INSTITUTUL DE MATEMATICA
AL ACADEMIEI ROMANE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

STABILITY AND ROBUST STABILIZATION TO LINEAR STOCHASTIC SYSTEMS DESCRIBED BY DIFFERENTIAL EQUATIONS WITH MARKOVIAN JUMPING AND MULTIPLICATIVE WHITE NOISE

by

VASILE DRAGAN and TOADER MOROZAN

Preprint nr. 17/1999

BUCURESTI

STABILITY AND ROBUST STABILIZATION TO LINEAR STOCHASTIC SYSTEMS DESCRIBED BY DIFFERENTIAL EQUATIONS WITH MARKOVIAN JUMPING AND MULTIPLICATIVE WHITE NOISE

by

VASILE DRAGAN *) and TOADER MOROZAN **)

November, 1999

*) Institute of Mathematics of the Romanian Academy, P.O.Box 1–764, RO–70700, Bucharest, Romania.
**) Institute of Mathematics of the Romanian Academy, P.O.Box 1–764, RO–70700, Bucharest, Romania.

STABILITY AND ROBUST STABILIZATION TO LINEAR STOCHASTIC SYSTEMS DESCRIBED BY DIFFERENTIAL EQUATIONS WITH MARKOVIAN JUMPING AND MULTIPLICATIVE WHITE NOISE

Vasile Dragan and Toader Morozan
Institute of Mathematics of the Romanian Academy,
P.O.Box. 1-764, RO-70700, Bucharest, Romania

ABSTRACT

In this paper we consider linear controlled stochastic systems subjected both to white noise disturbance and Markovian jumping.

Our aim is to provide a mathematical background in order to give unified approach for a large class of problems associated to linear controlled systems subjected both to multiplicative white noise perturbations and Markovian jumping. First we proof an Itô type formula. Our result extends the result of [24], to the case when the stochastic process $x(t)$ has not all moments bounded.

Necessary and sufficient conditions assuring the exponential stability in mean square for the zero solution of a linear stochastic system with multiplicative white noise and Markovian jumping are provided.

Some estimates for solutions of affine stochastic systems are derived and necessary and sufficient conditions assuring the stochastic stabilizability and stochastic detectability are given.

A stochastic version of Bounded Real Lemma is proven and several aspects of the problem of robust stabilization by state feedback for a class of linear systems with multiplicative white noise and Markovian jumping are investigated.

1 INTRODUCTION

The control of stochastic systems with multiplicative white noise received much attention in the last four decades. For the results concerning the stability for stochastic systems with state dependent noise we refer the readers to [1, 5, 21, 22, 23] and the references therein. The linear quadratic problem associated to a linear stochastic systems with multilicative white noise was investigated in [6, 16, 35, 36].

Robust stabilization for this class of stochastic systems with multiplicative white noise was intensively investigated in [8, 12, 17, 18, 28, 31].

There exists also a great number of papers in which the controlled systems with Markovian jumping are studied. Such systems can be used to represent many important physical systems subject to random failures and structure changes such as electric power systems [37], control system of a solar thermal central receiver [34], communications systems [2], aircraft flight control [26], control of nuclear power plants [33] and manufacturing systems [3]. For the results concerning the stability and optimal stabilization problem we refer the readers to [27, 20, 29, 25].

The robust stabilization problem for linear systems with Markovian jumping was studied in [30, 32, 9] and the references therein.

In this paper we consider linear controlled stochastic systems subjected both to white noise disturbance and Markovian jumping.

Such class of systems was considered in [24], where the problem of the existence of the bounded solution was discussed and in [15] where sufficient conditions concerning stability and boundedness of the solution are given and in [15] where the infinite horizon optimal control of linear stochastic systems with quadratic cost integrand is studied.

Our aim is to provide a mathematical background in order to give unified approach for a large class of problems associated to linear controlled systems subjected both to multiplicative white noise perturbations and Markovian jumping. The problem of exponential stability in mean square is investigated in connection with a class of linear positive operators which are defined on a finite dimensional Hilbert space adequately associated.

The paper is organised as follows:

The section 2 contains the list of notations used throughout the paper while the section 3 contains the proof of an Itô type formula. Our result extends the result of [24], to the case when the stochastic process $x(t)$ has not all moments bounded and this is the case when the process $x(t)$ is a solution of a system of stochastic differential equations whose inputs are non-anticipative stochastic processes which are in $L^2([t_0, T] \times \Omega)$.

