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by

RUXANDRA STAVRE

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RUXANDRA STAVRE *)

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*) Institute of Mathematics of the Romanian Academy,
P.O. Box 1-764, RO-70700, Bucharest, Romania.
E-mail RSTAVRE@IMAR.RO

A theoretical and numerical approach of a distributed control model for a free boundary problem

Ruxandra Stavre
Institute of Mathematics of the Romanian Academy,
P. O. Box 1-764, RO-70700, Bucharest, Romania,
fax: (401)2229826, e-mail: RSTAVRE@IMAR.RO

Summary. The flow of an incompressible fluid through a non-homogeneous dam is considered. A distributed control problem associated with this free boundary problem is studied. The aim of this paper is to minimize the total pressure of the fluid, the control being the permeability coefficient of the dam. The first order necessary conditions of optimality are derived for a family of regular control problems. A finite element approximation of the optimality system is introduced and the convergence of the proposed algorithms is studied. Some numerical results are discussed, for the case of the non-homogeneous rectangular dam.

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A theoretical and numerical approach of a control problem

1. Introduction

The flow of an incompressible fluid through a non-homogeneous dam with general geometry was studied, for instance, in: Alt (1979), Alt (1980), Friedman and Huang (1985), Stavre and Vernescu (1985), Stavre and Vernescu (1989). In Alt (1979), Friedman and Huang (1985), Stavre and Vernescu (1985) this free boundary problem was studied from a theoretical point of view, while in Alt (1980), Stavre and Vernescu (1989) numerical methods were used for solving it.

We introduce and study an optimal control model associated with this free boundary problem. We want to minimize the "total pressure" of the fluid in the dam, given by the functional:

$$J(k) = \int_D p(x,y)\,dx\,dy, \tag{1.1}$$

where the control k is the permeability coefficient of the dam, $D \subset \mathbb{R}^2$ is the cross-section of the dam and p, the pressure of the fluid. The purpose of the paper is to obtain the optimality system (the necessary conditions of optimality) and to approximate it in order to compute an optimal control k^*, characterized as a minimum point for the functional J, defined by (1.1).

Other optimal control models associated with the homogeneous dam problem were studied in Barbu (1984), Friedman and Yaniro (1985), Friedman, Huang and Yong (1987). In Friedman and Yaniro (1985), Friedman, Huang and Yong (1987) the control variable is the rate allowed to withdraw water from the bottom of the dam and in Barbu (1984) the control is the highest level of the fluid in the reservoirs.

The plan of the paper is as follows. In Section 2 we define the distributed control problem and we prove an existence result. The necessary conditions of optimality are deduced in the next section, by approximating the control problem by a family of control problems which are regular. Section 4 deals with the finite element approximation of the optimality system associated with the family of regularized control problems; the convergence of the proposed algorithms is also discussed. In the last section, some numerical results are presented, for the case of a non-homogeneous, rectangular dam.

2. The control problem

First we describe the mathematical formulation of the physical problem, introduced in Brezis, Kinderlehrer and Stampacchia (1978), Carrillo-Menendez and Chipot (1982) for the homogeneous dam and in Stavre and Vernescu (1985) for the non-homogeneous case.

The cross-section of the dam is denoted by D, where $D \subset \mathbb{R}^2$ is open, bounded, connected, with the boundary ∂D, which is locally a Lipschitz graph. The boundary is formed by three disjoint parts: S_1–the impervious part, S_2–the part in contact with the air and $S_3 = S_{3,1} \cup S_{3,2}$–the part in contact with the reservoirs ($S_{3,1}$, $S_{3,2}$ being the connected components of S_3).

We denote by h_i the level of the fluid in the reservoir with bottom $S_{3,i}$, $i = 1, 2$ and we define $f : S_2 \cup S_3 \mapsto \mathbb{R}$,

$$(2.1) \qquad f = \begin{cases} 0 & on\ S_2, \\ h_i - y & on\ S_{3,i} \quad i = 1, 2. \end{cases}$$

The variational formulation of the physical problem is (see Stavre and Vernescu (1985)):

$$(VP)_k \begin{cases} Find\ p_k \in H^1(D),\ p_k \geq 0\ a.e.\ in\ D,\ p_k = f\ on\ S_2 \cup S_3, \\ \int_D k(\nabla p_k \cdot \nabla \varphi + H(p_k)\dfrac{\partial \varphi}{\partial y})dxdy \leq 0\ \forall \varphi \in H^1(D),\ \varphi = 0\ on\ S_3,\ \varphi \geq 0\ on\ S_2, \end{cases}$$

where k is the permeability coefficient of the dam, p_k the corresponding pressure of the fluid and H, the Heaviside function. It is obvious that the pressure of the fluid in the dam depends on the function k.

We suppose that k is a control variable belonging to the following bounded, closed, convex set:

$$(2.2) \quad K = \{v \in H^1(D) / \|v\|_{H^1(D)} \leq r,\ \alpha \leq v \leq \beta\ a.e.\ in\ D,\ \frac{\partial v}{\partial y} \geq 0\ a.e.\ in\ D\},$$

where α, β, r are positive constants, with r large enough.

Since $(VP)_k$ has not, in general, a unique solution p_k (see Stavre and Vernescu (1985)), the correspondence $k \mapsto p_k$ is multi-valued.

We define:

$$P_k = \{p / \, p \text{ solution of } (VP)_k\}, \tag{2.3}$$

and we introduce the following problem:

$$\begin{cases} \text{Find } (k^*, p^*) \in K \times P_{k^*}, \\ \int_D p^* dxdy \leq \int_D p dxdy \ \forall (k, p) \in K \times P_k. \end{cases} \tag{2.4}$$

It is known from Stavre and Vernescu (1985) (Theorem 4.2) that there exists a unique solution $\bar{p}_k$ of $(VP)_k$ so that the boundary of each connected component of $\{\bar{p}_k > 0\}$ is in contact with at least a reservoir (S_3-connected solution). Hence, the correspondence $k \mapsto \bar{p}_k$ is uni-valued.

Lemma 2.1 *Let k_0 be an element of K and let $\bar{p}_{k_0}$, p_{k_0} be the S_3-connected solution of $(VP)_{k_0}$ and another solution of $(VP)_{k_0}$, respectively. Then:*

$$\int_D \bar{p}_{k_0} dxdy < \int_D p_{k_0} dxdy. \tag{2.5}$$

Proof. There exists at least a connected set $C_1 \subset \{p_{k_0} > 0\}$ so that $\partial C_1 \cap S_3 = \emptyset$. We denote by C the union of all the connected components of $\{p_{k_0} > 0\}$ with the above property and we define:

$$p^*_{k_0} = \begin{cases} p_{k_0} & \text{in } D - \bar{C}, \\ 0 & \text{in } \bar{C}. \end{cases} \tag{2.6}$$

It can be proved, as in Stavre and Vernescu (1985) (Theorem 3.7), that $p^*_{k_0}$ is a solution for $(VP)_{k_0}$. Moreover, from (2.6) it follows that $p^*_{k_0}$ is S_3-connected; hence $p^*_{k_0} = \bar{p}_{k_0}$. Since, from (2.6) we get $\bar{p}_{k_0} < p_{k_0}$ in C, the assertion of the lemma is obtained.

