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1 Introduction

It is well known that for a Noetherian ring R, an ideal I of R and M a finitely generated R-module, the local cohomology modules $H_I^i(M)$ are not always finitely generated. On the other hand if R is local and $\underline{m}$ its maximal ideal then $H_{\underline{m}}^i(M)$ are Artinan modules, which is the same thing to say that:

(i) $\mathrm{Supp}_R(H_{\underline{m}}^i(M)) \subseteq \{\underline{m}\}$, and

(ii) The vector space $\mathrm{Hom}_R(k, H_{\underline{m}}^i(M))$ has finite dimension over k where $k = R/\underline{m}$.

Taking account of these facts, Grothendieck [5] made the following conjecture:

Conjecture 1.1 *If I is an ideal of a local Noetherian ring R and M a finitely generated R-module, then* $\mathrm{Hom}_R(R/I, H_I^i(M))$ *is finitely generated.*

Later, Hartshorne [7] refined this conjecture and asked the following more general question:

Conjecture 1.2 *If I is an ideal of a Noetherian local ring R and M a finitely generated R-module, does it follow that* $\mathrm{Ext}^i_R(R/I, H^j_I(M))$ *are finitely generated for all* $i, j \geq 0$ *?*

In the same paper, Hartshorne gave the following general definition:

Definition 1.3 *If I is an ideal of a Noetherian ring R, then a module N will be called I-cofinite if it satisfies the following conditions:*

(i) $\mathrm{Supp}_R(N) \subseteq V(I)$

(ii) $\mathrm{Ext}^i_R(R/I, N)$ *is finitely generated for all* $i \geq 0$.

In Grothendieck's definition of cofiniteness, in (ii) it was asked only that $\mathrm{Hom}_R(R/I, N)$ to be finitely generated. Hartshorne's definition is motivated upon the fact that, if we have a short exact sequence of modules, in which two of them are I-cofinite, in the sense of Grothendieck, then does not result that the third has the same property, due to the presence of an Ext^1. Hartshorne's definition is fulfiled, for instance, in the case when the ideal I coincides with the maximal ideal of a local ring, since in this case not only the socle $\mathrm{Hom}_R(k, H^i_{\underline{m}}(M))$ is finitely generated but all $\mathrm{Ext}^j_R(k, H^i_{\underline{m}}(M))$ are finitely generated for all i and j, as we can see by using Matlis duality.

He also gave a counterexample to (1.1) which essentially is the following: let k be field and let $R = k[[X, Y, Z, U]]$ and $I = (X, U)R$. If we take $M = R/(XY - ZU)$ then $H^2_I(M)$ is not I-cofinite. In fact he proved much more than this, namely $\mathrm{Hom}_R(k, H^2_I(M))$ is not finitely generated. If we denote by $A = k[[X, Y, Z, U]]/(XY - ZU)$ and by $J = (X, U)A$ then the counterexample of Hartshorne says that $\mathrm{Hom}_A(A/J, H^2_J(A))$ is not finitely generated. So, even in the case of a complete intersection ring A, $H^i_J(A)$ can not be J-cofinite, for all $i \geq 0$. Nonetheless by using the derived category theory he proved that if R is complete regular local ring, then $H^i_I(M)$ is I- cofinite in two cases:

(i) I is a non-zero principal ideal, and

(ii) I is a prime ideal with $\dim(R/I) = 1$

In [10] Huneke and Koh generalized the above result to the case of a Gorenstein complete domain, and an arbitrary one-dimensional ideal.

In [3] Delfino extends the above result to a complete ring, under some restrictions and finally, Marley and Delfino [11] proved the general case:

Theorem 1.4 *Let R be a Noetherian local ring, I a dimension one ideal of R, and M a finitely generated Rmodule. Then $H_I^i(M)$ is I-cofinite for all i.*

The purpose of this paper is to give a new proof of this result. We are trying to avoid, as far as possible, the use of spectral sequence, a technique adopted both by Hartshorne and Huneke-Koh. The hard part of the proof is to show, in the case of a complete regular ring, that $H_I^{d-1}(M)$ is I-cofinite for a finitely generated module M. The reduction to a complete regular local ring uses the same idea as in [3] or [11].

The terminology used is standard and follows [9] and [13].