In section 4 we prove several results containing necessary and sufficient conditions assuring the exponential stability in mean square for the zero solution of a linear stochastic system with multiplicative white noise and Markovian jumping.

In section 5 we derive some estimates for solutions of affine stochastic systems and in section 6 some necessary and sufficient conditions assuring the stochastic stabilizability and stochastic detectability are given.

A stochastic version of Bounded Real Lemma is given in Section 7, while in section 8 we investigate several aspects of the problem of robust stabilization by state feedback for a class of linear systems with multiplicative white noise and Markovian jumping.

2 NOTATIONS AND PRELIMINARY REMARKS

The following notations will be used throughout this paper.

A. $\mathbf{R}^n$ is the real n-dimensional space.
$\mathbf{R}_+$ is the set of nonnegative real numbers.
$\mathbf{R}^{n\times m}$ is the set of all real $n \times m$ matrices.

I_n is the identity matrix in $\mathbf{R}^n$.

If X is a matrix (or a vector) X^* is the transpose of X; if A is a matrix $|A|$ is the operator norm of A and TrA is the trace of A.

In this paper $\mathcal{D} = \{1, 2, ..., d\}$.

If H is a matrix, then $H \geq 0$ means that H is symmetric positive semidefinite.

B. By $\mathcal{S}_n$ we denote the space of all $n \times n$ symmetric matrices and by $\mathcal{S}_n^d$ we denote the space of all $H = (H(1), ..., H(d))$ with $H(i) \in \mathcal{S}_n$.

$\mathcal{S}_n^d$ is a real Hilbert space with the inner product

$$\langle H, G\rangle = \sum_{i=1}^{d} Tr(H(i)G(i)).$$

The norm induced by this inner product is $|||H||| = \langle H, H\rangle^{1/2}$ for all $H \in \mathcal{S}_n^d$. On $\mathcal{S}_n^d$ we consider also the norm $|H| = \max\{|H(i)|; 1 \le i \le d\}, H \in \mathcal{S}_n^d$. We have

$$|H| \le |||H||| \le \sqrt{nd}|H|. \tag{2.1}$$

If $T : \mathcal{S}_n^d \to \mathcal{S}_n^d$ is a linear operator, then $||T||$ is the operator norm of T induced by the norm $|\cdot|$ on $\mathcal{S}_n^d$. If T is a linear operator on $\mathcal{S}_n^d$, T^* stands for its adjoint operator. If $H \in \mathcal{S}_n^d$, we say that H is positive and write $H \ge 0$ if $H(i) \ge 0$ for all $i \in \mathcal{D}$.$T : \mathcal{S}_n^d \to \mathcal{S}_n^d$ is called positive operator if $H \ge 0$ implies $TH \ge 0$. It is easy to see that if $T : \mathcal{S}_n^d \to \mathcal{S}_n^d$ is a linear positive operator, then

$$||T|| = |TJ| \tag{2.2}$$

where $J \in \mathcal{S}_n^d, J(i) = I_n, i \in \mathcal{D}$. If $H : I \to \mathcal{S}_n^d$ we shall say that H is uniformly positive if there exists $\delta > 0$ such that $H(t) \ge \delta J$ for all $t \in I$.

Lemma 2.1 *If $T : \mathcal{S}_n^d \to \mathcal{S}_n^d$ is a linear positive operator, then T^* is linear positive operator.*

Proof: Let $H \in \mathcal{S}_n^d, H \ge 0$ and $T^*H = \tilde{H} = (\tilde{H}(1), \tilde{H}(2), ..., \tilde{H}(d))$. Let $i \in \mathcal{D}$ be fixed and $x \in \mathbf{R}^n$ arbitrary. We take $G \in \mathcal{S}_n^d$ where $G = (G(1), G(2), ..., G(d))$ with $G(k) = 0$ if $k \ne i$ and $G(k) = xx^*$ if $k = i$.