We introduce another minimum problem:

$$\begin{cases} \text{Find } k^* \in K, \\ \int_D \bar{p}_{k^*} dxdy \leq \int_D \bar{p}_k dxdy, \ \forall k \in K, \end{cases} \tag{2.7}$$

and we prove, by using Lemma 2.1, the following:

Proposition 2.2 *(2.4) has a solution iff (2.7) has a solution.*

Proof. Let (k^*, p^*) be a solution of (2.4). Hence:

$$\int_D p^* dxdy \leq \int_D \bar{p}_{k^*} dxdy.$$

By using Lemma 2.1, it follows $p^* = \bar{p}_{k^*}$ and, from (2.4) for $p = \bar{p}_k$, $k \in K$, we obtain:

$$\int_D \bar{p}_{k^*} dxdy \leq \int_D \bar{p}_k dxdy \;\; \forall k \in K.$$

Conversely, if k^* is a solution of (2.7), we define $p^* = \bar{p}_{k^*} \in P_{k^*}$ and, by using again Lemma 2.1, the proof is achieved.

We shall study in the sequel the problem (2.7).

We define the functional $J : K \mapsto \mathbb{R}_+$,

$$J(k) = \int_D \bar{p}_k dxdy. \tag{2.8}$$

(2.7) can be written as the following control problem:

$$(CP) \begin{cases} Find\ k^* \in K, \\ J(k^*) = min\{J(k) \,/\, k \in K\}. \end{cases}$$

The last result of this section is an existence theorem.

Theorem 2.3 *(CP) has at least a solution.*

Proof. Let $\{k_n\}_{n\in\mathbb{N}} \subset K$ be a minimizing sequence. Since K is bounded in $H^1(D)$, closed and convex, it follows that $k_{n_s} \rightarrow k_0$ weakly in $H^1(D)$ when $s \rightarrow \infty$ and $k_0 \in K$.

Taking into account that $\{H(\bar{p}_{k_{n_s}})\}_{s\in\mathbb{N}}$ is bounded in $L^\infty(D)$ and, from $(VP)_{k_{n_s}}$, $\{\bar{p}_{k_{n_s}}\}_{s\in\mathbb{N}}$ is bounded in $H^1(D)$, we get, by passing to the limit on a subsequence in $(VP)_{k_{n_s}}$:

$$\int_D k_0(\nabla p_0 \cdot \nabla \varphi + \bar{H}\frac{\partial \varphi}{\partial y})dxdy \leq 0 \;\forall \varphi \in H^1(D),\; \varphi = 0 \;on\; S_3,\; \varphi \geq 0 \;on\; S_2, \tag{2.9}$$

where p_0 is the weak limit in $H^1(D)$ of a subsequence of $\{\bar{p}_{k_{n_s}}\}_{s\in\mathbb{N}}$ and $\bar{H}$ is the weak star limit in $L^\infty(D)$ of a subsequence of $\{H(\bar{p}_{k_{n_s}})\}_{s\in\mathbb{N}}$. With the same technique as in Stavre and Vernescu (1985), we get $\bar{H} = H(p_0)$ and, hence, p_0 verifies $(VP)_{k_0}$. Moreover, $\int_D p_0 dxdy = min\{J(k) \,/\, k \in K\}$. This equality and Lemma 2.1 imply $p_0 = \bar{p}_{k_0}$ and therefore, the theorem has been proved.

For simplicity we shall assume in the sequel that D has a geometry which ensures the uniqueness of the solution of $(VP)_k$, $\forall\, k \in K$ (for instance, S_1 given by $y = 0$, $x \in (0, a)$). We shall denote the unique solution of $(VP)_k$ by p_k.

In the next section, (CP) will be approximated by a family of regularized problems, for which we shall deduce the necessary conditions of optimality.

3. The optimality system

We introduce in the sequel the following family of regularized control problems:

$$(CP)_\varepsilon \begin{cases} For\ \varepsilon > 0,\ find\ k^*_\varepsilon \in K, \\ J_\varepsilon(k^*_\varepsilon) = min\{J_\varepsilon(k) \,/\, k \in K\}, \end{cases}$$

where $J_\varepsilon(k) = \int_D p^\varepsilon_k dxdy$, p^ε_k being a solution of:

$$(VP)^\varepsilon_k \begin{cases} p^\varepsilon_k \in H^1(D),\ p^\varepsilon_k = f\ on\ S_2 \cup S_3, \\ \int_D k(\nabla p^\varepsilon_k \cdot \nabla\varphi + H_\varepsilon(p^\varepsilon_k)\dfrac{\partial\varphi}{\partial y})dxdy = 0\ \forall\varphi = 0\ on\ S_2 \cup S_3, \end{cases}$$

with $H_\varepsilon(x) = \dfrac{x^{+2}}{x^2 + \varepsilon^2}$, $x^+ = max(x, 0)$.

Before studying the family of control problems $(CP)_\varepsilon$, we remark that $(VP)^\varepsilon_k$ is of the same type as (VP_ε) considered in Stavre and Vernescu (1985), but with a more regular function H_ε. We shall use the regularity of H_ε in the next section, for obtaining the convergence of a sequence of solutions of the discrete optimality system to a solution of the optimality system, associated with $(CP)_\varepsilon$.

The proof of the next theorem is similar to those of Theorems 3.1, 3.2 from Stavre and Vernescu (1985), therefore we shall omit it.

Theorem 3.1 *For any $\varepsilon > 0$, $k \in K$, there exists a unique solution p_k^ε of $(VP)_k^\varepsilon$ with the properties: $p_k^\varepsilon \in C(D \cup S_2 \cup S_3)$, $p_k^\varepsilon \geq 0$ in D.*

The sense of the approximation of $(VP)_k$ by the family $(VP)_k^\varepsilon$, $\varepsilon > 0$ is given by:

Theorem 3.2 *Let p_k^ε and p_k be the unique solution of $(VP)_k^\varepsilon$ and $(VP)_k$, respectively. Then $p_k^\varepsilon \to p_k$ weakly in $H^1(D)$, when $\varepsilon \to 0$.*

Proof. By choosing in $(VP)_k^\varepsilon$ $\varphi = p_k^\varepsilon - v$, $v \in H^1(D)$, $v = f$ on $S_2 \cup S_3$ and by taking into account the properties of k and H_ε, we obtain the boundedness in $H^1(D)$ of the sequence $\{p_k^\varepsilon\}_{\varepsilon>0}$. Moreover, $\{H_\varepsilon(p_k^\varepsilon)\}_{\varepsilon>0}$ is bounded in $L^\infty(D)$. Hence, we obtain, on a subsequence: $p_k^\varepsilon \rightharpoonup p$ weakly in $H^1(D)$, $H_\varepsilon(p_k^\varepsilon) \to \bar{H}$ weakly star in $L^\infty(D)$, when $\varepsilon \to 0$. Moreover, $p = f$ on $S_2 \cup S_3$, $p \geq 0$ a.e. in D, $0 \leq \bar{H} \leq 1$ a.e. in D.

