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Abstract

The aim of the present paper is to provide a state-space solution to the robust stabilisation problem with respect to the NLCF for two–time–scale systems. The construction of the robust controller uses the solutions of the control and filtering algebraic Riccati equations corresponding to the slow and to the fast dynamics of the nominal plant, respectively. These solutions do not depend upon the singular perturbation and therefore all computations are well–conditioned. It is also investigated the particular case when an ε-independent robust controller can be determined. Numerical examples illustrate the theoretical developments.

1 Introduction

Singularly perturbed systems have been intensively investigated over the last three decades. This interest is motivated by the wide area of applications in which such systems appear (see *e.g.* [12], [8]). More recently a special attention has been paid to the robust stabilization problem of two–time–scale systems for which a key role is played by the H^∞ theory (see *e.g.* [7], [9], [18] and the references therein). In the case of singularly perturbed systems the H^∞ control problem reveals specific aspects determined by the descomposition of the problem into slow and fast components. Different methods have been proposed to investigate this problem including the state–space solutions derived in [13], [15] and [16], the differential game–theoretic approach [10], [11] and frequency domain methods [6].

On the other hand the robust stabilization problem with respect to normalized left coprime factorization (NLCF) (see *e.g.* [9], [17]) has been frequently considered in the recent control literature. This is a very general type of unstructured uncertainty used in the robustness specifications ([19]) as well as in the loop-shaping procedures ([9]).

Although the robust stabilization problem with respect to the NLCF of the two–time–scale systems can be integrated in the general framework of H^∞ theory, there are some specific aspects which individualize this problem. The first is the fact that the computations become ill-conditioned for small values of the singular perturbation $\varepsilon > 0$. In order to avoid this feature the computations must be performed using the slow and the fast components of the nominal system without using its whole dynamics. The second specific aspect is the fact that the robustness radius with respect to the NLCF has an explicit formula in terms of the solutions of the control and filtering algebraic Riccati equations associated with the nominal plant.

The aim of the present paper is to provide a state-space solution to the robust stabilization problem with respect to the NLCF for two–time–scale systems. The construction of the robust controller uses the solutions of the control and filtering algebraic Riccati equations corresponding to the slow and to the fast dynamics of the nominal plant, respectively. These solutions do not depend upon the singular perturbation and therefore all computations are well–conditioned. It is also investigated the particular case when an ε-independent robust controller can be determined. Numerical examples illustrate the theoretical developments.

2 Problem statement and preliminaries

2.1 Problem statement

Consider the singularly perturbed system $G(\varepsilon)$ with the state-space equations:

$$\begin{aligned} \dot{x} &= A(\varepsilon)x + B(\varepsilon)u \\ y &= Cx \end{aligned} \tag{1}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; A(\varepsilon) = \begin{bmatrix} A_{11} & A_{12} \\ \frac{1}{\varepsilon}A_{21} & \frac{1}{\varepsilon}A_{22} \end{bmatrix}; B(\varepsilon) = \begin{bmatrix} B_1 \\ \frac{1}{\varepsilon}B_2 \end{bmatrix}; C = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

x_1 and x_2 are the state components corresponding to the slow and fast dynamics, respectively and $\varepsilon > 0$ is a small parameter. Let $\widetilde{M}(\varepsilon)$, $\widetilde{N}(\varepsilon)$ be the normalized left coprime factors of $G(\varepsilon)$, namely $\widetilde{M}(\varepsilon)$, $\widetilde{N}(\varepsilon)$ are stable, $G(\varepsilon) = \widetilde{M}^{-1}(\varepsilon)\widetilde{N}(\varepsilon)$ and $\widetilde{M}(\varepsilon)\widetilde{M}^*(\varepsilon) + \widetilde{N}(\varepsilon)\widetilde{N}^*(\varepsilon) = I$, $\widetilde{M}^*(\varepsilon)$, $\widetilde{N}^*(\varepsilon)$ denoting the adjoint of $\widetilde{M}(\varepsilon)$, $\widetilde{N}(\varepsilon)$ respectively. Then the robust stabilization problem with respect to NLCF consists in finding a controller $K(\varepsilon)$ stabilizing all perturbed systems $G_\Delta(\varepsilon) = \left(\widetilde{M}(\varepsilon) + \Delta_{\widetilde{M}}\right)^{-1}\left(\widetilde{N}(\varepsilon) + \Delta_{\widetilde{N}}\right)$ with $\Delta_{\widetilde{M}}$ and $\Delta_{\widetilde{N}}$ stable modelling uncertainty such that $\left\|\Delta_{\widetilde{M}} \;\; \Delta_{\widetilde{N}}\right\|_\infty < r$ where $r > 0$ is an imposed *robustness radius* (see *e.g.* [9], [17]).

2.2 Preliminaries

According to the results in [9] for the systems without singular perturbations, if a solution of the robust stabilization problem with respect to NLCF exists then the robust controller can be obtained in terms of the stabilizing solutions of the control and filtering algebraic Riccati equations:

$$A^TX + XA - XBB^TX + C^TC = 0 \tag{2}$$

$$AY + YA^T - YC^TCY + BB^T = 0, \tag{3}$$

respectively. In [9] it is also proved that the maximal robustness radius with respect to NLCF is given by $r_{max} = \gamma_{\min}^{-1}$ where $\gamma_{\min} = (1 + \rho(XY))^{\frac{1}{2}}$, $\rho(.)$ denoting the spectral radius.

Throughout the paper the following assumption is made:

Assumption A$_1$: *A_{22} in the state space representation (1) is invertible.*

Under the above assumption we can separate according to the theory of singular perturbations ([12]) the slow and the fast components of $G(\varepsilon)$, namely:

$$\begin{aligned} \dot{x}_s &= A_s x_s + B_s u_s \\ y_s &= C_s x_s + D_s u_s \end{aligned} \tag{4}$$

with:

$$\begin{aligned} A_s &= A_{11} - A_{12}A_{22}^{-1}A_{21} \\ B_s &= B_1 - A_{12}A_{22}^{-1}B_2 \\ C_s &= C_1 - C_2A_{22}^{-1}A_{21} \\ D_s &= -C_2A_{22}^{-1}B_2\,, \end{aligned}$$

and

$$\begin{aligned} \frac{dx_f}{d\tau} &= A_{22}x_f + B_2u_f \\ y_f &= C_2x_f\,, \end{aligned} \tag{5}$$

respectively, where $\tau = \frac{t}{\varepsilon}$.

