INSTITUTUL DE MATEMATICA
AL ACADEMIEI ROMANE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS

OF THE ROMANIAN ACADEMY

ISSN 0250 3638

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Preprint No. 29/1996

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November, 1996

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ON A NEHARI TYPE PROBLEM ON SPACES WITH INDEFINITE INNER PRODUCT

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Dedicated to Heinz Langer on the 60th anniversary of his birthday

We formulate a generalised Nehari type problem, in the sense of Adamyan-Arov-Kreĭn, on spaces with indefinite inner product. We adapt the approach of Ball-Helton and Treil-Volberg to view it as a problem of existence of invariant maximal nonnegative subspaces and, in a certain case, we characterise its solvability. Applications to bounds and singular values of generalised Hankel operators, to contractive intertwining dilations and to some interpolation problems of Carathéodory-Schur and Nevanlinna-Pick type for matrix-valued meromorphic functions are given. A direct proof of a theorem of Treil-Volberg is also included.

1. Introduction

According to a celebrated theorem of Z. Nehari [28], for an arbitrary function $f \in L^\infty$ the distance of f to H^∞ coincides with the operator norm of the Hankel operator $\Gamma_f = P_{H^2_-} M_f$, where H^2_- denotes the space $L^2 \ominus H^2$ and $M_f \in \mathcal{L}(L^2)$ denotes the multiplication operator associated to f. Moreover, this distance is actually a minimum, in the sense that there exists a function $g \in H^\infty$ such that $\operatorname{dist}(f, H^\infty) = \|f - g\|_\infty$.

Let S denote the forward shift on L^2. An equivalent formulation of the Nehari problem looks as follows: given a bounded linear operator $\Gamma: L^2 \to H^2_-$ such that $\Gamma S = P_{H^2_-} S \Gamma$, it is required to determine a function $f \in L^\infty$ such that $\Gamma = P_{H^2_-} M_f$ with $\|f\|_\infty = \|M_f\| = \|\Gamma\|$. Note that for any function $f \in L^\infty$ such that $\Gamma = P_{H^2_-} M_f$ we have $\|f\|_\infty \geq \|\Gamma\|$. On the other hand, since the class of multiplication operators on L^2 with functions in L^∞ coincides with the class of linear bounded operators on L^2 commuting with the operator S, if follows that the problem requires actually to determine those linear operators $M \in \mathcal{L}(L^2)$ such that $SM = MS$, $\Gamma = P_{H^2_-} M$, and $\|M\| \leq \|\Gamma\|$.

The latter formulation of the Nehari problem turns out to be a very general problem encompassing many interpolation problems and it also can be put in a form dealing with operator valued functions, in particular with matrix valued functions. For a pertinent survey of these facts, as well as other formulations of the Nehari problem, see C. Foiaş and A. Frazho [19].

The problem of characterisation of the singular values of compact Hankel operators was considered by V.M. Adamyan, D.Z. Arov and M.G. Kreĭn in [2]. Noting that the above formulation of the Nehari problem can be regarded as referring to the first singular number $s_0(\Gamma) = \|\Gamma\|$, this can be considered as a generalisation of the Nehari problem. In the first formulation of the Nehari problem mentioned above this corresponds to the determination of the distance $\operatorname{dist}(f, H_l^\infty)$ where H_l^∞ denotes the class of all meromorphic functions g which admit a multiplicative representation $h\varphi^{-1}$ where $h \in H^\infty$ and φ is a polynomial with at most l roots in the open unit disc $\mathbb{D}$.

In this paper we formulate a generalised Nehari type problem, in the sense of Adamyan-Arov-Kreĭn, on spaces with indefinite inner product. The approach we follow is an adaptation of the angular operator and shift invariant maximal nonnegative subspaces method initiaded by J.A. Ball and J.W. Helton in [8]. However, in the case we are dealing with this geometric interpretation with angular operator is lost. In addition, this forces a limitation of this method to the case when the added space is positive definite. To make things precise from the beginning let us fix some terminology and recall some background material from the theory of indefinite inner product spaces.

A Kreĭn space is by definition a complex vector space $\mathcal{K}$ endowed with an (indefinite) inner product $[\cdot,\cdot]$ such that there exists an operator $J\colon \mathcal{K} \to \mathcal{K}$, $J^{-1} = J$ with the property that the positive inner product $\langle\cdot,\cdot\rangle_J$,

$$\langle x, y\rangle_J = [Jx, y], \quad x, y \in \mathcal{K},$$

turns $(\mathcal{K}; \langle\cdot,\cdot\rangle_J)$ into a Hilbert space. The operator J is called a *fundamental symmetry*. Any fundamental symmetry J admits a Jordan decomposition $J = J^+ - J^-$. The spectral subspaces $\mathcal{K}^\pm = J^\pm\mathcal{K}$ are orthogonal with respect to the inner product $[\cdot,\cdot]$, $(\mathcal{K}^+, [\cdot,\cdot])$ is positive definite and $(\mathcal{K}^-, [\cdot,\cdot])$ in negative definite. The decomposition $\mathcal{K} = \mathcal{K}^+[+]\mathcal{K}^-$ is called a *fundamental decomposition*. The *strong topology* of the Kreĭn space $\mathcal{K}$ is given by an arbitrary fundamental symmetry, more precisely, by the corresponding positive definite inner product. It does not depend on which fundamental symmetry is chosen.

If $\mathcal{L}$ is a (closed) subspace of the Kreĭn space $\mathcal{K}$ then we denote by $\mathcal{L}^\perp$ its orthgonal. A decomposition $\mathcal{L} = \mathcal{L}_-[+]\mathcal{L}^0[+]\mathcal{L}_+$ exists, where $\mathcal{L}_-$ is negative definite, $\mathcal{L}_+$ is positive definite, and $\mathcal{L}^0 = \mathcal{L} \cap \mathcal{L}^\perp$ is the *isotropic subspace*. The cardinal numbers $\kappa_\pm(\mathcal{L}) = \dim(\mathcal{L}_\pm)$ are called the *positive/negative signatures* of $\mathcal{L}$ and they do not depend on the particular above decomposition. If these numbers are finite then they coincide with the number of positive/negative squares of the quadratic forms $\mathcal{L}_\pm \ni x \mapsto [x, x]$.

If $\mathcal{K}_i$, $i = 1, 2$, are Kreĭn spaces then we denote by $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ the vector space of bounded linear operators $T\colon \mathcal{K}_1 \to \mathcal{K}_2$. For such an operator we denote by $T^\sharp \in \mathcal{L}(\mathcal{K}_2, \mathcal{K}_1)$ its adjoint, more precisely,

$$[Tx, y] = [x, T^\sharp y], \quad x \in \mathcal{K}_1,\ y \in \mathcal{K}_2.$$

If fundamental symmetries J_1 and J_2 on $\mathcal{K}_1$ and, respectively, $\mathcal{K}_2$ are fixed then $T^\sharp = J_1 T^* J_2$.

Let $(\mathcal{K}, [\cdot,\cdot])$ be a Kreĭn space and $A \in \mathcal{L}(\mathcal{K})$ a selfadjoint operator, that is, $A = A^\sharp$. We consider a fundamental symmetry J in $\mathcal{K}$ and then the operator $G = JA$ is selfadjoint with respect to the Hilbert space $(\mathcal{K}; \langle\cdot,\cdot\rangle_J)$. If $G = G_+ - G_-$ is the Jordan decomposition of G then we denote by $\kappa_\pm(A)$ the dimension of the spectral subspace $\operatorname{cl} G_\pm\mathcal{K}$. For instance, if $\kappa_-(A)$ is finite then it coincides with the number of negative eigenvalues, counted with their multiplicities, of G.

Let $(\mathcal{K}, [\cdot,\cdot])$ a Kreĭn space and let $\mathcal{H}$ be a (closed) subspace of $\mathcal{K}$. We fix a fundamental symmetry J on $\mathcal{K}$, let $\langle\cdot,\cdot\rangle$ be the corresponding positive definite inner product, and let $G \in \mathcal{L}(\mathcal{H})$ be the Gram operator of $[\cdot,\cdot]$ with respect to $\langle\cdot,\cdot\rangle$ on $\mathcal{H}$, that is, G is selfadjoint with respect to the positive definite inner product $\langle\cdot,\cdot\rangle$ and

$$[x,y] = \langle Gx, y\rangle, \quad x,y \in \mathcal{H}. \tag{1.1}$$

Clearly we have $G = P_{\mathcal{H}}J|\mathcal{H}$, where $P_{\mathcal{H}}$ denotes the projection on $\mathcal{H}$ along $J\mathcal{H}^{\perp}$.

Consider $G = G_+ - G_-$ the Jordan decomposition of G, let $\mathcal{H}_+$ denote the spectral subspace corresponding to the nonnegative semiaxis $[0,+\infty)$ and let $\mathcal{H}_-$ be the spectral subspace corresponding to the negative semiaxis $(-\infty,0)$. Clearly we have the decomposition

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \tag{1.2}$$

and, if $x = x_+ + x_-$ and $y = y_+ + y_-$ are the corresponding representations of arbitrary vectors $x,y \in \mathcal{H}$, then

$$[x,y] = \langle G_+x_+, y_+\rangle - \langle G_-x_-, y_-\rangle.$$

In the following we will use the notions of positivity, negativity, neutrality, etc. with respect to the indefinite inner product space $(\mathcal{H}; [\cdot,\cdot])$, and fix the decomposition (1.2).

