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ON THE REGULARITY OF p-BOREL IDEALS

JÜRGEN HERZOG AND DORIN POPESCU

Introduction

In this paper we prove Pardue's conjecture on the regularity of principal p-Borel ideals. As a consequence we obtain an upper bound for the regularity of general p-Borel ideals.

Let K be field, and $I \subset S$ a graded ideal in the polynomial ring $S = K[x_1, \dots, x_n]$. Recall that the generic initial ideal $\text{Gin}(I)$ of I with respect to the reverse lexicographical order is Borel-fixed. This means that $\text{Gin}(I)$ is fixed under the action of the Borel group of the upper triangular invertible matrices acting linearly on the polynomial ring. By a theorem of Bayer and Stillman [2], the regularity of I and $\text{Gin}(I)$ coincide. This is one of the reasons why one is interested to compute the regularity of $\text{Gin}(I)$. In characteristic zero a Borel-fixed ideal is strongly stable, and so its regularity is simply the highest degree of a minimal generator. In positive characteristic however, Borel-fixed ideals are p-Borel (see 1.1 for the definition), and these are monomial ideals with a quite difficult combinatorial structure.

Monomials $u_1, \dots, u_m \in I$ of a p-Borel are called Borel generators of I, if I is the smallest p-Borel ideal containing $u_1, \dots, u_m$. In this case we write $I = \langle u_1, \dots, u_m \rangle$. The ideal I is called principal p-Borel if I has only one Borel generator. Pardue conjectured a formula for the regularity of a principal p-Borel ideal which only depends on the exponents of the Borel generator, see 1.4. In a paper by Aramova and Herzog [1] it was shown that Pardue's formula gives indeed a lower bound for the regularity. Some of the results in that paper have been later extended by Ene, Pfister and Popescu [5] to more general ideals. In the present paper we will show that Pardue's formula yields also an upper bound. Our method in proving this uses a criterion of Eisenbud, Reeves and Totaro [4] for determining the regularity of p-Borel ideals.

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1. p-Borel ideals

Throughout this paper we fix a field K, and let $S = K[x_1, \dots, x_n]$ be the polynomial in n indeterminates over K.

Let p be a prime number, and k and l be non-negative integers with p-adic expansion $k = \sum_i k_i p^i$ and $l = \sum_i l_i p^i$. We set $k \leq_p l$ if $k_i \leq l_i$ for all i.

Definition 1.1. *A monomial ideal $I \subset S$ is p-Borel, if the following condition holds: for each monomial $u \in I$, $u = \prod_i x_i^{\mu_i}$, one has $(x_i/x_j)^\nu u \in I$ for all i, j with $1 \leq i < j \leq n$ and all $\nu \leq_p \mu_j$.*

The significance of p-Borel principal ideals is given by

Proposition 1.2 (Pardue). *Suppose* $\operatorname{char} K = 0$, *and let* $I \subset S$ *be a monomial ideal. Then* I *is Borel-fixed if and only if* I *is* p*-Borel.*

We denote by $G(I)$ the unique minimal set of monomial generators of a monomial ideal I. It is easy to see (cf. [1]) that I is p-Borel if the conditions of 1.1 are satisfied for all $u \in G(I)$.

A principal p-Borel ideal can be explicitly described. We use the following standard notation: If J is a monomial ideal we let $J^{[p^i]}$ be the ideal generated by all monomials u^{p^i} with $u \in G(J)$. The ideal $J^{[p^i]}$ is called the p^ith Frobenius power of J. Note that we define the p^ith Frobenius power of monomial ideals in any characteristic.

Proposition 1.3 (Pardue). *Let* $u = \prod_i x_i^{\mu_i}$, *and let* $\mu_i = \sum_i \mu_{ij} p^j$ *be the p-adic expansion of* μ_i *for* $i = 1, \dots, n$. *Then*

$$\langle u \rangle = \prod_{i=1}^{n} \prod_{j} ((x_1, \dots, x_i)^{\mu_{ij}})^{[p^j]}$$

In particular, $\langle u \rangle = \prod_{i=1}^{n} \langle x_i^{\mu_i} \rangle$.

It follows from 1.3 that $\langle x^\mu \rangle = (x_1^{\mu_1})\langle x^\mu / x_1^{\mu_1} \rangle$, so that $\operatorname{reg}\langle x^\mu \rangle = \mu_1 + \operatorname{reg}\langle x^\mu / x_1^{\mu_1} \rangle$. Therefore, if we are interested in the regularity of the p-Borel principal ideal $\langle x^\mu \rangle$ we may assume that x_1 does not divide x^μ.