By applying the Stokes formula for $\varphi \in H^1(D)$, $\varphi = 0$ on S_3, $\varphi \geq 0$ on S_2 it follows, as in Stavre and Vernescu (1985):

$$(3.1) \int_D k(\nabla p_k^\varepsilon \cdot \nabla\varphi + H_\varepsilon(p_k^\varepsilon)\frac{\partial\varphi}{\partial y})dxdy \leq 0 \ \forall\varphi \in H^1(D),\ \varphi = 0 \text{ on } S_3,\ \varphi \geq 0 \text{ on } S_2.$$

By passing to the limit, on a subsequence, in (3.1), we get:

$$(3.2) \quad \int_D k(\nabla p \cdot \nabla\varphi + \bar{H}\frac{\partial\varphi}{\partial y})dxdy \leq 0 \ \forall\varphi \in H^1(D),\ \varphi = 0 \text{ on } S_3,\ \varphi \geq 0 \text{ on } S_2.$$

If we choose $\varphi \in \mathcal{D}(D)$ in (3.2) we obtain:

$$(3.3) \qquad div(k\nabla p) + \frac{\partial}{\partial y}(k\bar{H}) = 0 \text{ in } \mathcal{D}'(D)$$

and, since $k\bar{H} \in L^\infty(D)$, by using elliptic regularity (see Gilbarg and Trudinger (1977)), we deduce that $p \in C(D \cup S_2 \cup S_3)$.

In order to conclude that $\bar{H} = H(p)$, we have to prove:

$$(i)\ \bar{H} = 1 \text{ a.e. in } \{p > 0\},$$

$$(ii)\ \bar{H} = 0 \text{ a.e. in } D - \overline{\{p > 0\}},$$

$$(iii)\ mes(D \cap \partial\{p > 0\}) = 0.$$

We begin with the proof of the assertion (i). We have:

$$\int_{\{p>0\}} \bar{H}\,dxdy = \liminf_{\varepsilon\to 0} \int_{\{p>0\}} H_\varepsilon(p_k^\varepsilon)dxdy. \tag{3.4}$$

Let δ be a fixed positive number. It can be easily proved that:

$$\int_{\{p>0\}} H_\varepsilon(p_k^\varepsilon)dxdy > \int_{\{p>\delta\}\cap\{p_k^\varepsilon>\delta\}} dxdy - \frac{\varepsilon^2}{\delta^2+\varepsilon^2}\int_{\{p>\delta\}\cap\{p_k^\varepsilon>\delta\}} dxdy. \tag{3.5}$$

By passing to the inferior limit with $\varepsilon \to 0$ in (3.5) and combining this with (3.4), we get:

$$\int_{\{p>0\}} \bar{H}dxdy \geq \int_{\{p>\delta\}} dxdy \ \ \forall\,\delta > 0. \tag{3.6}$$

By passing to the limit in (3.6) with $\delta \to 0$ and by taking into account that $\bar{H} \leq 1$ a.e. in D, we obtain the assertion (i).

For obtaining (ii) and (iii) we need only (3.2) and (i); hence, the proof of (ii) and (iii) is that of Lemma 3.1 of Stavre and Vernescu (1989).

Since $\bar{H} = H(p)$, it follows from (3.2) that p is the unique solution of $(VP)_k$. We remark that the uniqueness of the solution of $(VP)_k$ gives the uniqueness of the weak limit point in $H^1(D)$ of the sequence $\{p_k^\varepsilon\}_{\varepsilon>0}$, which completes the proof.

Theorem 3.3 *For any $\varepsilon > 0$, $(CP)_\varepsilon$ has at least a solution.*

Proof. Let $\{k_n^\varepsilon\}_{n\in\mathbb{N}} \subset K$ be a minimizing sequence for J_ε. It follows that on a subsequence, denoted also by k_n^ε, we have: $k_n^\varepsilon \to k_\varepsilon^*$ weakly in $H^1(D)$, $k_n^\varepsilon \to k_\varepsilon^*$ weakly star in $L^\infty(D)$, $k_n^\varepsilon \to k_\varepsilon^*$ a.e. in D, when $n \to \infty$ and $k_\varepsilon^* \in K$. Moreover, $\lim_{n\to\infty} J_\varepsilon(k_n^\varepsilon) = min\{J_\varepsilon(k) \,/\, k \in K\}$.

By taking in $(VP)_{k_n^\varepsilon}^\varepsilon$ $\varphi = p_{k_n^\varepsilon}^\varepsilon - v$, with $v = f$ on $S_2 \cup S_3$ and by using the properties of k_n^ε and H_ε we obtain, on a subsequence: $p_{k_n^\varepsilon}^\varepsilon \to p^\varepsilon$ weakly in $H^1(D)$ and $H_\varepsilon(p_{k_n^\varepsilon}^\varepsilon) \to H_\varepsilon(p^\varepsilon)$ strongly in $L^2(D)$, when $n \to \infty$. By passing to the limit in $(VP)_{k_n^\varepsilon}^\varepsilon$ on a subsequence, when $n \to \infty$ it follows that p^ε satisfies $(VP)_{k_\varepsilon^*}^\varepsilon$ and hence, $p^\varepsilon = p_{k_\varepsilon^*}^\varepsilon$. This yields:

$$min\{J_\varepsilon(k) \,/\, k \in K\} = \lim_{n\to\infty}\int_D p_{k_n^\varepsilon}^\varepsilon dxdy = \int_D p_{k_\varepsilon^*}^\varepsilon dxdy = J_\varepsilon(k_\varepsilon^*).$$

We establish next the relation between the regularized minimum problems $(CP)_\varepsilon$, $\varepsilon > 0$ and the initial control problem.

Theorem 3.4 *For any $\varepsilon > 0$, let $k_\varepsilon^* \in K$ be a minimum point of J_ε. Then, any weak limit point in $H^1(D)$, k^*, of $\{k_\varepsilon^*\}_{\varepsilon>0}$ is a solution for (CP). Moreover:*

$$\lim_{\varepsilon\to 0} J_\varepsilon(k_\varepsilon^*) = min\{J(k)\,/\,k \in K\}. \tag{3.7}$$

Proof. From the definition (2.2) and $\{k_\varepsilon^*\}_{\varepsilon>0} \subset K$ it follows that there exists at least an element $k^* \in K$ such that, we have, on a subsequence: $k_\varepsilon^* \to k^*$ weakly in $H^1(D)$, $k_\varepsilon^* \to k^*$ weakly star in $L^\infty(D)$, $k_\varepsilon^* \to k^*$ a.e. in D, when $\varepsilon \to 0$.

From $(VP)_{k_\varepsilon^*}^\varepsilon$ we obtain, as before, the boundedness in $H^1(D)$ of $\{p_{k_\varepsilon^*}^\varepsilon\}_{\varepsilon>0}$.

Moreover $\{H_\varepsilon(p_{k_\varepsilon^*}^\varepsilon)\}_{\varepsilon>0}$ is bounded in $L^\infty(D)$. We can now extract subsequences such that $p_{k_\varepsilon^*}^\varepsilon \to p$ weakly in $H^1(D)$, $H_\varepsilon(p_{k_\varepsilon^*}^\varepsilon) \to \bar{H}$ weakly star in $L^\infty(D)$, when $\varepsilon \to 0$, with $p = f$ on $S_2 \cup S_3$, $p \geq 0$ in D, $0 \leq \bar{H} \leq 1$ a.e. in D.

