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NEW TYPES OF CODIMENSION-ONE AND -TWO BIFURCATIONS IN THE PLANE

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Abstract. By studying the two-dimensional FitzHugh-Nagumo (F-N) biodynamical system a double breaking saddle connection bifurcation was detected (Section 2). Numerical investigations of the bifurcation curves emerging from this point, in the parameter plane, allowed us to discover new types of codimension-one and -two bifurcations. They were coined by us saddle-node–saddle connection bifurcation and saddle-node–saddle with separatrix connection bifurcation respectively. The local bifurcation diagrams corresponding to these bifurcations are presented in Section 3. An analogy between the feature of bifurcation corresponding to the point of double homoclinic bifurcation and the point of double breaking saddle connection bifurcation is also presented in Section 3.

Keywords: codimention-one and -two bifurcations, dynamical systems, breaking saddle connection bifurcation, saddle-node bifurcation, FitzHugh-Nagumo.

AMS subject classification: 34A47

1. Some codimension-one and -two bifurcations for the F-N system

Consider the Cauchy problem $x(0) = x_0, y(0) = y_0$ for the FitzHugh-Nagumo (F-N) system [1]

$$\begin{aligned} \dot{x} &= c(x + y - \frac{x^3}{3}), \\ \dot{y} &= -\frac{1}{c}(x - a + by). \end{aligned} \tag{1.1}$$

Here $a, b, c \in \mathbf{R}$ are parameters, $x, y : \mathbf{R} \to \mathbf{R}$, $x = x(t), y = y(t)$ are the unknown functions, t is the independent variable and the dot over quantities stands for their rate of change. For $a = b = 0$, (1.1) becames the Van der Pol system having x as the main variable and y as the auxiliary Liénard variable.

The F-N model is associated with a two-dimensional time-continuous dynamical system. It has x and y as state variables and t plays the role of the time.

Since the transformation $(x, y) \to (-x, -y)$ corresponds to the phase space portraits for $-a$, only the case $a \geq 0$ will be considered. Our theoretical and numerical results, valid for fixed $c > 2$ and concerning the bifurcation for the F-N model, were sumarized in the global bifurcation diagram [2], [6]. In the following we present only those bifurcation manifolds concerning the new types of bifurcations. Their representation from Figure 1 is qualitative, due to the very small gap between some of the curves.

Thus, the saddle-node bifurcation takes place for values of the parameters (b, a) situated on the curves $S_{1,2}$ of equations

$$a = \pm\frac{2}{3}|b|\sqrt{(1 - \frac{1}{b})^3}, b \in (-\infty, 0) \cup [1, \infty). \tag{1.2}$$

The Hopf bifurcation takes place for values of the parameters (b, a) situated on the curves $H_{1,2}$ of equations

$$a = \pm\frac{b}{3}(-2 + \frac{3}{b} - \frac{b}{c^2})\sqrt{1 - \frac{b}{c^2}}, b \in (-c, c)\,. \tag{1.3}$$

The points of tangency of S_1, S_2 and H_1 and H_2 were denoted by Q_1, Q_2, Q_3, Q_4. At these points, the linearized system around the double equilibrium point has a double zero eigenvalue.

