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Abstract. All finite lattices of quasivarieties of commutative Moufang's loops with the nilpotence class ≤ 2 are described.

Introduction

The presented paper is dedicated to the investigation of the problem (analogous to the problem of M.I.Kargapolov [3]) of the description of the lattice of the quasivarieties of commutative Moufang's loops with the nilpotence class ≤ 2. In [5] the author have investigated this problem for the commutative Moufang's loops with the nilpotence class 2 with exponent 3. It was shown there that the lattice of subquasivarieties of any variety $\mathbf{M}$ of commutative Moufang's loops is finite or infinite but numerable and is finite if and only if $\mathbf{M}$ is generated by a finite group. In ths paper in the class $\mathbf{N}_2$ of commutative Moufang's loops with the nilpotence class ≤ 2 are described all quasivarieties which possessis a finit number of subquasivarieties. It is presented in an evident form the list of all loops (which are parts of cyclic groups and nonasociative loops generated by three elements) with the property that $\mathbf{K} \subseteq \mathbf{N}_2$ has only finitly many subquasivarieties if and only if $\mathbf{K}$ is generated by a finite set of loops of this list. In particular, if the finitely generated commutative Moufang's loop L with the nilpotence class 2 is not approximated by these descibed loops then the quasivariety generated by the loop L contains a conmtinuum set of distinct quasivarieties. There are also described some finite lattices and some of them are largly presented. It can be observed that in the simplest cases these lattices are not modular.

§1. Notations and preliminary rezults.

The commutative loop Moufang (see [1]) is called the algebra L with the unary operation $^{-1}$ and other binary one $\cdot$ in which for any $x, y, z \in L$ are valid relations:

$$xy = yx, xy \cdot zx = x(yz) \cdot x, x^{-1} \cdot xy = y.$$

We introduce the following notations: $F_n = F_n(x_1, \ldots, x_n)$ is the free commutative loop Moufang of rangul n generated by the free elements $x_1, \ldots, x_n$;

$\mathbf{N}_2$ is the class of all commutative loops Moufang with the nilpotence class ≤ 2, defined by the identity

$$(xy \cdot z)(x \cdot yz)^{-1} \cdot uv = ((xy \cdot z)(x \cdot yz)^{-1} \cdot u)v;$$

$N_{2,3^k}$ is the variety defined in N_2 by the identity

$$x^{3^k} = 1,$$

where the number $k \neq 0$ is natural;

$F_n(\mathbf{M}) = F_n(\mathbf{M}\ ; x_1, \dots, x_n)$ is the free commutative loop Moufang of the rang n of the variety $\mathbf{M} \subseteq N_2$;

$Q(L)$ is the quasivariety generated commutative loop Moufang L;

$L_q\mathbf{M}$ is the lattice of subquasivarieties of the quasivariety $\mathbf{M} \subseteq N_2$;

$T = Q(F_3)$, $T_k = Q(F_3(N_{2,3^k}))$;

$A \underset{\mathbf{M}}{*} B$ is the $\mathbf{M}$-free product of the loops A, B which belongs to the quasivariety $\mathbf{M} \subseteq N_2$;

Z_{p^k} is the cyclic group of ordinul puterea a k a numărului prim p; Z is the infinit cyclic group.

As usual the elements of the commutative loop Moufang $F_n(x_1, \dots, x_n)$ are called the word of the variables $x_1 \dots, x_n$ and the elements of F'_n are called asociative words.

We shall say that the loop L of the variety $\mathbf{M} \subseteq N_2$ has in $\mathbf{M}$ the reprezentation

$$L = lp(x_1, \dots, x_n || R = 1),$$

if $L \cong F_n(\mathbf{M}; x_1 \dots, x_n)/\overline{R}$, where $\overline{R}$ is the normal subloop in $F_n(\mathbf{M})$ generated by the set $R \subseteq F_n(\mathbf{M})$.

The loop is reffered as monolite if is finite generated and is not decomposable in the direct product of two nonunit ssubloops of it.

The exponent of the commutative loop Moufang will be called the least common multiple of the orders of all its elements.

Let $\mathbf{M}$ be a quasivariety of N_2, L be an arbitrary commutative loop Moufang. The least normal subloop H of L for which $L/H \in \mathbf{M}$ is denoted by $\mathbf{M}(L)$ and is called the cvasiverbal subloop of the loop L corresponding to the quasivariety $\mathbf{M}$.

Let us agree that the expression "the element $x \neq 1$ of the loop L is approximated by the loop K" is equivalent to the expression "there exists a morphism of loops $\varphi : L \to K$ such that $x^\varphi \neq 1$".

We shall need below the following criteria of belongness the proof of which we can found in [6].

The finitely generated commutative loop Moufang L belongs to the quasivariety generated by the class $\mathbf{K}$ *of commutative loops Moufang if and omly if the nonunit elements of L are approximated by the loops of* $\mathbf{K}$.

Let us recall some notions and rezults from [1] which in the sequel will be used sometimes without mentioning them explicitly.

Let L be a commutative loop Moufang. The asociator $[x, y, z]$ of the elements $x, y, \ z \in L$ are defined by the equality

$$[x, y, z] = (xy \cdot z)(x \cdot yz)^{-1}.$$

The asociator L' of the loop L is called the subloop generated by all asociators of L. If X, Y, Z are nonvoid subsets of L then the notation $[X, Y, Z]$ means the subloop of L, generated by all asociators of the form $[x, y, z]$, where $x \in X, \ y \in Y, \ z \in Z$. The set $Z(L) = \{x \in L \mid [x, y, z] = 1 \ \text{ for all } \ y, z \in L\}$ is called the center of the loop L. The subloop H of the commutative loop Moufang L is called normal, if

$$x \cdot yH = xy \cdot H$$

for any $x, y \in L$. The subloop and the normal subloop in L generated by the elements $a_1, \dots, a_n$ are denoted by $lp(a_1, \dots, a_n)$ si $lp(a_1, \dots, a_n)^L$ respectively. It is easy to observe, that the asociator L' and the center $Z(L)$ of the loop L are normal subloops and the factor-loop L/L' is an abelian group. We can easily convince ourselves that it is normal in L the subloop L^m generated by the powers of m's of all elements of L.

In any commutative loop Moufang are valid the identities

$$[x, y, z]^3 = 1, \tag{1}$$

$$[x, y, z] = [y, z, x] = [y, x, z]^{-1}, \tag{2}$$

and in the loops of the class $\mathbf{N}_2$ it is also valid the identity

$$[x \cdot y, z, t] = [x, z, t][y, z, t]. \tag{3}$$

(Theorem of Moufang) *If the asociator of three elements a, b, c of the commutative loop Moufang L is equal to unit thenthe subloop generated by the elements a, b, c is a group. In particular, any two elements of L generates an asociative subloop.*

Here below we shall formulate some lemmas of the papers [5] and [6].

Lema 1.1 *[5]. The nontrivial quasiidentity*

$$\&_{i=1}^{m} u_i(x_1, \ldots, x_n) = 1 \longrightarrow u(x_1, \ldots, x_n) = 1$$

which is valid in the commutative loop Moufang $F_3(\mathrm{N}_{2,3})$ is equivalent in the variety $\mathrm{N}_{2,3}$ with quasiidentity

$$\&_{l=1}^{m}([y_{3l-2}, y_{3l-1}, y_{3l}]a_l = [y_1, y_2, y_3]a_1)$$

$$\& \ H(y_1, \ldots, y_n) = 1 \longrightarrow [y_1, y_2, y_3]a_1 = 1,$$

where $3m \leq n, F_3(\mathrm{N}_{2,3}) \models](K(y_1, \ldots, y_n) = 1 \longrightarrow [y_1, y_2, y_3]a_1 = 1), lp(a_1, \ldots, a_m) \cup H \leq lp([y_i, y_j, y_k] : 1 \leq i < j < k \leq n, (i,j,k) \neq (3l-2, 3l-1, 3l), l = 1 \ldots, m)$.

Lemma 1.2 *[5 - 6]. Let* M *be a nonasociative quasivariety from the arbitrary* $\mathrm{N} \in \{\mathrm{N}_2, \mathrm{N}_{2,3^k}, k = 1, 2, \ldots\}, A, B$ *are loops of* M *represented in* N *such that*

$$A = lp(x_1, \ldots, x_n || M(x_1, \ldots, x_n) = 1),$$

$$B = lp(y_1, \ldots, y_m || M(x_1, \ldots, x_n) = 1),$$

where M, N are totalities of asociative words. If M is the normal subloop of $C = A \underset{\mathrm{N}_2}{*} B$ generated by some asociators of the form $[x_i, x_j, y_k]$ or $[x_i, y_j, y_k]$, then $C/H \in \mathrm{M}$.

Lemma 1.3 *[6].*

$F_{3_n}(\mathrm{N}_{2,3^k}$ *(resp.,* $\mathrm{N}_2); x_1, \ldots, x_{3n})/ \ lp([x_1, \ x_2, \ x_3] \ [x_4, \ x_5, \ x_6] \ldots [x_{3n-2}, x_{3n-1}, \ x_{3n}]) \in \mathrm{T}_k$ *(resp.,* T*)*

The following lemma is proved in the same way as lemma 4 of [5].

Lemma 1.4. *Let L be the N_2 - free product of the loops $F_{3n}^i(\mathrm{N}_{2,3^k}$ respectively, $\mathrm{N}_2)$; $x_1^i, \ldots, x_{3n}^i)$.*

$i = 1, \ldots, l$

with glued elements

$$a = \prod_{i=1}^{n}[x_{3i-2}^1, x_{3i-1}^1, x_{3i}^1] = \ldots = \prod_{i=1}^{n}[x_{3i-2}^l, x_{3i-1}^l, x_{3i}^l].$$

Then the element $a \in L$ can be represented via a product of a number $< n$ of the asociators.

We introduce the notations of some commutative loops Moufangs which will be used in the proofs.

$$M_{\infty\infty\infty} = F_3(x, y, z);$$

$$M_{r\infty\infty} = lp(x, y, z \mid\mid x^{3^r} = 1);$$

$$H_{rs\infty} = lp(x, y, z \mid\mid x^{3^r} = y^{3^s} = 1);$$

$$H_{rst} = lp(x, y, z \mid\mid x^{3^r} = y^{3^s} = z^{3^t} = 1)$$

$$(H_{00t} = Z_{3^t}, H_{00\infty} = Z, H_{000} = \{1\}),$$

where r, s, t are integer numbers and $0 \leq r \leq s \leq t$;

A_{mk} (respectively A_m) $= lp(a_{ij}, 1 \leq i \leq m, 1 \leq j \leq 3m+3)$ is a loop of the variety $\mathbf{N}_{2,3^k}$ (respectively $\mathbf{N}_2$)) which determinant relations are:

$$(4) \qquad \prod_{i=1}^{m+1} [a_{13i-2}, a_{13i-1}, a_{13i}] = \ldots = \prod_{i=1}^{m+1} [a_{m3i-2}, a_{m3i-1}, a_{m3i}],$$

$$(5) \qquad [a_{ij}, a_{kl}, a_{pr}] = 1, i \neq k \vee i \neq p \vee k \neq p, 3 < j, l, r \leq 3m+3;$$

$B_{lmk} = A^1_{mk}$ x ... x A^1_{mk} (respectively $B_{lm} = A^1_m$ x ... x A^1_m) is the cartesian product of l instances of the loop A_{mk} (respectively A_m);

$C_{lmk} = B_{lmk}/lp(a^i(a^i)^{-1},\ 1 \leq i \leq l)$ (respectively $C_{lm} = B_{lm}/lp(a^i(a^i)^{-1}$, $1 \leq i \leq l)$; where element a^i is the copy in the loop A^i_{mk} (respectively A^i_m) of the element

$$(6) \qquad a = \prod_{i=1}^{m+1} [a_{13i-2}, a_{13i-1}, a_{13i}]$$

of the loop A_{mk} (respectively A_m)

As it was shown in [5], the subquasivarieties of $\mathbf{N}_{2,3}$ are characterised by the asociator quasivarieties. We shall emphasize in $\mathbf{N}_2$ si în $\mathbf{N}_{2,3^k}$, for $k \geq 2$, those quasivarieties which have the same structure and are also characterised by asociator quasivarieties. We denote by $\mathbf{N}_{(2,3^k)}$ the subvariety of $\mathbf{N}_{2,3^k}$ defined by the quasidentities

$$(7) \qquad x^{3^{k-1}} = 1 \rightarrow [x, y, z] = 1,$$

$$(8) \qquad x^3 = \prod_{i=1}^{n} [x_{3i-2}, x_{3i-1}, x_{3i}] \rightarrow x^3 = 1,$$

for any natural number n. By $\mathbf{N}_{(2)}$ we denote the quasivariety of $\mathbf{N}_2$ defined by the quasiidentities

(9) $$x^9 = 1 \longrightarrow x^3 = 1,$$

(10) $$x^3 = 1 \longrightarrow [x, y, z] = 1,$$

(11) $$x^p = 1 \longrightarrow x = 1,$$

for all prime numbers $p \neq 3$.