For any $\varphi \in H^1(D)$, $\varphi = 0$ on S_3, $\varphi \geq 0$ on S_2 we obtain from $(VP)_{k_\varepsilon^*}^\varepsilon$, as in Theorem 3.2:

$$\int_D k_\varepsilon^*(\nabla p_{k_\varepsilon^*}^\varepsilon \cdot \nabla\varphi + H_\varepsilon(p_{k_\varepsilon^*}^\varepsilon)\frac{\partial\varphi}{\partial y})dxdy \leq 0. \tag{3.8}$$

By passing to the limit in (3.8), on a subsequence, with $\varepsilon \to 0$, we get:

$$\int_D k^*(\nabla p \cdot \nabla\varphi + \bar{H}\frac{\partial\varphi}{\partial y})dxdy \leq 0 \ \forall\varphi \in H^1(D),\ \varphi = 0 \ on\ S_3,\ \varphi \geq 0 \ on\ S_2. \tag{3.9}$$

We conclude, with the same proof as in Theorem 3.2, that $\bar{H} = H(p)$ and, hence, $p = p_{k^*}$, the unique solution of $(VP)_{k^*}$.

On the other hand we have $J_\varepsilon(k_\varepsilon^*) \leq J_\varepsilon(k)\ \forall\, k \in K,\ \forall\varepsilon > 0$, i.e.:

$$\int_D p_{k_\varepsilon^*}^\varepsilon dxdy \leq \int_D p_k^\varepsilon dxdy \ \forall\, k \in K,\ \forall\varepsilon > 0. \tag{3.10}$$

Taking $\varepsilon \to 0$ in (3.10) and using Theorem 3.2 and the weak convergence in $H^1(D)$ of $\{p_{k_\varepsilon^*}^\varepsilon\}_{\varepsilon>0}$ to p_{k^*} we obtain $J(k^*) \leq J(k)\ \forall\, k \in K$; hence, the first assertion of the theorem holds.

For proving (3.7), we first remark that the boundedness of $\{p^{\varepsilon}_{k^*_\varepsilon}\}_{\varepsilon>0}$ in $H^1(D)$ implies the boundedness of $\{J_\varepsilon(k^*_\varepsilon)\}_{\varepsilon>0}$ in $\mathbb{R}$.

If we suppose, by contradiction, that there exists two subsequences such that $J_{\varepsilon_s}(k^*_{\varepsilon_s}) \to l_1$, when $s \to \infty$ and $J_{\varepsilon_q}(k^*_{\varepsilon_q}) \to l_2$, when $q \to \infty$, with $l_1 \neq l_2$ we obtain, as before, that, on a subsequence, we have: $k^*_{\varepsilon_s} \to k_1$, $p^{\varepsilon_s}_{k^*_{\varepsilon_s}} \to p_{k_1}$ weakly in $H^1(D)$, when $s \to \infty$ and $k^*_{\varepsilon_q} \to k_2$, $p^{\varepsilon_q}_{k^*_{\varepsilon_q}} \to p_{k_2}$ weakly in $H^1(D)$, when $q \to \infty$.

From (3.10), for $\varepsilon = \varepsilon_s$, $k = k_2$ we get, as $s \to \infty$ $l_1 \le l_2$ and, for $\varepsilon = \varepsilon_q$, $k = k_1$ we get, as $q \to \infty$ $l_1 \ge l_2$; hence, a contradiction with $l_1 \neq l_2$.

Thus, (3.7) holds.

In the sequel, we shall derive the necessary conditions of optimality associated with $(CP)_\varepsilon$.

We first establish the following:

Lemma 3.5 For any k, $k_0 \in K$, $\varepsilon > 0$, we have:

$$J'_\varepsilon(k_0)\cdot(k-k_0) = \int_D q^\varepsilon dxdy, \tag{3.11}$$

where $q^\varepsilon \in H^1(D)$ is the unique weak solution of the problem:

$$\begin{cases} div(k_0\nabla q^\varepsilon) + \dfrac{\partial}{\partial y}(k_0 H'_\varepsilon(p^\varepsilon_{k_0})q^\varepsilon) \\ \quad = div((k_0-k)\nabla p^\varepsilon_{k_0}) + \dfrac{\partial}{\partial y}((k_0-k)H_\varepsilon(p^\varepsilon_{k_0})) \ in\ D, \\ q^\varepsilon = 0 \ on\ S_2\cup S_3, \\ k_0(\dfrac{\partial q^\varepsilon}{\partial n} + H'_\varepsilon(p^\varepsilon_{k_0})q^\varepsilon n_y) = (k_0-k)(\dfrac{\partial p^\varepsilon_{k_0}}{\partial n} + H_\varepsilon(p^\varepsilon_{k_0})n_y) \ on\ S_1, \end{cases} \tag{3.12}$$

where $\vec{n} = (n_x, n_y)$ is the outward unit normal to ∂D.

Proof. We begin by proving that the solution of (3.12) is unique. Let us suppose that there exists two solutions of (3.12), q^ε_1, q^ε_2 and let us define $Q^\varepsilon = q^\varepsilon_1 - q^\varepsilon_2$. $Q^\varepsilon \in H^1(D)$ satisfies the following variational problem:

$$\begin{cases} \displaystyle\int_D k_0(\nabla Q^\varepsilon\cdot\nabla\varphi + H'_\varepsilon(p^\varepsilon_{k_0})Q^\varepsilon\frac{\partial\varphi}{\partial y})dxdy = 0 \ \forall\varphi\in H^1(D),\ \varphi = 0 \text{ on } S_2\cup S_3, \\ Q^\varepsilon = 0 \ on\ S_2\cup S_3. \end{cases} \tag{3.13}$$

If $Q^\varepsilon \neq 0$ in D, we can suppose that $mes(\{Q^\varepsilon > 0\}) > 0$.

For $\delta > 0$ given, we take $\varphi = \dfrac{(Q^\varepsilon - \delta)^+}{Q^\varepsilon}$ in (3.13) and we obtain:

$\|ln(1 + \dfrac{(Q^\varepsilon - \delta)^+}{\delta})\|_{H^1(D)} \leq c$, the constant being independent of δ. When δ tends to 0 we obtain $Q^\varepsilon \leq 0$ a.e. in D. It follows that $Q^\varepsilon = 0$ in D and, hence, $q_1^\varepsilon = q_2^\varepsilon$ in D.

Let $t \in (0,1)$. We denote $(p^\varepsilon_{k_0+t(k-k_0)} - p^\varepsilon_{k_0})/t$ by q_t^ε; hence,

$$\lim_{t\to 0} \frac{J_\varepsilon(k_0 + t(k-k_0)) - J_\varepsilon(k_0)}{t} = \lim_{t\to 0} \int_D q_t^\varepsilon \, dx dy.$$

We prove next that $\{q_t^\varepsilon\}_{t\in(0,1)}$ is bounded in $H^1(D)$.

By computing $\dfrac{(VP)^\varepsilon_{k_0+t(k-k_0)} - (VP)^\varepsilon_{k_0}}{t}$ for $\varphi = q_t^\varepsilon$, we get:

$$\|q_t^\varepsilon\|_{H^1(D)} \leq c_1 \|q_t^\varepsilon\|_{L^2(D)} + c_2, \tag{3.14}$$

the constants c_1, c_2 being independent of t.

If $\{q_t^\varepsilon\}_{t\in(0,1)}$ is bounded in $L^2(D)$, we obtain, from (3.14), the boundedness of the sequence in $H^1(D)$.

Let us suppose that $\{q_t^\varepsilon\}_{t\in(0,1)}$ is unbounded in $L^2(D)$. For a subsequence, denoted again by $\{q_t^\varepsilon\}_{t\in(0,1)}$ we have $\lim_{t\to 0} \|q_t^\varepsilon\|_{L^2(D)} = \infty$.