2 Background material

We will use the following remarkable result, known as Hartshorne-Lichtenbaum vanishing theorem (HLVT in short):

Theorem 2.1 *Let $(R, \underline{m})$ a Noetherian ring of Krull dimension d and I an ideal of R. Then the following are equivalent:*

(i) $H_I^d(R) = 0$.

(ii) $\dim(\hat{R}/I\hat{R} + P) > 0$, *for all prime ideals P of the $\underline{m}$ - adic completion $\hat{R}$ such that* $\dim(\hat{R}/P) = d$.

Remark 2.2 For a complete domain the condition (ii) is equivalent to the fact that I is not $\underline{m}$-primary.

We shall use the following standard fact:

Proposition 2.3 *Let R be a Noetherian ring and I an ideal of R and m a natural number such that $H_I^i(R) = 0$ for all $i > m$. Then there is the following isomorphism:*

$$H_I^m(M) \simeq M \otimes_R H_I^m(R)$$

for all R-modules M.

Proof. The condition $H_I^i(R) = 0$ for all $i > m$ is equivalent to $H_I^i(M) = 0$ for all $i > m$ and all R-modules M (cf.[8]). This implies that the functor $H_I^m(-)$ is right exact and so by [13] (Th.3.33), we infer that $H_I^m(M) \simeq M \otimes_R H_I^m(R)$, for all R-modules M. ∎

To prove the finiteness of some Ext, we shall often check, by using Matlis duality, that the Matlis dual of it is an Artinian module, which is similar to checking that some Tor is an Artinian module.

In this sense the following proposition is fundamental:

Proposition 2.4 *Let R be a commutative ring, M and N two R-modules and I an injective R-module, then we have the following isomorphism:*

$$\mathrm{Hom}_R(\mathrm{Tor}_i^R(M, N), I) \simeq \mathrm{Ext}_R^i(M, \mathrm{Hom}_R(N, I))$$

In addition, if R is Noetherian and M is finitely generated, then

$$\mathrm{Hom}_R(\mathrm{Ext}_R^i(M, N), I) \simeq \mathrm{Tor}_i^R(M, \mathrm{Hom}_R(N, I))$$

A proof of this result can be found in [6] Prop.VI.5.1 or [14]

Lemma 2.5 *Let R be a Noetherian ring, S a multiplicative closed set such that $\underline{m} \cap S \neq \phi$ and N a R_S-module. Then* $\mathrm{Hom}_R(k, N) = 0$.

Proof. We have the following isomorphisms

$$\mathrm{Hom}_R(k, N) \simeq \mathrm{Hom}_R(k, \mathrm{Hom}_{R_S}(R_S, N)) \simeq \mathrm{Hom}_{R_S}(k \otimes_R R_S, N)$$

But the last module is zero since, by hypothesis we have $\underline{m} R_S = R_S$. ∎

Lemma 2.6 *Let A be a Noetherian ring and $A \xrightarrow{\varphi} B$ a faithfully flat A-algebra. Then for any finitely generated A-module N and each A-module M, such that $M \otimes_A coker(\varphi) = 0$, we have:*

$$\mathrm{Tor}_i^A(M, \mathrm{Ext}_A^j(N, coker(\varphi))) = 0$$

and

$$\mathrm{Tor}_i^A(M, \mathrm{Tor}_A^j(N, coker(\varphi))) = 0$$

for all $i, j \geq 0$.(For the second equality is not necesary to assume that N is finitely generated.)

Proof. Since B is a faithfully flat A-algebra we infer that A is a pure submodule of B and so $coker(\varphi)$ is a flat A-module. Now we have the following canonical isomorphisms: (see [1], Prop.7 and Prop.8 pg.108-109)

$$\begin{aligned}\mathrm{Tor}_i^A(M, \mathrm{Ext}_A^j(N, coker(\varphi))) &\simeq \mathrm{Tor}_i^A(M, \mathrm{Ext}_A^j(N, A) \otimes_A coker(\varphi)) \\ &\simeq \mathrm{Tor}_i^A(M \otimes_A coker(\varphi), \mathrm{Ext}_A^j(N, A))\end{aligned}$$

But the last module is zero by hypothesis. For the second equality a similar proof works. (In the above isomorphisms we used the flatness of $coker(\varphi)$). ■

The proof of the following lemma is straightforward, so we'll omit it.