Denote by $\lfloor * \rfloor$ the greatest integer function, and for $1 \leq k \leq n$ and $j \geq 0$ define

$$d_{kj}(\mu) = \sum_{i=1}^{k} \lfloor \frac{\mu_i}{p^j} \rfloor.$$

For each k such that $\mu_k \neq 0$, let $s_k = \lfloor \log_p \mu_k \rfloor$, and set

$$D_k = d_{ks_k}(\mu) p^{s_k} + (k-1)(p^{s_k} - 1).$$

Conjecture 1.4 (Pardue). *If* x_1 *does not divide* x^μ, *then*

$$\operatorname{reg}\langle x^\mu \rangle = \max_{k\colon \mu_k \neq 0} \{D_k\}.$$

In the following we will express the right hand side of 1.4 in different ways. The following easy fact can be found in [1].

Proposition 1.5. *Let* $\mathcal{S} = \{s_k \colon \mu_k \neq 0\}$, *let* $q_j = \max\{k \colon s_k = j\}$ *for each* $j \in \mathcal{S}$, *and set* $E_j = D_{q_j}$. *Then*

(i) $E_j = \sum_{i=j}^{s} (\sum_{k=2}^{q_j} \mu_{ki}) p^i + (q_j - 1)(p^j - 1)$ *for all* $j \in \mathcal{S}$;
(ii) $\max\{D_k \colon \mu_k \neq 0\} = \max\{E_j \colon j \in \mathcal{S}\}$;

We shall need still another reformulation of Pardue's formula for the regularity of a principal p-Borel ideal. Set $s = \max\{s_k \colon \mu_k \neq 0\}$, and for each t with $1 \leq t \leq s$ let $m_t = \max\{k \colon \mu_{kt} \neq 0\}$. Finally set

$$F_t = \sum_{i=t}^{s} (\sum_{k=2}^{n} \mu_{ki}) p^i + (m_t - 1)(p^t - 1) \quad \text{for all} \quad t = 1, \dots, s.$$

Proposition 1.6. *With the notation introduced we have*

$$\max_{1\le t\le s} F_t = \max_{j\in\mathcal{S}} E_j.$$

Proof. It is clear that $m_j \ge q_j$ for all $j \in \mathcal{S}$, so that $\max_{1\le t\le s} F_t \ge \max_{j\in\mathcal{S}} E_j$.

In order to show the opposite inequality we first prove the following claim: let $\mathcal{S}_t = \{j \in \mathcal{S}\colon j \ge t\}$ and $\mathcal{Q}_t = \{q_j\colon j \in \mathcal{S}_t\}$. (Note that $\mathcal{S}_t \neq \emptyset$, since $s \in \mathcal{S}_t$). Let $e \in \mathcal{S}_t$ such that $q_e = \max\{q_j \in \mathcal{Q}_t\}$. Then we claim that $F_e \ge F_t$.

Indeed, we have

$$F_e - F_t = -\sum_{i=t}^{e-1}\Big(\sum_{k=2}^{n}\mu_{ki}\Big)p^i + (m_e-1)(p^e-1) - (m_t-1)(p^t-1).$$

Since we assume that q_e is maximal in $\mathcal{Q}_t$ it follows that $\mu_{ki} = 0$ for $k > q_e$ and $i \ge t$. Thus $m_e = q_e \ge \max\{k\colon \mu_{kt} \neq 0\} = m_t$, because again $\mu_{kt} = 0$ for $k > q_e$.

Now it follows that

$$F_e - F_t = -\sum_{i=t}^{e-1}\Big(\sum_{k=2}^{q_e}\mu_{ki}\Big)p^i + (q_e-1)(p^e-1) - (m_t-1)(p^t-1).$$

Finally, since $\sum_{i=t}^{e-1}(\sum_{k=2}^{q_e}\mu_{ki})p^i = \sum_{k=2}^{q_e}(\sum_{i=t}^{e-1}\mu_{ki}p^i \le (q_e-1)(p^e-p^t)$, we get $F_e - F_t \ge (q_e-1)(p^t-1) - (m_t-1)(p^t-1) = (q_e-m_t)(p^t-1) \ge 0$. This concludes the proof of the claim.