Lema 1.5 *For any $l \geq 1$, the lattices of the nonasociative quasivarieties $\mathbf{N}_{(2,3^l)}$ and $\mathbf{N}_{(2)}$.*

Proof. According to the quasiidentities (7), (8) (respectively (9) - (11)) the determinant relations of any monolite nonasociative loop of $\mathbf{N}_{(2,3^l)}$ (respectively $\mathbf{N}_{(2)}$) are some equalities of some asociator words equal to unit (see lemma 5 of [6]). It is clear that the nonasocitive subquasivarieties of the quasivarieties $\mathbf{N}_{(2,3^l)}$ and $\mathbf{N}_{(2)}$ are generated by monolite loops.

We establish a reciprocal correspondence between nonasociative monolite loops of $\mathbf{N}_{(2,3^l)}$ and monolite nonasociative loops of $\mathbf{N}_{(2)}$ in the following way: we consider that to the monolite nonasociative loop $A^l \in \mathbf{N}_{(2,3^l)}$ with the generators $a_1^l, \ldots, a_n^l$ and the determinant relations (in $\mathbf{N}_{(2,3^l)}$) $R(a_1^l, \ldots, a_n^l) = 1$ corresponds the monolite asociative loop $A \in \mathbf{N}_{(2)}$ with the generators $a_1, \ldots, a_n$ and determinant relations (in $\mathbf{N}_2$) $R(a_1, \ldots, a_n) = 1$. We prove now the following staements:

1°. *If the monolite nonasociative loop $L = lp(x_1, \ldots, x_n)$ belongs to the quasivariety $\mathbf{N}_{(2,3^l)}$ (respectively, $\mathbf{N}_{(2)}$), then $Q(L/L') = Q(Z_{3^e})$ (respectively, $Q(L/L') = Q(Z)$).*

2°. *Let $A^l = lp(a_1^l, \ldots, a_n^l)$, $B^l = lp(b_1^l, \ldots, b_m^l)$ are monolite nonasociative loops of $\mathbf{N}_{2,3^l}$ and $A = lp(a_1, \ldots, a_n)$, $B = lp(b_1, \ldots, b_m)$ are coresponding loops of $\mathbf{N}_{(2)}$. If $A^l \in Q(B^l)$, then $A \in Q(B)$.*

The proof of the statement 1° rezults from the fact that $L/L' = lp(x_1 L')$ x ... x $lp(x_n L')$ and every cyclic subgroup $lp(x_i L')$ is isomorph with Z_3 (respectively, Z); so $Q(L/L') = Q(lp(x_i L'))$ or $Q(L/L') = Q(Z_{3^l})$ (respectively, $Q(L/L') = Q(Z)$).

We prove 2°. Let u^l be an arbitrary element of the loop A^l. If the corresponding element $u \notin A'$ then u is approximated by the loop B because $Q(Z) \subseteq Q(B)$

and according to 1°, $Q(A/A') = Q(Z)$. Let $1 \neq u \in A'$. Then u can be represented in the form:

$$u = \prod_{1 \leq i<j<k \leq n} [a_i, a_j, a_k]^{\alpha_{ijk}},$$

where $0 \leq \alpha_{ijk} < 3$. Suppose the element $u^l = \prod_{1 \leq i<j<k \leq n} [a_1^l, a_j^l, a_k^l]^{\alpha_{ijk}}$ is approximated by the loop B^l via morphism of loops $\varphi : A^l \to B^l$, defined by the applications $(a_i^l)^\varphi = (b_1^l)^{\beta_{1i}} \dots (b_m^l)^{\beta_{mi}} c_i^l$, $1 \leq i \leq n$, where $0 \leq \beta_{ji} < 3^l, j = 1, \dots, m, c_i^l \in (B^l)'$. We investigate the morphism of the loops $\Psi : A \to B$ defined as:

$$a_i^\psi = b_1^{\gamma_{1i}} \dots b_m^{\gamma_{mi}} c_i, \ 1 \leq i \leq n,$$

where $0 \leq \gamma_{ji}$ and $\gamma_{ji} = \beta_{ji} \bmod 3^l$, for $j = 1, \dots, m$, and to the element $c_i \in B'$ corresponds the element $c_i^l \in (B^l)'$.

We verify if $u^l \neq 1$. Firstly we observe that

$$(u^l)^\varphi = \prod_{1 \leq i<j<k \leq n} [(a_i^\varphi)^\varphi, (a_j^l)^\varphi, (a_k^l)^\varphi]^{\alpha_{ijk}},$$

$$u^\Psi = \prod_{1 \leq i<j<k \leq n} [a_i^\psi, a_j^\psi, a_k^\psi]^{\alpha_{ijk}}.$$

We substitute in these equalities $(a_i^l)^\varphi, a_i^\varphi$ with the expresions b_i^l, b_i and applying the identities (2) and (3), we obtain

$$(u^l)^\varphi = \prod_{1 \leq i<j<k \leq n} [b_i^l, b_j^l, b_k^l]^{\delta_{ijk}},$$

$$u^\theta = \prod_{1 \leq i<j<x \leq n} [b_i, b_j, b_k]^{\theta_{ijk}}$$

Taking into consideration the equalities $\gamma_{ij} = \beta_{ij} \ mod\ 3$ and the identity (1), we obtain $\delta_{ijk} =$

$\theta_{ijk} mod 3$. Because B^l and B have in their varieties the same determinant relations and $(u^l)^\varphi$, u^Ψ are written with the same words of the corresponding generators we obtain that $u^\Psi \neq 1$. The statement 2° is proved.

Now we establish a reciprocal corespondence between nonasociative subquasivarieties of $N_{(2,3^l)}$ and nonasociative subquasivarieties of $N_{(2)}$ in the following way. Let N^l be an arbitrary quasivariety of $N_{(2,3^l)}$, so $N^l = Q\{A_i^l, \ i \in I\}$, where A_i is the monolite nonasociative loop of $N_{2,3^l}$. Then we consider that the corresponding quasivariety $\mathbf{N} \subseteq \mathbf{N}_{(2)}$ is generated by corresponding loops, i.e. $\mathbf{N} = Q\{A_i, \ i \in I\}$, where A_i is the monolite nonasociative loop of $\mathbf{N}_{(2)}$ corresponding to the loop A_i^l. From 2° rezults that the established correspondence is independent of the choice of generated loops, is reciprocal and conserves the inclusion. The lemma is proved.

Lema 1.6. *For* $m = 1, 2, \ldots,$ *we have* $A_{mk} \in \mathbf{T}_k, A_m \in \mathbf{T}$.

Proof. The element $a \in A_{mk}$ (respectively, A_m) is approximated by the morphism of loops $\varphi : A_{mk} \to F_3(\mathbf{N}_{2,3k}; x, y, z)$ (respectively,) $A_m \to F_3(x, y, z))$ defined by the equalities

$$a_{11}^{\varphi} = x, \quad a_{12}^{\varphi} = y, \quad a_{13}^{\varphi} = z,$$

$$a_{21}^{\varphi} = x, \quad a_{22}^{\varphi} = y, \quad a_{23}^{\varphi} = z,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m_1}^{\varphi} = x, \quad a_{m_2}^{\varphi} = y, \quad a_{m_3} = z,$$

$$a_{ij}^{\varphi} = 1, \quad \forall i, \quad \forall j > 3.$$

Now we show that $A_{mk}/lp(a) \in \mathbf{T}_k$ (respectively $A_m/lp(a) \in \mathbf{T}$). According to the lemma 1.3, the loops K_i represented in $\mathbf{N}_{2,3^k}$ (respectively, $\mathbf{N}_2$) as follows.

$$K_i = lp(a_{i1}, \ldots, a_{i3m+3} \ || \prod_{j=1}^{m+1} [a_{i3j-2}, \ a_{i3j-1}, \ a_{i3j}] = 1]$$

are contained in $\mathbf{T}_k$ (respectively, $\mathbf{T}$). $A_{mk}/lp(a)$ (respectively, $A_m/lp(a)$ is a free product in the variety $\mathbf{N}_{2,3^k}$ (respectively, $\mathbf{N}_{(2)}$) of the loops K_i factored over the relations (5) and according to lemma 1.2 it belongs to the quasivariety $\mathbf{T}_k$ (respectively, $\mathbf{T}$). The lemma is proved.

§2. The first auxiliary rezult.

We introduce some notations of some loops of the variety $\mathbf{N}_{2,3^k}$ (respectively, $\mathbf{N}_2$) :

$B = B(n, V, k)$ *(respectively,* $B(n, V)) = lp(x, x_1, \ldots, x_{3n}) \ ||$ $[x, x_1^{\alpha_1} \ldots x_{3n}^{\alpha_{3n}}, x_1^{\beta_1} \ldots x_{3n}^{\beta_{3n}}]$,

$$\begin{pmatrix} \alpha_1 \ldots \alpha_{3n} \\ \beta_1 \ldots \beta_{3n} \end{pmatrix} \in V);$$

$H = H(n, V, x)$ (respectively, $H(n, V)$) is the factor-loop of the loop B over rel;ation

$$x^3 = \prod_{i=1}^{n} [x_{3i-2}, x_{3i-1}, x_{3i}]$$

H_m is the factor-loop of the N_2-free product of the loops B and $\mathcal{A}_{mk}$ (respectively, $\mathcal{A}_m$) over relation

$$x^3 = \prod_{i=1}^{n} [x_{3i-2}, x_{3i-1}, x_{3i}]a,$$

where a is defined by the equality (6).

Lema 2.1. $H_m \in Q(H)$.

Proof. The element $a \in H_m$ is approximatedd by the loop $F_3(N_{2,3^k}; u, v, w)$ (respectively, $F_3(u, v, w)$) via loop morphism $\varphi : H_m \to F_3(N_{2,3^k})$ (respectively, $H_m \to F_s$) defined by the equalities:

$$x_1^\varphi = u^{-1}, \quad x_2^\varphi = v, \quad x_3^\varphi = w,$$

$$x_4^\varphi = 1, \ldots, x_{3n}^\varphi = 1, x^\varphi = 1,$$

$$a_{m_1}^\varphi = u, a_{m_2}^\varphi = v, a_{m_3}^\varphi = w,$$

$$a_{ij}^\varphi = 1, i = 1, \ldots, m, j = 4, \ldots, 3m + 3.$$

Now we show that $H_m/lp(a) \in Q(H)$. Really, $H_m/lp(a) \cong M * (A_{mk}/lp(a)$) (respectively, $H * (A_m/lp(a))$, where $A_{mk}/lp(a) \in T_k$ (respectively, $A_m/lp(a) \in$ T). But $T_k \subseteq Q(H)$ (respectively, $T \subseteq Q(H)$), then in virture of lemma 1.2 we obtain $H_m/lp(a) \in Q(H)$. The lemma is proved.