Lemma 2.7 *Let $(R, \underline{m}, k)$ be a Noetherian local ring and*

$$M \longrightarrow N \longrightarrow L \longrightarrow P$$

an exact sequence of R-modules, where M and P are finitely generated. If T is a finitely generated R-module then $\mathrm{Hom}_R(T, N)$ is finitely generated if and only if $\mathrm{Hom}_R(T, L)$ has the same property.

3 The proof of (1.4) in the regular case

The reduction to the regular case is the same as in [3] or [11]. So, we prove the theorem only in the case when R is regular.

We need the following lemma:

Lemma 3.1 *Let $(R, \underline{m}, k)$ be a complete Gorenstein ring of Krull dimension d, and I a radical ideal of it such that $\dim(R/I) = 1$ and $H_I^d(R) = 0$. If $\{P_1, P_2, \ldots, P_n\}$ is the set of minimal prime ideals over I and $S = R \setminus \bigcup_{i=0}^{n} P_i$, then there exists the following two exact sequences*

$$0 \longrightarrow R \xrightarrow{\Phi_S} \widehat{R_S} \longrightarrow H^{\vee} \longrightarrow 0$$

which gives by localization

$$0 \longrightarrow R_S \xrightarrow{\Psi_S} \widehat{R_S} \longrightarrow (H^{\vee})_S \longrightarrow 0$$

where $H^{\vee}$ means the Matlis dual of $H_I^{d-1}(R)$ and Ψ_S is the canonical homomorphism of completion.

Proof. For any element f of S and any natural number n, we have the following exact sequence, which arises from Čech complex applied to the R-module R/I^n and to the principal ideal (f) :

$$0 \longrightarrow H^0_{(f)}(R/I^n) \longrightarrow R/I^n \xrightarrow{\Phi_f^n} (R/I^n)_f \longrightarrow H^1_{(f)}(R/I^n) \longrightarrow 0$$

By the base ring independence of local cohomology, applied to the canonical homomorphism $R \to R/I^n$ and to the ideal (f), we deduce the exact sequence:

$$0 \longrightarrow H^0_{(I^n,f)}(R/I^n) \longrightarrow R/I^n \xrightarrow{\Phi_f^n} (R/I^n)_f \longrightarrow H^1_{(I^n,f)}(R/I^n) \longrightarrow 0$$

Since f is an element of S it results that the radical of (I^n, f) is the maximal ideal and taking the direct limit over all elements f of S the above exact sequence becomes:

$$0 \longrightarrow H^0_{\underline{m}}(R/I^n) \longrightarrow R/I^n \xrightarrow{\Phi_S^n} (R/I^n)_S \longrightarrow H^1_{\underline{m}}(R/I^n) \longrightarrow 0$$

All modules of the above sequence define inverse sistems which satisfie Mittag-Leffler condition so, taking inverse limit, we obtain

$$0 \longrightarrow \varprojlim_n H^0_{\underline{m}}(R/I^n) \longrightarrow R \xrightarrow{\Phi_S} \widehat{R_S} \longrightarrow \varprojlim_n H^1_{\underline{m}}(R/I^n) \longrightarrow 0$$

By local duality we know that $H^i_{\underline{m}}(N) \simeq \mathrm{Hom}_R(\mathrm{Ext}^{d-i}_R(N,R),E)$, for any finitely generated R-module N, where E denotes the injective hull of k, so we get the following isomorphisms:

$$\begin{aligned} \varprojlim_n H^i_{\underline{m}}(R/I^n) &\simeq \varprojlim_n \mathrm{Hom}_R(\mathrm{Ext}^{d-i}_R(R/I^n,R),E) \\ &\simeq \mathrm{Hom}_R(\varinjlim_n \mathrm{Ext}^{d-i}_R(R/I^n,R),E) \\ &\simeq \mathrm{Hom}_R(H^{d-i}_I(R),E) \end{aligned}$$

Since $H^d_I(R) = 0$ we get the exact sequence:

$$0 \longrightarrow R \xrightarrow{\Phi_S} \widehat{R_S} \longrightarrow H^{\vee} \longrightarrow 0$$

It's clear that the homomorphism Φ_S is obtained as the composition of the continuous ring homomorphisms $R \to R_S \xrightarrow{\Psi_S} \widehat{R_S}$, where the first homomorphism is the localization with respect to S and Ψ_S is the completion homomorphism of the semilocal ring R_S with respect to the topology given by its Jacobson radical. ∎

We are ready to prove the main result of this section.