We define $Q_t^\varepsilon = \dfrac{q_t^\varepsilon}{\|q_t^\varepsilon\|_{L^2(D)}}$. It is obvious that $\|Q_t^\varepsilon\|_{L^2(D)} = 1$ and, from (3.14), that $\{Q_t^\varepsilon\}_{t\in(0,1)}$ is bounded in $H^1(D)$. Thus we can extract a subsequence such that $Q_t^\varepsilon \to Q^\varepsilon$ weakly in $H^1(D)$, when $t \to 0$.

Moreover, $\|Q^\varepsilon\|_{L^2(D)} = 1$.

By considering the problem satisfied by Q_t^ε and by passing to the limit with $t \to 0$, we obtain that Q^ε is the solution of (3.13), i.e. $Q^\varepsilon = 0$ in D, which contradicts $\|Q^\varepsilon\|_{L^2(D)} = 1$.

The sequence $\{q_t^\varepsilon\}_{t\in(0,1)}$ being bounded in $H^1(D)$, it follows that it has at least a weak limit point in $H^1(D)$, q^ε, which is the solution of (3.12).

From the uniqueness of the solution of (3.12), we obtain that the weak limit point of $\{q_t^\varepsilon\}_{t\in(0,1)}$ is unique, which completes the proof.

The main result of this section, the necessary conditions of optimality associated with $(CP)_\varepsilon$, is a consequence of the above lemma.

Theorem 3.6 For any $\varepsilon > 0$ let k_ε^* be a solution of $(CP)_\varepsilon$. Then, there exists the unique elements $p_{k_\varepsilon^*}^\varepsilon$, $Q_{k_\varepsilon^*}^\varepsilon \in H^1(D)$, which satisfy the optimality system:

$$
(OS)\left\{\begin{array}{l}
\left\{\begin{array}{l}
div(k_\varepsilon^*\nabla p_{k_\varepsilon^*}^\varepsilon) + \dfrac{\partial}{\partial y}(k_\varepsilon^* H_\varepsilon(p_{k_\varepsilon^*}^\varepsilon)) = 0 \text{ in } D,\\
p_{k_\varepsilon^*}^\varepsilon = f \text{ on } S_2 \cup S_3,\\
k_\varepsilon^*(\dfrac{\partial p_{k_\varepsilon^*}^\varepsilon}{\partial n} + H_\varepsilon(p_{k_\varepsilon^*}^\varepsilon)n_y) = 0 \text{ on } S_1,
\end{array}\right.\\
\left\{\begin{array}{l}
div(k_\varepsilon^*\nabla Q_{k_\varepsilon^*}^\varepsilon) - k_\varepsilon^* H_\varepsilon'(p_{k_\varepsilon^*}^\varepsilon)\dfrac{\partial Q_{k_\varepsilon^*}^\varepsilon}{\partial y} = 1 \text{ in } D,\\
Q_{k_\varepsilon^*}^\varepsilon = 0 \text{ on } S_2 \cup S_3,\\
k_\varepsilon^*\dfrac{\partial Q_{k_\varepsilon^*}^\varepsilon}{\partial n} = 0 \text{ on } S_1,
\end{array}\right.\\
\displaystyle\int_D (\nabla p_{k_\varepsilon^*}^\varepsilon \cdot \nabla Q_{k_\varepsilon^*}^\varepsilon + H_\varepsilon(p_{k_\varepsilon^*}^\varepsilon)\frac{\partial Q_{k_\varepsilon^*}^\varepsilon}{\partial y})(k - k_\varepsilon^*)dxdy \geq 0 \; \forall k \in K.
\end{array}\right.
$$

Proof. It is obvious that $(\mathrm{OS})_I$ has a unique solution, $p_{k_\varepsilon^*}^\varepsilon$, since it represents $(VP)_{k_\varepsilon^*}^\varepsilon$.

The uniqueness of the solution $Q_{k_\varepsilon^*}^\varepsilon$ of $(\mathrm{OS})_{II}$ is given by the general results of Chicco (1970).

We denote by $q^{*\varepsilon}$ the function given by Lemma 3.5, corresponding to $k_0 = k_\varepsilon^*$. It is obvious that:

$$\int_D q^{*\varepsilon}\,dxdy \geq 0. \tag{3.15}$$

By taking $\varphi = q^{*\varepsilon}$ in the variational formulation of $(\mathrm{OS})_{II}$, $\varphi = Q_{k_\varepsilon^*}^\varepsilon$ in the variational formulation of (3.12) for $k_0 = k_\varepsilon^*$ and by using (3.15), we obtain $(\mathrm{OS})_{III}$, which completes the proof.

The next section deals with the finite element approximation of (OS).

4. The approximation of the control system

Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\bar{D}$ and let $K_h \times V_h \times H_h$ be an internal approximation of $K \times V \times H$ (see Glowinski, Lions and Tremolieres (1981)), where:

$$(4.1)\quad \begin{cases} V = \{v \in H^1(D) \,/\, v = f \text{ on } S_2 \cup S_3\}, \\ H = \{v \in H^1(D) \,/\, v = 0 \text{ on } S_2 \cup S_3\}. \end{cases}$$

We consider the discrete optimality system:

$$(4.2)\quad \begin{cases} (k_h^*, p_h^*, Q_h^*) \in K_h \times V_h \times H_h, \\ \int_D k_h^*(\nabla p_h^* \cdot \nabla \varphi_h + H_\varepsilon(p_h^*)\frac{\partial \varphi_h}{\partial y})dxdy = 0 \;\forall \varphi_h \in H_h, \\ \int_D k_h^*(\nabla Q_h^* \cdot \nabla \varphi_h + H_\varepsilon'(p_h^*)\frac{\partial Q_h^*}{\partial y}\varphi_h)dxdy = -\int_D \varphi_h dxdy \;\forall \varphi_h \in H_h, \\ \int_D (\nabla p_h^* \cdot \nabla Q_h^* + H_\varepsilon(p_h^*)\frac{\partial Q_h^*}{\partial y})(k_h - k_h^*)dxdy \geq 0 \;\forall k_h \in K_h. \end{cases}$$

(4.2) approximates the optimality system (OS) in the sense given by the next theorem:

Theorem 4.1 *There exists a subsequence $\{(k_{h_m}^*, p_{h_m}^*, Q_{h_m}^*)\}_{m\in\mathbb{N}}$ such that: $k_{h_m}^* \to k_\varepsilon^*$ weakly in $H^1(D)$, $k_{h_m}^* \to k_\varepsilon^*$ weakly star in $L^\infty(D)$, $k_{h_m}^* \to k_\varepsilon^*$ a.e. in D, $p_{h_m}^* \to p_\varepsilon^*$ strongly in $H^1(D)$, $Q_{h_m}^* \to Q_\varepsilon^*$ strongly in $H^1(D)$, when $m \to \infty$ and $(k_\varepsilon^*, p_\varepsilon^*, Q_\varepsilon^*)$ is solution for (OS).*

Proof. The assertions of the theorem concerning $\{k_{h_m}^*\}_{m\in\mathbb{N}}$ are a consequence of the fact that $\{k_h^*\}_{h>0} \subset K_h \subset K$.