Continuing with the proof of the opposite inequality, we let $t \le s$ be the maximal number for which $F_t = \max_{1\le r\le s} F_r$. Let $e \in \mathcal{S}_t$ be chosen such that q_e is maximal in $\mathcal{Q}_t$. Then, according to our claim, we have $F_e \ge F_t$. By the choice of t this implies that $e = t$, so that in particular, $t \in \mathcal{S}_t$. Since q_t is maximal in $\mathcal{Q}_t$ it now follows that $\mu_{ki} = 0$ for $i \ge t$ and $k > q_t$. Consequently, $m_t = q_t$, and so $F_t = E_t$. □

Remark 1.7. Using the methods of [1] the following result was proved in [5]: Let $(I_t)_{1\le t\le s}$ be some stable ideals and $I = \prod_{t=1}^{s} I_t^{[p^{r_t}]}$ for some integers $0 \le r_1 < \cdots < r_s$. If I_j contains $x_{m(I_{j+1})}^{p^{r_{j+1}-r_j}-1}$ for all $1 \le j < s$ (we set $m(u) = \max\{j : x_j|u\}$ for a monomial u) and $m(I_{j+1}) = \max\{m(u) : u \in G(I_{j+1})\}$) then $\operatorname{reg}(I) = pa(I)$, where we $pa(I) = \max_{1\le t\le s}\{\sum_{i>t}^{s} p^{r_i}\max(I_i) + \max_{u\in G(I_t)}[p^{r_t}\deg(u) + (m(u)-1)(p^{r_t}-1)]\}$ Moreover if I_t has the form $I_t = \prod_{i=2}^{n}(x_1,\dots,x_i)^{\mu_{it}}$ with $0 \le \mu_{it} < p$ for all $t < s$, the above result gives $\operatorname{reg}(I) = \max_{1\le t\le s} F_t$. Hence the Pardue Conjecture holds in a special case, which can be also obtained directly from [1]. Trying to extend the equality $\operatorname{reg}(I) = \operatorname{pa}(I)$ for general products of p^i-th Frobenius powers of stable ideals one must consider first the following example which shows how tight Pardue's Conjecture is: Let $n = 3$, $p = 2$, $I_1 = (x_1,x_2)^2$, $I_2 = (x_1,x_2,x_3)$ and $I = I_1 I_2^{[2]}$. Then $\operatorname{pa}(I) = 4$, but $\operatorname{reg}(I) > 4$, because I is not stable (see 2.1 below).

2. The proof of Pardue's conjecture

In [1] it is shown that if $x^\mu \in S$ is a monomial which is not divisible by x_1, then

$$\operatorname{reg}\langle x^\mu\rangle \ge \max_{k\colon \mu_k\neq 0}\{D_k\}.$$

In the section we will prove the opposite inequality. Our proof is based on the following result [4]:

Proposition 2.1 (Eisenbud, Reeves, Totaro). *Let I be a p-Borel ideal with* $\max(I) = d$, *and let $e \geq d$ be the smallest integer such that $I_{\geq e}$ is stable. Then* $\operatorname{reg}(I) = e$.

2.1 needs some explanations: $\max(I) = \max\{\deg u \colon u \in G(I)\}$, and $I_{\geq e}$ is the ideal generated by all monomials $u \in I$ with $\deg u \geq e$. Finally, recall that, according to Eliahou and Kervaire [3], a monomial ideal I is stable if for all monomials (or equivalently all generators) u of I one has $(x_i/x_{m(u)})u \in I$ for all $i \leq m(u)$, where $m(u) = \max\{j \colon x_j | u\}$.

Recall from Section 1 that $\langle x^\mu \rangle = \prod_t I_t^{[p^t]}$ where $I_t = \prod_{i=2}^n (x_1, \dots, x_i)^{\mu_{it}}$ with $0 \leq \mu_{it} < p$. Thus the desired inequality will follow from 2.1 and

Theorem 2.2. *For given integers $0 \leq r_1 < \cdots < r_s$, and integers $0 \leq a_{tk} < p^{r_{t+1}-r_t}$ for $t = 1, \dots, s$ and $k = 1, \dots, m_t$ let $I_t = \Pi_{k=2}^{m_t}(x_1, \dots, x_k)^{a_{tk}}$ and $I = \Pi_{t=1}^s I_t^{[p^{r_t}]}$. Let $\delta_t = \sum_{i=t}^s p^{r_i} \max(I_i) + (m_t - 1)(p^{r_t} - 1)$ and $d = \max\{\delta_t | 1 \leq t \leq s\}$. Then $I_{\geq d}$ is stable.*

The proof needs some preparations.

Lemma 2.3. *Let $J = \prod_{k=2}^m (x_1, \dots, x_k)^{a_k}$ with $0 \leq a_k \leq p^r - 1$, and let $\eta \in J$ be a monomial such that $\deg \eta \geq 1 + \max\{m(\eta)(p^r - 1), \max(J) + p^r - 1\}$. Then there exists an integer t such that $\eta \in x_t^{p^r} J$.*

Proof. We reduce the problem to the case where $m(\eta) \leq m$. Since J is stable, η has the following Eliahou-Kervaire decomposition: $\eta = vw$ for monomials v and w with $v \in G(J)$ and $\min(w) \geq m(v)$. We may assume that $w \notin (x_t^{p^r})$ for all t. Then $w = w' x_{m+1}^{\beta_{m+1}} \cdots x_{m(\eta)}^{\beta_{m(\eta)}}$ with $\beta_i \leq p^r - 1$. Thus the element $\eta' = vw'$ has degree $\geq \deg \eta - (m(\eta) - m)(p^r - 1) \geq m(p^r - 1) + 1$. Since $m(\eta') \leq m$ and $\max(J) \leq (m - 1)(p^r - 1)$, we may replace η by η', and thus may as well suppose that $m(\eta) \leq m$.