Lemma 2.2. *The element $x^{3\alpha} \in H_m$, where $\alpha \neq 0$ mod 3, can not be represented as a product of $m - 1$ asociatori.*

Proof. Let

$$N = lp(x_1, x_2, \ldots, x_{3n}, a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, \ldots, a_{m1}, a_{m2}, a_{m3},$$

$$[x, a_{ij}, a_{kl}], \ 1 \leq i, k \leq m, \ 4 \leq j, l \leq 3m + 3)^{H_m} \subseteq H_m$$

Then H_m/N is a direct product with the reunion of the central subloops $x^3N = aN$ of the cyclic group $A = lp(xH)$ and of the subloop $D = lp(a_{ij}N, \ i = 1, \ldots, m, j = 1\varphi, \ldots, 3m + 3)$. According to lemma 1.4 the element $aN \in D$

can not be represented as a product of $m-1$ asociators. Since the subloop A is contained in the center of the loop H_m/N, results that aN can not be represented as a product of $m-1$ asociator and in the entire loop H_m/N. But $x^{3\alpha} = a^{\alpha}$ mod N, so $x^{3\alpha}$ can not written as the product of $m-1$ asociators. The lemma is proved.

Lema 2.3. *Let H_m^3 be the set of all cubes of all elements of H_m. Then* $H_m^3 \cap H_m' = \{x^{3\alpha}, 0 \leq \alpha < 3\}$.

Proof. Since the loop Moufang is diasociative and in any commutative loop Moufang is valid the identity (1), then it is easy to understand that H_m^3 is a subloop in H_m and is contained in the center of $Z(H_m)$, so it is also normal in H_m. From here and from the construction of the loop H_m it is clear that the nonunitary elements of the asociator H_m' which are contained in the subloop H_m^3 are $x^{\pm 3}$. The lemma is proved.

Lema 2.4. *For $r > 3^{3m^6}, m \geq n$, we have* $H_m \notin Q(M_2)$.

Proof. Let φ be an arbitrary morphism of loops of H_m în H_r. According to lemma 2.3 $x^3 \in H_m^3$ and $x^3 \in H_m'$, and from here $(x^3)^{\varphi} \in H_r^3, (x^3)^{\varphi} \in H_r'$, so $(x^3)^{\varphi} \in H_r^3 \cap H_r'$. The element x^3, and subsequently $(x^3)^{\varphi}$ according to the construction of the loop H_m is represented as a preoduct of $m+n+1 < r-1$ asociators. But in virture of the lemma .2 the nonunitary elements of $H_r^3 \cap H_r'$ can not be represented as a preoduct of $r-1$ asociators. Subsequently $(x^3)^{\varphi} = 1$ and so $H_m \notin Q(H_r)$. The lemma is proved.

Lema 2.5 *If $m > 3^{3r^6}$, $r \geq n$ then* $H_m \notin Q(H_r)$.

Proof. Let us denote

$$a_i = \prod_{i=2}^{m+1} [a_{i3j-2}, a_{i3j-1}, a_{i3j}], \quad i = 1, \ldots, m.$$

We suppose that the lemma is not true. Then for the element x^3 exists an morphism of loops $\varphi : H_m \to H_r$ such that $(x^3)^{\varphi} \neq 1$. Since the number of the generators of the loop H_r is $3r(r+1)+n+2 = t$, and from the condition of the lemma $n \leq r$ we obtain

$$|H_r'| \leq |F_t'| = 3^{\frac{1}{3}t(t-1)(t-2)} \leq 3^{r^6}$$

Subsequently, taking into consideration the condition $m > 3^{3r^6}$, we obtain that for some

$i \leq m$

$$lp(a_{i1}, \ldots, a_{i3m+3})^{\varphi} \subseteq lp(a_{j1}, \ldots, a_{j3m+3}, j = 1, \ldots, i-1, i+1, \ldots, m)^{\varphi}.$$

From here on the foundation of the equalities of (5) the equality $a_i^{\varphi} = 1$ is deduced. Then

$$(x^3)^{\varphi} = ([x_1, x_2, x_3] \ldots [x_{3n-2}, x_{3n-1}, x_{3n}]a)^{\varphi} =$$

$$([x_1, x_2, x_3] \ldots [x_{3n-2}, x_{3n-1}, x_{3n}][a_{i1}, a_{i2}, a_{i3}])^{\varphi},$$

so the element $(x^3)^{\varphi} \in H_r^3 \cap H_r'$ is represented as a product of $n+1$ asociators. Since the nonunit elements of $H_r^3 \cap H_r'$ can not be represented as a product of $r - 1 \geq n + 1$ asociators we obtain $(x^3)^{\varphi} = 1$, that can not be. The lemma is proved.

Lemma 2.6 (The first auxiliary rezult). *The lattice $L_qQ(H)$ is continous.*

Proof. In virture of the lemma 2.1 $Q(H)$ contains infinite many loops H_m, where m takes the values of the set of natural numbers. We construct the infinite sequence $\{m_i, i = 1, 2, \ldots\}$ of natural numbers in the following way: $m_1 = n + 2$, $m_{i+1} = 3^{3m_i^6} + 1$ for $i > 1$. Now we shall show that different subsets of loops of the set $\{M_{m_i}, i = 1, 2, \ldots\}$ generates different quasivarieties.

Let $\mathrm{M} = Q\{M_{m_i}, i \in I\}, \mathrm{N} = Q\{M_{m_j}, j \in J\}, I \neq J$. Suppose that $i \in I$ and $i \notin J$, so $M_{m_i} \in m$. We show that $M_{m_i} \notin \mathrm{N}$. Really, if it is not so then the element $x^3 \in M_m$ is approximated by a loop $M_{m_j}, j \in J$. According to the choosen sequence $\{m_i, i = 1, 2, \ldots\}$ and lemmas 2.4, 2.5, we obtained the contradiction. Lemma is proved.

§3. The second auxiliary result.

Let L be a loop of the variety $\mathbf{N}_{2,3}$ generated by a finite number of elements $x_1, \ldots, x_n$, and $\mathrm{T}_1(L)$ is its quasiverbal subloop which corresponds to the quasivariety T_1 with elemeents $u_1, \ldots, u_t$.

3.1. The construction of the loops L_m^i.

According to lemma 1.1 for any $u_i \in \{u_1, \ldots, u_t\}$ there can be taken such generators $z_1, \ldots, z_{n_i}$ of the loop L that the defining relations of the loop L have the form

$$[z_{1i}, z_{2i}, z_{3i}]v_{1}, = \ldots = [z_{3l_i-2,i}, z_{3l_i-1,i}, z_{3l_i,i}]v_{1i} = u_i,$$

$$H_i(z_{1i}, \ldots, z_{ni}) = 1,$$

where $H_i, v_{1i}, \ldots, v_{l,i}$ are contained in the subloop generated by the asociators: $[z_{ri}, z_{si}, z_{pi}]$, for all triplets (r, s, p) not contained in the set $\{(1,2,3), (4,5,6), \ldots, (3l_i - 2, 3l_i - 1, 3l_i)\}$.

In §1 the commutative loop Moufang B_{lmk} have been defined. We take the loop

$B_{l_i n_1} = A^1_{m1} \times \ldots \times A^{l_i}_{m1}$ and denote the generators of the subloop A^j_{m1}, $1 \leq j \leq l_i$, in correspondence with $a^{ji}_{11}, a^{ji}_{12}, \ldots, a^{ji}_{m,3m+3}$. Now we define the loop L^i_m as the factor–loop of the $N_{2,3}$ - free product of the loops $B_{l_i n1}$ and F_n ($N_{2,3;}$ $z_{1i}, \ldots, z_{ni}$) over the relations:

$$(11) \qquad H_i(z_{1i}, \ldots, z_{ni}) = 1,$$

$$(12) \qquad [a^{ji}_{rs}, a^{ki}_{pq}, z_{li}] = 1 (j = k \to l \notin \{3j-2, 3j-1, 3j\}),$$

$$(13) \qquad [a^{ji}_{rs}, z_{pi}, z_{qi}] = 1 (p \vee q \notin \{3j-2, 3j-1, 3j\}),$$

$$(14), \qquad a^{1i}[z_{1i}, z_{2i}, z_{3i}]v_{1i} = \ldots = a^{l_i i}[z_{3l_i-2,i}, z_{3l_i-1,i} z_{3l_i,i}]v_{l_i i}$$

where $a^{1i}, \ldots, a^{l_i i}$ are elements of the loops $A^{1i}_{m}, \ldots, A^{l_i i}_{m1}$ corresponding to the element a (see formula (4)) of the loop A_{m1}. Fix the notation $x_i = a^{1i}[z_{1i}, z_{2i}, z_{3i}]v_{1i}$.

3.2. The properties of the loops L^i_m.

Lemma 3.1. $L^i_m \notin T_1$.

Proof. Let $\varphi : L^i_m \to F_3(N_{2,3})$ be some morphism of loops. Two cases are possible:

1. $(a^{1i})^\varphi = \ldots = (a^{l_i i})^\varphi = 1$, so $([z_{1i}, z_{2i}, z_{3i}]v_{1i})^\varphi = \ldots =$

$$([z_{3l_i-2,i}, z_{3l_i-1,i}, z_{3l_i,i}]v^e_{l_i i})$$

We denote $N = lp(a^{1i}, \ldots, a^{l_i i}) \subset L^i_m$. Evidently we have $N \subset ker\varphi$ si $lp(z_{1i}, \ldots, _{l_i i})N|N \cong L$. Taking all these into consideration and also the fact that $[z_{1i}, z_{2i}, z_{3i}]v_{1i} \in \mathrm{T}_1(L)$ we deduced $([z_{1i}, z_{2i}, z_{3i}]v_{1i})^\varphi = 1$. This means that $x_i^\varphi = (a_1^{1i})^\varphi \cdot ([z_{1i}, z_{2i}, z_{3i}]v_{1i})^\varphi = 1$.

2. $(a^{ji})^\varphi \neq 1$ for some $j, 1 \leq j \leq l_i$.

Suppose for simplicity $j = 1$. Since $a^{1i} = \prod_{i=1}^{m+1} [a_{13i-2}^{1i}, a_{13i-1}^{1i}, a_{13i}^{1i}]$ (see formula (4)), then, we suppose that $[a_{11}^{1i}, a_{12}^{1i}, a_{13}^{1i}]^\varphi \neq 1$. In the commutative loop Moufang $F_3(\mathrm{N}_{2,3})$ is valid the universal formula:

$$\tau = ([x_1, x_2, x_3] \neq 1 \ \&\ [x_1, x_2, x_4] = 1 \ \&\ [x_1, x_3, x_4] = 1$$

$$\&\ [x_2, x_3, x_4] = 1 \rightarrow [x_4, x_5, x_6] = 1)$$

In the defining relations (13) of the loops L_m^i and in the relations of the loop B_{l,m_1} (see the construction of this loop) are the corresponding equalities:

(15) $$[a_{11}^{1i}, a_{12}^{1i}, z_{pi}] = 1, \ [a_{11}^{1i}, a_{13}^{1i}, z_{pi}] = 1, \ [a_{12}^{1i}, a_{13}^{1i}, z_{pi}] = 1$$

for all indices $p \notin \{1, 2, 3\}$;

(16) $$[a_{11}^{1i}, a_{12}^{1i}, a_{1r}^{2i}] = 1, \ [a_{11}^{1i}, a_{13}^{1i}, a_{1r}^{2i}] = 1, \ [a_{12}^{1i}, a_{13}^{1i}, a_{1r}^{2i}] = 1,$$

for all $r, 1 \leq r \leq 3m+3$.

From the inequality $[a_{11}^{1i}, a_{12}^{1i}, a_{13}^{1i}]^\varphi \neq 1$ and the equalities (13) on the basis of the formula τ we obtain $[z_{4i}, z_{5i}, z_{6i}]^\varphi = 1$ and $u_{2i}^\varphi = 1$(u_{2i} is a product of asociators $[x, y, z]$, where at least one of the variables x, y, z is a generator of the form z_{pi} for some $p \notin \{1, 2, 3\}$). From the inequality $[a_{11}^{1i}, a_{12}^{1i}, a_{13}^{1i}]^\varphi \neq 1$, from the inequalities (15) and from the formula τ reszults

$$[a_{11}^{2i}, a_{12}^{2i}, a_{13}^{2i}]^\varphi = 1, \ldots, [a_{13m+1}^{2i}, a_{13m+2}^{2i}, a_{13m+3}^{2i}]^\varphi = 1$$

Finally we obtain that in the first case $x_i^\varphi = 1$. This means that we can conclude that $x_i^\varphi = 1$, for any morphism of loops $\varphi : L_m^i \rightarrow F_3(\mathrm{N}_{2,3})$. The lemma is proved.