For $\varphi_{h_m} = p_{h_m}^* - v_{h_m}$, with $\{v_{h_m}\}_{m\in\mathbb{N}} \subset V_{h_m}$ a strongly convergent sequence in $H^1(D)$, $(4.2)_2$ gives the boundedness in $H^1(D)$ of $\{p_{h_m}^*\}_{m\in\mathbb{N}}$ and, hence, the existence of a weak limit point in $H^1(D)$, denoted p_ε^*. We can now pass to the limit, on a subsequence, in $(4.2)_2$ and we obtain that p_ε^* satisfies $(OS)_I$; therefore $p_\varepsilon^* = p_{k_\varepsilon^*}^\varepsilon$. From the uniqueness of the solution of $(OS)_I$ we deduce that $\{p_{h_m}^*\}_{m\in\mathbb{N}}$ has a unique limit point. We also obtain from $(OS)_I$:

$$(4.3)\quad \int_D k_\varepsilon^*(\nabla p_{k_\varepsilon^*}^\varepsilon \cdot \nabla \varphi_h + H_\varepsilon(p_{k_\varepsilon^*}^\varepsilon)\frac{\partial \varphi_h}{\partial y})dxdy = 0 \;\forall \varphi_h \in H_h.$$

By computing $(4.2)_2$–(4.3) for $h = h_m$, with $\varphi_{h_m} = p_{h_m}^* - v_{h_m}$, $\{v_{h_m}\}_{m\in\mathbb{N}} \subset$

V_{h_m}, $v_{h_m} \to v$ strongly in $H^1(D)$ when $m \to \infty$, we get:

$$(4.4)\quad \begin{aligned}&\int_D k^*_{h_m}|\nabla(p^*_{h_m} - p^\varepsilon_{k^*_\varepsilon})|^2 dxdy = -\int_D k^*_{h_m}\nabla(p^*_{h_m} - p^\varepsilon_{k^*_\varepsilon})\cdot\nabla p^\varepsilon_{k^*_\varepsilon} dxdy\\&+\int_D k^*_{h_m}\nabla(p^*_{h_m} - p^\varepsilon_{k^*_\varepsilon})\cdot\nabla v_{h_m} dxdy - \int_D (k^*_{h_m} - k^*_\varepsilon)\nabla p^\varepsilon_{k^*_\varepsilon}\cdot\nabla(p^*_{h_m} - v_{h_m})dxdy\\&-\int_D (k^*_{h_m} - k^*_\varepsilon)H_\varepsilon(p^*_{h_m})\frac{\partial}{\partial y}(p^*_{h_m} - v_{h_m})dxdy\\&-\int_D k^*_\varepsilon(H_\varepsilon(p^*_{h_m}) - H_\varepsilon(p^\varepsilon_{k^*_\varepsilon}))\frac{\partial}{\partial y}(p^*_{h_m} - v_{h_m})dxdy.\end{aligned}$$

By using the properties of $\{k^*_{h_m}\}_{m\in\mathbb{N}}$, $\{p^*_{h_m}\}_{m\in\mathbb{N}}$, $\{v_{h_m}\}_{m\in\mathbb{N}}$ and of the function H_ε it follows that for $m \to \infty$ the right member vanishes. Hence (4.4) gives $p^*_{h_m} \to p^\varepsilon_{k^*_\varepsilon}$ strongly in $H^1(D)$, when $m \to \infty$.

We prove next that $\{Q^*_{h_m}\}_{m\in\mathbb{N}}$ is bounded in $H^1(D)$. If $\{Q^*_{h_m}\}_{m\in\mathbb{N}}$ is bounded in $L^2(D)$, we obtain, from $(4.2)_3$, with $\varphi_{h_m} = Q^*_{h_m}$, the boundedness of $\{Q^*_{h_m}\}_{m\in\mathbb{N}}$ in $H^1(D)$. If $\{Q^*_{h_m}\}_{m\in\mathbb{N}}$ is not bounded in $L^2(D)$, we can extract a subsequence, denoted also by $\{Q^*_{h_m}\}_{m\in\mathbb{N}}$ with $\|Q^*_{h_m}\|_{L^2(D)} \to \infty$ when $m \to \infty$. We define $R_m = \dfrac{Q^*_{h_m}}{\|Q^*_{h_m}\|_{L^2(D)}}$. It is obvious that $\|R_m\|_{L^2(D)} = 1$ and $\{R_m\}_{m\in\mathbb{N}}$ is bounded in $H^1(D)$.

Multiplying $(4.2)_3$ for $h = h_m$ with $\dfrac{1}{\|Q^*_{h_m}\|_{L^2(D)}}$ we obtain:

$$(4.5)\quad \begin{aligned}&\int_D k^*_{h_m}(\nabla R_m\cdot\nabla\varphi_{h_m} + H'_\varepsilon(p^*_{h_m})\frac{\partial R_m}{\partial y}\varphi_{h_m})dxdy =\\&-\frac{1}{\|Q^*_{h_m}\|_{L^2(D)}}\int_D \varphi_{h_m} dxdy\ \forall\varphi_{h_m}\in H_{h_m}.\end{aligned}$$

For passing to the limit in (4.5), we use the properties: the embedding $H^1(D)\subset L^p(D)\ \forall\ 1\le p<\infty$ is compact, $k^*_{h_m}\to k^*_\varepsilon$ a.e. in D, $\varphi_{h_m}\to\varphi$ strongly in $H^1(D)$, on a subsequence $R_m\to R$ weakly in $H^1(D)$ and $H'_\varepsilon(p^*_{h_m})\to H'_\varepsilon(p^*_\varepsilon)$ strongly in $L^4(D)$. The last assertion is a consequence of the regularity of the function H_ε. Passing to the limit with $m\to\infty$ in (4.5), we get:

$$(4.6)\quad \int_D k^*_\varepsilon(\nabla R\cdot\nabla\varphi + H'_\varepsilon(p^*_\varepsilon)\frac{\partial R}{\partial y}\varphi)dxdy = 0.$$

Combining (4.6) with $R = 0$ on $S_2\cup S_3$, we obtain $R = 0$ in D i.e. a contradiction with $\|R\|_{L^2(D)} = 1$. Hence $\{Q^*_{h_m}\}_{m\in\mathbb{N}}$ is bounded in $H^1(D)$, which ensures the existence of a weak limit point in $H^1(D)$, Q^*_ε. Passing to the limit, as in (4.5), on a subsequence, in $(4.2)_3$ for $h = h_m$ we obtain that Q^*_ε is the unique solution of $(OS)_{II}$, i.e. $Q^*_\varepsilon = Q^*_{k^*_\varepsilon}$. From the uniqueness of the solution of $(OS)_{II}$ we obtain that $\{Q^*_{h_m}\}_{m\in N}$ has a unique weak limit point. With a similar technique as in the first part of the proof, we obtain $Q^*_{h_m} \to Q^*_\varepsilon$ strongly in $H^1(D)$. Finally, by passing to the limit in (4.2) for $h = h_m$ it follows that $(k^*_\varepsilon, p^*_\varepsilon, Q^*_\varepsilon)$ satisfies (OS), which completes the proof.