Let $\eta = x_1^{\alpha_1} \cdots x_{m(\eta)}^{\alpha_{m(\eta)}}$. We apply induction on $\max(J)$, and may assume that $a_m \neq 0$. If $\max(J) = 1$, then $J = (x_1, \dots, x_m)$. If $m(\eta) = 1$, then $\eta = x_1^{\alpha_1}$ with $\alpha_1 \geq \max(J) + p^r = p^r + 1$. In that case, $\eta = x_1^{p^r}\eta'$ with $\eta' = x_1^{\alpha_1 - p^r} \in J$. Suppose now that $m(\eta) \geq 2$. Then, since $\deg \eta \geq m(\eta)(p^r - 1) + 1$, it follows that $\eta \in (x_t^{p^r})$ for a certain $t \leq m(\eta)$, and so $\eta = x_t^{p^r}\eta'$ where η' is a monomial of degree $\geq (m(\eta) - 1)(p^r - 1) > 1$. Hence $\eta' \in J$, and so $\eta \in x_t^{p^r} J$.

Now suppose that $\max(J) > 1$. We will distinguish several cases. In the first case suppose that $\alpha_{m(\eta)} \geq p^r$. Let again $\eta = vw$ be the Eliahou-Kervaire decomposition of η. Then $\deg w = \deg \eta - \deg v \geq p^r > 0$. Hence, since $m(v) \leq \min(w)$, it follows that $x_{m(\eta)}^{p^r}$ divides w, and we are done.

Now we consider the case that $\alpha_{m(\eta)} \leq p^r - 1$, $m(\eta) \geq 3$ and $\alpha_{m(\eta)} \leq \sum_{i=m(\eta)}^m a_i$. We choose the maximal integer t, $m(\eta) \leq t \leq m$, such that $\alpha_{m(\eta)} \leq \sum_{i=t}^m a_i$, and write $\alpha_{m(\eta)} = \sum_{i=t+1}^m a_i + b_t$ with $1 \leq b_t \leq a_t$. Now set $\varphi = \eta / x_{m(\eta)}^{\alpha_{m(\eta)}}$. Observe that for all monomials $\rho \in J$ with $m(\rho) \leq m$ one has

$$\rho/x_{m(\eta)} \in \prod_{k=2}^{m-1}(x_1,\dots,x_k)^{a_k}(x_1,\dots,x_m)^{a_m-1}. \tag{1}$$

Applying (1) successively we see that $\varphi \in J''$ where

$$J'' = \prod_{k=2}^{t-1}(x_1,\dots,x_k)^{a_k}(x_1,\dots,x_t)^{a_t-b_t}.$$

We have

$\deg\varphi = \deg\eta - \alpha_{m(\eta)} \geq \deg\eta - (p^r-1) \geq 1+(m(\eta)-1)(p^r-1) \geq 1+m(\varphi)(p^r-1)$, and $\deg\varphi = \deg\eta - \alpha_{m(\eta)} \geq \max(J)+p^r-\alpha_{m(\varphi)} = \max(J'')+p^r$. Hence we may apply our induction hypothesis, and conclude that there exists an integer $q \leq m(\varphi)$ such that $\varphi \in x_q^{p^r}J''$. It follows that $\eta \in x_q^{p^r}J$.

Next we consider the case $\alpha_{m(\eta)} \leq p^r-1$, $m(\eta) \geq 3$ and $\alpha_{m(\eta)} > \sum_{i=m(\eta)}^m a_i$. Using again (1) we see that $\eta = x_{m(\eta)}^{\sum_{i=m(\eta)}^m a_i}\eta'$ with $\eta' \in \tilde{J}$ where $\tilde{J} = \prod_{k=2}^{m(\eta)-1}(x_1,\dots,x_k)^{a_k}$

Note that for any monomial $\rho \in J$ with $m(\rho) > m$ it follows that $\rho/x_{m(\eta)} \in J$. Applying this successively to η' we see that $\varphi = \eta/x_{m(\eta)}^{\alpha_{m(\eta)}}$ belongs to $\tilde{J}$. As in the second case it follows that $\deg\varphi \geq 1+m(\varphi)(p^r-1)$. Since on the other hand $\max(\tilde{J}) \leq (m(\eta)-2)(p^r-1)$, it also follows that $\deg\varphi \geq 1+(m(\eta)-1)(p^r-1) \geq \max\tilde{J}+p^r$. Applying the induction hypothesis to φ and $\tilde{J}$ yields the desired conclusion for η.