Proposition 3.2 *Let $(R,\underline{m},k)$ be a complete Gorenstein ring of Krull dimension d, I a radical ideal of it, such that $H^d_I(R) = 0$ and $\dim(R/I) = 1$. If M is a finitely generated R-module then $H^{d-1}_I(M)$ is I-cofinite.*

Proof. We have to show that $\mathrm{Ext}^i_R(R/I, H^{d-1}_I(M))$ is finitely generated for all $i \geq 0$, or, by using Matlis duality, this is similar to show that:

$$\mathrm{Ext}^i_R(R/I, H^{d-1}_I(M))^{\vee} \simeq \mathrm{Tor}^R_i(R/I, H^{d-1}_I(M)^{\vee})$$

is an Artinian module. By using a standard caracterisation of Artinian modules this is the same thing to prove the following two facts:

(i) $\mathrm{Hom}_R(k, \mathrm{Tor}_i^R(R/I, H_I^{d-1}(M)^\vee))$ is a finitely dimensional k-vector space.

(ii) $\mathrm{Supp}_R(\mathrm{Tor}_i^R(R/I, H_I^{d-1}(M)^\vee)) \subseteq \{\underline{m}\}$

Since $H_I^{d-1}(-)$ is right exact we have, by (2.3),

$$H_I^{d-1}(M)^\vee \simeq (M \otimes_R H)^\vee \simeq \mathrm{Hom}_R(M, H^\vee)$$

Firstly we'll prove (i). Since $\mathrm{Hom}_R(-, N_S) \simeq \mathrm{Hom}_{R_S}((-) \otimes_R R_S, N_S)$ for any R-module N and applying the functor $\mathrm{Hom}_R(M, -)$ to the first exact sequence of lemma (3.1), we get a long exact sequence of Ext's :

$$0 \longrightarrow M^* \longrightarrow \mathrm{Hom}_{R_S}(M_S, \widehat{R_S}) \xrightarrow{\alpha} H_I^{d-1}(M)^\vee \xrightarrow{\beta} \mathrm{Ext}_R^1(M, R) \longrightarrow \cdots$$

where M^* is $\mathrm{Hom}_R(M, R)$ and S is defined as in (3.1). The above sequence breaks up into short exact sequences from which we keep only the first two of them:

$$0 \longrightarrow M^* \longrightarrow \mathrm{Hom}_{R_S}(M_S, \widehat{R_S}) \longrightarrow U \longrightarrow 0$$

and

$$0 \longrightarrow U \longrightarrow H_I^{d-1}(M)^\vee \longrightarrow V \longrightarrow 0$$

where $U := Im(\alpha)$ and $V := Im(\beta)$. Note that V is finitely generated.

Tensoring the second exact sequence with R/I, we get a long exact sequence of Tor's :

$$\cdots \mathrm{Tor}_{i+1}^R(R/I, V) \to \mathrm{Tor}_i^R(R/I, U) \to \mathrm{Tor}_i^R(R/I, H_I^{d-1}(M)^\vee) \to \mathrm{Tor}_i^R(R/I, V) \cdots$$

Since $\mathrm{Tor}_i^R(R/I, V)$ are finitely generated for all $i \geq 0$, by (2.7) it results that the k-vector space $\mathrm{Hom}_R(k, \mathrm{Tor}_i^R(R/I, U))$ is finitely generated if and only if $\mathrm{Hom}_R(k, \mathrm{Tor}_i^R(R/I, H_I^{d-1}(M)^\vee))$ has the same property.

Tensoring with R/I the first exact sequence we obtain:

$$\mathrm{Tor}_{i+1}^R(R/I, M^*) \to \mathrm{Tor}_i^R(R/I, \mathrm{Hom}_{R_S}(M_S, \widehat{R_S})) \to \mathrm{Tor}_i^R(R/I, U) \to \mathrm{Tor}_i^R(R/I, M^*)$$

and using (2.7) we infer that $\mathrm{Hom}_R(k, \mathrm{Tor}_i^R(R/I, U))$ is finitely generated if and only if $\mathrm{Hom}_R(k, \mathrm{Tor}_i^R(R/I, \mathrm{Hom}_{R_S}(M_S, \widehat{R_S})))$ has this property. But, by (2.5), the last module is zero and (i) is completely proved.