We have $x^*\tilde{H}(i)x = \langle \tilde{H}, G\rangle = \langle T^*H, G\rangle = \langle H, TG\rangle = \sum_{i=1}^d TrH(i)\tilde{G}(i)$, where $TG = (\tilde{G}(1), ..., \tilde{G}(d))$. Since T is a linear positive operator then $\tilde{G}(i) \ge 0$. Since $H(i) \ge 0$ it remains to prove that $TrS_1S_2 \ge 0$, if $S_k \in \mathcal{S}_n, S_k \ge 0, k = 1, 2$. If $S_1 \ge 0$ then $S_1 = \sum_{j=1}^n \lambda_j e_j e_j^*$ where $\lambda_i \ge 0$ are eigenvalues of S_1 and $e_j, j = 1, ..., n$ is an orthonormal basis of orthonormal eigenvectors of S_1. We have

$$TrS_1S_2 = \sum_{j=1}^n Tr\lambda_j e_j e_j^* S_2 = \sum_{j=1}^n \lambda_j e_j^* S_2 e_j \ge 0$$

and the proof is complete. $\square$

C. By $\mathcal{M}_{n,m}^d$ we denote the linear space of $A = (A(1), A(2), ..., A(d))$ where $A(i) \in \mathbf{R}^{n\times m}$. On $\mathcal{M}_{n,m}^d$ we introduce the norm $|A| = \max_{i\in\mathcal{D}}\{|A(i)|\}$. Thus, $(\mathcal{M}_{n,m}^d, |\cdot|)$ is a finite dimensional Banach space. Sometimes $\mathcal{M}_{n,n}^d$ will be denoted $\mathcal{M}_n^d$. Obviously $\mathcal{S}_n^d \subset \mathcal{M}_n^d$.

D. Throughout this paper $\{\Omega, \mathcal{F}, \mathcal{P}\}$ is a given probability space; the argument $\omega \in \Omega$ will not be written.

$E[x|\mathcal{H}]$ denotes conditional expectation of x with respect to the σ-algebra $\mathcal{H}, \mathcal{H} \subset \mathcal{F}$ and $E[x|\eta(t) = i]$ stands for conditional expectation on the event $\eta(t) = i$; Ex stands for expectation of random variable x.

As usually, two random vectors x and y are identified if $x = y$ a.e. (almost everywhere).

$w(t) = (w_1(t), ..., w_r(t))^*, t \in R_+$ is a standard r-dimensional Wiener process on the given probability space (see [13]).

Throughout the paper $\eta(t), t \geq 0$ is a right continuous homogeneous Markov chain with state space the set $\mathcal{D}$ and the probability transition matrix $P(t) = [p_{ij}(t)] = e^{Qt}, t > 0$; here $Q = [q_{ij}]$ with $\sum_{j=1}^{d} q_{ij} = 0, i \in \mathcal{D}$ and $q_{ij} \geq 0$ if $i \neq j$.

Assume that $\mathcal{P}\{\eta(0)) = i\} > 0$, for all $i \in \mathcal{D}$.

For each $t \geq 0$ we denote $\mathcal{F}_t$ the smallest $\sigma-$algebra $\mathcal{F}_t \subset \mathcal{F}$, containing all sets $M \in \mathcal{F}$ with $\mathcal{P}(M) = 0$ and with respect to which all random vectors $\{w(s), 0 \leq s \leq t\}$ are measurable.

By $\mathcal{G}_t$ we denote the $\sigma-$algebra generated by $\eta(s), 0 \leq s \leq t$.

Throughout the paper we assume that $\sigma-$algebra $\mathcal{G}_t$ is independent of $\sigma-$algebra $\mathcal{F}_t$ for all $t \geq 0$. $\mathcal{H}_t$ stands for the smallest $\sigma-$algebra containing $\sigma-$ algebras $\mathcal{F}_t$ and $\mathcal{G}_t$.

By $L^2_{\eta,w}([t_0, \infty), \mathbf{R}^l), t_0 \geq 0$ we denote the space of all measurable functions $u : [t_0, \infty) \times \Omega \to \mathbf{R}^l$ with the properties: $u(t)$ is $\mathcal{H}_t$-measurable for every $t \geq t_0$ and

$$E[\int_{t_0}^{\infty} |u(s)|^2 ds|\eta(t_0) = i] < \infty, i \in \mathcal{D}.$$

Since for every $t \geq 0$, $\mathcal{H}_t$ contains all sets $M \in \mathcal{F}$ with $\mathcal{P}(M) = 0$ it is not difficult to verify that $L^2_{\eta,w}([t_0, \infty), \mathbf{R}^l)$ is a real Hilbert space with the inner product

$$< u, v >= \sum_{i=1}^{d} E[\int_{t_0}^{\infty} u^*(t)v(t)dt|\eta(t_0) = i]. \tag{2.3}$$

The space $L^2_{\eta,w}([t_0, T], \mathbf{R}^l), 0 \leq t_0 < T$ is defined in a similar way.