In order to solve (4.2), we propose the following algorithm: for $k_{h0} \in K_h$ given, for any $m \in N^*$ and for a suitable choice of a positive number ρ_m, we define $(k^*_{hm+1}, p^*_{hm}, Q^*_{hm}) \in K_h \times V_h \times H_h$ as a solution of the following problem:

$$(4.7)\begin{cases} \int_D k^*_{hm}(\nabla p^*_{hm}\cdot\nabla\varphi_h + H_\varepsilon(p^*_{hm})\frac{\partial\varphi_h}{\partial y})dxdy = 0\ \forall\varphi_h \in H_h, \\ \int_D k^*_{hm}(\nabla Q^*_{hm}\cdot\nabla\varphi_h + H'_\varepsilon(p^*_{hm})\frac{\partial Q^*_{hm}}{\partial y}\varphi_h)dxdy = -\int_D \varphi_h dxdy\ \forall\varphi_h \in H_h, \\ k^*_{hm+1} = \begin{cases} P_{K_h}(k^*_{hm} - \rho_m\frac{f_{hm}}{\|f_{hm}\|_{L^2(D)}}) & if\ \|f_{hm}\|_{L^2(D)} \neq 0, \\ k^*_{hm} & if\ \|f_{hm}\|_{L^2(D)} = 0. \end{cases} \end{cases}$$

where $f_{hm} = \nabla p^*_{hm}\cdot\nabla Q^*_{hm} + H_\varepsilon(p^*_{hm})\frac{\partial Q^*_{hm}}{\partial y}$ and P_{K_h} is the projection map of the internal approximation of $L^2(D)$ on K_h. The projection map can be defined since K is a closed, convex subset of $L^2(D)$.

Proposition 4.2 The sequence $\{(k^*_{hm}, p^*_{hm}, Q^*_{hm})\}_{m\in N}$ is convergent to a solution of $(4.2)_1-(4.2)_3$, for ρ_m chosen with $\rho_m \to 0$ when $n \to \infty$.

Proof. From $(4.7)_3$ and from the properties of the projection map it follows:

$$\|k^*_{hm+1} - k^*_{hm}\|_{L^2(D)} \leq \rho_m\ \forall\, m \in N.$$

Hence the sequence $\{k^*_{hm}\}_{m\in N}$ is strongly convergent in $L^2(D)$ to k^*_h. It can be proved as in Theorem 4.1 that $p^*_{hm} \to p^*_h$, $Q^*_{hm} \to Q^*_h$ strongly in $H^1(D)$ when $m \to \infty$, with (k^*_h, p^*_h, Q^*_h) satisfying $(4.2)_1-(4.2)_3$.

It is obvious that the interest is to obtain a sequence $\{k^*_{hm}\}_{m\in N}$ which approximate a discrete optimal control k^*_h. This is possible if we choose ρ_m such that

$$J_\varepsilon(P_{K_h}(k^*_{hm} - \rho_m\frac{f_{hm}}{\|f_{hm}\|_{L^2(D)}})) \leq J_\varepsilon(k^*_{hm}).$$

For any $m \in \mathbb{N}^*$, we define $\rho_m > 0$ such that: $\alpha \leq k^*_{hm} - \rho_m \frac{f_{hm}}{\|f_{hm}\|_{L^2(D)}} \leq \beta$; $\frac{\partial}{\partial y}(k^*_{hm} - \rho_m \frac{f_{hm}}{\|f_{hm}\|_{L^2(D)}}) \geq 0$ in D, $\|k^*_{hm} - \rho_m \frac{f_{hm}}{\|f_{hm}\|_{L^2(D)}}\|_{H^1(D)} \leq r$.

We finish this section with the remark that the nonlinear problem $(4.7)_1$ was solved in Stavre and Vernescu (1989) for a less regular H_ε.

5. Numerical results

Our numerical tests have been performed for $D = (0, a) \times (0, h_1)$. We are interested in comparing the results for different values of $k_{h0} \in K_h$ and for different grids of the domain D. Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\bar{D}$ such that $\bar{D} = \bigcup_{T \in \mathcal{T}_h} T$, the finite elements T being triangles, as in Fig. 1. Let Σ_h be the set of mesh points in $\bar{D}$. V_h, H_h and K_h of Section 4 are given by:

$$\begin{cases} V_h = \{v_h \in C^0(\bar{D}) \,/\, v_h(n_i) = f(n_i)\ \forall n_i \in \Sigma_h \cap (S_2 \cup S_3),\ v_h/_T \in P_1\ \forall T \in \mathcal{T}_h\}, \\ H_h = \{v_h \in C^0(\bar{D}) \,/\, v_h(n_i) = 0\ \forall n_i \in \Sigma_h \cap (S_2 \cup S_3),\ v_h/_T \in P_1\ \forall T \in \mathcal{T}_h\}, \\ K_h = K \cap \{v_h \in C^0(\bar{D}) \,/\, v_h/_T \in P_1\ \forall T \in \mathcal{T}_h\}. \end{cases}$$

Fig. 1 →

The aim of the first experiment is to compare the minimum values of the functional J_ε, the expressions of the pressure and the expressions of the permeability coefficient for different values of k_{h0}. The data common to all runs in the first experiment are: $a = 4$, $h_1 = 5$, $h_2 = 1.5$, $\alpha = 1$, $\beta = 50$, $r = 100$, $\varepsilon = 0.1$, for a mesh size $h = \Delta x = \Delta y = 0.25$. The following expressions of k_{h0} have been considered:

$$k^1_{h0}(x, y) = \begin{cases} 4, & if\ y \in [0, \frac{h_1}{2}), \\ 10, & if\ y \in [\frac{h_1}{2}, h_1], \end{cases} \qquad k^2_{h0}(x, y) = \begin{cases} 4, & if\ y \in [0, \frac{h_1}{2}), \\ 4(y - \frac{h_1}{2} + 1), & if\ y \in [\frac{h_1}{2}, h_1], \end{cases}$$

$$k^3_{h0}(x, y) = \begin{cases} 2, & if\ x \in [0, \frac{a}{2}), \\ 2(x - \frac{a}{2} + 1), & if\ x \in [\frac{a}{2}, a], \end{cases} \qquad k^4_{h0}(x, y) = 4.$$

In each case, the computed values of the functional J_ε decrease from one iteration to another. The CPU time for one iteration was 75 seconds and satisfactory convergence was obtained after 10-15 iterations. In all these cases, we obtained almost the same nodal values for the pressure; the minimum values of J_ε are contained in the interval [14.76; 14.82]. The expressions of the computed permeability coefficient which gives the minimum of J_ε, $k_{h_{min}}$, were different for different k_{h0}, but the ratio $k^l_{h_{min}}/k^l_{h0}$, $l = 1, ..., 4$, was almost the same. We give below the nodal values of $k_{h_{min}}$ corresponding to k^2_{h0}.