Let $\mathcal{M}$ be a nonnegative subspace of $\mathcal{H}$, that is, a closed linear manifold such that $[x,x] \geq 0$ for all $x \in \mathcal{M}$. With respect to the decomposition (1.2) this means

$$\langle G_+x_+, x_+\rangle \geq \langle G_-x_-, x_-\rangle, \quad x = x_+ + x_- \in \mathcal{M}. \tag{1.3}$$

As in the case of Kreĭn spaces this enables us to introduce an angular operator. Let $P_\pm$ denote the projection of $\mathcal{H}_\pm$ with respect to the decomposition (1.2). Clearly $P_\pm$ are orthogonal projections in the Hilbert space $\mathcal{H}$, in particular their norms are ≤ 1. Let us define an operator $K_{\mathcal{M}}\colon P_+\mathcal{M} \to \mathcal{H}_-$ by

$$K_{\mathcal{M}}\colon P_+x \mapsto P_-x, \quad x \in \mathcal{M}. \tag{1.4}$$

By (1.3) and taking into account that G_- is injective on $\mathcal{H}_-$, this definition is correct and

$$\mathcal{M} = \{x + K_{\mathcal{M}}x \mid x \in P_+\mathcal{M}\}. \tag{1.5}$$

Since $\mathcal{M}$ is closed, this implies that the operator $K_{\mathcal{M}}$ is closed. The operator $K_{\mathcal{M}}$ is called the *generalised angular operator* of the nonnegative subspace $\mathcal{M}$. Also note that in this general setting there is no reason to conclude that $P_+\mathcal{M}$ is a (closed) subspace. This anomaly is remedied if an extra condition is imposed, more precisely, the condition that in the Jordan decomposition of the Gram operator G the operator G_- has closed range or, equivalently, the condition that the spectrum of G has a gap $(-\varepsilon, 0)$. This condition is independent on which admissible positive definite inner product $\langle\cdot,\cdot\rangle$ we consider on the space $\mathcal{H}$, since, by changing it with another Gram operator, say B, we have $B = C^*GC$ for some boundedly invertible $C \in \mathcal{L}(\mathcal{H})$ (the inner products on the incoming Hilbert space $\mathcal{H}$ and the outgoing Hilbert space $\mathcal{H}$ are different) and this transformation preserves the topology of the spectrum.

The following result established in [35] (cf. [36], see also [20]) shows the possibility of handling generalised angular operators in a similar fashion as the angular operators in Kreĭn spaces.

LEMMA 1.1 *With the previous notation, assume that the operator G_- has closed range. Then:*

(1) *$\mathcal{M}$ is a nonnegative subspace of $\mathcal{H}$ if and only if $P_+\mathcal{M}$ is closed, $K_{\mathcal{M}}$ is bounded and the following inequality holds:*

$$K_{\mathcal{M}}^* G_- K_{\mathcal{M}} \leq P_{P_+\mathcal{M}} G_+ | P_+\mathcal{M}. \tag{1.6}$$

(2) *Let $\mathcal{M}$ and $\mathcal{N}$ be nonnegative subspaces. Then $\mathcal{M} \subseteq \mathcal{N}$ if and only if $K_{\mathcal{M}} \subseteq K_{\mathcal{N}}$, that is, $P_+\mathcal{M} \subseteq P_+\mathcal{N}$ and $K_{\mathcal{M}}x = K_{\mathcal{N}}x$ for all $x \in P_+\mathcal{M}$.*

(3) *For any nonnegative subspace $\mathcal{M}$ there exists a maximal nonnegative subspace $\widetilde{\mathcal{M}}$ such that $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$.*

(4) *A nonnegative subspace $\mathcal{M}$ is maximal if and only if $P_+\mathcal{M} = \mathcal{H}_+$.*

Another result that can be extended from Kreĭn spaces to subspaces of Kreĭn spaces under the condition that G_- has closed range refers to the existence of maximal nonnegative invariant subspaces for expansive operators. Starting with the work of Pontryagin [30] the problem of existence of semi-definite invariant subspaces played a key role in the development of the theory of operators on indefinie metric spaces. Major contributions appeared in the work of M.G. Kreĭn [24], H. Langer [27], and I. S. Iokhvidov [23]. Using a fixed point theorem of Ky Fan [16] and Glicksberg [21] and following an idea of Ky Fan [17], I.S. Iokhvidov [22] essentially proved (cf. [36]) the following result:

THEOREM 1.2 *Let $\mathcal{H}$ be a subspace of some Kreĭn space such that, with the above notation, the operator G_- has closed range. Let $V \in \mathcal{L}(\mathcal{H})$ be an operator subject to the following conditions:*

(i) *V is expansive, $[Vx, Vx] \geq [x, x]$ for all $x \in \mathcal{H}$.*

(ii) *The operator $P_{\mathcal{H}_+} V P_{\mathcal{H}_-}$ is compact.*

Then there exists a maximal nonnegative subspace $\mathcal{M}$ in $\mathcal{H}$ which is invariant under the operator V.

A recent important application of this theorem was provided by S. Treil and A. Volberg, [35], [36] to an abstract Nehari problem encompassing applications in the field of Hankel operators on weighted Bergman spaces. The main idea was to use Theorem 1.2 in conjuction with the angular operator approach used by J. A. Ball and J. W. Helton [8] to a generalised interpolation problem. This allows for a much broader range of applications, as already shown in [36].

In this paper we adapt the approach of Ball-Helton and Treil-Volberg to a Nehari type problem on spaces with indefinite inner products. We consider it as a problem of existence of invariant maximal nonnegative subspaces and, in a certain case, we characterise its solvability, cf. Theorem 2.4. Applications to bounds and singular values of generalised Hankel operators, to contractive intertwining dilations and to some interpolation problems of Carathéodory-Schur and Nevanlinna-Pick type for matrix-valued meromorphic functions are given. A direct proof of the main theorem of Treil-Volberg in [36] is also included.

2. A Generalised Nehari Type Problem in Kreĭn Spaces

In this section we reformulate in the framework of indefinite metric spaces an abstract Nehari problem and we indicate a situation when necessary and sufficient conditions can be found in order that the problem can be solved.

Let $\mathcal{G}_1$ be a Kreĭn space and let S_1 be a bounded operator in $\mathcal{G}_1$. Also, let $\mathcal{G}_2$ be another Kreĭn space such that $\mathcal{G}_2$ contains the space $\mathcal{H}_2$ as a *regular subspace* (that is, a subspace of $\mathcal{G}_2$ which is also a Kreĭn space with the induced indefinite inner product and the same strong topology).

We also consider S_2 a bounded operator in $\mathcal{G}_2$ and we assume that the subspace $\mathcal{G}_2 \cap \mathcal{H}_2^{\perp}$ is *invariant* under S_2. Following the idea in [36] we introduce

DEFINITION 2.1 With the above notation, a bounded operator $\Gamma\colon \mathcal{G}_1 \to \mathcal{H}_2$ is called an (S_1, S_2)*-Hankel operator* if $\Gamma S_1 = P_{\mathcal{H}_2} S_2 \Gamma$.

DEFINITION 2.2 With the above notation, let Γ be an (S_1, S_2)-Hankel operator, $\rho > 0$ and κ a cardinal number. The set $\mathrm{N}_\kappa(\Gamma; \rho)$ consists of those pairs $(M; \mathcal{E})$ subject to the following conditions:

(1) $\mathcal{E}$ is a subspace of $\mathcal{G}_1$ invariant under S_1 and of codimension at most κ;

(2) $M\colon \mathcal{E} \to \mathcal{G}_2$ is bounded, $[Mx, Mx] \leq \rho^2[x,x]$ for all $x \in \mathcal{E}$, and $MS_1|\mathcal{E} = S_2 M$;

(3) $\Gamma|\mathcal{E} = P_{\mathcal{H}_2} M$.

The problem that we address here is to determine the elements of the set $\mathrm{N}_\kappa(\Gamma; \rho)$. First we notice that

$$\mathrm{N}_\kappa(\Gamma; \rho) = \{\rho^{-1} M \mid M \in \mathrm{N}_\kappa(\rho^{-1}\Gamma; 1)\}. \tag{2.1}$$

As a conclusion, it is sufficient to determine the set $\mathrm{N}_\kappa(\Gamma; 1)$; in the following this set will be denoted by $\mathrm{N}_\kappa(\Gamma)$, for simplicity.

For the beginning we obtain a necessary condition of solvability of the generalised Nehari problem.

LEMMA 2.3 *Assume that the set* $\mathrm{N}_\kappa(\Gamma; \rho)$ *is nonvoid. Then*

$$\kappa_-(\rho^2 I - \Gamma^\sharp \Gamma) \leq \kappa + \kappa_-(\mathcal{G}_2 \cap \mathcal{H}_2^{\perp}).$$

Proof. Let $(M; \mathcal{E})$ be in $\mathrm{N}_\kappa(\Gamma; \rho)$. Then for all $x \in \mathcal{E}$ we have

$$\begin{aligned}\rho^2[x,x] - [\Gamma x, \Gamma x] &= \rho^2[x,x] - [P_{\mathcal{H}_2} Mx, P_{\mathcal{H}_2} Mx] \\ &= \rho^2[x,x] - [Mx, Mx] + [P_{\mathcal{G}_2 \cap \mathcal{H}_2^{\perp}} Mx, P_{\mathcal{G}_2 \cap \mathcal{H}_2^{\perp}} Mx].\end{aligned}$$

Taking into account that the quadratic form $\rho^2[x,x] - [Mx, Mx]$ is nonnegative on $\mathcal{E}$ and that the codimension of $\mathcal{E}$ is at most κ, from here we obtain the desired inequality. ∎

A general method to produce elements of $\mathrm{N}_\kappa(\Gamma)$ in case of Hilbert spaces was applied in [36], for $\kappa = 0$, following an idea of J.A. Ball and J.W. Helton [8] based on the existence of maximal nonnegative subspaces with some additional properties. We now adapt this construction in this more general setting of indefinite inner product spaces.

Let

$$\mathcal{K} = \mathcal{G}_1 \oplus \mathcal{G}_2 \tag{2.2}$$

on which we consider the indefinite inner product $[\cdot,\cdot]$ defined by

$$[x_1 + x_2, y_1 + y_2] = [x_1, y_1] - [x_2, y_2], \quad x_1, y_1 \in \mathcal{G}_1,\ x_2, y_2 \in \mathcal{G}_2.$$

Then $(\mathcal{K}, [\cdot,\cdot])$ becomes a Kreĭn space. Fix fundamental symmetries J_1, J_2 and J_2' on $\mathcal{G}_1$, $\mathcal{H}_2$ and $\mathcal{G}_2 \ominus \mathcal{H}_2$. On $\mathcal{K}$ we have the fixed fundamental symmetry J where, with respect to the decomposition

$$\mathcal{K} = \mathcal{G}_1 \oplus \mathcal{H}_2 \oplus (\mathcal{G}_2 \ominus \mathcal{H}_2),$$

the operator J has the representation

$$J = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & -J_2 & 0 \\ 0 & 0 & -J_2' \end{bmatrix}.$$

We consider the linear manifold $\mathcal{H}$ in $\mathcal{K}$

$$\mathcal{H} = \{x + \Gamma x \mid x \in \mathcal{G}_1\} \oplus (\mathcal{G}_2 \ominus \mathcal{H}_2). \tag{2.3}$$

Taking into account that $\mathcal{H}$ is the direct orthogonal sum of the graph of a bounded operator, hence a subspace, with another subspace, it follows that $\mathcal{H}$ itself is closed, that is, it is a subspace of $\mathcal{K}$. This implies that the Gram operator of $\mathcal{H}$ is $G = P_{\mathcal{H}} J | \mathcal{H}$.