3 ITÔ TYPE FORMULA

Let $\sigma : [t_0, T] \to \mathbf{R}^{n \times r}$ be a matrix valued function with the columns $\sigma_1(t), \ldots,$ $\sigma_r(t)$, $\sigma_k \in L^2_{\eta,w}([t_0,T], \mathbf{R}^n)$. The stochastic integral

$$z(t) = \int_{t_0}^{t} \sigma(s)dw(s), t \in [t_0, T]$$

is well-defined (see [Friedman]) because the $\sigma-$algebras $\mathcal{H}_t, t \geq t_0$ have the properties used in the theory of stochastic Itô integral: i.e. $\mathcal{H}_{t_1} \subset \mathcal{H}_{t_2}$ if $t_1 < t_2, \mathcal{F}_t \subset \mathcal{H}_t$ and $\mathcal{H}_t$ is independent of the σ-algebra generated by $\{w(t+h) - w(t), h > 0\}$ for every $t \geq 0$.

Hence, from Theorem 2.5 and Theorem 3.2 in [13], chapter 5, it follows, with probability one, that $z(t)$ is a continuous process, $z(t)$ is $\mathcal{H}_t$-adapted process, $E[z(t)|\mathcal{H}_{t_0}] = 0$,

$$E[|z(t)|^2|\mathcal{H}_{t_0}] = \sum_{j=1}^{r} E[\int_{t_0}^{t} |\sigma_j(s)|^2 ds|\mathcal{H}_{t_0}], t \in [t_0, T] \tag{3.1}$$

and we conclude that $z \in L^2_{\eta,w}([t_0,T], \mathbf{R}^n)$.

Let us consider $a \in L^2_{\eta,w}([t_0,T], \mathbf{R}^n)$ and a n-dimensional random vector ξ $\mathcal{H}_{t_0}$-measurable with $E|\xi|^2 < \infty$.

It will follow that

$$x(t) = \xi + \int_{t_0}^{t} a(s)ds + \int_{t_0}^{t} \sigma(s)dw(s), t \in [t_0, T] \tag{3.2}$$

is continuous with probability 1 and $x \in L^2_{\eta,w}([t_0,T], \mathbf{R}^n)$. If $x(t)$ verifies (3.2) we write

$$dx(t) = a(t)dt + \sigma(t)dw(t)$$

$t \in [t_0, T]$ and $x(t_0) = \xi$.

Theorem 3.1 *(A Itô type formula). Let ξ, a and σ be as above and let*

$$v(t, x, i) = x^* K(t, i)x + 2k^*(t, i)x + k_0(t, i)$$

where $K : [t_0, T] \times \mathcal{D} \to \mathcal{S}_n, k : [t_0, T] \times \mathcal{D} \to \mathbf{R}^n, k_0 : [t_0, T] \times \mathcal{D} \to \mathbf{R}$ are C^1-functions with respect to t. Then we have:

$$\begin{aligned} &E[(v(t, x(t), \eta(t)) - v(t_0, \xi, i))|\eta(t_0) = i] \\ &= E[\int_{t_0}^{t} \{x^*(s)\dot{K}(s, \eta(s))x(s) + 2\dot{k}^*(s, \eta(s))x(s) + \dot{k}_0(s, \eta(s)) \\ &+2[x^*(s)K(s, \eta(s)) + k^*(s, \eta(s))]a(s) + Tr(\sigma^*(s)K(s, \eta(s))\sigma(s)) \\ &+ \sum_{j=1}^{d} v(s, x(s), j)q_{\eta(s)j}\}ds|\eta(t_0) = i] \end{aligned} \tag{3.3}$$

for all $i \in \mathcal{D}, t \in [t_0, T]$ and for the stochastic process $x(t), t \in [t_0, T]$, which verifies (3.2).

Proof: The proof consists in three steps:

Step 1: Assume that ξ, a, σ satisfy the assumption in the statement and additionally ξ is a bounded random vector a, σ are bounded on $[t_0, T] \times \Omega$, and $a(t), \sigma(t)$ are, with probability one, right continuous functions on $[t_0, T]$.