Then the following additional assumption is made:

Assumption A$_2$: *the pairs (A_s, B_s), $(A_f, B_f)$are stabilizable and the pairs (C_s, A_s), (C_f, A_f) are detectable.*

Let us denote by X_s and Y_s the stabilizing solutions of the control and filtering Riccati equations associated to the slow dynamics (4), namely:

$$A_s^T X_s + X_s A_s - \left(X_s B_s + C_s^T D_s\right) S^{-1}\left(B_s^T X_s + D_s^T C_s\right) + C_s^T C_s = 0 \tag{6}$$

$$A_s Y_s + Y_s A_s^T - \left(Y_s C_s^T + B_s D_s^T\right) R^{-1}\left(C_s Y_s + D_s B_s^T\right) + B_s B_s^T = 0, \tag{7}$$

respectively, where $R := I + D_s D_s^T$, $S := I + D_s^T D_s$, and by X_f, Y_f the stabilizing solutions of the Riccati equations corresponding to the fast dynamics:

$$A_{22}^T X_f + X_f A_{22} - X_f B_2 B_2^T X_f + C_2^T C_2 = 0 \tag{8}$$

$$A_{22} Y_f + Y_f A_{22}^T - Y_f C_2^T C_2 Y_f + B_2 B_2^T = 0. \tag{9}$$

Recall that under assumption (A$_2$) these solutions exist and they are positive semidefinite.

Then the following result proved in [12] holds:

Proposition 1 *a) Under assumptions* (A_1) *and* (A_2) *there exists* $\hat{\varepsilon} > 0$ *such that for all* $\varepsilon \in (0, \hat{\varepsilon})$ *the Riccati equation (2) associated with the system (1) admits the stabilizing solution* $X(\varepsilon) \geq 0$ *with the following asymptotic structure:*

$$X(\varepsilon) := \begin{bmatrix} X_{11}(\varepsilon) & X_{12}(\varepsilon) \\ X_{12}^T(\varepsilon) & X_{22}(\varepsilon) \end{bmatrix} = \begin{bmatrix} X_s + \varepsilon \widehat{X}_{11}(\varepsilon) & \varepsilon \tilde{X}_{12} + \varepsilon^2 \widehat{X}_{12}(\varepsilon) \\ \varepsilon \tilde{X}_{12}^T + \varepsilon^2 \widehat{X}_{12}^T(\varepsilon) & \varepsilon X_f + \varepsilon^2 \widehat{X}_{22}(\varepsilon) \end{bmatrix} \tag{10}$$

where

$$\tilde{X}_{12} := -\left(A_{21}^T X_f + X_s A_{12} - X_s B_1 B_2^T X_f + C_1^T C_2\right)\left(A_{22} - B_2 B_2^T X_f\right)^{-1}$$

and $\left|\widehat{X}_{ij}(\varepsilon)\right| \leq c < \infty$ *for all* $\varepsilon \in (0, \hat{\varepsilon})$, $i, j = 1, 2$. *Moreover if* λ_{s_i}, $i = 1, ..., n_1$ *and* λ_{f_j}, $j = 1, ..., n_2$ *are the eigenvalues of* $A_s - B_s S^{-1}\left(B_s X_s + D_s^T C_s\right)$ *and* $A_{22} - B_2 B_2^T X_f$ *respectively, then for all* $\varepsilon \in (0, \hat{\varepsilon})$ *the matrix* $A(\varepsilon) - B(\varepsilon) B^T(\varepsilon) X(\varepsilon)$ *has* n_1 *eigenvalues* $\lambda_{s_i} + O(\varepsilon), i = 1, ..., n_1$ *and* n_2 *eigenvalues* $\frac{1}{\varepsilon}\lambda_{f_j} + O(\varepsilon)$, $j = 1, ..., n_2$ *where* $O(\varepsilon)$ *is a function such that* $\lim_{\varepsilon \to 0_+} O(\varepsilon) = 0$;

b) There exists $\tilde{\varepsilon} > 0$ *such that for all* $\varepsilon \in (0, \tilde{\varepsilon})$ *the filtering Riccati equation (3) associated with the system (1) has the stabilizing solution* $Y(\varepsilon) \geq 0$ *with the asymptotic structure:*

$$Y(\varepsilon) := \begin{bmatrix} Y_{11}(\varepsilon) & Y_{12}(\varepsilon) \\ Y_{12}^T(\varepsilon) & Y_{22}(\varepsilon) \end{bmatrix} = \begin{bmatrix} Y_s + \varepsilon \widehat{Y}_{11}(\varepsilon) & \tilde{Y}_{12} + \varepsilon \widehat{Y}_{12}(\varepsilon) \\ \tilde{Y}_{12}^T + \varepsilon \widehat{Y}_{12}^T(\varepsilon) & \frac{1}{\varepsilon}\left(Y_f + \varepsilon \widehat{Y}_{22}(\varepsilon)\right) \end{bmatrix} \tag{11}$$

where

$$\tilde{Y}_{12} := -\left(A_{12} Y_f + Y_s A_{21}^T - Y_s C_1^T C_2 Y_f + B_1 B_2^T\right)\left(A_{22}^T - C_2^T C_2 Y_f\right)^{-1}.$$

If μ_{s_i}, $i = 1, ..., n_1$ *and* μ_{f_j}, $j = 1, ..., n_2$ *are the eigenvalues of the matrices* $A_s - \left(B_s D_s^T + Y_s C_s^T\right) R^{-1} C_s$ *and* $A_f - Y_f C_2^T C_2$ *respectively, then for all* $\varepsilon \in (0, \tilde{\varepsilon})$ *the matrix* $A(\varepsilon) - Y(\varepsilon) C^T(\varepsilon) C(\varepsilon)$ *has* n_1 *eigenvalues* $\mu_{s_i} + O(\varepsilon), i = 1, ..., n_1$ *and* n_2 *eigenvalues* $\frac{1}{\varepsilon}\mu_{f_j} + O(\varepsilon)$, $j = 1, ..., n_2$.□

A direct consequence of the above proposition is the following corollary:

Corollary 1 *If* $r_{\max}(\varepsilon)$ *denotes the maximal stability radius of* $G(\varepsilon)$ *with respect to NLCF then*

$$\lim_{\varepsilon \to 0_+} r_{\max}(\varepsilon) = \min\left\{\left(1 + \rho(X_s Y_s)\right)^{-\frac{1}{2}}, \left(1 + \rho(X_f Y_f)\right)^{-\frac{1}{2}}\right\}. \square \tag{12}$$

A similar result to Proposition 1 has been proved in [3] for game-theoretic Riccati equations; a modified version of this result, useful in the next section is the following proposition:

Proposition 2 *Let:*

$$
\begin{aligned}
A(\varepsilon) &= \begin{bmatrix} A_{11} & A_{12} \\ \frac{1}{\varepsilon}A_{21} & \frac{1}{\varepsilon}A_{22} \end{bmatrix}; B_1(\varepsilon) = \begin{bmatrix} B_{11} \\ \frac{1}{\varepsilon}B_{12} \end{bmatrix}; B_2(\varepsilon) = \begin{bmatrix} B_{21} \\ \frac{1}{\varepsilon}B_{22} \end{bmatrix}; \\
C &= \begin{bmatrix} C_1 & C_2 \end{bmatrix}
\end{aligned}
$$

with A_{22} invertible and consider the slow dynamics:

$$
\begin{aligned}
A_s &= A_{11} - A_{12}A_{22}^{-1}A_{21} \\
B_{is} &= B_{i1} - A_{12}A_{22}^{-1}B_{2i} \\
C_s &= C_1 - C_2A_{22}^{-1}A_{21} \\
D_{is} &= -C_2A_{22}^{-1}B_{2i}, \ i = 1, 2.
\end{aligned}
$$

Assume that the game-theoretic Riccati equations associated to the slow component above and to the fast dynamics, namely:

$$A_s^T Z_s + Z_s A_s - \left(Z_s B_s + C_s^T D_s\right) \tilde{S}^{-1} \left(B_s^T Z_s + D_s^T C_s\right) + C_s^T C_s = 0$$

and

$$A_{22}^T Z_f + Z_f A_{22} + Z_f \left(B_{21}B_{21}^T - B_{22}B_{22}^T\right) Z_f + C_2^T C_2 = 0$$

with

$$B_s := \begin{bmatrix} B_{1s} & B_{2s} \end{bmatrix}, D_s := \begin{bmatrix} D_{1s} & D_{2s} \end{bmatrix}, \tilde{S} := \begin{bmatrix} D_{1s}^T D_{1s} - I & D_{1s}^T D_{2s} \\ D_{2s}^T D_{1s} & D_{2s}^T D_{2s} \end{bmatrix}$$

have the stabilizing solutions $Z_s \geq 0$ and $Z_f \geq 0$, respectively. Then there exists $\hat{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \hat{\varepsilon})$ the game-theoretic Riccati equation

$$A^T(\varepsilon) Z + ZA(\varepsilon) + Z\left(B_1(\varepsilon) B_1^T(\varepsilon) - B_2(\varepsilon) B_2^T(\varepsilon)\right) Z + C^T C = 0$$

has the stabilizing solution $Z(\varepsilon) \geq 0$ with the following asymptotic structure:

$$Z(\varepsilon) = \begin{bmatrix} Z_s + \varepsilon\tilde{Z}_{11}(\varepsilon) & \varepsilon Z_{12} + \varepsilon^2 \tilde{Z}_{12}(\varepsilon) \\ \varepsilon Z_{12}^T + \varepsilon^2 \tilde{Z}_{12}^T(\varepsilon) & \varepsilon Z_f + \varepsilon^2 \tilde{Z}_{22}(\varepsilon) \end{bmatrix} \tag{13}$$

where

$$\begin{aligned} Z_{12} &= -\left(\left(A_{21} + \left(B_{12}B_{11}^T - B_{22}B_{21}^T\right) Z_s\right)^T + Z_s A_{12}^T + C_1^T C_2\right) \\ &\times \left(A_{22} + \left(B_{12}B_{12}^T - B_{22}B_{22}^T\right) Z_f\right)^{-1} \end{aligned}$$

and

$$\left|\tilde{Z}_{ij}(\varepsilon)\right| \leq c < \infty \quad \text{forall } \varepsilon \in (0, \hat{\varepsilon})$$

Moreover if λ_{s_i}, $i = 1, ..., n_1$ *and* λ_{f_j}, $j = 1, ..., n_2$ *are the eigenvalues of* $A_s + B_s\tilde{S}^{-1}\left(B_s Z_s + D_s^T C_s\right)$ *and* $A_{22} + \left(B_{21}B_{21}^T - B_{22}B_{22}^T\right) Z_f$ *respectively, then for all* $\varepsilon \in (0, \hat{\varepsilon})$ *the matrix* $A(\varepsilon) - \left(B_1(\varepsilon) B_1^T(\varepsilon) - B_2(\varepsilon) B_2^T(\varepsilon)\right) Z(\varepsilon)$ *has* n_1 *eigenvalues* $\lambda_{s_i} + O(\varepsilon), = 1, ..., n_1$ *and* n_2 *eigenvalues* $\frac{1}{\varepsilon}\lambda_{f_j} + O(\varepsilon)$, $j = 1, ..., n_2$.□

The following result is proved in [14]:

Proposition 3 *Consider the two-time-scale system::*

$$\begin{aligned} \dot{x}_1 &= A_{11}(\varepsilon) x_1 + A_{12}(\varepsilon) x_2 + B_1(\varepsilon) u \\ \varepsilon \dot{x}_2 &= A_{21}(\varepsilon) x_1 + A_{22}(\varepsilon) x_2 + B_2(\varepsilon) u \\ y &= C_1(\varepsilon) x_1 + C_2(\varepsilon) x_2 + D(\varepsilon) u \end{aligned}$$

where $A_{ij}(\varepsilon), B_i(\varepsilon), C_i(\varepsilon), D(\varepsilon)$ *are* $\boldsymbol{C}^1$ *functions with respect to* ε. *Assume that* $A_{22}(0)$ *is invertible and the matrices* $A_{22}(0)$ *and* $A_{11}(0) - A_{12}(0) A_{22}^{-1}(0) A_{21}(0)$ *are stable. Then there exists an* $\hat{\varepsilon} > 0$ *such that for all* $\varepsilon \in (0, \hat{\varepsilon})$ *we have*

a) The above system is stable;

b) If $G(\varepsilon)$ *is the transfer matrix of the system and* $G^0(\varepsilon)$ *is the transfer matrix of*

the system

$$\begin{aligned} \dot{x}_1 &= A_{11}(0) x_1 + A_{12}(0) x_2 + B_1(0) u \\ \varepsilon \dot{x}_2 &= A_{21}(0) x_1 + A_{22}(0) x_2 + B_2(0) u \\ y &= C_1(0) x_1 + C_2(0) x_2 + D(0) u \end{aligned}$$

then there exists $\alpha > 0$ *such that*

$$\left\|G(\varepsilon) - G^0(\varepsilon)\right\|_\infty < \alpha\varepsilon$$

for all $\varepsilon \in (0, \hat{\varepsilon})$.□

Another useful result for the next developments is ([10]):

Proposition 4 *Let $T(\varepsilon)$ be a two-time-scale system with both slow and fast components T_s and T_f, respectively, assumed stable. Then:*

$$\lim_{\varepsilon \to 0_+} \|T(\varepsilon)\|_\infty = \max\left\{\|T_s\|_\infty, \|T_f\|_\infty\right\}. \square$$

3 Robust controller with respect to the NLCF

The main result of this section is given by the following theorem:

Theorem 1 *Under the assumptions (A_1) and (A_2) for any $\gamma > \gamma_o$ where*

$$\gamma_o = \left[1 + \max\left\{\rho(X_s Y_s), \rho(X_f Y_f)\right\}\right]^{\frac{1}{2}}, \tag{14}$$

there exists an $\hat{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \hat{\varepsilon})$ the two-time-scale controller:

$$\begin{aligned} K(\varepsilon) \quad : &= (A_k(\varepsilon), B_k(\varepsilon), C_k) \\ &= \left(\begin{bmatrix} A_{k11} & A_{k12} \\ \frac{1}{\varepsilon}A_{k21} & \frac{1}{\varepsilon}A_{k22} \end{bmatrix}, \begin{bmatrix} B_{k1} \\ \frac{1}{\varepsilon}B_{k2} \end{bmatrix}, \begin{bmatrix} C_{k1} & C_{k2} \end{bmatrix} \right) \end{aligned} \tag{15}$$

with:

$$\begin{aligned} A_{k11} &= A_{11} - Y_s C_1^T C_1 - B_1\left(B_1^T Z_s + B_2^T Z_{12}^T\right) \\ A_{k12} &= A_{12} - Y_s C_1^T C_2 - B_1 B_2^T Z_f \\ A_{k21} &= A_{21} - Y_f C_2^T C_1 - B_2\left(B_1^T Z_s + B_2^T Z_{12}^T\right) \\ A_{k22} &= A_{22} - Y_f C_2^T C_2 - B_2 B_2^T Z_f \\ B_{k1} &= Y_s C_1^T \\ B_{k2} &= Y_f C_2^T \\ C_{k1} &= -\left(B_1^T Z_s + B_2^T Z_{12}^T\right) \\ C_{k2} &= -B_2^T Z_f \end{aligned}$$

where

$$\begin{aligned} Z_s \quad : &= -\gamma^2\left(\left(1-\gamma^2\right)I + X_s Y_s\right)^{-1} X_s \\ Z_{12} \quad : &= -\gamma^2\left(1-\gamma^2\right)\left(\left(1-\gamma^2\right)I + X_s Y_s\right)^{-1} \\ & \left(\tilde{X}_{12} - \left(\tilde{X}_{12} Y_f + X_s \tilde{Y}_{12}\right)\left(\left(1-\gamma^2\right)I + X_f Y_f\right)^{-1} X_f\right) \\ Z_f \quad : &= -\gamma^2\left(\left(1-\gamma^2\right)I + X_f Y_f\right)^{-1} X_f, \end{aligned}$$

stabilizes all $G_\Delta(\varepsilon) = \left(\widetilde{M}(\varepsilon) + \Delta_{\widetilde{M}}\right)^{-1}\left(\widetilde{N}(\varepsilon) + \Delta_{\widetilde{N}}\right)$ *with* $\Delta_{\widetilde{M}}$ *and* $\Delta_{\widetilde{N}}$ *stable modelling uncertainty such that* $\left\|\Delta_{\widetilde{M}} \;\; \Delta_{\widetilde{N}}\right\|_\infty < \gamma^{-1}$.

Proof It is a know fact (see *e.g.* [9]) that the robust stabilization for a nominal system $G := (A, B, C)$ with respect to the NLCF can be reduced to an H^∞ control problem corresponding to the following generalized system

$$\begin{aligned} \dot{x} &= Ax - Hu_1 + Bu_2 \\ y_1 &= \begin{bmatrix} C \\ 0 \end{bmatrix} x + \begin{bmatrix} I \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ I \end{bmatrix} u_2 \\ y_2 &= Cx + u_1 \end{aligned} \tag{16}$$

where

$$H := -YC^T \tag{17}$$

with Y denoting the stabilizing positive definite solution of the filtering algebraic Riccati equation (3). Moreover, according to [1], a controller K is a solution to the H^∞ control problem for (16) if and only if it is also an H^∞ controller for a certain outer system (Theorem 3 in [1]) associated with the generalized system; in our case this outer system has the following state-space equations:

$$\begin{aligned} \dot{x} &= \left(A - \gamma^{-\frac{1}{2}} HF_1\right) x - \gamma\left(\gamma^2 - 1\right)^{-\frac{1}{2}} Hu_1 + Bu_2 \\ y_1 &= -\gamma^{-\frac{1}{2}} F_2 x \\ y_2 &= \left(C + \gamma^{-\frac{1}{2}} F_1\right) x + \gamma\left(\gamma^2 - 1\right)^{-\frac{1}{2}} u_1 \end{aligned} \tag{18}$$

with

$$\begin{aligned} F_1 &:= \gamma^{\frac{1}{2}}\left(\gamma^2 - 1\right)^{-1}\left(C - H^T Z\right) \\ F_2 &:= -\gamma^{-\frac{1}{2}} B^T Z \\ Z &:= -\gamma^2\left(\left(1 - \gamma^2\right) I + XY\right)^{-1} X \end{aligned} \tag{19}$$

where X and Y are the stabilizing positive semidefinite solutions of (2) and (3) respectively.

When coupling the controller $K := (A_k, B_k, C_k)$ to (18) one obtains the resulting system:

$$\begin{aligned} \dot{x} &= \left(A - \gamma^{-\frac{1}{2}} HF_1\right) x + BC_k x_k - \gamma\left(\gamma^2 - 1\right)^{-\frac{1}{2}} Hu_1 \\ \dot{x}_k &= B_k\left(C + \gamma^{-\frac{1}{2}} F_1\right) x + A_k x_k + \gamma\left(\gamma^2 - 1\right)^{-\frac{1}{2}} B_k u_1 \\ y_1 &= -\gamma^{-\frac{1}{2}} F_2 x + C_k x_k\,, \end{aligned} \tag{20}$$

where x_k denotes the state of the controller. We shall show in the following that for all $\varepsilon \in (0, \hat{\varepsilon})$ and for A_k, B_k, C_k given by (15) the system (20) is stable and its H^∞ norm is less than γ.