We remark that we can write

$$\mathcal{H} = \mathcal{H}_0 \oplus (\mathcal{G}_2 \ominus \mathcal{H}_2), \tag{2.4}$$

where $\mathcal{H}_0 = \{x + \Gamma x \mid x \in \mathcal{G}_1\}$ is the graph of Γ. Letting $G_0 = P_{\mathcal{H}_0} J | \mathcal{H}_0$, with respect to the decomposition (2.4) we have

$$G = \begin{bmatrix} G_0 & 0 \\ 0 & -J_2' \end{bmatrix}. \tag{2.5}$$

$\mathcal{H}_0$ is also a subspace of $\mathcal{K}$, and of $\mathcal{H}$ as well.

Let us remark that $\kappa_-(\mathcal{H}_0) = \kappa_-(I - \Gamma^\sharp \Gamma)$. To see this, just note that for arbitrary $x \in \mathcal{G}_1$ we have

$$[x + \Gamma x, x + \Gamma x] = [x, x] - [\Gamma x, \Gamma x] = [(I - \Gamma^\sharp \Gamma)x, x].$$

Consider now the Jordan decomposition $G_0 = G_{0+} - G_{0-}$ of the Gram operator G_0 and let $\mathcal{H}_{0-} = \operatorname{cl} \mathcal{R}(G_{0-})$ and $\mathcal{H}_{0+} = \mathcal{H}_0 \ominus \mathcal{H}_{0-}$. Therefore

$$\operatorname{rank} G_{0-} = \dim \mathcal{H}_{0-} = \kappa_-(\mathcal{H}_0) = \kappa_-(I - \Gamma^\sharp \Gamma). \tag{2.6}$$

Further, letting

$$G_+ = G_{0+} \oplus J_2'^-, \quad G_- = G_{0-} \oplus J_2'^+,$$

where $J_2' = J_2'^+ - J_2'^-$ is the Jordan decomposition of J_2', it follows from (2.5) that $G = G_+ - G_-$ is the Jordan decomposition of G, and $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is the corresponding spectral decomposition, where

$$\mathcal{H}_+ = \mathcal{H}_{0+} \oplus (\mathcal{G}_2 \ominus \mathcal{H}_2)_-, \quad \mathcal{H}_- = \mathcal{H}_{0-} \oplus (\mathcal{G}_2 \ominus \mathcal{H}_2)_+. \tag{2.7}$$

Finally, with respect to the decomposition (2.2) of the Kreĭn space $\mathcal{K}$ we define

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}. \tag{2.8}$$

We remark that the space $\mathcal{H}$ is invariant under S.

In this framework we can try to construct elements of $\mathrm{N}_\kappa(\Gamma)$ as follows:

Step 1 Let $\mathcal{L}$ be an $\mathcal{H}$-maximal nonnegative subspace invariant under S, and such that

$$\mathcal{E} = P_{\mathcal{G}_1}\mathcal{L} \quad \text{is a closed subspace of} \quad \mathcal{G}_1. \tag{2.9}$$

Step 2 Define the mapping

$$M\colon \mathcal{E} = P_{\mathcal{G}_1}\mathcal{L} \ni P_{\mathcal{G}_1}f \mapsto P_{\mathcal{G}_2}f, \quad f \in \mathcal{L}, \tag{2.10}$$

and assume that M is correctly defined and bounded.

We can verify that the pair $(M;\mathcal{E})$ satisfies most of the properties of an element in $\mathrm{N}_\kappa(\Gamma)$. To see this we remark that

$$\mathcal{L} = \{P_{\mathcal{G}_1}f + P_{\mathcal{G}_2}f \mid f \in \mathcal{L}\} = \{x + Mx \mid x \in \mathcal{E}\},$$

so, if $x \in \mathcal{E}$, then $S(x + Mx) \in \mathcal{L}$ or $S_1x = y$ and $S_2Mx = My$ for some $y \in \mathcal{E}$. Therefore, $S_1\mathcal{E} \subset \mathcal{E}$ and $S_2M = MS_1|\mathcal{E}$.

Since $\mathcal{L}$ is nonnegative, it follows for $x = P_{\mathcal{G}_1}f$, $f \in \mathcal{L}$, that

$$[Mx, Mx] = [MP_{\mathcal{G}_1}f, MP_{\mathcal{G}_1}f] = [P_{\mathcal{G}_2}f, P_{\mathcal{G}_2}f] \le [P_{\mathcal{G}_1}f, P_{\mathcal{G}_1}f] = [x, x],$$

so that, M is a contraction.

Since $\mathcal{L} \subseteq \mathcal{H}$ and (2.4), we have

$$\Gamma P_{\mathcal{G}_1}f = P_{\mathcal{H}_2}f, \quad f \in \mathcal{L} \tag{2.11}$$

and then, for arbitrary $x = P_{\mathcal{G}_1}f \in \mathcal{E}$ we have

$$\Gamma x = P_{\mathcal{H}_2}f = P_{\mathcal{H}_2}P_{\mathcal{G}_2}f = P_{\mathcal{H}_2}MP_{\mathcal{G}_1}f = P_{\mathcal{H}_2}Mx,$$

and hence $P_{\mathcal{H}_2}M = \Gamma|\mathcal{E}$ holds.

It is clear that we have difficulties to deal with the property $\operatorname{codim}_{\mathcal{G}_1}\mathcal{E} \le \kappa$ in this general framework, as well as with verification of the conditions involved in the constructions of *Step 1* and *Step 2*. We indicate here a situation general enough to include some applications and for which we can perform the previous construction in order to obtain the existence of at least one element in $\mathrm{N}_\kappa(\Gamma)$.

THEOREM 2.4 *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be Krein spaces and let $\mathcal{H}_2$ be a regular subspace of $\mathcal{G}_2$ such that the subspace $\mathcal{G}_2 \cap \mathcal{H}_2^\perp$ is positive definite. Assume that S_1 is an expansive operator on $\mathcal{G}_1$ and let S_2 be a contraction on $\mathcal{G}_2$. Let Γ be an (S_1, S_2)-Hankel operator, κ a cardinal number and $\rho > 0$ be such that $\kappa_-(\rho^2 I - \Gamma^\sharp\Gamma) < \infty$. Then the set $\mathrm{N}_\kappa(\Gamma;\rho)$ is nonvoid if and only if $\kappa_-(\rho^2 I - \Gamma^\sharp\Gamma) \le \kappa$.*

Proof. As noted before, it is sufficient to prove the result for $\rho = 1$. From Lemma 2.3 and taking into account that $\mathcal{G}_2 \cap \mathcal{H}_2^\perp$ is positive definite, in order for there to exist solutions of the problem $\mathrm{N}_\kappa(\Gamma)$ it is necessary that $\kappa_-(I - \Gamma^\sharp \Gamma) \leq \kappa$.

Conversely, assume that $\kappa_-(I - \Gamma^\sharp \Gamma) \leq \kappa$. We divide the proof in three steps.

1. *We show that there exists an $\mathcal{H}$-maximal nonnegative subspace $\mathcal{L}$, invariant under S.* Since S_1 is expansive and S_2 is contractive, for any vector $x = x_1 + x_2$, $x_1 \in \mathcal{G}_1$ and $x_2 \in \mathcal{G}_2$ we have

$$[Sx, Sx] = [S_1 x_1, S_1 x_1] - [S_2 x_2, S_2 x_2] \geq [x_1, x_1] - [x_2, x_2] = [x, x],$$

that is, S is expansive.

Since $\dim \mathcal{H}_{0-} = \kappa_-(I - \Gamma^\sharp \Gamma) < \infty$, it follows from (2.6) that the operator G_- has closed range, with G defined as in (2.5) (and $J_2' = I$, due to the hypothesis that the subspace $\mathcal{G}_2 \cap \mathcal{H}_2^\perp$ is positive definite). We now take into account the decomposition (2.4) of $\mathcal{H}$ and get

$$\begin{aligned} P_{\mathcal{H}_+} S P_{\mathcal{H}_-} &= P_{\mathcal{H}_{0-}} S (P_{\mathcal{H}_{0-}} + P_{\mathcal{G}_2 \ominus \mathcal{H}_2}) \\ &= P_{\mathcal{H}_{0-}} S P_{\mathcal{H}_{0-}} + P_{\mathcal{H}_{0+}} S P_{\mathcal{G}_2 \ominus \mathcal{H}_2}. \end{aligned}$$

Since $\mathcal{G}_2 \ominus \mathcal{H}_2$ is invariant under S_2 and $\mathcal{H}_{0+} \subseteq \mathcal{G}_1 \oplus \mathcal{H}_2$ we also have that $P_{\mathcal{H}_{0+}} S P_{\mathcal{G}_2 \ominus \mathcal{H}_2} = 0$. Therefore

$$P_{\mathcal{H}_+} S P_{\mathcal{H}_-} = P_{\mathcal{H}_{0-}} S P_{\mathcal{H}_{0-}},$$

and hence $\operatorname{rank} P_{\mathcal{H}_+} S P_{\mathcal{H}_-} \leq \dim \mathcal{H}_{0-} = \kappa_-(I - \Gamma^\sharp \Gamma) < \infty$, in particular, the operator $P_{\mathcal{H}_+} S P_{\mathcal{H}_-}$ is compact.

The assumptions of Theorem 1.2 are verified and hence there exists an $\mathcal{H}$-maximal nonnegative subspace $\mathcal{L}$ invariant under S.

2. *We show that $\mathcal{E} = P_{\mathcal{G}_1} \mathcal{L}$ is closed and* $\operatorname{codim}_{\mathcal{G}_1} \mathcal{E} = \kappa_-(I - \Gamma^\sharp \Gamma)$.