Under these assumptions, applying Theorem 6.3 in [13], we deduce that,

$$\sup_{t \in [t_0, T]} E|x(t)|^{2k} < \infty,$$

for all $k \in \mathbf{N}, k \geq 1$. We can write:

$$\begin{aligned} & v(t+h, x(t+h), \eta(t+h)) - v(t, x(t), \eta(t)) \\ = {} & v(t+h, x(t+h), \eta(t+h)) - v(t, x(t), \eta(t+h)) + v(t, x(t), \eta(t+h)) \\ & -v(t, x(t), \eta(t)) = \sum_{j=1}^{d} \chi_{\eta(t+h)=j}(v(t+h, x(t+h), j) - v(t, x(t), j)) \\ & \qquad + v(t, x(t), \eta(t+h)) - v(t, x(t), \eta(t)), \end{aligned}$$

where χ_M is the indicator function of the set M.

For each fixed $j \in \mathcal{D}$, we can apply the Itô formula (see [13]) and obtain

$$\begin{aligned} v(t+h, x(t+h), j) - v(t, x(t), j) &= \int_t^{t+h} m_j(s) ds \\ &+ 2 \int_t^{t+h} (x^*(s) K(s, j) + k^*(s, j)) \sigma(s) dw(s) \end{aligned}$$

where $m_j(s) = x^*(s) \dot{K}(s, j) x(s) + 2\dot{k}^*(s, j) x(s) + \dot{k}_0(s, j) + 2x^*(s) K(s, j) a(s) + 2k^*(s, j) a(s) + Tr(\sigma^*(s) K(s, j) \sigma(s)), j \in \mathcal{D}$.

Using Lemma 1 in [24], we deduce that

$$E[\chi_{\eta(t+h)=j} \int_t^{t+h} [x^*(s) K(s, j) + k^*(s, j)] \sigma(s) dw(s) | \mathcal{H}_t] = 0.$$

Hence $E[\chi_{\eta(t+h)=j} \int_t^{t+h} (x^*(s) K(s, j) + k^*(s, j)) \sigma(s) dw(s) | \eta(t_0) = i] = 0$ and finally we deduce

$$\begin{aligned} & E[(v(t+h, x(t+h), \eta(t+h)) - v(t, x(t), \eta(t+h)) | \eta(t_0) = i] \\ & \qquad = \sum_{j=1}^{d} E[\chi_{\eta(t+h)=j} \int_t^{t+h} m_j(s) ds | \eta(t_0) = i]. \end{aligned} \tag{3.4}$$

It is obvious that $m_j(s)$ is, with probability one, right continuous and hence we have:

$$\lim_{h\searrow 0}\frac{1}{h}\int_t^{t+h} m_j(s)ds = m_j(t),\ \ t\in[t_0,T), j\in\mathcal{D}.$$

Since $\eta(t)$ is right continuous we can write:

$$\lim_{h\searrow 0}\frac{1}{h}\chi_{\eta(t+h)=j}\int_t^{t+h} m_j(s)ds = \chi_{\eta(t)=j}m_j(t). \tag{3.5}$$

On the other hand, since $\sup_{t\in[t_0,T]}E|x(t)|^4<\infty$ we obtain that there exists $\beta>0$ (not depending upon t,h) such that:

$$E|\frac{1}{h}\chi_{\eta(t+h)=j}\int_t^{t+h} m_j(s)ds|^2\le\beta.$$

Thus, from (3.4) and (3.5) it follows:

$$\lim_{h\searrow 0}\frac{1}{h}E[(v(t+h,x(t+h),\eta(t+h))-v(t,x(t),\eta(t+h)))|\eta(t_0)=i]$$
$$=\sum_{j=1}^{r}E[\chi_{\eta(t)=j}m_j(t)|\eta(t_0)=i]=E[\tilde{m}(t)|\eta(t_0)=i], t\in[t_0,T), i\in\mathcal{D}. \tag{3.6}$$

where

$$\tilde{m}(t)=x^*(t)\dot{K}(t,\eta(t))x(t)+2k^*(t,\eta(t))x(t)+\dot{k}_0(t,\eta(t))$$
$$+2[x^*(t)K(t,\eta(t))+k^*(t,\eta(t))]a(t)+Tr(\sigma^*(t)K(t,\eta(t))\sigma(t))$$

Further, we can write:

$$E[(v(t,x(t),\eta(t+h))-v(t,x(t),\eta(t)))|\eta(t_0)=i]$$
$$=E[(\sum_{j=1}^{d}\chi_{\eta(t+h)=j}v(t,x(t),j)-v(t,x(t),\eta(t)))|\eta(t_0)=i] \tag{3.7}$$
$$=\sum_{j=1}^{d}E[v(t,x(t),j)E[\chi_{\eta(t+h)=j}|\mathcal{H}_t]|\eta(t_0)=i]-E[v(t,x(t),\eta(t))|\eta(t_0)=i].$$