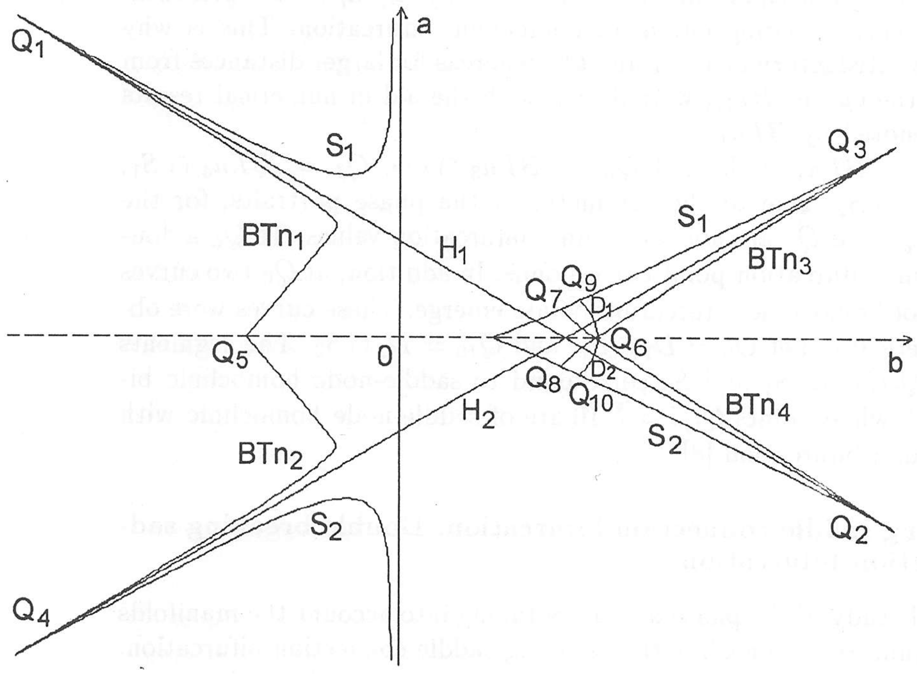


Fig. 1. Some bifurcation manifolds for the F-N system

The following theorem holds.

Theorem 1.1 [1] [7] *The system (1.1) with c fixed has at Q_1 resp. Q_3 a codimension-two bifurcation of Bogdanov-Takens type. At Q_1, resp. Q_3 one curve of homoclinic bifurcation values emerges. Its approximation around Q_1, resp. Q_3, is given by the curve BT_1*

$$a = \frac{7b^2 - 10bc^2 + 3c^2}{15c^3}\sqrt{\frac{7b^2 + 5bc^2 - 12c^2}{5b}}, \tag{1.4}$$

for $-c < b < 0$, resp. BT_3

$$a = \frac{7b^2 + 10bc^2 - 17c^2}{15c^3}\sqrt{\frac{-7b^2 + 5bc^2 + 2c^2}{5b}}, \tag{1.5}$$

for $0 < b < c$.

Thus, one equilibrium (among the two equilibria) corresponding to Q_1 or Q_3 is a Bogdanov-Takens bifurcation point. By symmetry, this result holds for Q_2 and Q_4.

[1] The proof of this th. will be given in Sec. 4.

In fact, the given equations for $BT_{1,3}$ are only asymptotic representations of the corresponding curves of homoclinic bifurcation. This is why their form is satisfactory near Q_1 and Q_3, whereas for larger distances from Q_1 and Q_3 the curves $BT_{1,3}$ were drawn with the aid of numerical results and were denoted by $BTn_{1,3}$.

Let $Q_5 = BTn_1 \cap Ob$ and $Q_6 = BTn_3 \cap Ob$, $Q_7 = BTn_4 \cap S_1$, $Q_8 = BTn_3 \cap S_2$. Due to the symmetry of the phase portraits, for the case $a = 0, Q_5$ and Q_6 are not homoclinic bifurcation values. To Q_6 a double homoclinic bifurcation point corresponds. In addition, at Q_6 two curves D_1 and D_2 of homoclinic bifurcation points emerge. These curves were obtained numerically. Let $Q_9 = D_1 \cap S_1$, and $Q_{10} = D_2 \cap S_2$. The segments Q_7Q_9 and Q_8Q_{10} of S_1 and S_2 correspond to saddle-node homoclinic bifurcation [4], while points $Q_i, i = \overline{7,10}$ are of saddle-node homoclinic with separatrix loop bifurcation [8].

2. Breaking saddle connection bifurcation. Double breaking saddle connection bifurcation

A numerical study of the phase dynamics taking into account the manifolds of saddle equilibria emphasizes the breaking saddle connection bifurcation.