From the defining relations (12) - (15) of the loop L_m^i we observe that the subloop

$$lp(a_{jr}^{si}, 1 \leq s \leq l_i, 1 \leq j \leq m, 1 \leq r \leq 3m+3)$$

is a subloop of the type $B_{l_i m_1}$. This subloop in the below lemmas will means the loop $B_{l_i m 1}$.

Lemma 3.2. *The elements of the normal subloop* $B^{L'_m}_{l,m_1}$ *generated by the loop* B_{l,m_1} *is approximated in* L^i_m *by the loop* $F_3(\mathbf{N}_{2,3})$.

Proof. Note $N = lp(x_i, [z_{si}, z_{ri}, z_{pi}], a_{ji}$, for any triplet $(s,r,p) \notin \{(1,2,3), \ldots, (3l_i - 2, 3l_i - 1, 3l_i)$ and any $j > 3l_i) \subset L^i_m$.

From the defining relations of the loop L^i_m we observe that $N \cap B^{L'_m}_{l,m1} = 1$. Then the lemma will be proved if we shall show that $L^i_m/N \in \mathbf{T}_1$. From the defining relations of the loops L^i_m *si* L^i_m/N results that L^i_m/N represents a cartesian product of l_i isomorphic copies of the loop

$$A = (A_{m1} \ {}_{\mathbf{N}_{2,3}} A_{m_1 N_{2,3}} F_3(\mathbf{N}_{2,3}; z_1, z_2, z_3)) \ /lp(a[z_1, z_2, z_3])$$

It remains to prove that $A \in \mathbf{T}_1$. We show that $A/lp(a) \in \mathbf{T}_1$. Really,

$$A/lp(a) = A_{m_1}/lp(a) \ \overset{*}{_{\mathbf{N}_{2,3}}} F_3(\mathbf{N}_{2,3}; z_1, z_2, z_3)/lp([z_1, z_2, z_3]),$$

evidently

$$F_3(\mathbf{N}_{2,3}; z_1, x_2, z_3)/lp([z_1, z_2, z_3]) \in \mathbf{T}_1,$$

and from the proof of lemma 1.6, $A_{m1}/lp(a) \in \mathbf{T}_1$, and then accordingly to lemma 1.2 $A/lp(a) \in \mathbf{T}_1$. Now we show that the elements of $lp(a) \subset A$ are approximated by the loop $F_3(\mathbf{N}_{2,3}; x, y, z)$ via morphism of loops $\varphi : A \rightarrow F_3(\mathbf{N}_{2,3}; x, y, z)$ defined as:

$$a^{\varphi}_{11} = x, \ a^{\varphi}_{12} = y, \ a^{\varphi}_{13} = z,$$

$$a^{\varphi}_{21} = x, \ a^{\varphi}_{22} = y, \ a^{\varphi}_{23} = z,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a^{\varphi}_{m1} = x, a^{\varphi}_{m2} = y, a^{\varphi}_{m3} = z,$$

$$z^{\varphi}_1 = x, \ z^{\varphi}_2 = y, z^{\varphi}_3 = z^{-1},$$

$$a^{\varphi}_{ij} = 1, \ i = 1, \ldots, m, j = 4, 5, \ldots, 3m + 3.$$

So $A/lp(a) \in \mathbf{T}_1$ and the elements of $lp(a)$ are approximated by $F_3(\mathbf{N}_{2,3}; x, y, z)$ from where we obtain $A \in \mathbf{T}_1$ and lemma is proved.

Lemma 3.3. $L^i_m \in Q(L)$.

Proof results from the isomorphism $L^i_m/B^{L'_m}_{l,m1} \cong L$ and lemma 3.2.

Lemma 3.4. *The element* $x_i \in L^i_m$ *is approximated by the element* $c_i = a^{1i} = \ldots = a^{l_i}$ *of the loop* $C_{l,m1}$.

Proof. Evidently, that

$$L^i_m/lp(z_{1i},\dots,z_{ni})^{L^i_m} \cong C_{l,}m_1.$$

Then as a result of composition of the morphisms of loops

$$L^i_m \to L^i_m/lp(z_{1i},\dots,z_{ni})^{L^i_m} \to C_{l,m1}.$$

we obtained what we need.

Lemma 3.5. *The element $x_i \in L^i_m$ can not be represented as a product of $m-1$ asociators.*

Proof. According to lemmas 3, 4, it is sufficient to show this for the element

$$c_i = a^{1i} = \dots = a^{l_i i}$$

of the loop C_{l,m_1}.

Let

$$N = lp(a^{1i}_{11}, a^{1i}_{12}, a^{1i}_{13}, a^{1i}_{21}, a^{1i}_{22}, a^{1i}_{23}, \dots, a^{1i}_{m1}, a^{1i}_{m2}, a^{1i}_{m3}, \dots, a^{l_i i}_{11}, a^{l_i i}_{12},$$
$$a^{l_i i}_{13}, a^{l_i i}_{21}, a^{l_i i}_{22}, a^{l_i i}_{23}, \dots, a^{l_i i}_{m1}, a^{l_i i}_{m2}, a^{l_i i}_{m3})C_{l,m1}$$

According to lemma 1.3 the element c_iN of the loop $C_{l,m1}/N$ can not be represented as a product of $m-1$ asociators. So this is true also for c_i in $C_{l,}m_1$ *si* x_i *in* L^i_m. The lemma is proved.

3.3. The construction of the loops L_m.

Let the loop $L \in \mathrm{N}_{2,3}$ is generated by the elements $z_1,\dots,z_n$, and let $\{u_1,\dots,u_t\}$ be the set of nonunitary elements of the quasiverbal subloop $\mathbf{T}_1(L)$. As it was shown in p.31 for any element $u_i, 1 \le i \le t$, is chosen the system of generators $z_{1i},\dots,z_{ni}$ of the loop L such that the relations hold

$$[z_{1i}, z_{2i}, z_{3i}]v_{1i} = \dots = [z_{3l_i-2,i}, z_{3l_i-1,i}, z_{3l_i,i}]v_{l,i} = u_i, H_i(z_{1i},\dots,z_{ni}) = 1$$

If we would consider that the loops $L(z_1,\dots,z_n)$ si $L(z_{1i},\dots,z_{ni})$ are differnt, this would means that there exists the isimorphism $\varphi_i : L(z_1,\dots,z_n) \to L(z_{1i},\dots,z_{ni})$ such that

$$u_i^{\varphi_i} = [z_{1i}, z_{2i}, z_{3i}]v_{1i} = \dots = [z_{3l_i-2}, i, z_{3l_i-1i}, z_{3l_i-2}]v_{l_i i},$$

$$H(z_1,\dots,z_n)^{\varphi_i} = H(z_{1i}\dots,z_{ni}).$$

In a natural way we have the isomorphism of the loops

$$\psi : L^i_m/B^{L^i_m}_{l,m1} \to L(z_{1i},\dots,z_{ni}).$$

Since the elements $z_1^{\varphi_i},\dots,z_n^{\varphi_i}$ generates the loop $L(z_{1i},\dots,z_{ni})$ the conjugable classes $z_{1i}^{\varphi_i\psi^{-1}},\dots,z_{ni}^{\varphi_i\psi^{-1}}$ generates the factor–loop $L^i_m/B^{L^i_m}_{l,m_1}$. In each of these conjugable classes we choose one instance of $z^i_1,\dots,z^i_n$ such that $z^i_1,\dots,z^i_n \in lp(z_{1i},\dots,z_{ni}) \subset L^i_m$. Then $L^i_m = lp(z^i_1,\dots,z^i_n,B_{l,m1})$. Now denote for simplicity

$$B_i = B_{l,m1}, x_j = z^1_j\dots z^s_j \in L^1_m \times \dots \times L^s_m.$$

At the begining for an arbitrary numeration of the elements $u_1,\dots,u_t \in \mathbf{T}_1(L)$, we define the commutative loop Moufang $L_m(1,\dots,s$ in the following way:

$$L_m(1,\dots,s) = lp(x_1,\dots,x_n,B_i,i=1,\dots,s) \subset L^1_m \times \dots \times L^s_m$$

It is clear that $L_m(1,\dots,s)$ is the subdirect product of the loops $B^1_m,\dots,B^s_m$. Now we choose the numeration of the elements $u_1,\dots,u_t$ and the number s such that the conditions are verified:

a) $L_m(1), L_m(1,2),\dots,L_m(1,\dots,s) \notin \mathbf{T}_1$;

b) $L_m(1,\dots,s,k) \in \mathbf{T}_1$ pentru toti k, $s < k \leq t$.

Such choice is possible because by lemma 3.1. $L_m(1) = L^1_m \notin \mathbf{T}_1$. The loop L_m is now understanding that it is the loop $L_m(1,\dots,s)$ that verifies the conditions a) si b).

3.4. The properties of the loops L_m.

Lemma 3.6. $L_m \in Q(L)$.

The proof results from the definition of the loop L_m (it is a subdirect product of the loops L^i_m) and lemma 3.3.

Lemma 3.7. $\mathbf{T}_1(L_m) \cap (B_1 \times \dots \times B_s)^{L_m} = 1$

Proof. There are to show that every nonunit element of

$$(B_1 \times \dots \times B_s)^{L_m} = B_1^{L_m} \times \dots \times B_s^{L_m}$$

is approximated by the loop $F_3(\mathrm{N}_{2,3})$. To do this it is sufficient to investigate the nontrivial projection of the element onto the component i and to use lemma 3.2.

Lemma 3.8. $\mathrm{T}_1(L_m) \subseteq \{1, u_1 f_1, \ldots, u_s f_s\}$ *for some* $f_1, \ldots, f_s \in (B_1 \times \ldots \times B_s)^{L_m} \subset L_m$, and $u_i = u_i(x_1, \ldots, x_n) \in \mathrm{T}_1(L(x_1, \ldots, x_n))$

Proof. Let $a \in \mathrm{T}_1(L_m)$. The correspondence $x_j(B_1 \times \ldots \times B_s)^{L_m} \longleftrightarrow x_j \longleftrightarrow z_j$ is extended up to the isomorphism of the loop $L_m/(B_1 \times \ldots \times B_s)^{L_m} \simeq L(x_1, \ldots, x_n) \simeq L(z_1, \ldots, z_n)$. That is why, according to lemma 3.7, $a = u_k(x_1, \ldots, x_n) \cdot f$, where $u_k \in \{u_1, \ldots, u_t\}, f \in (B_1 \times \ldots \times B_s)^{L_m}$. Suppose that lemma is not true, i.e. $k > s$, and construct the commutative loop Moufang $L_m(1, \ldots, s, k) = lp(y_1, \ldots, y_m, B_i, i = 1, \ldots, s, k)$, unde $y_i = x_j z_j^k$. Now we consider that the initial generators $z_1, \ldots, z_n$ ai buclei L are chosen in correspondence to the loop L_n^k, i.e.

$$u_k(z_1, \ldots, z_n) = [z_1, z_2, z_3]v_1 = \ldots =$$

$$[z_{3l_k-2}, z_{3l_k-1}, z_{3l_k}]v_{3l_k}, \quad H_k(z_1, \ldots, z_n) = 1 \tag{17}$$

Then in the projections of the loop $L_m(1, \ldots, s, k)$ onto each component L_m^i we have the equalities

$$[z_1^i, z_2^i, z_3^i]v_1^i = \ldots = [z_{3l_k-2}^i, z_{3l_k-1}^i, z_{3l_k}^i]v_{l_k}^i \ mod\ B_i^{L_m^i} \tag{18}$$

In particular in the projection on the component k according to the definition we have

$$[z_1^k, z_2^k, z_3^k]v_1 a^{1k} = \ldots = [z_{3l_k-2}^k, z_{3l_k-1}^k, z_{3l_k}^k]v_{l_k} a^{l_k,k}, \tag{19}$$

where $a^{1k}, \ldots, a^{l_k,k} \in B_m^k$. In projections of the first s components, by multiplying of the equalities (18) we obtain

$$[x_1, x_2, x_3]v_1 b_1 = \ldots = [x_{3l_k-2}, x_{3l_k-1}, x_{3l_k}]v_{l_k} b_{l_k}, \tag{20}$$

where $b_1, \ldots, b_{l_k} \in B_m^k$.