Since σ-algebra $\mathcal{F}_t$ is independent of $\mathcal{G}_s$ for all $t,s\in[t_0,T]$ it is easy to verify that:

$$E[\chi_{\eta(t+h)=j}|\mathcal{H}_t]=E[\chi_{\eta(t+h)=j}|\mathcal{G}_t]=E[\chi_{\eta(t+h)=j}|\eta(t)]=p_{\eta(t),j}(h). \tag{3.8}$$

Hence from (3.7) and (3.8) we have

$$E[(v(t,x(t),\eta(t+h))-v(t,x(t),\eta(t)))|\eta(t_0)=i]$$
$$=E[\sum_{j\neq\eta(t)}(v(t,x(t),j)-v(t,x(t),\eta(t)))p_{\eta(t)j}(h)|\eta(t_0)=i].$$

Recall that $P(h) = [p_{ij}(h)] = e^{Qh}, h > 0$ with $\sum_{j=1}^{d} q_{ij} = 0$. Applying Lebesque's Theorem we obtain that

$$\lim_{h\searrow 0} \frac{1}{h} E[(v(t,x(t),\eta(t+h)) - v(t,x(t),\eta(t))|\eta(t_0) = i] \tag{3.9}$$

$$= \sum_{j=1}^{d} E[v(t,x(t),j)q_{\eta(t)j})|\eta(t_0) = i].$$

Combining (3.6) with (3.9) we conclude that

$$\lim_{h\searrow 0} \frac{1}{h} E[(v(t+h),x(t+h),\eta(t+h)) - v(t,x(t),\eta(t)))|\eta(t_0) = i]$$

$$= E[(\tilde{m}(t) + \sum_{j=1}^{d} v(t,x(t),j)q_{\eta(t)j})|\eta(t_0) = i].$$

Denote

$$G_i(t) = E[v(t,x(t),\eta(t))|\eta(t_0) = i], i \in \mathcal{D}$$

and

$$h_i(t) = E[(\tilde{m}(t) + \sum_{j=1}^{d} v(t,x(t),j)q_{\eta(t)j})|\eta(t_0) = i].$$

Since $\sup_{t\in[t_0,T]} E(\tilde{m}(t) + \sum_{j=1}^{r} v(t,x(t),j)q_{\eta(t)j})^2 < \infty$ it follows that $h_i(t)$ is right continuous and therefore

$$\lim_{h\searrow 0} \frac{1}{h} \int_t^{t+h} h_i(s)ds = h_i(t), t \in [t_0,T).$$

Hence

$$\lim_{h\searrow 0} \frac{1}{h}(G_i(t+h) - G_i(t) - \int_t^{t+h} h_i(s)ds) = 0, t \in [t_0,T), i \in \mathcal{D}. \tag{3.10}$$

Since the process $\eta(t)$ is continuous in probability (see [7]) it follows that $v(t,x(t),\eta(t))$ is continuous in probability.

Having $\sup_{t\in[t_0,T]} E|v(t,x(t),\eta(t))|^2 < \infty$ it follows that $G_i(t), i \in \mathcal{D}$ is a continuous function and thus from (3.10) we conclude that

$$G_i(t) - G_i(t_0) = \int_{t_0}^{t} h_i(t)dt, t \in [t_0,T], i \in \mathcal{D}$$

and so the equality (3.3) holds.

Step 2: Assume that ξ is $\mathcal{H}_{t_0}$-measurable and $E|\xi|^2 < \infty$, and a, σ are bounded on $[t_0,T] \times \Omega$, $a(t), \sigma(t)$ are $\mathcal{H}_t$-adapted. Let

$$\begin{aligned} \xi_k &= \xi\chi_{|\xi|\le k}, \\ a_k(t) &= k\int_{max\{t-\frac{1}{k},t_0\}}^{t} a(s)ds, \\ \sigma_k(t) &= \int_{\max\{t-\frac{1}{k},t_0\}}^{t} \sigma(s)ds. \end{aligned}$$