In order to prove that (20) is stable let us consider the nonsingular transformation:

$$T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$$

and denote by $e(t) := x(t) - x_k(t)$. Then (20) can be written in the equivalent form:

$$\begin{aligned} \dot{x} &= \left(A - \gamma^{-\frac{1}{2}} H F_1 + B C_k\right) x - B C_k e + \gamma \left(\gamma^2 - 1\right)^{-\frac{1}{2}} H u_1 \\ \dot{e} &= \left(A - \gamma^{-\frac{1}{2}} (H + B_k) F_1 - B_k C + B C_k - A_k\right) x - (B C_k - A_k) e \\ &\quad -\gamma \left(\gamma^2 - 1\right)^{-\frac{1}{2}} (H + B_k) u_1 \\ y_1 &= \left(-\gamma^{-\frac{1}{2}} F_2 + C_k\right) x - C_k e \end{aligned} \tag{21}$$

According to Proposition 1, there exists $\hat{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \hat{\varepsilon})$ the stabilizing solutions $X(\varepsilon)$ and $Y(\varepsilon)$ of (2) and (3) have the expressions (10) and (11), respectively. Then one obtains the following structure of H defined by (17):

$$H(\varepsilon) = \begin{bmatrix} H_1(\varepsilon) \\ \frac{1}{\varepsilon} H_2(\varepsilon) \end{bmatrix} = -\begin{bmatrix} Y_s C_1^T + O(\varepsilon) \\ \frac{1}{\varepsilon} Y_f C_2^T + O(\varepsilon) \end{bmatrix}. \tag{22}$$

Similarly, direct algebraic computations give the following expressions for $F_1(\varepsilon)$ and $F_2(\varepsilon)$:

$$\begin{aligned} F_1(\varepsilon) &= \gamma^{\frac{1}{2}} \left(\gamma^2 - 1\right)^{-1} \left[C_1 (I + Y_s Z_s) + C_2 \left(Y_{12}^T Z_s + Y_f Z_{12}^T\right) + O(\varepsilon) \right. \\ &\qquad \left. C_2 (I + Y_f Z_f) + O(\varepsilon)\right] \\ F_2(\varepsilon) &= \gamma^{-\frac{1}{2}} \left[\; B_1^T Z_s + B_2^T Z_{12}^T + O(\varepsilon) \quad B_2^T Z_f + O(\varepsilon) \;\right]. \end{aligned} \tag{23}$$

Let us denote

$$L(\varepsilon) := H(\varepsilon) F_1(\varepsilon) = \begin{bmatrix} L_{11} + O(\varepsilon) & L_{12} + O(\varepsilon) \\ \frac{1}{\varepsilon} L_{21} + O(\varepsilon) & \frac{1}{\varepsilon} L_{22} + O(\varepsilon) \end{bmatrix}. \tag{24}$$

Then replace (15) in (21) together with (22)-(24) obtaining thus the following two-time-scale structure for (21):

$$\begin{aligned} \dot{z}_1 &= \mathcal{A}_{11}(\varepsilon) z_1 + \mathcal{A}_{12}(\varepsilon) z_2 + \mathcal{B}_1(\varepsilon) u_1 \\ \dot{z}_2 &= \frac{1}{\varepsilon} \mathcal{A}_{21}(\varepsilon) z_1 + \frac{1}{\varepsilon} \mathcal{A}_{22}(\varepsilon) z_2 + \frac{1}{\varepsilon} \mathcal{B}_2(\varepsilon) u_1 \\ y_1 &= \mathcal{C}_1 z_1 + \mathcal{C}_2(\varepsilon) z_2 \end{aligned} \tag{25}$$

where z_1 includes the slow components of x and e, z_2 incorporates the fast components of x and e, and

$$\begin{aligned}
\mathcal{A}_{11}(\varepsilon) &= \begin{bmatrix} A_{11} - \gamma^{-\frac{1}{2}} L_{11} - B_1 \left(B_1^T Z_s + B_2^T Z_{12}^T \right) + O(\varepsilon) & B_1 \left(B_1^T Z_s + B_2^T Z_{12}^T \right) \\ O(\varepsilon) & A_{11} - Y_s C_1^T C_1 \end{bmatrix} \\
\mathcal{A}_{12}(\varepsilon) &= \begin{bmatrix} A_{12} - \gamma^{-\frac{1}{2}} L_{12} - B_1 B_2^T Z_f + O(\varepsilon) & B_1 B_2^T Z_f \\ O(\varepsilon) & A_{12} - Y_s C_1^T C_2 \end{bmatrix} \\
\mathcal{A}_{21}(\varepsilon) &= \begin{bmatrix} A_{21} - \gamma^{-\frac{1}{2}} L_{21} - B_2 \left(B_1^T Z_s + B_2^T Z_{12}^T \right) + O(\varepsilon) & B_2 \left(B_1^T Z_s + B_2^T Z_{12}^T \right) \\ O(\varepsilon) & A_{21} - Y_f C_2^T C_1 \end{bmatrix} \\
\mathcal{A}_{22}(\varepsilon) &= \begin{bmatrix} A_{22} - \gamma^{-\frac{1}{2}} L_{22} - B_2 B_2^T Z_f + O(\varepsilon) & B_2 B_2^T Z_f \\ O(\varepsilon) & A_{22} - Y_f C_2^T C_2 \end{bmatrix} \\
\mathcal{B}_1(\varepsilon) &= \gamma \left(\gamma^2 - 1 \right)^{-\frac{1}{2}} \begin{bmatrix} Y_s C_1^T + O(\varepsilon) \\ O(\varepsilon) \end{bmatrix} \\
\mathcal{B}_2(\varepsilon) &= -\gamma \left(\gamma^2 - 1 \right)^{-\frac{1}{2}} \begin{bmatrix} Y_f C_2^T + O(\varepsilon) \\ O(\varepsilon) \end{bmatrix} \\
\mathcal{C}_1 &= \left(B_1^T Z_s + B_2^T Z_{12}^T \right) \begin{bmatrix} \left(\gamma^{-1} + 1 \right) I & I \end{bmatrix} \\
\mathcal{C}_2(\varepsilon) &= B_2^T Z_f \begin{bmatrix} \left(\gamma^{-1} + 1 \right) I + O(\varepsilon) & I \end{bmatrix}
\end{aligned} \tag{26}$$

The fast dynamics of (26) has the state matrix $\mathcal{A}_{22}(0)$ which is triangular and its eigenvalues are:

$$\Lambda\left(\mathcal{A}_{22}(0)\right) = \Lambda\left(A_{22} - \gamma^{-\frac{1}{2}} L_{22} - B_2 B_2^T Z_f \right) \cup \Lambda\left(A_{22} - Y_f C_2^T C_2 \right),$$

where $\Lambda(\cdot)$ denotes the spectrum of $(\cdot)$. Since $\gamma > \gamma_o$, Z_f given by (15) is the stabilizing solution of the game-theoretic Riccati equation ([9])

$$A_{22}^T Z_f + Z_f A_{22} - Z_f B_2 B_2^T Z_f + C_2^T C_2 - \left(1 - \gamma^2\right)^{-1} \left(I + Z_f Y_f\right) C_2^T C_2 \left(I + Y_f Z_f\right) = 0$$

which fact implies that the matrix

$$A_{22} - \left(1 - \gamma^2\right)^{-1} Y_f C_2^T C_2 - \left(B_2 B_2^T + \left(1 - \gamma^2\right)^{-1} Y_f C_2^T C_2 Y_f \right) Z_f$$

is stable. Direct computations using (22), (23), (25) reveal that the matrix above is just $A_{22} - \gamma^{-\frac{1}{2}} L_{22} - B_2 B_2^T Z_f$ and hence this matrix is stable. On the other hand, according to the assumptions of the theorem, Y_f is the stabilizing solution of (10) and hence $A_{22} - Y_f C_2^T C_2$ is stable. Thus we conclude that $\mathcal{A}_{22}(0)$ is stable.