Since $\mathcal{L}$ is $\mathcal{H}$-maximal nonnegative subspace, by Proposition 1.1 there exists the generalised angular operator $K_\mathcal{L} \colon \mathcal{H}_+ \to \mathcal{H}_-$ such that

$$\mathcal{L} = \{x + K_\mathcal{L} x \mid x \in \mathcal{H}_+\}.$$

Taking into account of (2.7) we get

$$P_{\mathcal{G}_1} \mathcal{L} + \mathcal{H}_{0-} \supseteq P_{\mathcal{G}_1} (\mathcal{L} + \mathcal{H}_{0-}) \supseteq \mathcal{G}_1. \tag{2.12}$$

We claim now that the operator $P_{\mathcal{G}_1}$ is injective also when restricted to the subspace $\mathcal{L} + \mathcal{H}_{0-}$. Indeed, let $l \in \mathcal{L}$ and $h \in \mathcal{H}_{0-}$ be such that $P_{\mathcal{G}_1}(l + h) = 0$, equivalenty $P_{\mathcal{G}_1} l = -P_{\mathcal{G}_2} h$. Taking into account of (2.4) it follows that $l = (x + \Gamma x) + g_2$ for some $g_2 \in \mathcal{G}_2 \ominus \mathcal{H}_2$ and $x = -P_{\mathcal{G}_1} h$. But, by the construction of the space $\mathcal{H}_{0-}$ we have $h = x + \Gamma x$ where $x = -P_{\mathcal{G}_1} h$, and hence $l = -h + g_2$. Now remark that the subspaces $\mathcal{H}_{0-}$ and $\mathcal{G}_2 \ominus \mathcal{H}_2$ are negative subspaces and orthogonal with respect to the inner product $[\cdot, \cdot]$ of $\mathcal{K}$ and hence the vector $l = -h + g_2$ is either negative or null. But l is nonnegative, as any other vector in $\mathcal{L}$, and hence $l = 0$ and $h = 0$. The claim is proved.

Since $\mathcal{L}$ is a nonnegative subspace and $\mathcal{H}_{0-}$ is a negative subspace it follows that the sum $\mathcal{L} + \mathcal{H}_{0-}$ is direct and, taking into account that $P_{\mathcal{G}_1}$ is injective on $\mathcal{L} \dot{+} \mathcal{H}_{0-}$, from (2.12) we get

$$\mathcal{E} \dot{+} P_{\mathcal{G}_1} \mathcal{H}_{0-} = \mathcal{G}_1,$$

which proves that the codimension of $\mathcal{E}$ in $\mathcal{G}_1$ is exactly $\dim \mathcal{H}_{0-} = \kappa_-(I - \Gamma^\sharp \Gamma)$.

We now prove that $\mathcal{E}$ is closed. First consider the subspace $\mathcal{H}'_+ = \ker(P_{\mathcal{H}_{0-}} K_{\mathcal{L}}) \subseteq \mathcal{H}_+$ and remark that $\operatorname{codim}_{\mathcal{H}_+} \mathcal{H}'_+ \leq \dim \mathcal{H}_{0-} = \kappa$. Define the subspace of $\mathcal{L}$

$$\mathcal{L}' = \{x + K_{\mathcal{L}} x \mid x \in \mathcal{H}'_+\},$$

and note that since $K_{\mathcal{L}} \mathcal{H}'_1 \subseteq \mathcal{G}_2 \ominus \mathcal{H}_2$ it follows $P_{\mathcal{G}_1} \mathcal{L}' = P_{\mathcal{G}_1} \mathcal{H}'_+$. Since $\mathcal{H}'_+$ is a subspace of $\mathcal{H}_+$ it follows that

$$\mathcal{H}'_+ = \{f + \Gamma f \mid f \in P_{\mathcal{G}_1} \mathcal{H}'_+\}.$$

Since $\mathcal{H}'_+$ is closed and Γ is bounded it follows that $P_{\mathcal{G}_1} \mathcal{H}'_+ = P_{\mathcal{G}_1} \mathcal{L}'$ is closed. Taking into account that $\operatorname{codim}_{\mathcal{E}} P_{\mathcal{G}_1} \mathcal{L}' \leq \kappa < \infty$ it follows that the linear manifold $\mathcal{E}$ is closed, too.

3. *The mapping defined by* (2.10) *is a well-defined bounded operator.*

As a consequence of the injectivity of the operator $P_{\mathcal{G}_1} | \mathcal{L}$, which was proved at step 2, we get that the operator M is correctly defined and closed. Since its domain $\mathcal{E}$ is also closed, then the closed graph theorem implies that M is bounded. ∎

REMARK 2.5 Theorem 2.4 shows a bit more than it is stated, more precisely, under the assumptions of Theorem 2.4 and assuming that $\kappa_-(\rho^2 I - \Gamma^\sharp \Gamma) \leq \kappa$ it follows that for any solution $(M; \mathcal{E})$ of the problem $\mathrm{N}_\kappa(\Gamma; \rho)$ we have $\operatorname{codim}_{\mathcal{G}_1} \mathcal{E} = \kappa_-(\rho^2 I - \Gamma^\sharp \Gamma)$. ∎

As a by-product of the above approach we can show that the correspondence defined as in (2.9) and (2.10) between pairs $(M; \mathcal{E})$ and subspaces $\mathcal{L}$ provides a parametrization of $\mathrm{N}_\kappa(\Gamma)$ by $\mathcal{H}$-maximal nonnegative subspaces invariant under S, similar as the parametrization of the generalised interpolation problem obtained by J.A. Ball and J.W. Helton in [8].

THEOREM 2.6 *Assume the conditions of Theorem 2.4 hold and $\kappa_-(I - \Gamma^\sharp \Gamma) \leq \kappa$. Then the correspondence as in* (2.9) *and* (2.10) *is bijective between the set $\mathrm{N}_\kappa(\Gamma)$ and the set of all $\mathcal{H}$-maximal nonnegative subspaces invariant under S.*

Proof. Assume the conditions of Theorem 2.4 hold. That is, $\mathcal{G}_2 \cap \mathcal{H}_2^\perp$ is positive definite, S_1 is an expansive operator, S_2 is a contraction and Γ is an (S_1, S_2)-Hankel operator such that $\kappa_-(I - \Gamma^\sharp \Gamma) < \infty$. Let, in addition, $\kappa_-(I - \Gamma^\sharp \Gamma) \leq \kappa$.

First we show that if $M: \mathcal{E} (\subseteq \mathcal{G}_1) \to \mathcal{G}_2$ is a bounded contraction, $\operatorname{codim}_{\mathcal{G}_1} \mathcal{E} \leq \kappa$ and $\Gamma | \mathcal{E} = P_{\mathcal{H}_2} M$, then

$$\mathcal{L} = \{x + Mx \mid x \in \mathcal{E}\}, \tag{2.13}$$

is an $\mathcal{H}$-maximal nonnegative subspace invariant under S.

Since M is contractive we readily check that $\mathcal{L}$ is nonnegative. In order to prove that $\mathcal{L}$ is a subspace of $\mathcal{H}$, pick $f = x + Mx$ for some vector $x \in \mathcal{E}$. Then we have

$$\Gamma P_{\mathcal{G}_1} f = \Gamma P_{\mathcal{G}_1}(x + Mx) = \Gamma x = P_{\mathcal{H}_2} Mx = P_{\mathcal{H}_2}(x + Mx) = P_{\mathcal{H}_2} f.$$

In view of the definition of $\mathcal{H}$, this implies that $\mathcal{L} \subseteq \mathcal{H}$.

From Lemma 1.1 it follows that there exists an $\mathcal{H}$-maximal nonnegative subspace $\widetilde{\mathcal{L}} \supseteq \mathcal{L}$. Then, as in the proof of Theorem 2.4, we get that $P_{\mathcal{G}_1} \widetilde{\mathcal{L}}$ is a subspace of $\mathcal{G}_1$ of codimension κ. Since $P_{\mathcal{G}_1} \widetilde{\mathcal{L}} \supseteq P_{\mathcal{G}_1} \mathcal{L} = \mathcal{E}$ is a subspace of codimension in $\mathcal{G}_1$ at most κ it follows that $P_{\mathcal{G}_1} \widetilde{\mathcal{L}} = \mathcal{E}$, in particular $\mathcal{L} = \widetilde{\mathcal{L}}$ is an $\mathcal{H}$-maximal nonnegative subspace and $\operatorname{codim}_{\mathcal{G}_1} \mathcal{E} = \kappa$.

Finally, we show that $\mathcal{L}$ is invariant under S.

If f is an arbitrary vector in $\mathcal{L}$, then $f = x + Mx$ for some $x \in \mathcal{E}$. Consequently,

$$Sf = S(x + Mx) = S_1x + S_2Mx = S_1x + MS_1x \in \mathcal{L}.$$

The fact that $\mathcal{H}$-maximal nonnegative subspaces $\mathcal{L}$ invariant under S produce solutions of the problem $\mathrm{N}_\kappa(\Gamma)$ is already proved in Theorem 2.4. It remains to notice that these two correspondences are inverse one to the other. ∎

3. Bounds and Singular Numbers of Generalised Hankel Operators

3.1 Minus-operators. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Kreĭn spaces and $T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. The operator T is called *minus-operator* if there exists $\mu \geq 0$ such that

$$[Tx, Tx] \leq \mu[x, x], \quad x \in \mathcal{K}_1. \tag{3.1}$$

In this case, two numbers are associated to the minus-operator T,

$$\mu_+(T) = \sup_{[x,x]=1} [Tx, Tx], \quad \mu_-(T) = \inf_{[x,x]=-1} -[Tx, Tx]. \tag{3.2}$$

An operator $T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ is called *strong minus-operator* if T is a minus-operator and $\mu_+(T) > 0$.

If $\mathcal{K}_1$ and $\mathcal{K}_2$ are positive definite, that is, Hilbert spaces, then all operators in $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ are minus operators and, in this case, $\mu_+(T) = \|T\|$ and $\mu_-(T) = +\infty$. Clearly, all nontrivial operators are strong minus-operators.

If $\mathcal{K}_1$ and $\mathcal{K}_2$ are negative definite, then again all operators in $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ are minus-operators, $\mu_+(T) = -\infty$ and $\mu_-(T) = \gamma(T)$, the *minimum modulus* of the operator T.

