. According to [10], Lemma 1.3., if $H^0(L) \neq 0$, then $a, b \geq 0$, which is equivalent to the inequalities $H_\Gamma(1,1) \geq 0$, $H_\Lambda(1,1) \geq 0$. If $a,b > 0$, then L is ample (cf. [10], Lemma 1.3.) and $h^0(L) = abq = a_1b_1 > 0$, so it remains to study the cases $a = 0,\ b > 0$ and $a > 0,\ b = 0$.

Here we shall compute $H^0(L)$ for $a = 0,\ b > 0$. Before stating our result, let us introduce the following notion:

Definition 5.1. Let $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda}) \in \mathcal{P}$. Any holomorphic function $\theta : \mathbb{C}^2 \to \mathbb{C}$ so that:

$$\theta(g(u,z)) = e_g(u,z)\theta(u,z), \ \forall g \in G, \ u, z \in \mathbb{C} \tag{76}$$

is called a *θ-function* for the data $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$.

It is easy to see that there is a natural one-to-one correspondence between θ-functions for $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$ and sections of $L(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$.

Proposition 5.2. *If $c_1(L) = bE$, $b > 0$ then $h^0(L) \neq 0$ if and only if α_Γ is identically equal to* 1.

In this case, $b \in \mathbb{Z}$ and there is a natural isomorphism: $H^0(L) \cong H^0(L(H_\Lambda, \alpha_\Lambda))$, where $L(H_\Lambda, \alpha_\Lambda)$ is the line bundle over $\mathbb{C}/\Lambda$ associated to the hermitian form H_Λ and the multiplicator α_Λ.

Proof. The equality $a = 0$ is equivalent to $H_\Gamma = 0$ and then $\alpha_\Gamma : \Gamma \to U(1)$ is a morphism of groups with $\alpha_\Gamma(\xi\gamma) = \alpha_\Gamma(\gamma)$, $\forall \gamma \in \Gamma$. On the other hand, from Remark 4.1. and Remark 4.2., we may suppose that α_Λ is $U(1)$–valued. Moreover, since $H_\Gamma = 0$ then:

$$e_g(u,z) = \alpha_\Gamma(\gamma)\alpha_\Lambda(\lambda)e^{\pi H_\Lambda(u,\lambda) + \frac{\pi}{2}H_\Lambda(\lambda,\lambda)}$$

for both types of hyperelliptic surfaces.

Claim 1. If α_Γ is identically equal to 1 then $E_\Lambda(\Lambda \times \Lambda) \subset \mathbb{Z}$ and

$$\alpha_\Lambda(\lambda + \lambda') = \alpha_\Lambda(\lambda)\alpha_\Lambda(\lambda')e^{\pi i E_\Lambda(\lambda,\lambda')}.$$

Proof of Claim 1. For the case when S is of first type , this is nothing else than the definition. If S is of second type, then $H_\Gamma = 0$ implies that $1 = \alpha_\Gamma(1) = e^{-2\pi i E_\Lambda(c,\alpha)}$ so $E_\Lambda(c,\alpha) \in \mathbb{Z}$ i.e. $E_\Lambda(\Lambda \times \Lambda) \subset \mathbb{Z}$. Because $E_\Lambda(c,\alpha) \in \mathbb{Z}$, we apply (72) to get $\alpha_\Lambda(\lambda + \lambda') = \alpha_\Lambda(\lambda)\alpha_\Lambda(\lambda')e^{\pi i E_\Lambda(\lambda,\lambda')}$.

Claim 2. The condition $b \in \mathbb{Z}$ is equivalent to $E_\Lambda(\Lambda \times \Lambda) \subset \mathbb{Z}$.

Now, we turn back to the proof of Proposition 5.2.

"$\Longrightarrow$". If $h^0(L) > 0$, then there exists a θ–function for $(0, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$,say θ, non-identically zero. Then, $\forall u, z \in \mathbb{C}$, $\gamma \in \Gamma$, $\lambda \in \Lambda$, θ must satisfy:

$$\theta(u + \lambda, \xi^{l_5}z + \gamma + l_3 d) = \alpha_\Gamma(\gamma)\alpha_\Lambda(\lambda)e^{\pi H_\Lambda(u,\lambda) + \frac{\pi}{2}H_\Lambda(\lambda,\lambda)}\theta(u,z). \tag{77}$$

If we take $\lambda = 0$ in (77), it follows that:

$$\theta(u, z + \gamma) = \alpha_\Gamma(\gamma)\theta(u,z), \ \forall u, z \in \mathbb{C}, \ \gamma \in \Gamma. \tag{78}$$

Since α_Γ is $U(1)$–valued, then we can apply maximum priciple in (78) to conclude that θ does not depend on z i.e. $\theta(u,z) = \theta(u)$, $\forall u, z \in \mathbb{C}$. The condition (78) implies also that α_Γ must be identically equal to 1. Moreover, (77) becomes

$$\theta(u + \lambda) = \alpha_\Lambda(\lambda)e^{\pi H_\Lambda(u,\lambda) + \frac{\pi}{2}H_\Lambda(\lambda,\lambda)}\theta(u). \tag{79}$$

From (79) and *Claim 1.* we deduce that θ is in fact a θ-function for the data $(H_\Lambda, \alpha_\Lambda)$ with respect to the lattice Λ.

"$\Longleftarrow$". We apply again *Claim 1.* and then we can choose $\theta \in H^0(H_\Lambda, \alpha_\Lambda)$. It is easy to see that if we define $\theta(u, z) = \theta(u)$, then θ is also a θ-function for the data $(0, H_\Lambda, 1, \alpha_\Lambda)$.

For the final part of proposition, we apply *Claim 2.* and [9], Chapter I □.

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