Consider now the slow dynamics of (25) which state matrix is

$$\mathcal{A}_s := \mathcal{A}_{11}(0) - \mathcal{A}_{12}(0)\,\mathcal{A}_{22}^{-1}\mathcal{A}_{21}(0) \,. \tag{27}$$

Using (26) one obtains that A_s has the following asymptotic structure:

$$\mathcal{A}_s = \begin{bmatrix} \mathcal{A}_{s11} & * \\ 0 & \mathcal{A}_{s22} \end{bmatrix}$$

where $\star$ denotes an irrelevant entry and

$$\begin{aligned} \mathcal{A}_{s11} &= A_{11} - \gamma^{-\frac{1}{2}}L_{11} - B_1\left(B_1^T Z_s + B_2^T Z_{12}^T\right) \\ &\quad - \left(A_{12} - \gamma^{-\frac{1}{2}}L_{12} - B_1 B_2^T Z_f\right)\left(A_{22} - \gamma^{-\frac{1}{2}}L_{22} - B_1 B_2^T Z_f\right)^{-1} \\ &\quad \times \left(A_{21} - \gamma^{-\frac{1}{2}}L_{21} - B_2\left(B_1^T Z_s + B_2^T Z_{12}^T\right)\right) \\ \mathcal{A}_{s22} &= A_{11} - Y_s C_1^T C_1 - \left(A_{12} + Y_s C_1^T C_2\right)\left(A_{22} - Y_f C_2^T C_2\right)^{-1}\left(A_{21} - Y_f C_2^T C_1\right) \end{aligned} \tag{28}$$

It is easy to check that $\mathcal{A}_{s11}$ coincides with the slow dynamics of the two-time-scale system having the state matrix:

$$\mathcal{M}(\varepsilon) = A(\varepsilon) - \gamma^{-\frac{1}{2}}L(\varepsilon) + \gamma^{-\frac{1}{2}}B(\varepsilon)\,F_2(\varepsilon) \,. \tag{29}$$

On the other hand, for ε close to zero, the solutions of the Riccati equations (2) and (3) have the asymptotic structure (10) and (11) respectively. Moreover since $Z(\varepsilon)$ given by (13) is the stabilizing solution to the game-theoretic Riccati equation:

$$\begin{aligned} &A^T(\varepsilon)\,Z(\varepsilon) + Z(\varepsilon)\,A(\varepsilon) - Z(\varepsilon)\,B(\varepsilon)\,B^T(\varepsilon)\,Z(\varepsilon) + C^T C \\ &-\left(1-\gamma^2\right)^{-1}\left(I + Z(\varepsilon)\,Y(\varepsilon)\right)C^T C\left(I + Y(\varepsilon)\,Z(\varepsilon)\right) = 0 \,, \end{aligned}$$

it follows that $A(\varepsilon) - \left(1-\gamma^2\right)^{-1}Y(\varepsilon)\,C^T C\left(I + Y(\varepsilon)\,Z(\varepsilon)\right) - B(\varepsilon)\,B^T(\varepsilon)\,Z(\varepsilon)$ is stable. Using (19) one can check that this matrix is just $\mathcal{M}(\varepsilon)$ defined by (29) which fact shows that $\mathcal{M}(\varepsilon)$ is stable for all $\varepsilon \in (0, \hat{\varepsilon})$. Since X_s, Y_s, X_f, Y_f are the stabilizing solutions of the Riccati equations (6), (7), (8) and (9) respectively, it results that the limit of the slow modes of $\mathcal{M}(\varepsilon)$ have negative real parts and therefore $\mathcal{A}_{s11}$ is stable (for more details see [3]).

In order to prove that $\mathcal{A}_{s22}$ is stable we notice that this matrix is just the state matrix of slow dynamics of the system with the state matrix $A(\varepsilon) + H(\varepsilon)\,C$, which fact results by direct computations. The asymptotic structure of the eigenvalues of

$A(\varepsilon) + H(\varepsilon)C$ given in Proposition 1 shows that $\mathcal{A}_{s22}$ is stable. Thus we proved that the slow dynamics of (25) is stable.

Since the two-time-scale system (25) has both slow and fast components stable we conclude, according to the Klimusev-Krasovski theorem that (25) is stable for all $\varepsilon \in (0, \hat{\varepsilon})$.

We prove now that $K(\varepsilon)$ given by (15) is a γ-attenuating controller for the generalized system (18). To this end consider the modified controller $\mathcal{K}(\varepsilon) = (\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k)$ where

$$\begin{aligned} \mathcal{A}_k &= A + H(\varepsilon)C - BB^T Z(\varepsilon) \\ \mathcal{B}_k &= -H(\varepsilon) \\ \mathcal{C}_k &= -B^T Z(\varepsilon) \end{aligned} \tag{30}$$

with $H(\varepsilon)$ and $Z(\varepsilon)$ defined by (17) and (19) respectively where $X(\varepsilon)$ and $Y(\varepsilon)$ have the asymptotic structures given by (10) and (11), respectively. When coupling the controller (30) to the system (18) one obtains the following two-time-scale resulting system:

$$\begin{aligned} \dot{x}_{r1} &= A_{r11}(\varepsilon)x_{r1} + A_{r12}(\varepsilon)x_{r2} + B_{r1}(\varepsilon)u_1 \\ \dot{x}_{r2} &= \frac{1}{\varepsilon}A_{r21}(\varepsilon)x_{r1} + \frac{1}{\varepsilon}A_{r22}(\varepsilon)x_{r2} + \frac{1}{\varepsilon}B_{r2}(\varepsilon)u_1 \\ y_1 &= C_{r1}(\varepsilon)x_{r1} + C_{r2}(\varepsilon)x_{r2} + D_r(\varepsilon)u_1 . \end{aligned} \tag{31}$$

Direct computations show that the system obtained by coupling $K(\varepsilon)$ to (18) has the state-space equations:

$$\begin{aligned} \dot{x}_{r1} &= A_{r11}(0)x_{r1} + A_{r12}(0)x_{r2} + B_{r1}(0)u_1 \\ \dot{x}_{r2} &= \frac{1}{\varepsilon}A_{r21}(0)x_{r1} + \frac{1}{\varepsilon}A_{r22}(0)x_{r2} + \frac{1}{\varepsilon}B_{r2}(0)u_1 \\ y_1 &= C_{r1}(0)x_{r1} + C_{r2}(0)x_{r2} + D_r(0)u_1 . \end{aligned} \tag{32}$$

Then according to Proposition 3 it results that for ε small enough the H^∞ norm of (32) is less than γ.