It is clear that b_1 can be taken as equal to f_1, and then $a = [x_1, x_2, x_3]v_1 b_1$. Finally in the loop $L_m(1, \ldots, k)$ we have the equality

$$[y_1, y_2, y_3]v_1 b_1 a^{1k} = \ldots = [y_{3l_k-2}, y_{3l_k-1}, y_{3l_k}]v_{l_k} b_{l_k} a^{l_k,k} \tag{21}$$

We denote by

$$x = [y_1, y_2, y_3]v_1 b_1 a^{1k} \in L_n(1, \ldots, s, k)$$

and show that x is not approximated by the commutative loop Moufang $F_3(N_{2,3})$. Let λ: $L_m(1, \ldots, s, k) \to F_3(\mathrm{N}_{2,3})$ be some morphism of loops. The following cases are possible:

1) $(a^{jk})\lambda \neq 1$ for some j. For simplicity let $j = 1$. Then by formula τ (see the proof of the lemma 3.1) and from the defining relations of the loop L_m^1 we deduce the equality $(a^{2k})^\lambda$, and from the defining relations of the loop $L_m(1, \ldots, s, k)$ we deduce the equalities $v_2^\lambda = 1, [y_4, y_5, y_6]^\lambda = 1, b_2^\lambda = 1$. This means that $x^\lambda = 1$.

2) $(a^{jk})^\lambda = 1$ for all $j = 1, \ldots, l_k$. We denote

$$N = lp(a^{jk}, j = 1, \ldots, l_k), \quad A = lp(y_1, \ldots, y_n, B_1, \ldots, B_s) \subset L(1, \ldots, s, k).$$

Then evidently the application θ :

$$y_j N \to x_j, \ \ j = 1, \ldots, n, \ \ bN \to \bar{b}, \ \ b \in B_i, \ \ \bar{b} \in \overline{B}_i,$$

where $\overline{B}_i$ is the same loop B_i, but from $L(1, \ldots, s)$, $\bar{b}$ is the corresponding element to b, is extended up to an isomorphism between AN/N and $L(1, \ldots, s)$.

Let the following application is given $\overline{\lambda}$: $AN/N \to F_3(\mathrm{N}_{2,3})$ such that $(yN)^{\overline{\lambda}} = y^{\overline{\lambda}}$ for any $y \in A \cdot \overline{\lambda}$ is a morphism of loops, since $N \subset kerN$. This means that $\overline{\lambda}\theta^{-1} : L(1, \ldots, s) \to F_3(\mathrm{N}_{2,3})$ is a morphism of loops. Since $a \in \mathrm{T}_1(L(1, \ldots, s))$ we have

$$1 = a^{\theta^{-1}\overline{\lambda}} = ([x_1, x_2, x_3]u_1 b_1)^{\theta^{-1}\overline{\lambda}} = ([y_1, y_2, y_3]u_1 b_1 N)^{\overline{\lambda}} =$$

$$([y_1, y_2, y_3]u_1 b_1)^\lambda = ([y_1, y_2, y_3]u_1 b_1 a^{1k})^\lambda = x^\lambda.$$

So, $x^\lambda = 1$ for for any morphism of loops $\overline{\lambda}$: $L_m(1, \ldots, s, k) \to F_3(\mathrm{N}_{2,3})$. Subsequently, $L_m(1, \ldots, s, k) \notin \mathrm{T}_1$, which contradicts the minimality of the number s. According to obtained contradiction results that the statement of the lemma is tru.

Lemma 3.9. *The loop $L_m = L_m(1, \ldots, s)$ is approximated by the loops $C_{l_1 m_1}, \ldots, C_{l_s m_1}$. In particular, the element $u_i f_i \in \mathrm{T}_1(L_m) \subset \{1, u_i, f_i, \ldots, u_s f_s\}$ is approximated by the element $c_i = (a^{1i} = \ldots a^{l_i i})$ of the loop $C_{l_s m_1}$.*

Proof. It is sufficient to investigate the projection of the element $u_i f_i$ onto i projection and to apply lemma 3.4.

Lemma 3.10. *The nontrivial element of $T_1(L_m)$ can not be represented in the product of $m-1$ asociators.*

Proof. By lemma 3.9., the element $1 \neq a \in T_1(L_m)$ is approximated by the element $c_i \in C_{l,n_1}$. We investigate the normal subloop

$$N = lp(a^{ji}_{r1}, a^{ji}_{r2}, a^{ji}_{r3}, \quad 1 \leq j \leq l_i, \quad 1 \leq r \leq m)$$

of the loop.

$$C_{l,n_1} = lp(a^{ji}_{rq}, 1 \leq j \leq l_i, 1 \leq r \leq m, 1 \leq q \leq 3m+3)$$

It is easy to see that N is such subloop of C_{l,n_1} that the factor - loop $C_{l,m_1}/N$ and the element $c_i N \in C_{l,m_1}$ verifies the conditions of lemma 1.4. Subsequently, by this lemma 1.4 the element $c_i N$ can not be decomposed as the product of $m-1$ asociators in the loop $C_{l,m_1}/N$. This means that this is also true for c_i în C_{l,m_1} and a in L_m.

Lemma 3.11. $T_1(L_m) \subset N = lp(lp(y_1, \ldots, y_n)', a^{ji}, i = 1, \ldots, s, j = 1, \ldots, l_i)$

Proof. Denote $A = L_m/N, B = lp(B_1, \ldots, B_s)N/N, C = lp(y_1, \ldots, y_n)N/N$. Evidently that $C \in T_1$. Using the same rationaments of 2) of lemma 1.6 there can be easily also understand that $B \in T_1$, A is the factor - loop of the $N_{2,3}$ - free product of the loops B and C pover relations:

$$[a^{ij}_{rq}N, y_k N, y_l N] = 1, [a^{ji}_{rq}N, a^{ki}_{pt}N, y_l N] = 1$$

for the totaly of indices j, i, r, q, k, p, t, e indicated in the defining relations of the loops L^i_k si L_m. According to lemma 1.2 $A \in T_1$, where from $T_1(L_m) \in N$

Lemma 3.12. Let $m > 3^{n^3}$. *Then* $|L_m| < 3^{m^{12}}$

Proof. Denote by $l(A)$ the number of generators (independent moduluIo A') of the loop $A \in N_{2,3}$. Then

$$|A| \leq |F_l(N_{2,3}) \leq 3^{l+\frac{1}{6}l(l-1)(1-2)} \leq 3^{l^3},$$

where $l = l(A)$. We shall use this formula. Since

$$l_i < n < m, \quad l(L) = n < m, \quad s < |T_1(L)| < |L| < 3^{n+\frac{1}{6}(n-1)(n-2)}.$$

$$l(A_{m1}) = m(3m+3) = 3m(m+1),$$

results that

$$l(L_m) = l(L) + l(B_1) + \ldots + l(B_s) = l(L) + l(A_{m1})l_1 +$$

$$\ldots + l(A_{m1}) \cdot l_s < m + 3m^2(m+1) < m^4.$$

where from we have

$$|L_m| < 3^{(m^4)^3} = 3^{m^{12}}$$

With this the proof of lemma is completed.

Lemma 3.13 *Let $m > 3^{n^3}$. Then every element of L'_m can be represented as a product of m^8 asociators.*

Proof. If the commutative loop Moufang $A \in \mathrm{N}_{2,3}$ is generated by l elements then we in a simple way can understand that any element of A' can be written as a product of a number $< l^2$ of asociators. Taking this into consideration and the equality $l(L_m) < m^4$ we conclude that lemma is true.

Lemma 3.14. *Lete $m \geq 3^{n^3}, r \geq 3^{m^{12}}, 1 \neq u \in \mathrm{T}_1(L_m)$. Then u is not approximated by the loop L_r.*

Proof. Suppose the contrary, i.e. there is the mo4rphism of loops $\varphi : L_m \longmapsto L_r$, such that $u^\varphi \neq 1$. It is clear that $u^\varphi \in \mathrm{T}_1(L_r)$. According to lemma 3.13 the element u and this means also that u^φ in L_r can be written in the form of a product of m^8 asociators. On the other hand, according to lemma 3.10 u^φ can not be represented in L_r in the form of a product of $r-1$ asociators. But $r - 1 \geq 3^{m^{12}} - 1 > m^8$, that can not be. The lemma is proved.

Lemma 3.15. *Let $m \geq 3^{r^{12}}, r \geq 3^{n^3}, 1 \neq u \in \mathrm{T}_1(L_m)$. Then u is not approximated by the loop L_r*

Proof. According to lemma 3.11,

$$u = \prod_{\substack{i=1\ldots s \\ j=1\ldots,l_i}} (a^{ji})^{\alpha_{ij}} y,$$

where $0 \leq \alpha_{ij} < 3, y \in lp(y_1, \ldots, y_n)'$ and for every $q = 1, \ldots, m$ we have

$$a^{ji} = [a^{ji}_{q1}, a^{ji}_{q2}, a^{ji}_{q3}][a^{ji}_{q4}, a^{ji}_{q5}, a^{ji}_{q6}] \ldots [a^{ji}_{q,3m+1}, a^{ji}_{q,3m+2}, a^{ji}_{q,3m+3}]$$

Denote $I = \{(i,j) : \alpha_{ij} \neq 0\}, g^{ji}_q = [a^{ji}_{q4}, a^{ji}_{q5}, a^{ji}_{q6}] \ldots [a^{ji}_{q,3m+1}, a_{q,3m+2}, a^{j,}_{q,3_m+3}$ Now suppose the contrary, i.e. there is the morphism of loops $\varphi : L_m \to L_r$, such that $u^\varphi \neq 1$. Two cases are possible:

1) There exists a pare $(j, i) \in I$, such that $(g_q^{ji})^\varphi = 1$ for every $q \in \{1, \ldots, m\}$. Since $a^{ji} \in B_1 \times \ldots \times B_s$, and in the defining relations $B_1 \times \ldots \times B_s$ are (recall that $B_i = \prod_{j=1}^{l_i} A^j_{m_1}$; see also the definition of the loop $A^j_{m_1}$)) the equalities

$[a^{ji}_{qt}, a^{ji}_{q't'}, a^{ji}_{q''t''}] = 1$, *for all* $(j, i, q, t) \neq (j', i', q', t')$ *or* $(j, i, q, t) \neq (j'', i'', q'', t'')$, *or* $(j', i', q', t') \neq (j'', i'', q'', t'')$ *if* $t, t', t'' > 3$,

then the order of the subloop $lp((g_q^{ji})^\varphi, q = 1, \ldots, m) \subset L_r$ *is greater than m, this means that it is greater than the order of the loop* L_r *itself, because according to lemma 3.12 and to the condition* $|L_r| < 3^{r^{12}} \leq m$

2) For every pare $(j, i) \in I$ *there exists an* $q \in \{1, \ldots, m\}$ *such that* $(g_q^{ji})^\varphi = 1$. *Using again the relations of the loops* $B_1 \times \ldots \times B_s, B_i$ *and* A_{m_1}, u^φ *can be written in the form:*

$$u^\varphi = [\prod_{(i,j)} (a^{ji}_{q_1})^{\varphi\alpha_{ij}}, (a^{ji}_{q_2})^\varphi, (a^{ji}_{q_3})^\varphi] y^\varphi$$

Since $y^\varphi (y \in lp(y_1, \ldots, y_r) = lp([y_i, y_j, y_k], 1 \leq i < j < k \leq n)$ *can be written in the form of the product of* n^2 *asociators, than* u^φ *is represented as the product of* $n^2 + 1 < 3^{n^3} \leq r$ *asociators. We obtained the contradiction with lemma 3.10. The lemma is proved.*

Lemma 3.16 (The second auxiliary result). *Let L is a finite loop of the variety* $0\mathbf{N}_{2,3}$. *If* $L \notin \mathbf{T}_1$, *then* $Q(L)$ *contains a continuous number of different subquasivarieties.*

Proof. Accoding tom lemma 3.6. the quasivariety $Q(L)$ contains an infinite of commutative loops Maufang L_m, where m takes the values

$$m_1 = 3^{n^3}, m_{i+1} = 3^{m_i^{15}}$$

for $i > 1$. Analogically, as it was explained in the proof of lemma 2.6, using lemmas 3.14 si 3.15, we obtain that different subsets of this subsequence of loops generates different quasivarieties. The statement of lemma now follows.