It remains to consider the case $\alpha_{m(\eta)} \leq p^r-1$ and $m(\eta) \leq 2$. If $m(\eta)=1$, then $\alpha_1 \geq p^r$, a contradiction. Therefore $\eta = x_1^{\alpha_1}x_2^{\alpha_2}$ with $\alpha_2 \neq 0$ and $\alpha_1+\alpha_2 \geq \max\{2p^r-1, \max J+p^r\}$. It follows that $\alpha_1 \geq p^r$. Then the element $\eta' = x_1^{\alpha_1-p^r}x_2^{\alpha_2}$ belongs to $(x_1,x_2)^{\max J}$ which is contained in J, and so $\eta = x_1^{p^r}\eta' \in x_1^{p^r}J$. □

Corollary 2.4. *Let $J = \prod_{k=2}^m(x_1,\dots,x_k)^{a_k}$ where $0 \leq a_k \leq p^r-1$ for $k = 2,\dots,m$, and let q be a positive integer and $\eta \in J$ a monomial with $m(\eta) < q$ and $\deg\eta \geq 1+max\{(q-1)(p^r-1), \max(J)\}$. Then there exists $t \leq m(\eta)$ such that $x_q^{p^r-1}\eta \in x_t^{p^r}J$.*

Proof. Let $\eta' = x_q^{p^r-1}\eta$. We have $\deg\eta' \geq 1+\max\{m(\eta')(p^r-1), \max(J)+p^r-1\}$ since $m(\eta') = q$. Thus by 2.3 there exists an integer $t \leq m(\eta') = q$ such that $\eta' \in x_t^{p^r}J$, and hence $x_q^{p^r-1}\eta \in x_t^{p^r}J$. Since p^r-1 is the maximal power of x_q which divides η', we have $t \neq q$ and so $t \leq m(\eta)$. □

Lemma 2.5. *Let $J = \prod_{k=2}^m(x_1,\dots,x_k)^{a_k}$ with $0 \leq a_k \leq p^{r-e}-1$ for $k = 2,\dots,m$ and integers $0 \leq e < r$. Let $I = J^{[p^e]}$ and $\eta \in I$ a be monomial such that $\deg\eta \geq 1+\max\{m(\eta)(p^r-1), \max(I)+p^r-p^e+m(\eta)(p^e-1)\}$. Then there exists $t \leq m(\eta)$ such that $\eta \in x_t^{p^r}I$.*

Proof. We may write $\eta = v^{p^e}w$, $v \in G(J)$ and $w = \sigma_1^{p^e}\sigma_0$, where σ_0 and σ_1 are monomials, and $\sigma_0 \notin (x_1^{p^e},\dots,x_{m(\eta)}^{p^e})$. Thus $\deg\sigma_0 \leq m(\eta)(p^e-1)$ and the monomial

$\eta' = v\sigma_1$ belongs to J. Since $\eta = \eta'^{p^e}\sigma_0$ it follows that

$$\begin{aligned} p^e \deg \eta' &= \deg \eta - \deg \sigma_0 \geq \deg \eta - m(\eta)(p^e - 1) \\ &\geq 1 + \max\{m(\eta)(p^r - p^e), p^e \max(J) + p^r - p^e\}. \end{aligned}$$

Therefore $\deg \eta' \geq (1/p^e) + \max\{m(\eta)(p^{r-e} - 1), \max(J) + p^{r-e} - 1\}$. Since $\deg \eta'$ is an integer we get $\deg \eta' \geq 1 + \max\{m(\eta)(p^{r-e} - 1), \max(J) + p^{r-e} - 1\}$. Note that $m(\eta') \leq m(\eta)$. Therefore by Lemma 2.3 there exists an integer t, $t \leq m(\eta') \leq m(\eta)$ such that $\eta' \in x_t^{p^{r-e}} J$. Thus $\eta \in x_t^{p^r} I$. □

Applying 2.5 recursively we get

Corollary 2.6. *With the hypotheses of* 2.5 *suppose in addition that* $\deg \eta \geq cp^r + 1 + \max\{m(\eta)(p^r - 1), \max(I) + p^r - p^e + m(\eta)(p^e - 1)\}$ *for some integer* $c \geq 0$. *Then there exists a monomial* σ *of degree* $c + 1$ *such that* $m(\sigma) \leq m(\eta)$ *and* $\eta \in \sigma^{p^r} I$.