A distinct situation corresponds to the case when $\mathcal{K}_1$ is indefinite, that is, it contains positive vectors as well as negative vectors. In this case, according to a result of M.G. Kreĭn and Y.L. Shmulyan [26], an operator $T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ is minus-operator if and only if $[Tx, Tx] \leq 0$ for all $[x, x] \leq 0$, $x \in \mathcal{K}_1$. Moreover, if T is a minus-operator then $\mu_+(T) \leq \mu_-(T)$ and a real number μ satisfies (3.1) if and only if μ lies in the interval $[\mu_+(T), \mu_-(T)]$. A recent consideration of this bounds, for the finite dimensional case, was done by A. Ben-Artzi and I. Gohberg [9].

Let now $\mathcal{G}_1$ and $\mathcal{G}_2$ be Kreĭn spaces and consider linear operators $S_1 \in \mathcal{L}(\mathcal{G}_1)$ and $S_2 \in \mathcal{L}(\mathcal{G}_2)$. An (S_1, S_2)-*multiplier* is, by definition, an operator $M \in \mathcal{L}(\mathcal{G}_1, \mathcal{G}_2)$ intertwining the operators S_1 and S_2, that is, $MS_1 = S_2M$. Assume, in addition, that $\mathcal{H}_2$ is a Kreĭn subspace of $\mathcal{G}_2$ such that $\mathcal{G}_2 \cap \mathcal{H}_2^\perp$ is invariant under S_2 and positive definite. If M is an (S_1, S_2)-multiplier then $\Gamma_M = P_{\mathcal{H}_1}M$ is an (S_1, S_2)-Hankel operator. In addition, if M is a minus-operator and $\mu \in \mathbb{R}$ then it follows from the positive definiteness of $\mathcal{G}_2 \cap \mathcal{H}_2^\perp$ that

$$\mu[x,x] - [Mx, Mx] = \mu[x,x] - [\Gamma_M x, \Gamma_M x] - [P_{\mathcal{G}_2 \cap \mathcal{H}_2^\perp} Mx, P_{\mathcal{G}_2 \cap \mathcal{H}_2^\perp} Mx] \leq \mu[x,x] - [\Gamma_M x, \Gamma_M x],$$

and hence the interval $[\mu_+(M), \mu_-(M)]$ is contained in the interval $[\mu_+(\Gamma_M), \mu_-(\Gamma_M)]$. This shows that, if $\Gamma \in \mathcal{L}(\mathcal{G}_1, \mathcal{H}_2)$ is an (S_1, S_2)-Hankel minus-operator, that is, Γ is a minus-operator and $\Gamma S_1 = P_{\mathcal{H}_2} S_2 \Gamma$ (see (2.1)), then

$$\bigcup_{MS_1 = S_2M,\ \Gamma = P_{\mathcal{H}_2}M} [\mu_+(M), \mu_-(M)] \subseteq [\mu_+(\Gamma), \mu_-(\Gamma)]. \tag{3.3}$$

All these considerations are more or less trivial consequences of the definitions. The interesting part of this discussion is that Theorem 2.4 shows that if Γ is a strong minus-operator then the inclusion converse to (3.3) holds, too.

THEOREM 3.1 *Let $\Gamma \in \mathcal{L}(\mathcal{G}_1, \mathcal{H}_2)$ be an (S_1, S_2)-Hankel strong minus-operator and assume that the Kreĭn subspace $\mathcal{G}_2 \cap \mathcal{H}_2^\perp$ is positive definite. Then, for any $\mu \in [\mu_+(\Gamma), \mu_-(\Gamma)]$ there exists an (S_1, S_2)-multiplier M such that $\Gamma = P_{\mathcal{H}_2} M$ and $[Mx, Mx] \leq \mu[x, x]$ for all $x \in \mathcal{G}_1$.*

3.2 Bounds for Hankel Operators. In the paper [18] the following number is associated to an (S_1, S_2)-Hankel operator Γ in the Hilbert space case:

$$\mu(\Gamma) = \min\{\|M\| \mid M \in \mathcal{L}(\mathcal{G}_1, \mathcal{G}_2),\ MS_1 = S_2 M,\ \Gamma = P_{\mathcal{H}_2} M\},$$

and it is remarked that if S_1 and S_2 are isometric, the problem of computing $\mu(\Gamma)$ is equivalent to the lifting of commutants and that $\mu(\Gamma) = \|\Gamma\|$. As a consequnce of Theorem 1.1 in [36] (which is a particular case of Theorem 2.4) it follows that if S_1 is expansive and S_2 is contractive, then again $\mu(\Gamma) = \|\Gamma\|$. For this reason, it might present some interest to have an "elementary" and more constructive proof of it, i.e. a proof that is not based on Theorem 1.2.

THEOREM 3.2 [36] *Assume that $\mathcal{G}_i$ are Hilbert spaces, $i = 1, 2$, $\Gamma \in \mathcal{L}(\mathcal{G}_1, \mathcal{H}_2)$ is an (S_1, S_2)-Hankel operator, S_1 is expansive and S_2 is contractive. Then $\mu(\Gamma) = \|\Gamma\|$.*

Proof. Without restricting the generality we assume $\|\Gamma\| \leq 1$. Let us write

$$S_2 = \begin{bmatrix} \widetilde{S}_2 & Q \\ 0 & R \end{bmatrix}$$

with respect to the decomposition $\mathcal{G}_2 = \mathcal{H}_2^\perp \oplus \mathcal{H}_2$. The fact that Γ is an (S_1, S_2)-Hankel operator means that $R\Gamma = \Gamma S_1$. We search for a contraction $M = [T \quad \Gamma]$ with the property that $TS_1 = \widetilde{S}_2 T + Q\Gamma$. The fact that the problem of finding an element in $N(\Gamma)(= N_0(\Gamma))$ reduces to the solvability of a Lyapunov equation was already remarked in [18] and it explains the connection with the existence of maximal non-negative S-invariant subspaces. Thus, the following remark is well-known. Let $\mathcal{K}$ be a Krein space and $\mathcal{K} = \mathcal{K}^- + \mathcal{K}^+$ is a fundamental decomposition. To each maximal non-negative subspace $\mathcal{L}$ there corresponds its angular operator $T \in \mathcal{L}(\mathcal{K}^-, \mathcal{K}^+)$, $\|T\| \leq 1$, such that $\mathcal{L} = G(T)$. If S is an operator on $\mathcal{K}$ with the block-matrix representation

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}$$

with respect to the decomposition $\mathcal{K} = \mathcal{K}^- + \mathcal{K}^+$, then $\mathcal{L}$ is S-invariant if and only if

$$TS_{11} = S_{12} + S_{22}T,$$

which is an equation of the type encountered in the search for an element in $N(\Gamma)$.

Now, we show that the equation

$$TS_1 = \widetilde{S}_2 T + Q\Gamma$$

admits a solution provided that S_1 is expansive and S_2 is contractive. Without loss of generality, we can assume that Γ and $\widetilde{S}_2$ are strict contractions (indeed, if this is not the case, we can perturb these operators and then use a subsequence with the original operators as limits). Since $R\Gamma = \Gamma S_1$ and R is a contraction, it follows that

$$S_1^*\Gamma^*\Gamma S_1 = \Gamma^* R^* R\Gamma \leq \Gamma^*\Gamma,$$

so

$$I - S_1^*\Gamma^*\Gamma S_1 \geq I - \Gamma^*\Gamma.$$

Since S_1 is expansive,

$$S_1^*S_1 - S_1^*\Gamma^*\Gamma S_1 \geq I - S_1^*\Gamma^*\Gamma S_1 \geq I - \Gamma^*\Gamma,$$

therefore $S_1^*D_\Gamma^2 S_1 \geq D_\Gamma^2$, where $D_\Gamma = (I - \Gamma^*\Gamma)^{1/2}$. Consequently, we can define the contraction X by the formula $XD_\Gamma S_1 h = D_\Gamma h$ on the closure of the range of $D_\Gamma S_1$ and by $Xh = 0$ on the orthogonal of that space. We now search for T of the form $T = YXD_\Gamma$, with YX a contraction. Then, we must have

$$YXD_\Gamma S_1 = \widetilde{S}_2 YXD_\Gamma + Q\Gamma,$$

or

$$Y = \widetilde{S}_2 YX + Q\Gamma D_\Gamma^{-1}.$$

Since $\widetilde{S}_2$ is assumed to be a strict contraction, this equation admits the unique solution

$$Y = Q\Gamma D_\Gamma^{-1} + \widetilde{S}_2 Q\Gamma D_\Gamma^{-1}X + \widetilde{S}_2^2 Q\Gamma D_\Gamma^{-1}X^2 + \ldots.$$

It remains to show that YX is a contraction. To that end, define $C_0 = Q\Gamma D_\Gamma^{-1}X$ and for $n \geq 1$,

$$C_n = Q\Gamma D_\Gamma^{-1}X + \ldots + \widetilde{S}_2^n Q\Gamma D_\Gamma^{-1}X^n.$$

Then, since S_1 is expansive and S_2 and Γ are contractions, for all $h \in \mathcal{G}_1$ we have

$$\|\begin{bmatrix} C_0 D_\Gamma \\ \Gamma \end{bmatrix} S_1 h\|^2 = \|\begin{bmatrix} Q\Gamma \\ R\Gamma \end{bmatrix} h\|^2 = \|S_2 \begin{bmatrix} 0 \\ \Gamma h \end{bmatrix}\|^2$$

$$\leq \|\Gamma h\|^2 \leq \|h\|^2 \leq \|S_1 h\|^2.$$

This implies that

$$\|C_0 D_\Gamma S_1 h\|^2 \leq \|D_\Gamma S_1 h\|^2,$$

so C_0 is a contraction on the closure of the range of $D_\Gamma S_1$. Since on the orthogonal of that space C_0 is zero, it follows that C_0 is a contraction.

Suppose now that C_k are contractions for $1 \leq k \leq n-1$. Since S_2 and $\begin{bmatrix} C_{n-1}D_\Gamma \\ \Gamma \end{bmatrix}$ are contractions and S_1 is expansive, for $h \in \mathcal{G}_1$ we have

$$\left\| \begin{bmatrix} C_n D_\Gamma \\ \Gamma \end{bmatrix} S_1 h \right\|^2 = \left\| S_2 \begin{bmatrix} C_{n-1} D_\Gamma \\ \Gamma \end{bmatrix} h \right\|^2 \leq \|S_1 h\|^2.$$

Therefore, C_n has to be a contraction and the proof is concluded. ∎

3.3 Singular Numbers of Hankel Operators. We consider Hilbert spaces $\mathcal{G}_1$ and $\mathcal{G}_2$ and the remainder of the notation is as in Section 2. Let Γ be an (S_1, S_2)-Hankel operator. Let $\{s_k(\Gamma)\}_{k\geq 0}$ be the sequence of the singular numbers of Γ and note that

$$s_k(\Gamma) = \min\{\rho > 0 \mid \kappa_-(\rho^2 I - \Gamma^*\Gamma) \leq k\}, \quad k \geq 0.$$

This observation and Theorem 2.4 imply the following characterisation of singular numbers of generalised Hankel operators.