§4. The basic result.

Theorem. *The lattice of subquasivarieties* $\mathbf{K}$ *of commutative loops Moufang with the nilpotence class* ≤ 2 *is finite if and only if the quasivariety* $\mathbf{K}$ *is generated by a finite set of loops of the type* $H_{\infty\infty\infty}, H_{r\infty\infty}, H_{rs\infty}, H_{rst}, Z_{p^m}$, *where the prime number* $p \neq 3$.

Proof. Sufficiency. Let Σ be the set of all loops of the indicated types in the formulation of the theorem, and L be the direct product of a finite number of loops of Σ. Evidently, $\mathbf{K} = Q(L)$ containes only a finite number of loops, each of them is generated by three elements. So it is sufficiet to show that every subquasivariety in $\mathbf{K}$ is determined by loops generated by three elements. According to the aparness criteria this will be true if every finitely generated loop $B \in \mathbf{K}$ is approximated by itself subloops which are generated by three elements.

We show that it is really the case. If $b \in B$ and $b \notin B'$, thon it is obvious b is approximated by a cyclic subgroup. Let $b \in B'$. Since H_{rst}, $H_{\infty st}$, $H_{\infty\infty t}$, $H_{\infty\infty\infty}$ are respectively approximated by the loops $Z_{3^t} \times F_3(\mathrm{M}_{2,3}), Z_{3^t} \times Z \times F_3(\mathrm{N}_{2,3}), Z_{3^t} \times Z \times F_3(\mathrm{N}_{2,3}), Z \times F_3(\mathrm{N}_{2,3})$, then b is approximated by the loop $F_3(\mathrm{N}_{2,3})$. Let the application $\varphi : B \to F_3(\mathrm{N}_{2,3}; x, y, z)$ be a morphism of loops such that $b^\varphi = [x, y, z]$. Then $B/B'Ker\varphi$ is a product of 3 cyclic subgroups of order 3. According to thje theorem 8.1.1. of [2], there can be taken such generators $b_1B', \ldots, b_mB'$ of the loop B/B' that:

$$B/B' = lp(b_1B') \times \ldots \times lp(b_mB'),$$

$$Ker\varphi \cdot B'|B' = lp(b_1^3B') \times lp(b_2^3B') \times lp(b_3^3B') \times lp(b_4B') \times \ldots \times lp(b_mB')$$

We denote $C = lp(b_3, \ldots, b_m) \subset B$ and show that

$$lp(b_1, b_2, b_3), \cap C^B = 1.$$

Obviously that $lp(b_1, b_2, b_3)^\varphi = F_3(\mathrm{N}_{2,3})$ and $(C^B)^\varphi \subseteq F_3(\mathrm{N}_{2,3})'$. Then $lp(b_1, b_2, b_3) \cap C^B \subseteq Ker\varphi \cdot lp(b_1, b_2, b_3)'$. This means that every element $x \in lp(b_1, b_2, b_3) \cap C^B$ can be represented in the form $x = y[b_1, b_2, b_3]^\alpha$ for some $y \in Ker\varphi$ and $\alpha < 3$. Since $x \in lp(b_1, b_2, b_3)$ si $lp(b_1, b_2, b_3) \cap Ker\varphi = lp(b_1, b_2, b_3)^3$, it results $y = z^3$ for a fixed $z \in lp(b_1, b_2, b_3)$. So $x = z^3[b_1, b_2, b_3]^\alpha$. Beside of this, taking into consideration that commutative group B/B' is the direct product of the cyclic subgroups $lp(b_iB'), i = 1, \ldots, m$, we observe that $lp(b_1, b_2, b_3) \cap C^B \subseteq B'$. In particular, $x \in B'$.

Then according to quasiidentities (8) (evidently they are true in any loop of the set Σ), we have $z^3 = 1$, and subsequently $[b_1, b_2, b_3]^\alpha \in B' \cap C^B \subset Ker\varphi$. But $[b_1, b_2, b_3]^\alpha \in Ker\varphi$ only in the case when $\alpha = 0$, since as it was observed $[b_1, b_2, b_3]^\varphi \neq 1$. It means that $lp(a_1, a_2, a_3) \cap C^B = 1$, what was to show.

Since $b^{\varphi} = [x, y, z]$ si $[b_1, b_2, b_3]^{\varphi} = [x, y, z)^{\beta}$, $\beta \neq 0$ mod 3, results that $b = [b_1, b_2, b_3]^{\gamma}$ mod $Ker\varphi$ for some $y \neq 0$ mod 3. Then $b^{-1}[b_1, b_2, b_3]^{\gamma} \in Ker\varphi \cap B'$. But from the representation of the group B/B' and from the inequality $[b_1, b_2, b_3]^{\varphi} \neq 1$ it can be seen that

$$Ker\varphi \cap B' = lp([b_i, b_j, b_k], 1 \leq i < j < k \leq m, (i, j, k) \neq (1, 2, 3)) \subset C^B.$$

We obtain

$$b^{-1}[b_1, b_2, b_3]^{\gamma} \in C^B \;\; si \;\; b = [b_1, b_2, b_3]^{\gamma} mod C^B.$$

It remains to investigate the natural morphism of loops $\psi : B \to B/C^B$. We have

$$b^{\psi} = ([b_1, b_2, b_3]^{\gamma})^{\psi} = [b_1, b_2, b_3]^{\gamma} C^B \neq C^B,$$

i.e. the element b is approximated by the loop $lp(b_1, b_2, b_3) \subset B$, what was to be shown.

Necessitation. Let **K** be the quasivariety which contains only a finite number of subquasigroups and by absurdity we suppose that **K** is not generated by a fibite set of finite nonisomorphic loops of Σ. It is clear that **K** contains only a finite number of nonisomorphic loops of the set Σ. This meanbs that there is a finitely generated loop $L \in$ **K** such that the quasivariety $Q(L)$is not generated only by loops of Σ. In this case we shall show that $Q(L) \subset$ **K** contains continuous many different subquasivarieties and so we obtain the contradiction.

Consider the choosen L such that in K there are no such loops as L with number of generators less than the number of generators of this loop L. this is why if L is finite L can be considered with the exponent equal to the power of the number 3 (i.e. L is a 3 - loop). We investigate the cases.

1. For some elements $a, b_1, \ldots, b_{3n}$ din L

$$a^3 = \prod_{i=1}^{n} [b_{3i-2}, b_{3i-1}, b_{3i}] \neq 1$$

Denote by V the set of matrices of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 \ldots \alpha_n \\ \beta_1 & \beta_2 \ldots \beta_n \end{pmatrix},$$

which verifies the condition

$$[a, b_1^{\alpha_1} \ldots b_{3n}^{\alpha_{3n}}, b_1^{\beta_1} \ldots b_{3n}^{\beta_{3n}}] = 1,$$

where

$0 \leq \alpha_i, < 3$ and $0 \leq \beta_i < 3$, and investigate the commutative loop Moufang

$$B = B(n, V, k)\ (respectively,\ \ B(n, V)) = lp(x, y_1, \ldots, y_{3n} \ ||$$

$$x^3 = \prod_{i=1}^{n} [y_{3i-2}, y_{3i-1},$$

$$y_{3i}], [x, y^{\alpha_1}, \ldots y_{3n}^{\alpha_{3n}}, y_1^{\beta_1} \ldots y_{3n}^{\beta_{3n}}] = 1, \ \left(\begin{array}{c} \alpha_1, \ldots, \alpha_{3n} \\ \beta_1, \ldots, \beta_{3n} \end{array} \right) \in V),$$

where 3^k (respectively 0) is the exponent of the loop L (and the relations are given in the variety $\mathbf{N}_{2,3^k}$ (respectively, $\mathbf{N}_2$)). We show that the loop $B \in Q(L)$. Denote

$$T = lp(x^3, [x, y, z], \quad \text{for all elements} \quad y, z \in B) \subset B.$$

Any element of the subloop T is approximated by L via morphism of loops of B in L defined by the applications

$$x \to a, y_i \to b_i, \ \ i = 1, \ldots, 3n.$$

$B/T \cong Z_3 \times D$, where

$$Z_3 \cong lp(xT), D = F_{3n}(\mathbf{N}_{2,3^k}$$

$$(respectively,\ \ \mathbf{N}_2); y_1 \ldots, y_{3n})/lp(\prod_{i=1}^{n} [y_{3i-2}, y_{3i-1}, y_{3i}]).$$

Obviously $Z_3 \in Q(L)$, and according to lemma 1.3 $D \in Q(L)$, This means that $B/T \in Q(L)$. Where from following the criteria of apartness we have $B \in Q(L)$, what was to be proved. In virture of lemma 2.6 there are a continuous different subquasivarieties in $Q(B)$, so this is true also for $Q(L)$.

2. In L there are valid the quasiidentities

$$x^3 = \prod_{i=1}^{n} [x_{3i-2}, x_{3i-1}, x_{3i}] \to x^3 = 1, \ \ n = 1, 2, \ldots$$

The elements of the asociator L' are not approximated by the loop $F_3(\mathbf{N}_{2,3})$. Really, if this is not so then according to the apartness criterion

$$L \in Q(L/L' \times F_3(\mathbf{N}_{2,3})),$$

which contradicts the supposition. subsequently not all elements of L' are approximated by the loop $F_3(\mathbf{N}_{2,3})$.

According to the theorem 8.1.1. din [2], the loop L in the variety $\mathbf{N}_2$ can be represented as

$$L = lp(x_1, \ldots, x_n \parallel x_i^{s_i} = r_i, i = 1, \ldots, l, r_j = 1, j = l+1, \ldots, m),$$

where $r_i \in F_n(x_1, \ldots, x_n)'$, and s_i are some convinient numbers. We shall show that for every $i \leq l$ the number s_i is devided via 3 and according to the quasiidentities of the condition 2, $x^{s_i} = 1$ and $r_i = 1$. Really, let in the contrary, for some $i \leq l$ s_i is not devided via 3. If $r_i = 1$, than according to the identity (1), $x_i \in Z(L)$ *si* $lp(x_i) \cap L' = 1$. It results from here that $L = lp(x_i) \times lp(x_1 \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ that can not be. Let now $r_i \neq 1$, then according to identities (1) and (3) we have

$$[x_i, y, z] = [x_i^{s_i}, y, z] \quad \text{or} \quad [x_i, y, z] = [x_i^{s_i}, y, z]^{-1}$$

for any $y, z \in L$. Where from we obtain that $x_i \in Z(L)$. According to the Dik'stheorem [4] the application $x_i \to r_i, x_j \to x_j, j \neq i, j = 1, \ldots, n$, is extended up to an morphism of loops $\varphi : L \to L$. Since $x_i \in Z(L)$, results that $Ker\varphi = lp(x_i^3)$. But x_i^{φ} can be excluded from the set of generators of the loop L^{φ}. Since $lp(x_1^3) \cap L' = 1$ and $L/L' = lp(x_1 L') \times \ldots \times lp(x_1 L') \in Q(L)$ we obtain that L is approximated by the loop $L^{\varphi} \times lp(x_i L')$, which contradicts the choice of the loop L.