2.24	2.64	3.13	3.43	3.63	3.78	3.89	3.97	4.03	4.07	4.09	4.08	4.06	3.99	3.82	3.48	3.28
2.55	2.92	3.24	3.48	3.66	3.80	3.90	3.98	4.03	4.07	4.10	4.10	4.09	4.05	3.95	3.79	3.54
2.88	3.09	3.33	3.52	3.68	3.80	3.90	3.98	4.03	4.07	4.10	4.10	4.09	4.05	3.96	3.85	3.73
3.08	3.23	3.40	3.56	3.70	3.81	3.90	3.98	4.03	4.07	4.10	4.10	4.09	4.05	3.98	3.86	3.89
3.22	3.33	3.47	3.60	3.72	3.82	3.90	3.98	4.03	4.07	4.10	4.10	4.09	4.05	3.99	3.97	4.03
3.32	3.41	3.53	3.64	3.74	3.83	3.90	3.98	4.03	4.07	4.10	4.10	4.09	4.05	4.04	4.08	4.17
3.40	3.47	3.57	3.67	3.76	3.84	3.90	3.99	4.03	4.07	4.10	4.10	4.09	4.06	4.09	4.17	4.26
3.46	3.52	3.61	3.70	3.78	3.85	3.91	3.99	4.04	4.08	4.11	4.11	4.09	4.09	4.09	4.17	4.26
3.50	3.55	3.64	3.72	3.80	3.86	3.92	3.99	4.04	4.08	4.11	4.11	4.09	4.09	4.09	4.17	4.26
3.52	3.57	3.67	3.75	3.82	3.87	3.92	4.00	4.05	4.08	4.11	4.11	4.10	4.09	4.09	4.18	4.26
3.53	3.59	3.69	3.77	3.84	3.91	3.92	4.00	4.05	4.08	4.11	4.11	4.10	4.09	4.09	4.18	4.26
4.54	4.61	4.72	4.79	4.85	4.89	4.92	4.95	4.97	4.99	5.00	5.02	5.03	5.03	5.03	5.02	5.01
5.58	5.66	5.75	5.81	5.86	5.90	5.92	5.95	5.97	5.98	6.00	6.01	6.02	6.02	6.02	6.01	5.01
6.64	6.71	6.78	6.83	6.87	6.91	6.93	6.95	6.97	6.98	6.99	7.00	7.01	7.01	7.01	7.02	7.03
7.70	7.76	7.82	7.86	7.89	7.92	7.94	7.96	7.97	7.98	7.99	8.00	8.00	8.01	8.03	8.03	8.06
8.76	8.81	8.85	8.88	8.91	8.93	8.95	8.96	8.97	8.98	8.98	8.98	8.98	8.99	9.02	9.05	9.07
9.82	9.86	9.89	9.91	9.92	9.94	9.96	9.97	9.98	9.99	9.98	9.98	9.98	10.00	10.03	10.06	10.07
10.87	10.90	10.92	10.93	10.94	10.95	10.97	10.99	11.00	11.00	11.00	11.00	11.00	11.01	11.03	11.05	11.06
11.93	11.95	11.94	11.94	11.95	11.96	11.98	12.01	12.03	12.04	12.05	12.05	12.05	12.04	12.03	12.04	12.03
12.98	12.99	12.98	12.97	12.96	12.96	12.96	12.96	12.96	12.96	12.97	12.98	13.00	13.02	13.04	13.03	13.02
14.05	14.14	14.25	14.35	14.44	14.51	14.58	14.62	14.64	14.62	14.54	14.43	14.29	14.16	14.07	14.02	14.00

As it can be seen, the differences between $k_{h_{min}}$ and k_{h0} are greater near S_3. There exists a good reason for this: the pressure of the fluid, which must be minimized, has the greatest values near the boundary in contact with the reservoirs.

The purpose of the second experiment is to compare the minimum values of J_ε for two different grids. We took: $a = 1.5$, $h_1 = 2.5$, $h_2 = 1.2$, $\alpha = 1$, $\beta = 50$, $r = 100$, $\varepsilon = 0.1$ and $k_{h0} = k_{h0}^2$. The two different values of the mesh size h were 0.25 and 0.1. In the first case the minimum of J_ε was 2.21554 and in the second one, 2.22281.

For all the examples, the stopping test was:

$$|k_{hm+1}(n_i) - k_{hm}(n_i)| \leq 0.01 \; \forall n_i \in \Sigma_h.$$

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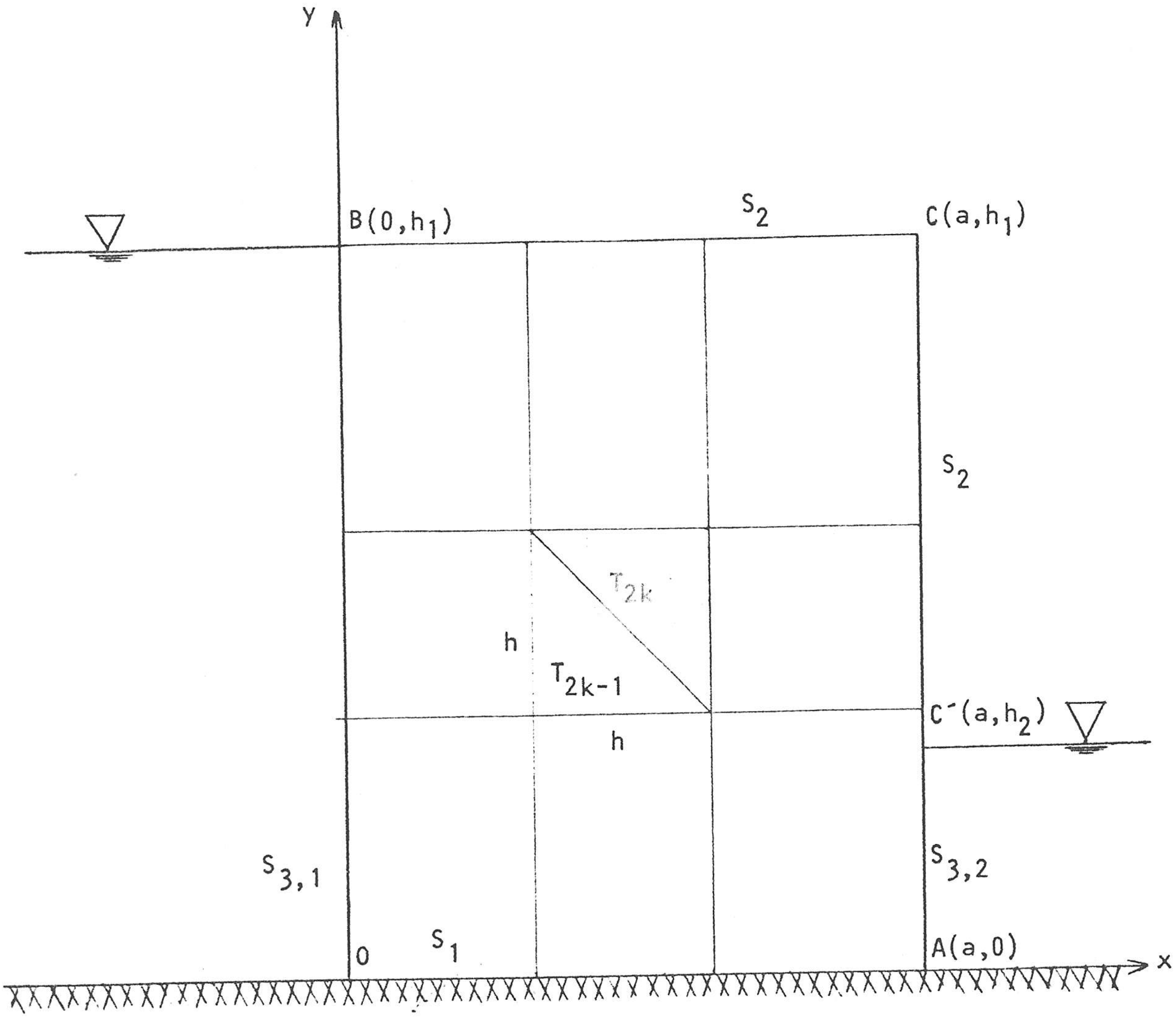

Fig. 1 The finite element mesh