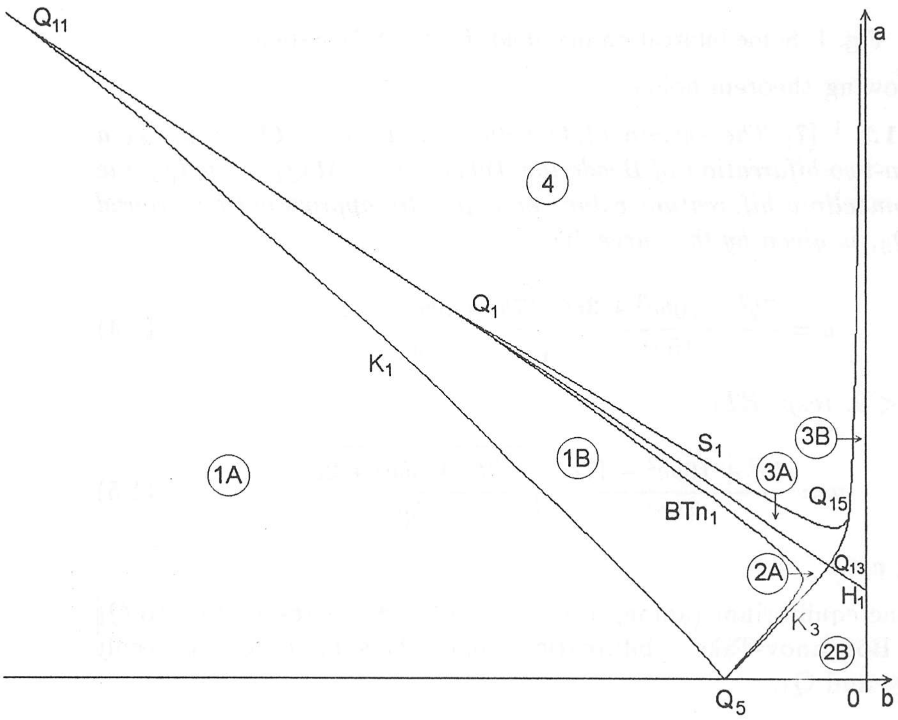


Fig. 2. The curves $K_{1,3}$

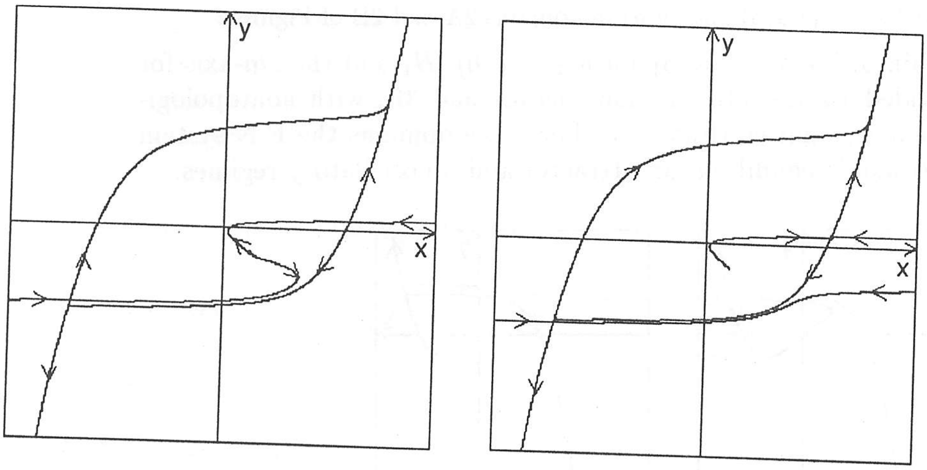
It takes place for (b, a) situated on the curves K_i $i = \overline{1,4}$ (Figure 2) passing through the point Q_5, already introduced in Section 1 as the point at which the curves $BTn1$ and $BTn3$ intersect the Ob-axis (Figure 1).

The curves K_2, K_4 are symmetric to K_1, K_2 with respect to the Ob-axis.

The representation from Figure 2 is also qualitative.

Consider now the following important points $Q_{11} = K_1 \cap S_1$, $Q_{13} = K_3 \cap H_1$, $Q_{15} = K_3 \cap S_1$ and denote by Q_{12}, Q_{14}, Q_{16} the symmetric points of Q_{11}, Q_{13}, Q_{15} with respect to the Ob-axis. Denote by (b_i, a_i) the coordonates of Q_i, $i = \overline{1,16}$.