So the numbers $s_i, i = 1, \ldots, l$, are devided by 3, so the loop L has the representation

$$L = lp(x_1, \ldots, x_n \parallel x_i^{s_i} = 1, v_i = 1, i = 1, \ldots, l; v_j = 1, j = l+1, \ldots, m)$$

We investigate the commutative loop Moufang represented in the variety $\mathbf{N}_{2,3^k}$ (respectively, $\mathbf{N}_2$) as

$$B = lp(x_1, \ldots, x_n \parallel v_i = 1, i = 1, \ldots, m).$$

It is clear that $B \in Q(L)$ and not all elements of B' are approximated by the loop $F_3(\mathbf{N}_{2,3})$. According to lemma 1.5, the lattice of subquasivarieties of $Q(B)$ is isomorphic to the lattice of subquasivarieties of $Q(B/B^3)$. According to lemma 3.16 $Q(B/B^3)$ continuous many different subquasivarieties. Subsequently $Q(B)$ and so also $Q(L)$ contain continuous many different subquasivarieties. The theorem is proved.

According to the apartness criteria it results from the theorem the following

Corollary. *Let L be a finite generated commutative loop Moufang with the class of nilpotence 2. Then the lattice of the subqvasivarieties of $L_qQ(L)$ is either finite or continuous, but $L_qQ(L)$ is finite if and only if L is the subdirect product of some loops belonging to the same finite set of loops of Σ.*

§5. The desciption of the lattice $L_qQ(Z_{3^k} \times F_3(\mathrm{N}_{2,3}))$.

As it was observed, any quasivariety M generated by the commutative loop Moufang with the exponent 3^k, which contains finitely many subquasivarieties, is contained in the quasivariety $Q(Z_{3^k},\ F_3(\mathrm{N}_{2,3}))$ and $\mathrm{M} = Q(H_{r_1 s_1 t_1}, \ldots, H_{r_n s_n t_n})$ for some r_i, s_i, t_i.

We introduce the notations:

$$\boxed{\begin{matrix} r_1 & r_2 & \ldots & r_n \\ s_1 & s_2 & \ldots & s_n \\ t_1 & t_2 & \ldots & t_n \end{matrix}} = Q(H_{r_1 s_1 t_1}, \ldots, H_{r_n s_n t_n}),$$

$$\varphi_{rst} = (x^{3^r} = 1 \ \& \ y^{3^s} = 1 \ \& \ z^{3^t} = 1 \rightarrow [x,y,z] = 1), 1 \leq r \leq s \leq t.$$

Proposition 1. *The quasiidentity φ_{rst} is true in the commutative loop Moufang $H_{r's't'}$ if anf only if at least one of the inequalities fulffils $r < r', s < s', t < t'$.*

Proof. Really, let φ_{rst} is true in $H_{r's't'}$ and suppose that $r \geq r', s \geq s', t \geq t'$. Then in $H_{r's't'}$ the identity is true $z^{3^t} = 1$, so also the identity $[x,y,z] = 1$ do, that can not be.

Let now be true one of the inequalities $r < r', s < s', t < t'$, for instance $r < r'$, and let for $x = a, y = b, z = c$, where the elements a, b, c belongs to the loop $H_{r's't'} = lp(x,y,z \ \| \ x^{3^{r'}} = 1, y^{3^{s'}} = 1, z^{3^{t'}} = 1)$, the left side of the quasiidentity φ_{rst} is true, so $a^{3^r} = 1, b^{3^s} = 1, c^{3^t} = 1$. Since a, b, c can be represented by the form

$$a = x^{\alpha_1} \cdot y^{\beta_1} z^{\gamma_1} a', \quad b = x^{\alpha_2} \cdot y^{\beta_2} z^{\gamma_2} b', c = x^{\gamma_3} \cdot y^{\beta_3} z^{\alpha_3} c',$$

where

$$0 \leq \alpha_i < 3^{r'}, 0 \leq \beta_i < 3^{s'}, 0 \leq \gamma_i < 3^{t'}, a', b', c' \in H'_{r's't'}.$$

We obtain the equalities

$$x^{\alpha_1 3^r} = 1, \quad x^{\alpha_2 3^r} = 1, \quad x^{\alpha_3 3^r} = 1.$$

and from here

$$\alpha_1 3^r = 0 mod 3^{r'}, \alpha_2 3^r = 0 mod 3^{r'}, \alpha_3 3^r = 0 mod 3^{r'}.$$

Taking into consideration the inequality $r < r'$ we have at the same time $\alpha_1, \alpha_2, \alpha_3$ are divisible via 3, so the elements $x^{\alpha_1}, x^{\alpha_2}, x^{\alpha_3}$ belongs to the central subloop $H^3_{r's't'} \subset H_{r's't'}$. Then we have

$$[a, b, c] = [x^{\alpha_1} y^{\beta_1} z^{\alpha_1}, x^{\alpha_2} y^{\beta_2} z^{\alpha_2}, x^{\alpha_3} y^{\beta_3} z^{\alpha_3}] =$$

$$[y^{\beta_1} z^{\alpha_1}, y^{\beta_2} z^{\alpha_2}, y^{\beta_3} z^{\alpha_3}] = 1$$

Subsequently we can conclude that φ_{rst} is true in $H_{r's't'}$. The proposition is proved.

Proposition 2. *If the subquasivariety* M *is contained in* $Q(Z_{3^k}, F_3(\mathrm{N}_{2,3}))$ *then*

$$\mathrm{M} = \boxed{\begin{matrix} r_1 & r_2 & \dots & r_n \\ s_1 & s_2 & \dots & s_n \\ t_1 & t_2 & \dots & t_n \end{matrix}}$$

for some $r_i, s_i, t_i \leq n$ *the condition is verified* $r_1 \leq r_2 \leq \ldots \leq r_n$ *and one of the following:*

a)

$$r_i < (\leq) r_{i+1},$$
$$s_i \leq (<) s_{i+1},$$
$$t_i > t_{i+1};$$

b)

$$r_i < r_{i+1},$$
$$s_i > (\geq) s_{i+1},$$
$$t_i \geq (>) t_{i+1};$$

c)

$$r_i < (\leq) r_{i+1},$$
$$s_i > s_{i+1},$$
$$t_i \leq (<) t_{i+1}.$$

Proof. As it was observed

$$M = \boxed{\begin{array}{cccc} r_1 & r_2 & \dots & r_n \\ s_1 & s_2 & \dots & s_n \\ t_1 & t_2 & \dots & t_n \end{array}}$$

where r_i, s_i, t_i are not determined in a unic way. We consider the system $\{H_{r_i s_i t_i}, i = 1, \dots, n\}$ those posible system which contain the least number of nonasociative loops. It is clear that we can suppose that $r_1 \leq r_2 \leq \dots \leq r_n$, also we observe that the choosen system contains a single cyclic group $H_{oos} = Z_{3^s}$ if the exponent of the quasivarieties **M** is greater than the exponent of each nonasociatice loop and in the contrary case the system does not contain groups.

To prove that r_i, s_i, t_i verifies the condition a), or b), or c), is sufficient to show that that in each of these conditions there can not be two equalities but two inequalities impluies the third. We verify this fact for the condition a) (analogically it is verified for conditions b), c)).

Suppose that $r_i = r_{i+1}, s_i = s_{i+1}$. If $0 = r_i = r_{i+1}$, we obtain the contradiction with the number of groups of the system. Let $0 \neq r_i = r_{i+1}$, then

$$Q(H_{r_i s_i t_i}, H_{r_{i+1}, s_{i+1} t_{i+1}}) = Q(H_{r_i s, min(t_i, t_{i+1})}, H_{00 \max_i t_i})$$

contradicts the condition of minimality of the number of nonasociative loops of the system.

Also there can not be the equalities

$$r_i = r_{i+1}, t_i = t_{i+1} \quad \text{sau} \quad s_i = s_{i+1}, t_i = t_{i+1}$$

We prove now that is true the implication

$$r_i < r_{i+1} \ \& \ s_i \leq s_{i+1} \rightarrow t_i > t_{i+1}$$

(The implications

$$r_i \leq r_{i+1} \ \& \ s_i < s_{i+1} \rightarrow t_i > t_{i+1},$$

$$r_i < r_{i+1} \ \& \ t_i > t_{i+1} \rightarrow s_i \leq (<) s_i + 1$$

are proved analogically.

Really, if $t_{i+1} \leq t_i$, then

$$Q(H_{r_is_it_i}, H_{r_{i+1}s_{i+1}t_{i+1}}) = Q(H_{r_is_it_i}, H_{00t_{i+1}})$$

contradicts the minimality of the number of nonasociative loops. The proposition is proved.

Proposition 3. *The quasivariety*

$$\boxed{\begin{array}{ccc} r_1 & \dots & r_n \\ s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{array}}$$

is contained in the quasivariety

$$\boxed{\begin{array}{ccc} r_i & \dots & r_n \\ s_i & \dots & s_n \\ t_i & \dots & t_n \end{array}}$$

if and only if $\max\limits_i(r'_i, s'_i, t'_i) \geq \max\limits_i(r_i, s_i, t_i)$ *and for every* $r_i \neq 0$ *there is a triplet*

$\begin{pmatrix} r'_j \\ s'_j \\ t'_j \end{pmatrix}$, *such that* $r_i \geq r'_j > 0, s_i \geq s'_j$ *and* $t_i \geq t'_j$.

Proof. *The sufficiency .* For every i and $j(i)$,

$$H_{r_is_it_i} \in Q(H_{r'_js'_jt'_j}, Z_{p^{max(r'_i,s'_i,t'_i)}})$$

if $r_i \neq 0$

The necessitation. $max(r'_i, s'_i, t'_i) \geq max\ (r_i, s_i, t_i)$ is obviously true since the exponent of the first quasivariety is not greater than the exponent of the second. Suppose that that the second condition is not fulfiled, i.e. for some $i \leq n$ and some $j \leq m$ or $r_i < r'_j$ or $s_i < s_i$, or $t_i < t'_j$. Then the quasiidentity $\varphi_{r_is_it_i}$ is false in $H_{r_is_it_i}$, and is true according to the proposition 1, in the loop $H_{r'_js'_jt'_j}$. The proposition is proved.

Corollary. *If*

$$\boxed{\begin{array}{ccc} r_1 & \dots & r_n \\ s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{array}} = \boxed{\begin{array}{ccc} r'_1 & \dots & r'_n \\ s'_1 & \dots & s'_n \\ t'_1 & \dots & t'_n \end{array}},$$

then the matrices

$$\begin{pmatrix} r_1 \dots r_n \\ s_1 \dots s_n \\ t_1 \dots t_n \end{pmatrix} \quad si \quad \begin{pmatrix} r'_1 \dots r'_n \\ s'1 \dots s'_n \\ t'_1 \dots s'_n \end{pmatrix}$$

coincide.

Proof. Really, accoding to the condition we obtain

$$\max_i(r_i, s_i, t_i) = \max_j(r_j, s_j, t_j).$$

If $r_i \neq 0$, according to proposition 3,

$$r_i \leq r'_j, s_i \leq s'_j, t_i \leq t'_j,$$

$$r'_j \leq r_k, s'_j \leq s_k, t'_j \leq t_k$$

for some indices j, k. Where from we obtain the inequalities

$$r_i \leq r_k, s_i \leq s_k, t_i \leq t_k,$$

which are true, according to the proposition 2, only in the case when $i = k$. Subsequently

Let N a quasivariety defined by the identities of (i), (ii) and quasiidentities of (iii) - (v) and suppose that $\mathrm{N} \neq Q(F_3(\mathrm{N}_{2,3^k}))$.

$$r_i = r'_j, s_i = s'_j, t_i = t'_j,$$

what was to be proved.

Observation 1. *For $k > 1$ the quasiidentities of the loop $F_3(\mathrm{N}_{2,3^k})$ has the following basis*

(i) $x^{3^k} = 1$,

(ii) $[[x, y, z], u, v] = 1$,

(iii) *the asociator quasiidentities of the loop* $F_3(\mathrm{N}_{2,3})$,

(iv) $x^3 = \prod_{i=1}^{n} [x_{3i-2}, x_{3i-1}, x_{3i}] \rightarrow x^3 = 1, n = 1, 2, \dots,$

(v) $x^{3^{k-1}} = 1 \rightarrow [x, y, z] = 1$.