Lemma 2.7. *Let* $J = \prod_{k=2}^{m} (x_1, \ldots, x_k)^{a_k}$ *with* $0 \leq a_k \leq p^{r-e} - 1$ *for* $k = 2, \ldots, m$ *and integers* $0 \leq e < r$. *Let* $I = J^{[p^e]}$, q *a positive integer and* $\eta \in J$ *a monomial with* $m(\eta) < q$ *and* $\deg \eta \geq cp^r + 1 + \max\{(q-1)(p^r - 1), \max(I) + (q-1)(p^e - 1)\}$ *for some integer* $c \geq 0$. *Then there exists a monomial of degree* $c + 1$ *such that* $x_q^{p^r - 1}\eta \in \sigma^{p^r} I$ *and* $m(\sigma) \leq m(\eta)$.

Proof. Set $\eta' = x_q^{p^r - 1}\eta$. Then

$$\deg \eta' \geq cp^r + 1 + \max\{m(\eta')(p^r - 1), \max(I) + p^r - p^e + m(\eta')(p^e - 1)\}.$$

So by Corollary 2.6 there exists a monomial σ with $m(\sigma) \leq m(\eta')$ and $\deg \sigma = c+1$ such that $\eta' \in \sigma^{p^r} I$. Finally we have $m(\sigma) \leq m(\eta)$ because x_q does not divide σ since it appears only with power $p^r - 1$ in σ. □

Lemma 2.8. *Let* $I_t = \prod_{k=2}^{m_t} (x_1, \ldots, x_k)^{a_{tk}}$ *with* $0 \leq a_{tk} \leq p^{e_{t+1} - e_t} - 1$, $1 \leq t \leq s$, *and integers* $0 \leq e_1 < \cdots < e_s < r = e_{s+1}$. *Let* $I = \prod_{t=1}^{s} I_t^{[p^{e_t}]}$, q *a positive integer and* $\eta \in I$ *a monomial with* $m(\eta) < q$ *and* $\deg \eta \geq cp^r + 1 + \max_{1 \leq t \leq s+1}\{\sum_{i=t}^{s} p^{e_i} \max(I_i) + (q-1)(p^{e_t} - 1)\}$ *for some integer* $c \geq 0$. *(Here* $\sum_{i=t}^{s} p^{e_i} \max(I_i) = 0$ *for* $t = s+1$*). Then there exists a monomial* σ *with* $\deg \sigma = c + 1$, $m(\sigma) \leq m(\eta)$ *and* $x_q^{p^r - 1}\eta \in \sigma^{p^r} I$.

Proof. We apply induction on s. The case $s = 1$ is given in 2.7. Let $d_j = \sum_{i=j}^{s} p^{e_i} \max(I_i) + (q-1)(p^{e_j} - 1)$, $1 \leq j \leq s+1$, and let $t \leq s$ be maximal integer such that $d_t = \max\{d_j : 1 \leq j \leq s\}$. Then we have $d_j < d_t$ for $t < j \leq s$ and $d_j \leq d_t$ for $j < t$.

We now distinguish two cases. In case $t > 1$, write $\eta = \eta' \Pi_{i=t}^{s} v_i^{p^{e_i}}$ with $v_i \in G(I_i)$ and $\eta' \in I' = \Pi_{i=1}^{t-1} I_i^{[p^{e_i}]}$. We have $\deg(\eta') = \deg(\eta) - \sum_{i=t}^{s} p^{e_i} \max(I_i) \geq cp^r + 1 + d_t - \sum_{i=t}^{s} p^{e_i} \max(I_i) = cp^r + 1 + (q-1)(p^{e_t} - 1)$. Choose the maximal integer $\varepsilon \geq 0$ such that $\deg(\eta') \geq cp^r + 1 + \varepsilon p^{e_t} + (q-1)(p^{e_t} - 1)$. As $d_t = \max\{d_i : 1 \leq i \leq s\}$, we see that η' satisfies the necessary inequalities and so by induction hypothesis there exists a monomial τ with $\deg(\tau) = cp^{r-e_t} + \varepsilon + 1$, $m(\tau) \leq m(\eta') \leq m(\eta)$ and $x_q^{p^{e_t} - 1}\eta' = \tau^{p^{e_t}}\eta''$ for some $\eta'' \in I'$. Note that by the choice of ε, we have $\deg(\eta') \leq$

$cp^r+1+\varepsilon p^{e_t}+q(p^{e_t}-1)$, and so $p^{e_t}\deg(\tau)+\deg(\eta'')\leq cp^r+1+\varepsilon p^{e_t}+(q+1)(p^{e_t}-1)$. Hence $\deg(\eta'')\leq 1-p^{e_t}+(q+1)(p^{e_t}-1)=q(p^{e_t}-1)$.