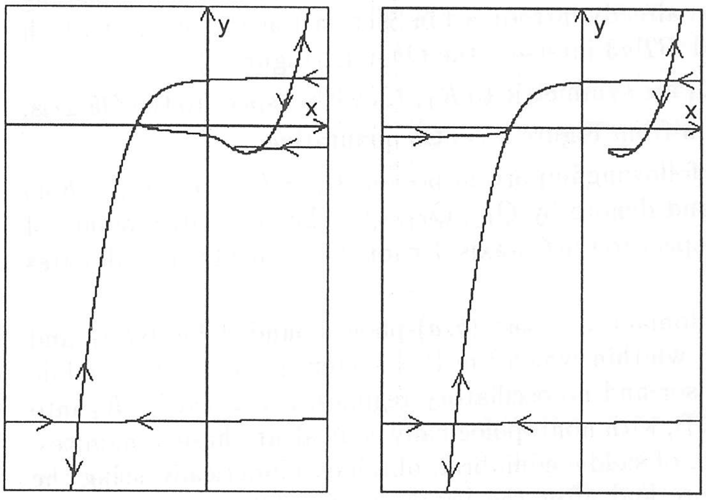
In this way, the domain 1 of the (b, a)-plane bounded by BTn_1 and S_1 for $b \in (-\infty, -c)$, whithin which the F-N system possesses two saddle equilibria and a repulsor and no oscillatory regimes, is divided by K_1 into two domains 1A and 1B, with nontopologically equivalent phase dynamices. Some global manifolds of saddle equilibria, obtained numerically using the soft DIECBI [3], before and after the breaking of saddle connection, are presented in Figure 3.



a) $a = 1.418$, $b = -3.2$ b) $a = 1.418$, $b = -3.1$

Fig. 3. Stable and unstable manifolds of the saddle-points for $c = 5$, $x, y \in (-3, 3)$ and (b, a) in domains 1A and 1B of Figure 2

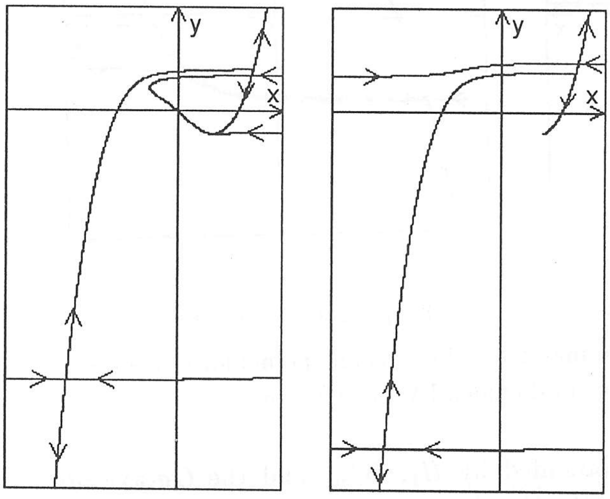
Symilarly, the domain 2, bounded by H_1, BT_{n1} and the Oa-axis for $a \in (0, 1)$, is divided by K_3 into the domains 2A and 2B. For (b, a) in 2A and 2B, the F-N system possesses two saddle equilibria, a repulsor and an attractive limit cycle. The phase portraits of 2A are not topologicaly equivalent to those of 2B (Figure 4).



a) $a = 1.418,\ b = -0.62437$ b) $a = 1.418,\ b = -0.6243675$

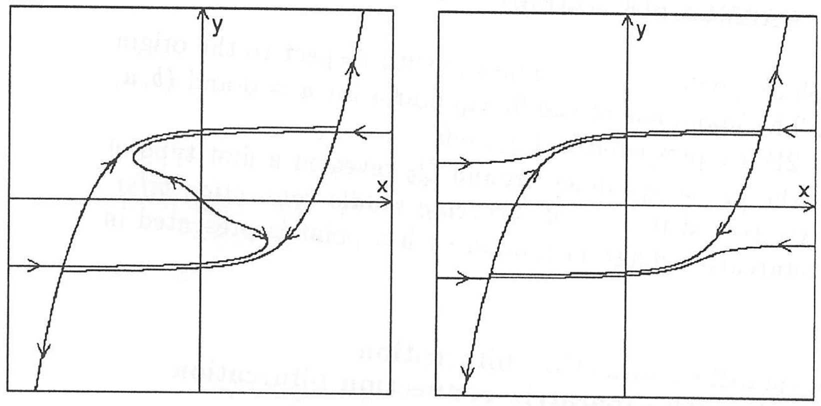
Fig. 4. Stable and unstable manifolds of the saddle-points for $c = 5$, $x \in (-5, 3), y \in (-9, 3)$ and (b, a) in domains 2A and 2B of Figure 2

The domain 3, bounded by S_1 for $b \in (-c, 0)$, H_1 and the Oa-axis for $a > 1$, is divided by K_3 into the domains 3A and 3B, with nontopologically equivalent dynamices (Figure 5). For these domains the F-N system possesses two saddle equilibria, an attractor and no oscillatory regimes.