It is obvious that a_k and σ_k are continuous (with probability one), bounded on $[t_0, T] \times \Omega$, and $\mathcal{H}_t$-adapted. From Lebesgue's Theorem it follows that

$$\lim_{k\to\infty} \int_{t_0}^{T} (|a_k(t) - a(t)|^2 + |\sigma_k(t) - \sigma(t)|^2)dt = 0$$

and applying the Lebesgue bounded convergence theorem we have

$$\lim_{k\to\infty} E \int_{t_0}^{T} (|a_k(t) - a(t)|^2 + |\sigma_k(t) - \sigma(t)|^2)dt = 0. \tag{3.11}$$

From Lebesgue's Theorem it follows that

$$\lim_{k\to\infty} E|\xi_k - \xi|^2 = 0.$$

It is easy to verify that $\sup_{t\in[t_0,T]} E|x(t)|^2 < \infty$ and

$$\sup_{t\in[t_0,T]} E|x_k(t) - x(t)|^2 \;\le\; 3E[|\xi_k - \xi|^2 + (T - t_0)\int_{t_0}^{T} |a_k(t) - a(t)|^2 dt + r\int_{t_0}^{T} |\sigma_k(t) - \sigma(t)|^2 dt],$$

$k \ge 1$, where

$$x_k = \xi_k + \int_{t_0}^{t} a_k(s)ds + \int_{t_0}^{t} \sigma_k(s)dw(s).$$

Applying the result of Step 1 for each $k \ge 1$ we obtain

$$E[(v(t, x_k(t), \eta(t)) - v(t_0, \xi_k, i))|\eta(t_0) = i] =$$
$$E\{\int_{t_0}^{t} [x_k^*(s)\dot{K}(s, \eta(s))x_k(s) + 2\dot{k}^*(s, \eta(s))x_k(s) + +\dot{k}_0(s, \eta(s))$$
$$+2(x_k^*(s)K(s, \eta(s)) + k^*(s, \eta(s)))a_k(s) + Tr(\sigma_k^*(s)K(s, \eta(s))\sigma_k(s))$$
$$+\sum_{j=1}^{d} v(s, x_k(s), j)q_{\eta(s)j}]ds|\eta(t_0) = i\}. \tag{3.12}$$

Taking the limit for $k \to \infty$ we conclude that (3.3) holds.

Step 3: Consider now that ξ, a, σ verify the general assumptions in the statement. Define

$$\bar{a}_k(t) = a(t)\chi_{|a(t)|\le k}$$
$$\bar{\sigma}_k(t) = \sigma(t)\chi_{|\sigma(t)|\le k}.$$

Applying Lebesque's Theorem it follows that $\bar{a}_k$ and $\bar{\sigma}_k$ verify a equality of type (3.11). On the other hand it can be proved that

$$\sup_{t\in[t_0,T]} E|\bar{x}_k(t) - x(t)|^2 \le 2E[\int_{t_0}^{T} (T - t_0)|\bar{a}_k(t) - a(t)|^2 + r|\bar{\sigma}_k(t) - \sigma(t)|^2 dt]$$

where $\bar{x}_k(t) = \xi + \int_{t_0}^t \bar{a}_k(s)ds + \int_{t_0}^t \bar{\sigma}_k(s)dw(s)$.

Now, applying the results from Step 2 for $\xi, \bar{a}_k, \bar{\sigma}_k, \bar{x}_k$ we obtain an equality of type (3.12) with $\xi_k, a_k, \sigma_k, x_k$ replaced by $\xi, \bar{a}_k, \bar{\sigma}_k, \bar{x}_k$.

Taking again the limit for $k \to \infty$ we conclude that (3.3) holds and the proof is complete.

Remark In [24] was also proved a Itô type formula as (3.3) for a class of nonlinear functions $v(t, x, i)$ which contains as a particular case the functions which are quadratic in x while the process $x(t)$ is a solution of a system of stochastic differential equations with Markovian jumping, $x(t_0) = x_0, x_0 \in \mathbf{R}^n, x(t)$ satisfying $\sup\{E|x(t)|^{2p}, t \in [t_0, T]\} < \infty$ for every $p \geq 1$ and all $T > t_0 \geq 0$. In this case, the Itô type formula follows easily by using the reasoning in the first step of the proof of Theorem 3.1. The particular case of the function $v(t, x, i)$ considered in Theorem 3.1 was choosen in order to be sure that the reasonings in steps 2 and 3 of the proof are valid, when the process $x(t)$ is in the general situation described in (3.2).