THEOREM 3.3 *Assume that S_1 is expansive and S_2 is contractive and let $\Gamma \in \mathcal{L}(\mathcal{G}_1, \mathcal{H}_2)$ be an (S_1, S_2)-Hankel operator. Then for all intergers $k \geq 0$ we have*

$$s_k(\Gamma) = \min\{\|M\| \mid M \in \mathcal{L}(\mathcal{E}, \mathcal{G}_2),\ \operatorname{codim}_{\mathcal{G}_1}\mathcal{E} \leq k,\ S_1\mathcal{E} \subseteq \mathcal{E},$$
$$MS_1|\mathcal{E} = S_2 M,\ \Gamma = P_{\mathcal{H}_2} M\}.$$

This theorem can be viewed as an abstract form of the operator valued version of the celebrated theorem of V.M. Adamyan, D.Z. Arov and M.G. Kreĭn ([2] and [1]). Following closely the approach in [36] it can be shown that it also contains the characterisation of the singular values for the matrix valued version of the four-block problem.

4. Contractive Intertwining Dilations

We show that a certain generalisation of the contractive intertwining dilation problem as in [20] can be obtained from the Nehari type problem considered in Section 2.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Kreĭn spaces and consider two operators $T_i \in \mathcal{L}(\mathcal{H}_i)$, $i = 1, 2$. We assume that for $i = 1, 2$ there exists pairs $(V_i; \mathcal{G}_i)$, subject to the following conditions:

(a$_i$) $\mathcal{G}_i$ is a Kreĭn space extension of $\mathcal{H}_i$;

(b$_i$) $V_i \in \mathcal{L}(\mathcal{G}_i)$ is a dilation of T_i, that is, $P_{\mathcal{H}_i} V_i = T_i P_{\mathcal{H}_i}$.

As a consequence of assumption (b$_i$) it follows that $\mathcal{G}_i \cap \mathcal{H}_i^\perp$ is invariant under the operator V_i, $i = 1, 2$.

Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be an operator intertwining the operators T_1 and T_2, that is $AT_1 = T_2 A$. The set of *contractive intertwining dilations* of A, denoted by $\mathrm{CID}_\kappa(A; T_1, T_2)$, consists of pairs $(A_\infty, \mathcal{E})$ subject to the following properties:

(1) $\mathcal{E}$ is a subspace of $\mathcal{G}_1$ of codimension at most κ and invariant under V_1;

(2) $A_\infty \in \mathcal{L}(\mathcal{E}, \mathcal{G}_2)$ is contraction, that is $[A_\infty x, A_\infty x] \leq [x, x]$ for all $x \in \mathcal{E}$;

(3) $P_{\mathcal{H}_2} A_\infty = A P_{\mathcal{H}_1} | \mathcal{E}$;

(4) $A_\infty V_1 | \mathcal{E} = V_2 A_\infty$.

Simply by inspecting the definitions we obtain.

LEMMA 4.1 *Let A, T_1, V_1, etc. be as above and denote $\Gamma = AP_{\mathcal{H}_1}:\mathcal{G}_1 \to \mathcal{H}_2$. Then Γ is a (V_1, V_2)-Hankel operator and*

$$\mathrm{CID}_\kappa(A; V_1, V_2) = \mathrm{N}_\kappa(\Gamma).$$

As a consequence of this equality and Theorem 2.4 we obtain the following result proved in [20] (for similar investigations see [4], [5]).

THEOREM 4.2 *If both of the subspaces $\mathcal{G}_i \cap \mathcal{H}_i^\perp$, $i = 1, 2$, are positive definite, V_1 is expansive, V_2 is a contraction and A is a quasi-contraction, that is $\kappa_-(I - A^\sharp A) < \infty$, then the set $\mathrm{CID}_\kappa(A; T_1, T_2)$ is nonvoid if and only if $\kappa_-(I - A^\sharp A) \le \kappa$.*

Conversely, under the conditions of Theorem 4.2 it is easy to see that each $\mathrm{N}_\kappa(\Gamma)$ can be realised as a set $\mathrm{CID}_\kappa(A; V_1, V_2)$. Thus, let Γ be an (S_1, S_2)-Hankel operator. Define $T_1 = V_1 = S_1$, $T_2 = P_{\mathcal{H}_2} S_2$, $V_2 = S_2$ and $A = \Gamma$. Then it is readily checked that, under the conditions in Theorem 4.2, we have $\mathrm{N}_\kappa(\Gamma) = \mathrm{CID}_\kappa(A; V_1, V_2)$.

The Hilbert space version of Theorem 4.2 for $\kappa = 0$ was obtained in [36] and it was shown there to contain the commutant lifting theorem of D. Sarason and Sz.-Nagy and Foiaş ([32], [33], [34]). In [20] it was mentioned that some of the indefinite variants of the commutant lifting theorem as in [11], [12], [13], [14] are also consequences of Theorem 4.2.

5. A Carathéodory-Schur Type Problem for Meromorphic Functions

We illustrate the applicability of Theorem 4.2 to an interpolation problem of Carathéodory-Schur type for meromorphic matrix valued functions. In a slightly different form it was formulated by M.G. Kreĭn and H. Langer in [25].

Let m, n be nonnegative integer numbers and denote by $M_{n,m}$ the set of $n \times m$ matrices with complex entries, identified with $\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$. We denote by $H^\infty(M_{n,m})$ the space of all functions $F: \mathbb{D} \to M_{n,m}$ which are analytic and uniformly bounded in $\mathbb{D}$,

$$\|F\|_\infty = \sup_{z \in \mathbb{D}} \|F(z)\| < \infty.$$

Let $\alpha \in \mathbb{D}$. We consider the Möbius transformation

$$b_\alpha(z) = \frac{|\alpha|}{\alpha} \frac{\alpha - z}{1 - \overline{\alpha} z}, \quad z \in \mathbb{D},$$

which maps conformally the unit disk into itself. A *Blaschke-Potapov cell* of order $q \le n$ is by definition a square matrix of order n

$$B_\alpha(z) = \begin{bmatrix} I_r & 0 & 0 \\ 0 & b_\alpha(z) I_q & 0 \\ 0 & 0 & I_s \end{bmatrix},$$

where $n = r + q + s$. A *Blaschke-Potapov product of finite order* is by definition a finite product of analytic functions, each one being unitary equivalent with a Blaschke-Potapov cell. The order of a Blaschke-Potapov product is the sum of the orders of all its factors.

Since the functions b_α maps $\partial\mathbb{D}$ into itself, a Blaschke-Potapov product is always of norm one and the corresponding multiplication operator is isometric $H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^n)$.

Let l be a nonnegative integer. By $\mathcal{S}_l(M_{n,m})$ we denote the generalised Schur class of functions G which can be represented $G = F\Psi^{-1}$ where $F \in H^\infty(M_{m,n})$, $\|F\|_\infty \leq 1$ and $\Psi \in H^\infty(M_n)$ is a Blaschke-Potapov product of order $\leq l$. If we impose the additional condition that no zero of the function G conincides with some zero of the Blaschke-Potapov product Ψ then this representation of functions in $\mathcal{S}_l(M_{m,n})$ is unique, modulo unitary equivalence. Such a factorization is called *right coprime*.

In the following we use a theorem whose proof, for the scalar case, can be found e.g. in [19]. The matrix valued version follows in a similar way but uses the theorem on the structure of the matrix valued inner functions (see e.g. [31]) and the theorem of Beurling-Lax characterising the shift invariant subspaces in $H^2(\mathbb{C}^n)$.

THEOREM 5.1 *A subspace $\mathcal{E} \in H^2(\mathbb{C}^n)$ is shift invariant and of codimension $l < \infty$ if and only if $\mathcal{E} = \Psi H^2(\mathbb{C}^n)$ where $\Psi \in H^\infty(M_n)$ is a Blaschke-Potapov product of order l.*

We are now in a position to introduce the variant of Carathéodory-Schur type problem we are interested in.

DEFINITION 5.2 Let $C = \{C_l\}_{l=1}^k \subset \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ be a sequence of $n \times m$ complex matrices. For $\kappa \in \mathbb{N}$ we define the set C-S$_\kappa(C)$ consisting of all meromorphic matrix valued functions $G \in \mathcal{S}_\kappa(\mathbb{C}^m, \mathbb{C}^n)$ such that the first $k+1$ "Taylor coefficients" at 0 of G coincide, respectively, with $C_0, C_1, \ldots C_k$. More precisely, if the function G has the representation $G = F\Psi^{-1}$, where $F \in H^\infty(M_{m,n})$ and $\Psi \in H^\infty(M_n)$ is a Blaschke-Potapov product of order $\leq \kappa$, then the first $k+1$ Taylor coefficients at 0 of G coincide, respectively, with the first Taylor coefficients of the analytic matrix-valued function $(C_0 + zC_1 + \cdots z^k C_k)\Psi(z)$.

We note that in the above definition a function $G \in$ C-S$_\kappa(C)$ is not necessarily analytic at 0.

Associated to the data $C = \{C_0, C_1, \ldots, C_k\}$ there is the following lower triangular block-matrix of Toeplitz type

$$T_C = \begin{bmatrix} C_0 & 0 & 0 & \cdots & 0 \\ C_1 & C_0 & 0 & \cdots & 0 \\ C_2 & C_1 & C_0 & \cdots & 0 \\ \vdots & & & & \\ C_k & C_{k-1} & C_{k-2} & \cdots & C_0 \end{bmatrix}. \tag{5.1}$$

THEOREM 5.3 *If κ is finite then the problem* C-S$_\kappa(C)$ *has solutions if and only if the number of negative eigenvalues, counted with their multiplicities, of the matrix $I - T_C^* T_C$ does not exceed κ.*

Proof. We consider the Hilbert space $\mathcal{G}_1 = H^2(\mathbb{C}^m)$ and the forward shift operator $V_1 \in \mathcal{L}(H^2(\mathbb{C}^m))$, $(V_1 g)(z) = zg(z)$, for all $g \in H^2(\mathbb{C}^m)$ and $z \in \mathbb{D}$. Let $\mathcal{H}_1$ be the subspace of $H^2(\mathbb{C}^m)$ of polynomials with coefficients in $\mathbb{C}^m$ of degree not exceeding k and define $T_1 = P_{\mathcal{H}_1} V_1 | \mathcal{H}_1$. Since $\mathcal{G}_1 \ominus \mathcal{H}_1$ is invariant under V_1 it follows that V_1 is a dilation of T_1.