a) $a = 1.5,\ b = -0.6$ b) $a = 1.5,\ b = -0.5$

Fig. 5. Stable and unstable manifolds of the saddle-points for $c = 5$, $x \in (-5, 3), y \in (-11, 3)$ and (b, a) in domains 3A and 3B of Figure 2



b) $a = 0, b = -1.95$ b) $a = 0, b = -1.8$

Fig. 6. Stable and unstable manifolds of the saddle-points for $c = 5$, $x, y \in (-3, 3)$ and (b, a) in domains 1A and 2B of Figure 2

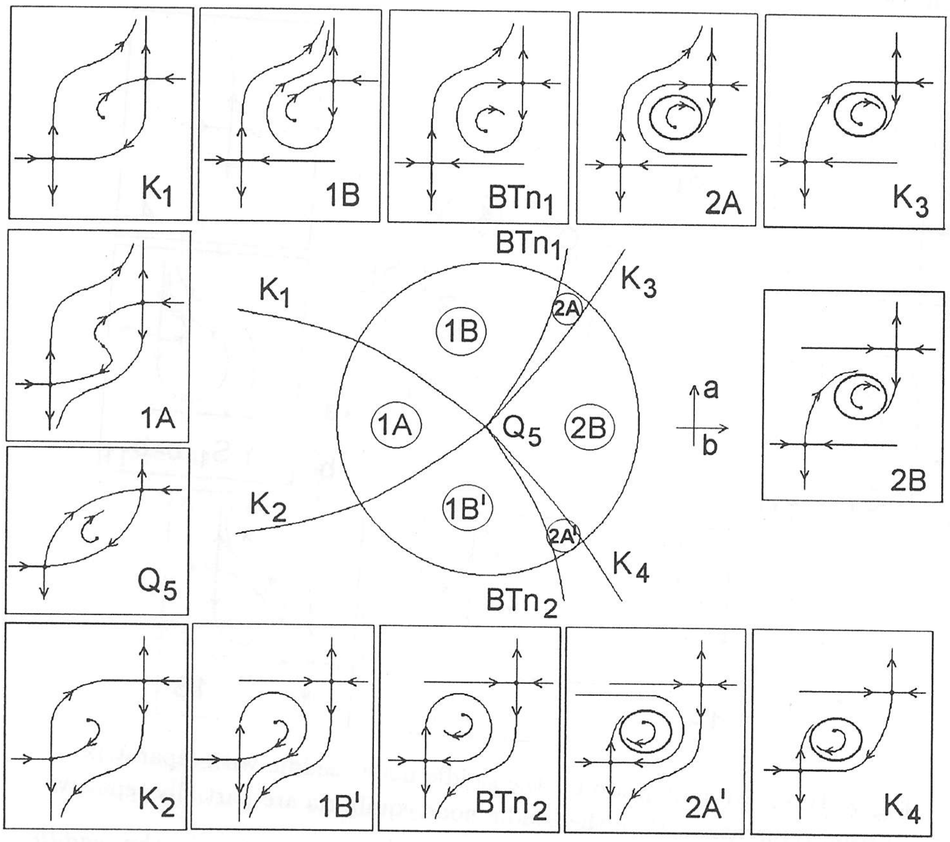


Fig. 7. The double breaking saddle connection bifurcation at Q_5

For $a = 0$, the phase portraits are symmetric with respect to the origin of the (x, y)-plane. The manifolds of saddle equilibria for $a = 0$ and (b, a) in domains 1A and 2B are presented in Figure 6.

The analysis of the phase dynamics around Q_5 revealed a first type of novel bifurcation. We coined it a *double breaking saddle connection bifurcation*. The local bifurcation diagram around such a point is presented in Figure 7.

3. Saddle-node–saddle connection bifurcation. Saddle-node–saddle with separatrix connection bifurcation

Our numerical investigations around Q_{11}, emphasized another two novel types of bifurcations, not yet met by us in the literature. One is of codimension-one, corresponding to points of S_1, for $b < b_{11}$, and another of codimension-two corresponding to the point Q_{11}.