4 A particular case: the ε-independent robust controller

In the present section we consider an important particular case namely the situation when the fast dynamics of $G(\varepsilon)$ is stable. We shall first investigate the robustness properties with respect to the NLCF in this case and then we shall describe a procedure to determine an ε-independent optimal robust controller.

4.1 The robustness radius with respect to NLCF of stable two–time–scale systems

The following result shows that if the fast dynamics is stable then one can stabilize $G(\varepsilon)$ by using an ε-independent controller which robustness properties with respect to the NLCF are investigated . In fact this result extends of the result deduced in [15] where the case $C_2 (j\omega I - A_{22})^{-1} B_2 \equiv 0$ is considered.

Theorem 2 *If the fast dynamics of $G(\varepsilon)$ is stable then any stabilizing controller for its slow dynamics is also a stabilizing controller for $G(\varepsilon)$ for $\varepsilon \to 0_+$. Moreover in this case, the robustness radius of $G(\varepsilon)$ with respect to the NLCF for is*

$$\lim_{\varepsilon\to 0_+} \tilde{r}_{\max} = \min\left\{ (1+\rho(X_sY_s))^{-\frac{1}{2}}, \left(1+\|G_f\|_\infty\right)^{-1} \right\}, \qquad (33)$$

where G_f denotes the fast dynamics of $G(\varepsilon)$.

Proof Let K_s be a stabilizing controller for the slow dynamics of $G(\varepsilon)$. Simple computations show that by coupling this controller to the generalized system (18) one obtains a two-time-scale resulting system which fast dynamics is

$$\mathcal{T}_f := \left(A_{22}, Y_f C_2^T, C_2, I\right). \qquad (34)$$

Moreover the slow dynamics of the resulting system coincides with the dynamics obtained by coupling the controller to the slow dynamics of (18). It follows that under the assumptions in the statement both fast and slow dynamics are stable and therefore the resulting two–time–scale system is stable; thus we conclude that K_s is a stabilizing controller for $G(\varepsilon)$ when $\varepsilon \to 0_+$.

On the other hand if we denote by $\mathcal{T}(\varepsilon)$ and $\mathcal{T}_s$ the transfer matrices of the resulting system and of its slow dynamics respectively, according to Proposition 4 it follows that

$$\lim_{\varepsilon\to 0_+} \tilde{r}_{\max} = \min\left\{ \|\mathcal{T}_s\|_\infty^{-1}, \|\mathcal{T}_f\|_\infty^{-1} \right\} \qquad (35)$$

From (34) it results that $\mathcal{T}_f = \widetilde{M}_f^{-1}$, where $\widetilde{M}_f$ is the invertible factor from the NLCF of G_f. Then $\widetilde{M}_f\widetilde{M}_f^* + \widetilde{N}_f\widetilde{N}_f^* = I$ gives that $I + G_f G_f^* = \widetilde{M}_f^{-1}\left(\widetilde{M}_f^{-1}\right)^*$ and therefore

$$\left\|\widetilde{M}_f^{-1}\right\|_\infty = \left\|\left[\begin{array}{cc} I & G_f \end{array}\right]\right\|_\infty = 1 + \|G_f\|_\infty . \qquad (36)$$

Since the robustness radius of T_s equals $(1+\rho(X_sY_s))^{-\frac{1}{2}}$, (33) from the statement directly follows from (35) and (36) and the proof ends. □

Remark 1 *Theorem 2 shows that if the fast component of $G(\varepsilon)$ is stable then one can use an ε-independent controller determined for the slow dynamics with*

$$\gamma > \max\left\{(1+\rho(X_sY_s))^{\frac{1}{2}}, (1+\|G_f\|_\infty)\right\}.$$

Remark 2 *Since the robustness radius $\tilde{r}_{\max}$ given by (33) corresponds to the case when ε-independent stabilizing controllers are considered, it results that $\tilde{r}_{\max} \leq r_{\max}$ for $\varepsilon \to 0_+$ where $r_{\max}$ given by (12) is obtained by two-time-scale controllers. Comparing (33) with (35) one obtains that*

$$1+\|G_f\|_\infty \geq (1+\rho(X_fY_f))^{\frac{1}{2}}.$$

The above inequality together with (12) and (33) reveals in fact the deterioration of the robustness radius when ε-independent controllers are used for the robust stabilization of $G(\varepsilon)$.

Remark 3 *Like a consequence of Theorem 2 we can investigate the influence of the small time constants induced by the sensors and actuators delay upon the robustness radius of a given plant. Indeed, let $G := (A, B, C, D)$ be a nominal system without singular perturbations and assume that the sensors introduce a small delay $\varepsilon > 0$, that is their transfer function are $H_{sens}(s) = 1/(\varepsilon s+1)$. Then the state equations of G together the dynamics of the sensors are*

$$\begin{aligned} \dot{x} &= Ax + Bu \\ \varepsilon\dot{z} &= Cx - z + Du. \end{aligned} \tag{37}$$

The fast dynamics of (37) is $G_f = (-I, D, I, 0)$; then from Theorem 2 it results that the following two cases can occur:

a) If $D = 0$ then $\|G_f\|_\infty = 0$ and, according to Theorem 2 the robustness radius with respect to the NLCF of (37) for $\varepsilon \to 0_+$ coincides with the one of G since in this case $G_f = 0$ and therefore

$$(1+\rho(X_sY_s))^{-\frac{1}{2}} < (1+\|G_f\|_\infty)^{-1}.$$

b) If $D \neq 0$ the robustness properties of (37) with respect to the NLCF can be worse than the robustness properties of G; this happens if

$$1+\rho(D^TD) > (1+\rho(X_sY_s))^{\frac{1}{2}}.$$

Remark 4 *Another particular situation can appear when a loop-shaping procedure is used. Indeed, in order to improve the sensitivity and its complementary performances, the weighting functions $W_1(s)$ and $W_2(s)$ respectively, are chosen such that the 'shaped' system $G_{sh} := W_2 G W_1$ satisfies the desired loop shape ([9]). Since usually the magnitude of complementary sensitivity is low at high frequency, the weighting function $W_2(s)$ introduces a fast dynamics such that G_{sh} can be regarded as a two-time-scale system. By choosing for example*

$$W_2(s) = C_w (sI - A_w)^{-1} B_w + D_w \text{ with } A_w = -\frac{1}{\varepsilon} I \text{ and } \varepsilon > 0,$$

and assuming for simplicity that $W_1(s) = I$, one obtains that $G_{sh} = W_2 G$ has the following state-space equations:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ \dot{w} &= B_w C x - \frac{1}{\varepsilon} w + B_w D u \\ y &= D_w C x + C_w w + D_w D u \end{aligned} \tag{38}$$

which denotes for small values of ε a two-time-scale system with the fast dynamics

$$G_f = (-I, 0, C_w, D_w D) = D_w D.$$

Thus it follows that if $D_w = 0$ or $D = 0$ then $\tilde{r}_{\max}$ given by (33) equals $(1 + \rho(X_s Y_s))^{-\frac{1}{2}}$ and therefore, according to Remark 1 the ε-independent robust controller with respect to the NLCF for the slow dynamics of (38) is also a robust controller for G_{sh}.