Set $\rho=\tau\Pi_{i=t}^{s}v_i^{p^{e_i-e_t}}$; then $\rho\in\tilde{I}=\Pi_{i=t}^{s}I_i^{[p^{e_i-e_t}]}$, and $p^{e_t}\deg(\rho)=\deg(x_q^{p^{e_t}-1}\eta)-\deg(\eta'')\geq cp^r+p^{e_t}+\max\{d_t,d_{s+1}\}-q(p^{e_t}-1)=cp^r+1+a$ for $a=p^{e_t}b$ and $b=\max\{\sum_{j=i}^{s}p^{e_j-e_t}\max(I_j)+(q-1)(p^{e_i-e_t}-1)\colon t\leq i\leq s+1\}$. Therefore $\deg(\rho)\geq cp^{r-e_t}+1/p^{e_t}+b$. Using that $\deg\rho$ is an integer we get $\deg(\rho)\geq cp^{r-e_t}+1+b$. Thus we may apply the induction hypothesis to $\rho,\tilde{I}$, and hence there exists a monomial σ with $m(\sigma)\leq m(\rho)\leq m(\eta)$, $deg(\sigma)=c+1$ such that $x_q^{p^{r-e_t}-1}\rho\in\sigma^{p^{r-e_t}}\tilde{I}$. Hence $x_q^{p^r-1}\eta=x_q^{p^r-p^{e_t}}\rho^{p^{e_t}}\eta''\in\sigma^{p^r}I$.

Finally we consider the case $t=1$, and write $\eta=\varphi\Pi_{i=2}^{s}v_i^{p^{e_i}}$, $v_i\in G(I_i)$ and $\varphi\in I_1^{[p^{e_1}]}$. We have $\deg(\varphi)=\deg(\eta)-\sum_{i=2}^{s}p^{e_i}\max(I_i)\geq cp^r+1+d_1-\sum_{i=2}^{s}p^{e_i}\max(I_i)=cp^r+1+p^{e_1}\max(I_1)+(q-1)(p^{e_1}-1)$. But $t=1$ implies $d_2<d_1$, and so $p^{e_1}\max(I_1)+(q-1)(p^{e_1}-1)>(q-1)(p^{e_2}-1)$. Thus φ satisfies the condition of 2.7. As in the previous case we choose the maximal integer $\varepsilon\geq 0$ such that $\deg(\varphi)\geq cp^r+1+\varepsilon p^{e_2}+(q-1)(p^{e_2}-1)$. Then by 2.7 there exists a monomial γ with $\deg(\gamma)=cp^{r-e_2}+1+\varepsilon$, $m(\gamma)\leq m(\eta)$ and $x_q^{p^{e_2}-1}\varphi=\gamma^{p^{e_2}}\varphi'$ for some $\varphi'\in I_1^{[p^{e_1}]}$, and we see as above that $\deg(\varphi')\leq q(p^{e_2}-1)$.

Set $\psi=\gamma'\Pi_{i=2}^{s}v_i^{p^{e_i-e_2}}$; then $\psi\in\hat{I}=\Pi_{i=2}^{s}I_i^{[p^{e_i-e_2}]}$, $m(\psi)\leq m(\eta)$ and $p^{e_2}\deg(\psi)=\deg(x_q^{p^{e_2}-1}\eta)-\deg(\varphi')\geq cp^r+p^{e_2}+\max\{d_j\colon 2\leq j\leq s+1\}-q(p^{e_2}-1)=cp^r+1+\max\{\sum_{i=j}^{s}p^{e_i}\max(I_i)+(q-1)(p^{e_j}-p^{e_2})\colon 2\leq j\leq s+1\}$. Since $\deg\psi$ is an integer we get $\deg(\psi)\geq cp^{r-e_2}+1+\max\{\sum_{i=j}^{s}p^{e_i-e_2}\max(I_i)+(q-1)(p^{e_j-e_2}-1)\colon 2\leq j\leq s+1\}$. Thus we may apply our induction hypothesis to ψ and $\hat{I}$ and conclude that $x_q^{p^{r-e_2}-1}\psi\in\nu^{p^{r-e_2}}\hat{I}$ for some monomial ν with $\deg(\nu)=c+1$ and $m(\nu)\leq m(\psi)\leq m(\eta)$. This yields the desired conclusion. □

We are now in the position to prove Theorem 2.2.