Now we'll prove (ii). Since:

$$\mathrm{Supp}_R(\mathrm{Hom}_R(\mathrm{Tor}_i^R(R/I, H_I^{d-1}(M)^{\vee})) \subseteq V(I)$$

to prove (ii) it will be sufficient to show that:

$$\mathrm{Tor}_i^{R_S}(R_S/I_S, \mathrm{Hom}_{R_S}(M_S, (H^{\vee})_S)) = 0$$

where we made the identifications:

$$((H_I^{d-1}(M))^{\vee})_S \simeq \mathrm{Hom}_R(M, H^{\vee})_S \simeq \mathrm{Hom}_{R_S}(M_S, (H^{\vee})_S)$$

Since $R_S \xrightarrow{\Psi_S} \widehat{R_S}$ is faithfully flat and since

$$R_S/I_S \otimes_{R_S} (H^{\vee})_S \simeq R_S/I_S \otimes_{R_S} coker(\Psi_S) = 0$$

(because $R_S/I_S \simeq \widehat{R_S}/\widehat{I_S}$), we may apply lemma (2.6) and (ii) is proved. ∎

Remark 3.3. Using a similar proof as above, and applying lemma (2.6), we can prove also that $\mathrm{Tor}_j^R(M, H)$ and $\mathrm{Ext}_R^j(M, H)$ are I-cofinite for all $j \geq 0$. We must observe that only $\mathrm{Hom}_R(M, H)$ and $\mathrm{Ext}_R^1(M, H)$ are non-zero modules, because $\mathrm{id}_R(H) = 1$, as we can see easily. In fact these are isomorphic with $\varinjlim_n \mathrm{Ext}_R^{d-1}(M/I^n M, R)$ and, respectively $\varinjlim_n \mathrm{Ext}_R^d(M/I^n M, R)$ as it can be seen by using some collapsing spectral sequence. This modules are, in Hartshorne's language of [7], nothing else but $H^{d-1}(D_I(M))$ and, respectively, $H^d(D_I(M))$.

We are ready to give the proof of (1.4) in the case when R is regular.

Proof. By faithfully flatness of the $\underline{m}$-adic completion $\hat{R}$, it's clear that a R-module T is I-cofinite if and only if $T \otimes_R \hat{R}$ is $I\hat{R}$-cofinite, and since:

$$\mathrm{Ext}_R^i(R/I, H_I^j(M)) \otimes_R \hat{R} \simeq \mathrm{Ext}_{\hat{R}}^i(\hat{R}/I\hat{R}, H_{I\hat{R}}^j(\widehat{M}))$$

we may assume that R is complete. On the other hand by, [3] or by [10], we can assume that I is a radical ideal. We'll proceed by induction on $\mathrm{pd}_R(M)$. If M is free then $H_I^i(M) = 0$ for all $i \neq d-1$ (since $\mathrm{depth}_I(R) = d-1$ and by HLVT we have $H_I^d(R) = 0$). By (3.2) $H_I^{d-1}(R)$ is I-cofinite and so in this case we are done.

Now, take a finite presentation of M : $0 \to N \to F \to M \to 0$, where F is a free module of finite rank and $\mathrm{pd}_R(N) < \mathrm{pd}_R(M)$. If we apply $\Gamma_I(-)$ to this exact sequence, and taking account that $H_I^i(F) = 0$ for all $i \neq d-1$ we'll obtain an exact sequence:

$$0 \longrightarrow H_I^{d-2}(M) \longrightarrow H_I^{d-1}(N) \xrightarrow{\mu} H_I^{d-1}(F) \longrightarrow H_I^{d-1}(M) \longrightarrow 0$$

and the isomorphisms $H_I^i(M) \simeq H_I^{i+1}(N)$ for all $i < d-2$. By induction we infer that $H_I^i(M)$ is I-cofinite for all $i < d-2$. If we break up the above exact sequence into two short exact sequence:

$$0 \longrightarrow H_I^{d-2}(M) \longrightarrow H_I^{d-1}(N) \longrightarrow Im(\mu) \longrightarrow 0$$

and

$$0 \longrightarrow Im(\mu) \longrightarrow H_I^{d-1}(F) \longrightarrow H_I^{d-1}(M) \longrightarrow 0$$

and since, by (3.2), $H_I^{d-1}(M)$ is I-cofinite, it results that $Im(\mu)$ is I-cofinite and finally we deduce that $H_I^{d-2}(M)$ is I-cofinite. ■

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