Proof. Let N be a quasivariety defined by the identities of (i), (ii) and quasiidentities of (iii), (iv), (v) and suppose that $\mathrm{N} \neq Q(F_3(\mathrm{N}_{2,3^k}))$. We investigate a finitely generated loop $L \in \mathrm{N}, L \notin Q(F_3(\mathrm{N}_{2,3}))$. Suppose that

$L = F_n/H$. Let $M = F'_n \cap H$. Then there is such element $u \in F'_n$ and $u \notin M$, which is not approximated by the loop $F_3(N_{2,3^k})$. Subsequently the quasiidentity $M = 1 \to u = 1$ is true in the loop $F_3(N_{2,3})$ and is false in F_n/M. But this is not true since in F_n/M there are true the quasiidentities (iii).

Corollary. *The quasivariety L generated by a finite commutative loop Moufang contains a continuous set of subquasivarieties if and only if in some 3 - subloop of L is false one of the quasiidentities:*

a) $$x^3 = \prod_{i=1}^{n}[x_{3i-2}, x_{3i-1}, x_{3i}] \to x^3 = 1, i = 1, 2, \ldots,$$

b) theasociator quasiidentities ofthe loop $F_3(N_{2,3})$.

Observation 2. *For $k \geq 2$, the lattice $L_q Q(Z_{3^k} \times F_3(N_{2,3}))$ is not modular*

.

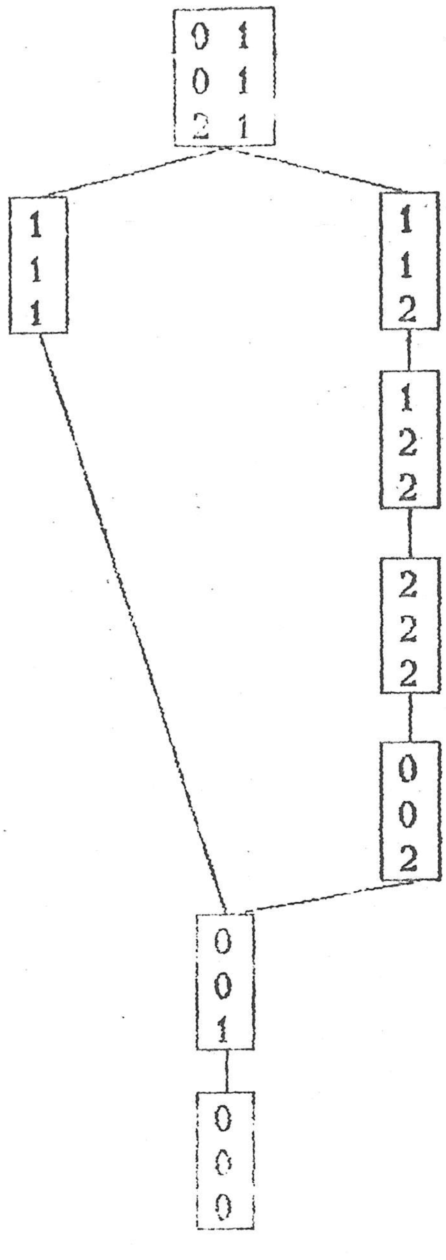
Really that is the case since we easily convince ourselves that the quasivarieties

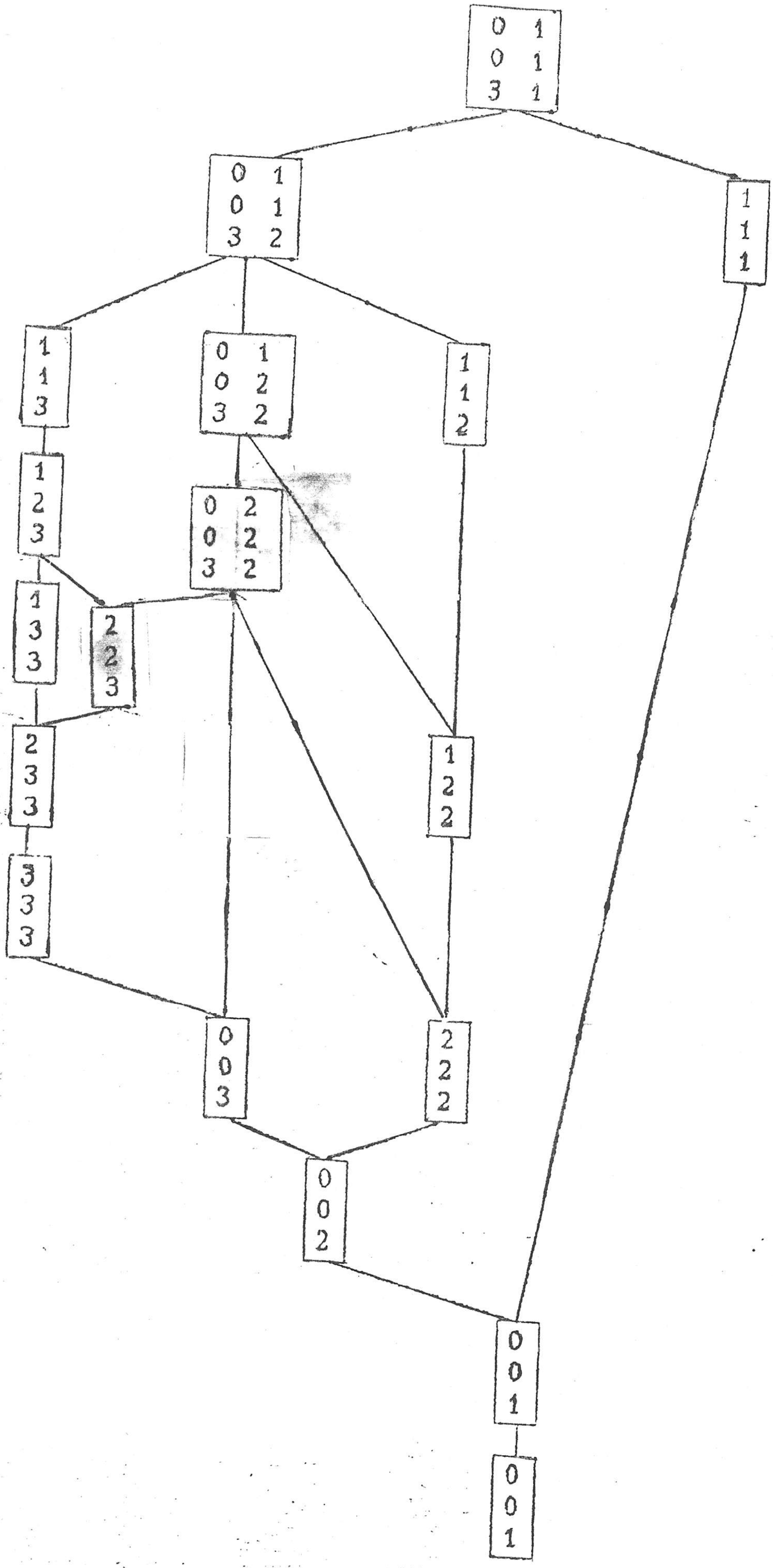
$$Q(Z_{3^2} \times F_3(N_{2,3})), Q(F_3(N_{2,3})), Q(Z_{3^2}), Q(Z_3)$$

formed in the lattice $L_q Q(Z_{3^k} \times F_3(N_{2,3})), k \geq 2$, a nonmodular sublattice of five elements.

Below we illustrate the aspect of the lattice $L_q(Q(Z_{3^k} \times F_3(N_{2,3}))$ for $k = 1, 2, 3$.


1 1 1
1 1 1
0 0 0


0 1
0 1
2 1
1 1 1
1 1 2
1 2 2
2 2 2
0 0 2
0 0 1
0 0 0



0 1
0 1
3 1
0 1
0 1
3 2
1
1
1
1
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3
0 1
0 2
3 2
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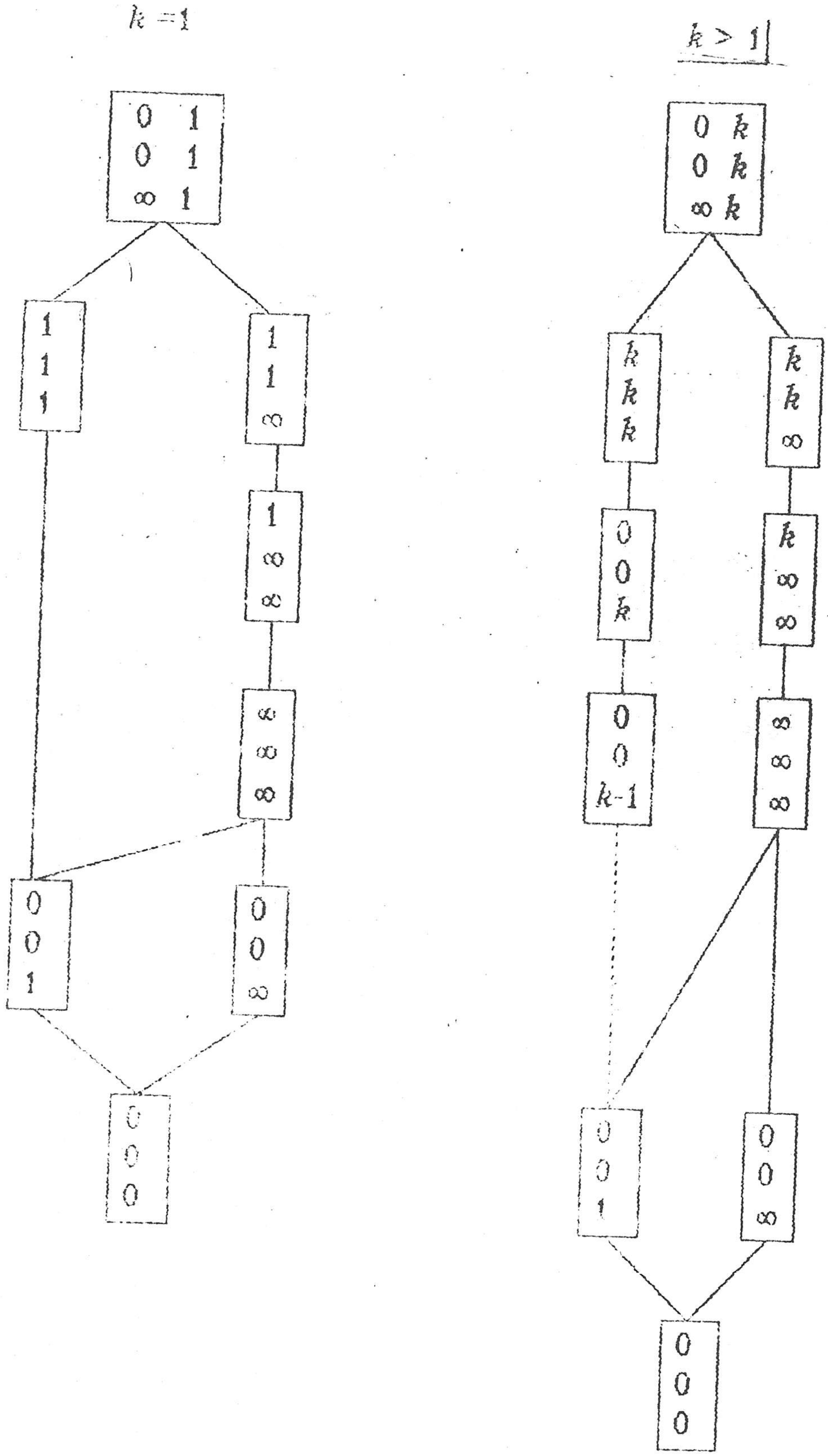
1

Observation 3. *For any natural number k the lattice $L_q(Z \times F_3(N_{2,3^k}))$ is finite, nonmodular and has the aspect:*



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