Proof. [Proof of 2.2] Let $\rho=\Pi_{t=1}^{s}u_t^{p^{r_t}}w$, $u_t\in G(I_t)$ and w a monomial such that $\deg(\rho)=d$. Let $j<m(\rho)$. We must show that $x_j\rho/x_{m(\rho)}\in I_{\geq d}$. Apply induction on s, case $s=0$ being trivial. If $m(\rho)=m(\eta)$, $\eta=\Pi_{t=1}^{s-1}u_t^{p^{r_t}}w$, then we may apply the induction hypothesis because $\deg(\eta)=d-\deg(u_s^{p^{r_s}})\geq\max\{\delta_t'\colon 1\leq t\leq s-1\}$, for $\delta_t'=\sum_{i=t}^{s-1}p^{r_i}\max(I_i)+(m_t-1)(p^{r_t}-1)$. By induction hypothesis $I'=\Pi_{t=1}^{s-1}I_t^{[p^{r_t}]}$ has $I'_{\geq\deg(\eta)}$ stable and so $x_j\eta/x_{m(\eta)}\in I'_{\geq\deg(\eta)}$. Hence $x_j\rho/x_{m(\rho)}=(x_j\eta/x_{m(\eta)})u_s^{p^{r_s}}\in I_{\geq d}$.

We may suppose from now on $m(\rho)=m(u_s)>m(\eta)$. Set $q=m(u_s)$, $d_t=\sum_{i=t}^{s-1}p^{r_i}\max(I_i)+(q-1)(p^{r_t}-1)$. We have $\delta_t'\geq d_t$ if and only if $m_t\geq q$. In particular $\delta_s'\geq d_s$, since $m_s\geq q$. If $m_t<q$, then $\max(I_t)\leq(q-1)(p^{r_{t+1}-r_t}-1)$ since $a_{tk}\leq p^{r_{t+1}-r_t}-1$, and so $d_{t+1}-\delta_t'=(q-1)(p^{r_{t+1}}-1)-(m_t-1)(p^{r_t}-1)-p^{r_t}\max(I_t)>(q-1)(p^{r_{t+1}}-p^{r_t})-p^{r_t}\max(I_t)\geq 0$. Thus $\delta_t'<d_{t+1}$ if $m_t<q$. The same argument shows that $d_t\leq d_{t+1}$ if $m_t<q$.

By backwards induction on j we now show that

$$\max\{d_t\colon j\leq t\leq s\}\leq\max\{\delta_t'\colon j\leq t\leq s\}. \tag{2}$$

We have already seen that (2) holds for $j=s$. Suppose the inequality holds for $j+1$. If $m_j\geq q$ then $\delta_j'\geq d_j$, and so (2) is implied by the induction hypothesis. If

$m_j < q$, then $\delta'_j < d_{j+1}$, $d_j \leq d_{j+1}$, and hence $\delta'_j < d_{j+1} \leq \max\{d_t \colon j+1 \leq t \leq s\} \leq \max\{\delta'_t \colon j+1 \leq t \leq s\}$. Hence $\max\{\delta'_t \colon j \leq t \leq s\} = \max\{\delta'_t \colon j+1 \leq t \leq s\} \geq \max\{d_t \colon j+1 \leq t \leq s\} = \max\{d_t \colon j \leq t \leq s\}$, as desired.

Now since by (2) we have $\deg(x_j\eta) = 1 + \deg(\eta) = 1 + \deg(\rho) - p^{r_s}\max(I_s) = 1 + \max\{\delta'_t \colon 1 \leq t \leq s\} \geq 1 + \max\{d_t \colon 1 \leq t \leq s\}$, we may apply Lemma 2.8 for $s-1$, $x_j\eta \in I'$ and q. Then there exists $e \leq m(\eta)$ such that $x_q^{p^{r_s}-1}(x_j\eta) = x_e^{p^{r_s}}\eta'$ for some $\eta' \in I'$. Thus $x_j\rho/x_{m(\rho)} = (x_e u_s/x_{m(u_s)})^{p^{r_s}}\eta'$ belongs to I, because I_s is stable. □

For a monomial u, we set $\text{pa}(u) = \max_{k \colon \mu_k \neq 0}\{D_k\}$, if u is not a multiple of x_1. Otherwise $u = x_1^{\mu_1}v$ such that $v \notin (x_1)$, and we set $\text{pa}(u) = \mu_1 + \text{pa}(v)$ (cf. Section 1). By our main theorem we have $\text{reg}\langle u\rangle = \text{pa}(u)$. More generally we get

Corollary 2.9. *Let I be a p-Borel ideal with Borel generators $u_1, \dots, u_m$. Then*

$$\text{reg}(I) \leq \max\{\text{pa}(u_1), \dots, \text{pa}(u_m)\},$$

and equality holds if I is principal p-Borel.

Proof. For each $i = 1, \dots, m$, the ideal $\text{reg}\langle u_i\rangle_{\geq d}$ is stable for $d \geq \text{pa}(u_i)$. Thus $I_{\geq d}$ is stable for $d \geq \max\{\text{pa}(u_1), \dots, \text{pa}(u_m)\}$. Therefore the assertion follows from 2.1. □

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