Similarly, let $\mathcal{G}_2 = H^2(\mathbb{C}^n)$ and V_2 the forward shift operator on $H^2(\mathbb{C}^n)$. Denote $\mathcal{H}_2$ the subspace of $H^2(\mathbb{C}^n)$ consisting of all polynomials with coefficients in $\mathbb{C}^n$ of degree not exceeding k and let $T_2 = P_{\mathcal{H}_2} V_2 | \mathcal{H}_2$. Since $\mathcal{G}_2 \ominus \mathcal{H}_2$ is invariant under V_2 it follows that V_2 is a dilation of T_2.

Let us consider now the matrix-valued polynomial $C(z) = C_0 + zC_1 + \cdots + z^k C_k$ and denote by $M_C \in \mathcal{L}(H^2(\mathbb{C}^m), H^2(\mathbb{C}^n))$ the multiplication operator with $C \in H^\infty(\mathbb{C}^m, \mathbb{C}^n)$. Then define $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ by $A = P_{\mathcal{H}_1} M_C | \mathcal{H}_2$. Since M_C is a multiplication operator it intertwines the operators V_1 and V_2. Therefore

$$AT_1 = P_{\mathcal{H}_2} M_C P_{\mathcal{H}_1} V_1 | \mathcal{H}_1 = P_{\mathcal{H}_2} M_C V_1 | \mathcal{H}_1 = P_{\mathcal{H}_2} V_2 M_C | \mathcal{H}_1 = T_2 A.$$

These relations show that the problem $\mathrm{CID}_\kappa(A; T_1, T_2)$ makes sense. In the following we prove that the problem $\mathrm{CID}_\kappa(A; T_1, T_2)$ has solutions if and only if the same does C-$\mathrm{S}_\kappa(C)$.

Indeed, let $(A_\infty, \mathcal{E}) \in \mathrm{CID}_\kappa(A; T_1, T_2)$. Since $\mathcal{E}$ is a shift invariant subspace of $H^2(\mathbb{C}^m)$ of codimension at most $\kappa < \infty$, from Theorem 5.1 it follows that $\mathcal{E} = \Psi H^2(\mathbb{C}^m)$ for some Blaschke-Potapov product $\Psi \in H^\infty(\mathbb{C}^m)$ of finite order. Taking into account that $A_\infty V_1 | \mathcal{E} = V_2 A_\infty$ we get

$$A_\infty \Psi V_1 h = A_\infty V_1 \Psi h = V_2 A_\infty \Psi h, \quad h \in H^2(\mathbb{C}^m),$$

that is, letting $F = A_\infty \Psi$ we have $FV_1 = V_2 F$ and hence F is a multiplication operator with some function $F \in H^\infty(\mathbb{C}^m, \mathbb{C}^n)$. Since A_∞ is contractive and the multiplication operator with the function Ψ is isometric we have

$$\|Fh\| = \|A_\infty \Psi h\| \leq \|\Psi h\| = \|h\|, \quad h \in H^2(\mathbb{C}^m),$$

that is, F is contractive. These relations prove that A_∞ is the multiplication operator with a function in $\mathcal{S}_\kappa(\mathbb{C}^m, \mathbb{C}^n)$.

We now take into account that $P_{\mathcal{H}_2} A_\infty = A P_{\mathcal{H}_1} | \mathcal{E}$. For arbitrary $h \in H^2(\mathbb{C}^m)$ we have

$$P_{\mathcal{H}_2} M_F h = P_{\mathcal{H}_2} A_\infty \Psi h = A P_{\mathcal{H}_1} M_\Psi h = P_{\mathcal{H}_2} M_{C\Psi} h.$$

This proves that the first $k+1$ Taylor coefficients at 0 of G coincide, respectively, with the first Taylor coefficients of the analytic matrix-valued function $(C_0 + zC_1 + \cdots z^k C_k)\Psi(z)$ and hence, we have a solution of the problem C-$\mathrm{S}_\kappa(C)$.

The converse implication that once we have a solution of the problem C-$\mathrm{S}_\kappa(C)$ we have also a solution of the problem $\mathrm{CID}_\kappa(A; T_1, T_2)$ is straighforward and we omit the details.

We note now that we can identify $\mathcal{H}_1$ with a direct sum of $k+1$ copies of $\mathbb{C}^m$ and, similarly, $\mathcal{H}_2$ can be identified with the direct sum copies of $k+1$ copies of $\mathbb{C}^n$. With these identifications it is easy to see that the operator A coincides with the operator T_C as in (5.1). The proof is now concluded as an application of Theorem 4.2. ∎

6. Quasi-Multipliers and an Interpolation Problem of Nevanlinna-Pick Type

Let Ω be a set of complex numbers and $m \in \mathbb{N}$. Consider a matrix valued positive definite kernel $K : \Omega \times \Omega \to \mathcal{L}(\mathbb{C}^m)$. Then, e.g. see N. Aronszajn [6], there exists $\mathcal{H}(K)$ a *Hilbert space with reproducing kernel* K, that is, $\mathcal{H}(K)$ consists of vector valued functions $h : \Omega \to \mathbb{C}^m$ such that

$$\langle f, K_\lambda x \rangle_{\mathcal{H}(K)} = \langle f(\lambda), x \rangle_{\mathbb{C}^m}, \quad f \in \mathcal{H}(K),\ \lambda \in \Omega,\ x \in \mathbb{C}^m \tag{6.1}$$

where $K_\lambda(\mu) = K(\lambda,\mu)$. The association between reproducing kernel Hilbert spaces and positive definite kernels is bijective. Also, $\mathcal{H}(K) = \text{c.l.s.}\{K_\lambda x \mid \lambda \in \Omega,\ x \in \mathbb{C}^m\}$.

A subset $\Omega_1 \subseteq \Omega$ is a *set of uniqueness* for $\mathcal{H}(K)$ if whenever two functions $f, g \in \mathcal{H}(K)$ coincide on Ω_1 it follows that they coincide on the whole set Ω. This is equivalent to say that $\text{c.l.s.}\{K_\lambda x \mid \lambda \in \Omega_1,\ x \in \mathbb{C}^m\} = \mathcal{H}(K)$.

Let now $K'\colon \Omega \times \Omega \to \mathcal{L}(\mathbb{C}^n)$ be another matrix valued positive definite kernel on Ω and $\mathcal{H}(K')$ the corresponding reproducing kernel Hilbert space. A matrix valued function $\Phi\colon \Omega \to \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ is called a *multiplier* if $\Phi h \in \mathcal{H}(K')$ for all $h \in \mathcal{H}(K)$. Denoting by $M_\Phi\colon \mathcal{H}(K) \to \mathcal{H}(K')$ the corresponding multiplication operator, by the closed graph theorem it follows that M_Φ is bounded. Let $\|\Phi\| = \|M_\Phi\|$ denote the *norm* of the multiplier Φ. Following J. Agler and N.J. Young [3] we introduce the following definition.

DEFINITION 6.1 Let κ be a nonnegative integer. A function $G\colon \Omega_G \to \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ is called a *κ-quasimultiplier* if $G = F\Psi^{-1}$, where:

(a) F and Ψ are multipliers on Ω such that the multiplication operator M_Ψ is isometric;
(b) Ψ is invertible everywhere on $\Omega_G \subseteq \Omega$ with the exception of at most κ points in Ω;
(c) $M_\Psi \mathcal{H}(K)$ has codimension at most κ as a subspace of $\mathcal{H}(K)$.

As some examples in [3] show, the following assumption is natural.

(A0) Any subset $\Omega_1 \subseteq \Omega$ such that $\Omega \setminus \Omega_1$ is finite is a set of uniqueness for $\mathcal{H}(K)$.

If G is a quasimultiplier (that is, G is a κ-quasimultiplier for some $\kappa < \infty$) then a multiplication operator $M_G\colon \mathcal{E} \to \mathcal{H}(K')$ can be defined, where $\mathcal{E} = \Psi\mathcal{H}(K)$ is a subspace of $\mathcal{H}(K)$ of codimension at most κ. The norm of the quasimultiplier G is defined by $\|G\| = \|M_G\|$. Thanks to the assumption (A0) it can be proven that this definition is correct.

Our interest is related to a problem of quasimultiplier extensions of multipliers defined on subsets. More precisely, let Δ be a subset of Ω and consider the reproducing kernel Hilbert spaces

$$\mathcal{H}(K|\Delta) = \text{c.l.s.}\{K_\lambda x \mid \lambda \in \Delta,\ x \in \mathbb{C}^m\}, \tag{6.2}$$

and

$$\mathcal{H}(K'|\Delta) = \text{c.l.s.}\{K'_\lambda y \mid \lambda \in \Delta,\ y \in \mathbb{C}^n\}. \tag{6.3}$$

Let also $\varphi\colon \Delta \to \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ be a multiplier on Δ, that is, $\varphi h \in \mathcal{H}(K'|\Delta)$ for all $h \in \mathcal{H}(K|\Delta)$, and let ρ be a positive real number. It is required to determine those quasimultipliers G of norm $\|G\| \leq \rho$ extending φ, more precisely, for some representation $G = F\Psi^{-1}$ as in Definition 6.1 we have

$$\varphi(\lambda)\Psi(\lambda) = F(\lambda), \quad \lambda \in \Delta. \tag{6.4}$$

The approach we follow for the above mentioned problem makes use of Theorem 4.2. However, as shown also by P. Quiggin [29] and J. Agler and N.J. Young [3], some further assumptions are needed. These are:

(A1) The forward shift operators of multiplication by the independent variable are bounded on $\mathcal{H}(K)$ and, respectively, $\mathcal{H}(K')$.