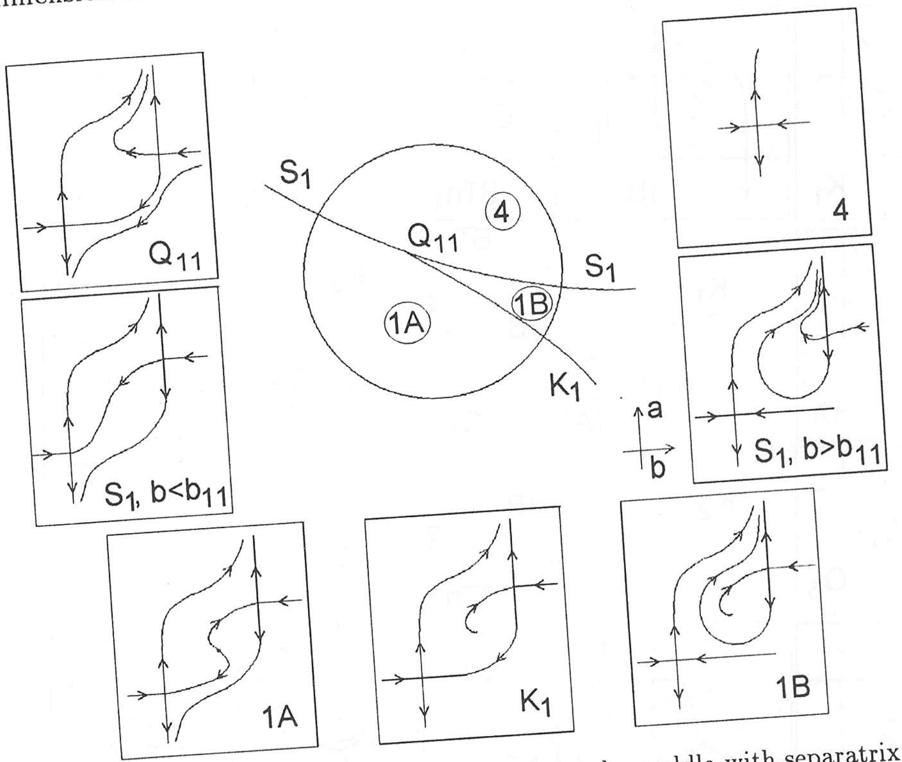


Fig. 8. Bifurcation diagram for the saddle-node–saddle with separatrix connection bifurcation. Here the saddle-node equilibria are partially repulsive

This new codimension-one bifurcation was coined by us the *saddle-node–saddle connection bifurcation*. This name was suggested by the fact that at the points of S_1, for $b < b_{11}$, a saddle-node is connected to a saddle.

The new codimension-two bifurcation, corresponding to Q_{11}, was coined by us the *saddle-node–saddle with separatrix connection bifurcation.* Its difference from the saddle-node–saddle connection bifurcation consists of the fact that in the case of Q_{11} the connection of the saddle-node with the saddle is a separatrix while in the case of the points of S_1, for $b < b_{11}$, the connection of the saddle-node with the saddle is not a separatrix.

The local bifurcation diagram around Q_{11} is presented in Figure 8.

A similar situation takes place around Q_{15}. However in this case there exist an attractive saddle-node. The local bifurcation diagram around Q_{15} is presented in Figure 9.