4.2 An optimal robust controller with respect to the NLCF

According to Theorem 2, two situations may occur concerning the robustness radius of $G(\varepsilon)$ with stable fast dynamics:

a) The case, when $(1 + \rho(X_s Y_s))^{-\frac{1}{2}} > (1 + \|G_f\|_\infty)^{-1}$; in this situation an optimal robust controller for $G(\varepsilon)$ when $\varepsilon \to 0_+$ can be obtained by determining the robust controller with respect to the NLCF corresponding to the slow dynamics G_s of $G(\varepsilon)$ for $\gamma = 1 + \|G_f\|_\infty$. Such a controller can be determined since the maximum robustness radius G_s equals $(1 + \rho(X_s Y_s))^{-\frac{1}{2}}$.

b) If $(1 + \rho(X_s Y_s))^{-\frac{1}{2}} < (1 + \|G_f\|_\infty)^{-1}$, then (33) gives that

$$\lim_{\varepsilon \to 0_+} \tilde{r}_{\max} = (1 + \rho(X_s Y_s))^{-\frac{1}{2}}.$$

In this case, when $\varepsilon \to 0_+$, the optimal robust controller for $G(\varepsilon)$ coincides with the optimal robust controller for G_s . In order to determine such a controller for G_s, one can use a known result proved in [9] which states that an optimal robust controller with respect to the NLCF is given by $K = UV^{-1}$ where U and V are the stable solution of the optimal one-block Nehari problem:

$$\min_{U,V \in RH^\infty} \left\| \begin{bmatrix} -\tilde{N}_s^* \\ \tilde{M}_s^* \end{bmatrix} + \begin{bmatrix} U \\ V \end{bmatrix} \right\|_\infty \tag{39}$$

Different explicit solutions of this Nehari problem can found in [5] and [2].

5 A numerical example

In order to illustrate the results derived in the previous sections, consider the short-period dynamics of the F-8 fighter at altitude $h = 20000\, ft$ and speed $V = 620\, ft/s$ ([4]) having the state-space realization (A, B, C, D) with

$$\begin{aligned} A &= \begin{bmatrix} -0.84 & 1 \\ -4.8 & -0.49 \end{bmatrix}; B = \begin{bmatrix} -0.11 \\ -8.7 \end{bmatrix}; \\ C &= \begin{bmatrix} -16.17 & 0 \\ 0 & 1 \end{bmatrix}; D = \begin{bmatrix} -2.12 \\ 0 \end{bmatrix}, \end{aligned}$$

where the state are: α-the angle of attack and q-the pitch rate, the control is the elevator deflection δ_e and the measured outputs are the normal acceleration a_n and the pitch rate q. Assume that the dynamics of the accelerometer and of the gyro are approximated by first order low-pass filter with the time constant ε. Then the short-period dynamics together with the sensors dynamics gives a two-time-scale system having the same structure as (37). The optimal level of attenuation given by (14) is $\gamma_o = 5.2039$. Then using (15) we determined for $\gamma = 1.1 \cdot \gamma_o$ the ε-dependent robust controllers corresponding to $\varepsilon = 0.01; 0.001; 0.0001$. In order to evaluate the robustness properties of these controllers we determined the generalized system (16) corresponding to the two-time-scale system (37). The robustness radius provided by these controllers is given by $\tilde{r} = \left(\|T_{y_1u_1}\|_\infty\right)^{-1}$ where $T_{y_1u_1}$ denotes the transfer matrix of the resulting system obtained by coupling the controller to the generalized system. The values of $\|T_{y_1u_1}\|_\infty$ for different values of ε are shown in Table 1.

Table 1: ε-dependent controllers

ε	0.01	0.001	0.0001
$\|T_{y_1u_1}\|_\infty$	5.8692	5.6892	5.6888

Since in this numerical example the fast dynamics of the two-time-scale system is stable we can determine an ε-independent optimal robust controller with respect to NLCF. Numerical computations give that $(1+\rho(X_sY_s))^{\frac{1}{2}} = 5.2039$ and $1+\|G_f\|_\infty = 3.1200$ and hence we are in the Case (b) discussed in Section 4.2. Then we determined the optimal robust controller with respect to the NLCF for the slow component by solving the one-block Nehari problem (39). This controller has the realization (A_k, B_k, C_k, D_k) where

$$\begin{aligned} A_k &= -7.6717; B_k = \begin{bmatrix} 2.2854 & -0.5244 \end{bmatrix}; \\ C_k &= -0.0725; D_k = \begin{bmatrix} -0.1643 & 1.0323 \end{bmatrix}. \end{aligned}$$

It stabilizes the generalized system (16) and the H^∞ norm of the resulting system for different values of ε are given in Table 2.

Table 2: ε-independent controllers

ε	0.01	0.001	0.0001
$\|T_{y_1u_1}\|_\infty$	6.0250	5.2999	5.2143

The above results show as expected, that for $\varepsilon \to 0_+$, an ε-independent controller provides a robustness radius of the two-time-scale system close to its optimum.

6 Concluding remarks

In this paper, the problem of robust stabilization with respect to the NLCF of singularly perturbed systems has been studied. The construction of the robust controller uses the stabilizing solutions of the Riccati equations associated with the slow and fast dynamics of the nominal system, respectively; thus the computations do not depend on the small parameter ε. Based on these solutions, an upper bound of the robustness radius is determined. It is shown that in the general case the robust controller is also a two-time-scale system. If the fast dynamics is stable then an ε-independent robust controller can be determined. This controller is just this is just the controller corresponding to the slow dynamics which achieves a robustness radius less or equal than $\min\left\{(1+\rho(X_sY_s))^{-\frac{1}{2}}, (1+\|G_f\|_\infty)^{-1}\right\}$.

Numerical results show that the obtained controllers provide robustness properties close to the specifications particularly for small values of ε, where in fact the ill-conditioned computations appear when the usual formulae for the H^∞ controller are used. This feature is due to the fact that the constructions of the robust controllers with respect to NLCF use asymptotic expansions for the solutions of the Riccati equations involved.

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