(A2) Any subspace $\mathcal{E}$ of $\mathcal{H}(K)$ invariant under the forward shift operator and of codimension at most $\kappa < \infty$ admits a representation $\mathcal{E} = M_\Theta \mathcal{H}(K)$, with $\Theta\colon \Omega \to \mathcal{L}(\mathbb{C}^m)$ an isometric multiplier which is invertible for all but at most κ points of Ω.

(A3) Every bounded linear operator $M\colon \mathcal{H}(K) \to \mathcal{H}(K')$ which intertwines the forward shift operators is of the form $M = M_F$ for some multiplier $F\colon \Omega \to \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$.

THEOREM 6.2 *Assume that* (A0), (A1), (A2) *and* (A3) *hold and, in addition, that the forward shift operator* V_1 *on* $\mathcal{H}(K)$ *is expansive and that the forward shift operator* V_2 *on* $\mathcal{H}(K')$ *is contractive. Let* Δ *be a subset of* Ω, φ *a multiplier on* Δ, κ *a nonnegative integer and* ρ *a positive real number. Then, there exists a* κ*-quasimultiplier* G *of norm* $\leq \rho$ *extending* φ, *in the sense of* (6.4), *if and only if the kernel* $\big(\rho^2 - \overline{\varphi}(\mu)\varphi(\lambda)\big)K(\lambda,\mu)$ *has at most* κ *negative squares on* Δ.

Proof. Without restricting the generality we will assume $\rho = 1$. We consider the Hilbert spaces $\mathcal{G}_1 = \mathcal{H}(K)$ and $\mathcal{G}_2 = \mathcal{H}(K')$. Then define $\mathcal{H}_1 = \mathcal{H}(K|\Delta)$, see (6.2). Note that

$$V_1^* K_\lambda = \overline{\lambda} K_\lambda, \quad \lambda \in \Delta,$$

and hence defining $T_1^* = V_1^*|\mathcal{H}_1$ we obtain that $P_{\mathcal{H}_1} V_1 = T_1 P_{\mathcal{H}_1}$, that is, V_1 is a dilation of T_1.

Similarly, define $\mathcal{H}_2 = \mathcal{H}(K'|\Delta)$, cf. 6.3, and the operator $T_2 \in \mathcal{L}(\mathcal{H}_2)$ $T_2^* = V_2^*|\mathcal{H}_2$. Since

$$V_2^* K'_\lambda = \overline{\lambda} K'_\lambda, \quad \lambda \in \Delta,$$

it follows that $P_{\mathcal{H}_2} V_2 = T_2 P_{\mathcal{H}_2}$, that is, V_2 is a dilation of T_2.

We define now the operator $A = M_\varphi \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and since

$$T_1^* A^* K'_\lambda = T_1^* \overline{\varphi} K_\lambda = A^* T_2^* K'_\lambda, \quad \lambda \in \Delta,$$

it follows that A intertwines the operators T_1 and T_2.

From what is proved until now we conclude that the set $\mathrm{CID}_\kappa(A; T_1, T_2)$ makes sense. Moreover, for arbitray complex valued function $\{\alpha_\lambda\}_{\lambda\in\Delta}$ with finite support, $\operatorname{supp}\{\alpha_\lambda \mid \lambda \in \Delta\} < \infty$, and arbitrary vectors $x, y \in \mathbb{C}^m$ we have

$$\langle (I - A^*A) \sum_{\lambda\in\Delta} \alpha_\lambda K_\lambda x, \sum_{\mu\in\Delta} \alpha_\mu K_\mu y\rangle_{\mathcal{H}(K)} = \sum_{\lambda,\mu\in\Delta} \alpha_\lambda \overline{\alpha}_\mu \langle \big(1 - \overline{\varphi}(\mu)\varphi(\lambda)\big) K(\lambda,\mu) x, y\rangle_{\mathbb{C}^m},$$

which proves that $\kappa_-(I - A^*A)$ coincides with the number of negative squares of the matrix valued kernel $\big(1 - \overline{\varphi}(\mu)\varphi(\lambda)\big)K(\lambda,\mu)$.

Assume now that the number of negative squares of the kernel $\big(1 - \overline{\varphi}(\mu)\varphi(\lambda)\big)K(\lambda,\mu)$ is $\leq \kappa$. By Theorem 4.2 there exists a solution $(A_\infty, \mathcal{E}) \in \mathrm{CID}_\kappa(A; T_1, T_2)$. From the assumption (A2) it follows that $\mathcal{E} = M_\Psi \mathcal{H}(K)$, where $\Psi: \Omega \to \mathcal{L}(\mathbb{C}^m)$ is a multiplier such that M_Ψ is isometric. Taking into account that $A_\infty V_1|\mathcal{E} = V_2 A_\infty$ we get

$$A_\infty M_\Psi V_1 h = A_\infty V_1 \Psi h = V_2 A_\infty \Psi h, \quad h \in \mathcal{H}(K),$$

that is, letting $F = A_\infty \Psi$ we have $FV_1 = V_2 F$. Using the assumption (A3) it follows that F is the multiplication operator with some function $F: \Omega \to \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$. Since A_∞ is contractive and M_Ψ is isometric we have

$$\|Fh\| = \|A_\infty \Psi h\| \leq \|\Psi h\| = \|h\|, \quad h \in \mathcal{H}(K),$$

that is, F is contractive. Therefore, A_∞ is the multiplication operator with a κ-quasimultiplier $F\Psi^{-1}$. In addition, from $P_{\mathcal{H}_2} A_\infty = A P_{\mathcal{H}_1}$, for all $\lambda, \mu \in \Delta$, $x \in \mathbb{C}^m$, and $y \in \mathbb{C}^n$ we have

$$\langle A_\infty \Psi K_\lambda x, K'_\mu y\rangle_{\mathcal{H}(K')} = \langle P_{\mathcal{H}_2} A_\infty \Psi K_\lambda x, K'_\mu y\rangle_{\mathcal{H}(K'|\Delta)} = \langle A P_{\mathcal{H}_1} \Psi K_\lambda x, K'_\mu y\rangle_{\mathcal{H}(K'|\Delta)}$$

$$= \langle P_{\mathcal{H}_1} \Psi K_\lambda x, A^* K'_\mu y \rangle_{\mathcal{H}(K|\Delta)} = \langle \Psi K_\lambda x, A^* K'_m u y \rangle_{\mathcal{H}(K|\Delta)} = \langle \varphi(\mu) \Psi(\mu) K(\lambda, \mu) x, y \rangle_{\mathbb{C}^n}.$$

On the other hand,

$$\langle A_\infty \Psi K_\lambda x, K'_\mu y \rangle_{\mathcal{H}(K')} = \langle F(\mu) K(\lambda, \mu) x, y \rangle_{\mathbb{C}^n},$$

and hence $\varphi(\mu)\Psi(\mu) = F(\mu)$ for all $\mu \in \Delta$.

The converse implication can be obtained by tracking back on the the same lines of the proof as above. ∎

Theorem 6.2 can be applied to a variant of the Nevanlinna-Pick interpolation problem which was first introduced by M.G. Kreĭn and H. Langer [25] in a slighly different form. The present formulation was considered first by J.A. Ball [7].

DEFINITION 6.3 Let $\mathbf{z} = \{z_l\}_{l=1}^k$ be a sequence of complex numbers in the open unit disc $\mathbb{D}$ and let $\mathbf{Z} = \{Z_l\}_{l=1}^k \subset \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ be a sequence of $n \times m$ complex matrices. For $\kappa \in \mathbb{N}$ we define the set N-P$_\kappa(\mathbf{z}; \mathbf{Z})$ consisting of all meromorphic matrix valued functions $G \in \mathcal{S}_\kappa(\mathbb{C}^m, \mathbb{C}^n)$ such that $G(z_l) = Z_l$, for all $l = 1, 2, \ldots, k$, more precisely, if the function G has the representation $G = F\Psi^{-1}$, where $F \in H^\infty(M_{m,n})$ and $\Psi \in H^\infty(M_n)$ is a Blaschke-Potapov product of order $\leq \kappa$, then the condition $G(z_l) = Z_l$ has to be interpreted as $F(z_l) = Z_l \Psi(z_l)$.

We note that in the above definition, it may happen z_l to be poles of the function G for some $l \in \{1, 2, \ldots, k\}$.

Associated to the data $\mathbf{z}$ and $\mathbf{Z}$ there is the *Pick matrix* $P(\mathbf{z}; \mathbf{Z})$ defined by

$$P(\mathbf{z}; \mathbf{Z}) = \left[\frac{I - Z_l^* Z_p}{1 - \bar{z}_l z_p} \right]_{p,l=1}^k. \tag{6.5}$$

Under the additional condition that the Pick matrix in nonsingular, the following result was first obtained by J.A. Ball [7]. In the more general case of the bitangential Nevanlinna-Pick problem it was obtained by J.A. Ball and J.W. Helton in [8]. In the following we show that, in the case of the variant of the Nevanlinna-Pick problem as in Definition 6.3, it can be obtained as a consequence of Theorem 6.2 by letting $K(\lambda, \mu) = I_m/(1 - \bar{\mu}\lambda)$ and, respectively, $K'(\lambda, \mu) = I_n/(1 - \bar{\mu}\lambda)$ to be the matrix valued Schur kernels. In this case $\mathcal{H}(K) = H^2(\mathbb{C}^m)$ and $\mathcal{H}(K') = H^2(\mathbb{C}^n)$, in particular, the forward shift operators are isometric on these spaces and hence the assumption (A1) is satisfied. Theorem 5.1 shows that the assumption (A2) is also satisfied. As it is well-known, the assumption (A3) holds, too. We consider $\Delta = \{z_i\}_{i=1}^k \in \mathbb{D}$ and let the muliplier φ be defined by $\varphi(z_i) = Z_i$ for all $i = 1, 2, \ldots, k$. Then note that the Pick matrix is the matrix representation of the kernel $(1 - \overline{\varphi}(\mu)\varphi(\lambda))K(\lambda, \mu)$, $\lambda, \mu \in \Delta$. Therefore, from Theorem 6.2 we obtain the following

THEOREM 6.4 *If κ is finite then the Nevanlinna-Pick problem* N-P$_\kappa(\mathbf{z}; \mathbf{Z})$ *has solutions if and only if the number of negative eigenvalues, counted with their multiplicities, of the Pick matrix $P(\mathbf{z}; \mathbf{Z})$ does not exceed κ.*

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AMS Classification Subjects: 47A20, 47A57, 47B50.