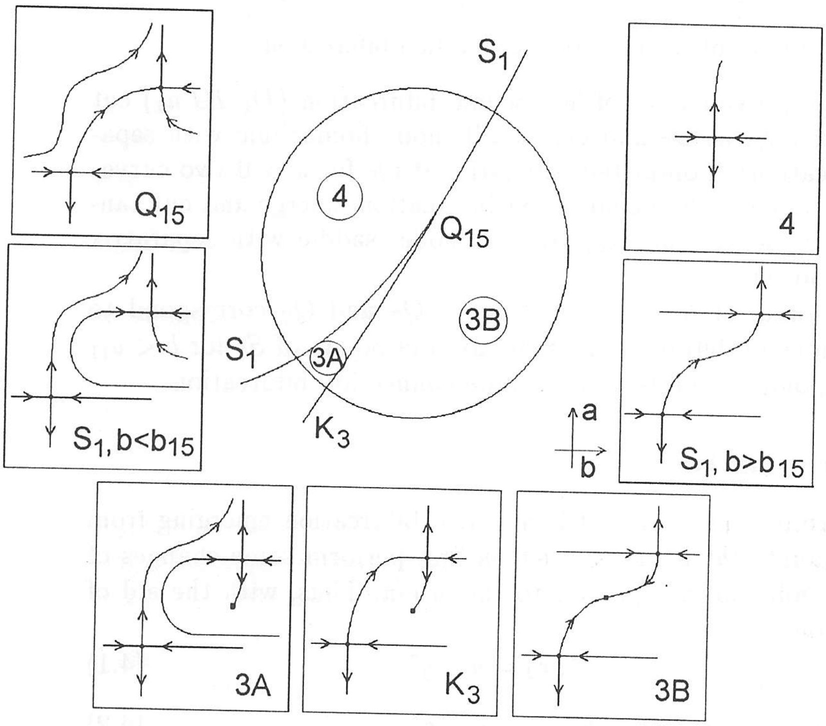


Fig. 9. Bifurcation diagram for the saddle-node–saddle with separatrix connection bifurcation. Here the saddle-node equilibria are partially attractive

The points of S_1 with $b < b_{11}$ or $b > b_{15}$ correspond to saddle-node–saddle connection bifurcation. A schematic representation of this new type of codimension-one bifurcation is given in Figure 10 for the two situations when the saddle-node is partially repulsive or partially attractive.

Let us remark a stricking analogy between the feature of bifurcation corresponding to the point Q_6 (double homoclinic bifurcation) and Q_5 (double saddle breaking connection).

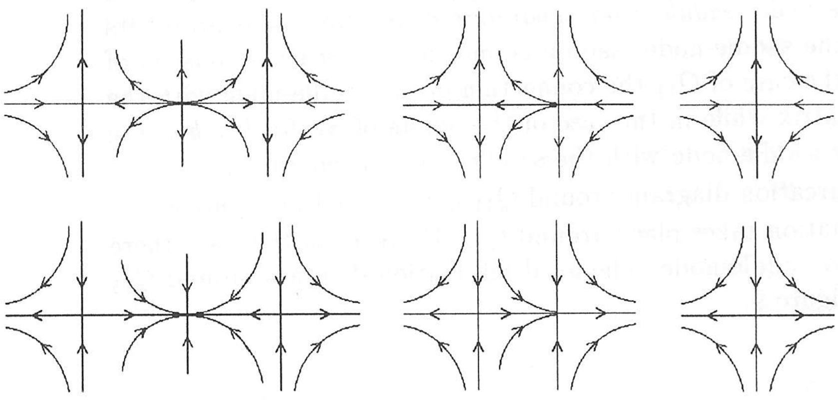


Fig. 10. Saddle-node–saddle connection bifurcation

For $a > 0$, at Q_6 two curves of homoclinic bifurcation (D_1, BTn_1) cut tangently the curve S_1 at Q_7 and Q_9 (saddle-node homoclinic with separatrix loop bifurcation). Completely similarly, at Q_5 for $a > 0$ two curves K_1 and K_3 of breaking saddle connection bifurcation emerge and cut tangently the curve S_1 at Q_{11} and Q_{15} (saddle-node–saddle with separatrix connection bifurcation).

Finally, the points of S_1 situated between Q_7 and Q_9 correspond to saddle-node homoclinic bifurcations. Similaly, the points of S_1 for $b < b_{11}$ or $b > b_{15}$ correspond to saddle-node–saddle connection bifurcation.

4. Proof of theorem 1.1

In order to determine the curves of homoclinic bifurcation emerging from the bifurcation points (b^*, a^*, x^*, y^*), let us first perform some changes of variables transforming such a point into the origin. Thus, with the aid of the transformations

$$x_1 = x - x^*, x_2 = y - y^*, \tag{4.1}$$

$$\alpha_1 = b - b^*, \alpha_2 = a - a^*, \tag{4.2}$$

the system (1.1) becomes

$$\dot{x}_1 = E_1 x_1 + c x_2 - E_2 x_1^2 - c x_1^3/3, \qquad \dot{x}_2 = -\frac{1}{c} x_1 + E_3 x b^2 \tag{4.3}$$

where

$$E_1 = c(1 - (x^*)^2),\ E_2 = c x^*,\ E_3 = -\frac{1}{c}(\alpha_1 + b^*). \tag{4.4}$$

Recall that at Q_1 we have $b^* = -c$, $a^* = \frac{2}{3}(c+1)\sqrt{1 + \frac{1}{c}}$ and $x^* = \sqrt{1 + \frac{1}{c}}$.

The linearized system around the equilibrium $(x_1, x_2) = (0, 0)$ has the eigenvalues

$$\lambda_{1,2} = [E_1 + E_3 \pm \sqrt{(E_1 + E_3)^2 - 4}]/2 \tag{4.5}$$

In the region of the (b, a)-plane where $(E_1 + E_3)^2 - 4 < 0$ we have

$$\lambda_{1,2}(\alpha) = \mu(\alpha) \pm i\omega(\alpha) \tag{4.6}$$

where

$\mu(\alpha) = \frac{1}{2c}[c^2(1 - (x^*)^2) - (\alpha_1 + b^*)]$,

$\omega(\alpha) = \frac{1}{2c}\sqrt{4c^2 - [c^2(1 - (x^*)^2) + (\alpha_1 + b^*)]^2}$.

Following the lines in [5] we shall use several tranformations which will associate (4.3) with its normal form. Thus, the first liniar transformation is

$$y_1 = \frac{1}{c}x_1, \qquad y_2 = -\frac{1}{c}x_1 + x_2 \tag{4.7}$$

such that the system (4.3) becomes

$$\begin{aligned} \dot{y}_1 &= (1 + E_1)y_1 + y_2 - cE_2y_1^2, \\ \dot{y}_2 &= (E_3 - E_1 - 2)y_1 + (E_3 - 1)y_2 + cE_2y_1^2. \end{aligned} \tag{4.8}$$

Using the nonlinear transformation

$$U_1 = y_1, \; U_2 = (1 + E_1)y_1 + y_2 - cE_2y_1^2, \tag{4.9}$$

the system (4.8) becomes

$$\begin{aligned} \dot{U}_1 &= U_2, \\ \dot{U}_2 &= (-E_1E_3 - 1)U_1 + (E_1 + E_3)U_2 + cE_2E_3U_1^2 - 2cE_2U_1U_2 \end{aligned} \tag{4.10}$$

With the transformation

$$U_1 = V_1 + \delta, \; U_2 = V_2, \tag{4.11}$$

where δ is chosen so that the coefficient of V_2 vanish, i.e. $\delta = \dfrac{E_1 + E_2}{2cE_2}$, the system (4.10) becomes

$$\dot{V}_1 = V_2, \qquad \dot{V}_2 = h_{00} + h_{10}V_1 + h_{20}\frac{V_1^2}{2} + h_{11}V_1V_2, \tag{4.12}$$

where

$h_{00} = [E_3(E_3^2 - E_1^2) - 2(E_1 + E_3)]/(4cE_2)$, $h_{10} = E_3^2 - 1$,

$h_{20} = 2cE_2E_3$, $h_{11} = -2cE_2$.

For a new time scale

$$t = \left|\frac{B(\alpha)}{A(\alpha)}\right|\tau \tag{4.13}$$

and a new tranformation

$$\eta_1 = \frac{A(\alpha)}{B^2(\alpha)}V_1,\ \eta_2 = sign(\frac{B(\alpha)}{A(\alpha)})(\frac{A^2(\alpha)}{B^3(\alpha)}V_2 \tag{4.14}$$

the system (4.12) has the normal form

$$\dot{\eta}_1 = \eta_2,\ \dot{\eta}_2 = \beta_1 + \beta_2\eta_1 + \eta_1^2 + s\eta_1\eta_2 \tag{4.15}$$

where

$A(\alpha) = \frac{h_{20}(\alpha)}{2}$, $B(\alpha) = h_{11}(\alpha)$, $\beta_1(\alpha) = \frac{B^4(\alpha)}{A^3(\alpha)}h_{00}(\alpha)$;

$\beta_2(\alpha) = \frac{B^2(\alpha)}{A^2(\alpha)}h_{10}(\alpha)$; $s = sign\frac{B(0)}{A(0)} = -1$.

Thus, at Q_1 a Bogdanov-Takens bifurcation takes place and in the (α_1, α_2)-parameter space a curve of homoclinic bifurcation

$$25A(\alpha)h_{00} + 6h_{10}^2(\alpha) = 0 \tag{4.16}$$

emerges for $h_{10} < 0$. The condition $h_{10} < 0$ implies $-1 < E_3 < 1$ and taking into account that at Q_1 we have $b^* = -c$, it follows $-c < b < c$. In terms of the initial variables, (4.16) reads $(x^*)^2 = 1 - \frac{7}{b} \pm \frac{7}{5}(-\frac{b}{c^2} + \frac{1}{b})$ and, taking into account that x^* is an equilibrium point, (1.4) follows.

Similarly, starting with the point Q_3, (1.5) is obtained.

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