4 STABILITY OF LINEAR STOCHASTIC SYSTEM DESCRIBED BY DIFFERENTIAL EQUATIONS WITH MARKOVIAN JUMPING AND MULTIPLICATIVE WHITE NOISE

A. Consider the linear system:

$$dx(t) = A_0(t, \eta(t))x(t)dt + \sum_{j=1}^{r} A_j(t, \eta(t))x(t)dw_j(t). \tag{4.1}$$

Throughtout this section we suppose that $A_j(\cdot, i), 0 \leq i \leq r$ are bounded on $\mathbf{R}_+$ and continuous matrix valued functions.

By the standard procedure of succesive aproximation and by using the properties of stochastic integral, it is easy to obtain the existence and uniqueness of the solution $x(t, t_0, x_0), t \geq t_0 \geq 0, x_0 \in \mathbf{R}^n$ of system (4.1) having the properties $x(t_0) = x_0, x(\cdot, t_0, x_0) \in L^2_{\eta,w}([t_0, T], \mathbf{R}^n)$ for all $T > t_0, x(\cdot, t_0, x_0)$ is continuous with probability one. Moreover, it can be proved that

$$\sup_{t \in [t_0, T]} E|x(t, t_0, x_0)|^{2p} < \infty,$$

for all $p \in \mathbf{N}, p \geq 1$.

By $\Phi(t,s), t \geq s$ we denote the fundamental random matrix of solutions associated to system (4.1).

Hence, $x(t,t_0,x_0) = \Phi(t,t_0)x_0, t \geq t_0$. Let $\tilde{\Phi}(t,t_0), t \geq t_0$ be the fundamental random matrix solution associated to the stochastic differential equation:

$$dz(t) = [-A_0^*(t,\eta(t)) + \sum_{k=1}^{r}(A_k^*(t,\eta(t)))^2]z(t)dt - \sum_{k=1}^{r} A_k^*(t,\eta(t))z(t)dw_k(t).$$

Using the Jtô formula (see [13]) we obtain:

$$\Phi(t,t_0)\tilde{\Phi}^*(t,t_0) = \tilde{\Phi}^*(t,t_0)\Phi(t,t_0) = I_n, a.e. t \geq 0$$

hence the matrix $\Phi(t,t_0)$ is invertible and $\Phi^{-1}(t,t_0) = \tilde{\Phi}^*(t,t_0)$.

B. On the Hilbert space $\mathcal{S}_n^d$ we define the linear operator $L(t) : \mathcal{S}_n^d \to \mathcal{S}_n^d$ by

$$(L(t)H)(i) = A_0(t,i)H(i) + H(i)A_0^*(t,i) + \sum_{k=1}^{r} A_k(t,i)H(i)A_k^*(t,i) + \sum_{j=1}^{d} q_{ji}H(j),$$

$H \in \mathcal{S}_n^d, i \in \mathcal{D}, t \geq 0$.

On the space $\mathcal{S}_n^d$ we consider the linear differential equation

$$\frac{dS(t)}{dt} = L(t)S(t). \tag{4.2}$$

By $S(t,t_0,H)$ we denote the solution of the equation (4.2) with the initial condition $S(t_0,t_0,H) = H, H \in \mathcal{S}_n^d$.

Let $T(t,t_0)$ be the linear evolution operator associated to the equation (4.2).

We have $S(t,t_0,H) = T(t,t_0)(H)$. It is easy to see that $T(t,s) = (T(s,t))^{-1}$. $T(s,s) = \tilde{J}$ ($\tilde{J}$ being the identity operator on the Hilbert space $\mathcal{S}_n^d$).

We have also

$$\begin{aligned} \frac{d}{dt}T(t,s) &= L(t)T(t,s) \\ \frac{d}{dt}T^*(t,s) &= T^*(t,s)L^*(t) \\ \frac{d}{dt}T^*(s,t) &= -L^*(t)T^*(s,t). \end{aligned} \tag{4.3}$$

It is not difficult to check that

$$(L^*(t)H)(i) = A_0^*(t,i)H(i) + H(i)A_0(t,i) + \sum_{k=1}^{r} A_k^*(t,i)H(i)A_k(t,i) + \sum_{j=1}^{d} q_{ij}H(j),$$

$H \in \mathcal{S}_n^d, i \in \mathcal{D}, t \geq 0$.

Remark. If $A_k(t,i) = A_k(i), k \in \{0,1,...,r\}, i \in \mathcal{D}, t \in \mathbf{R}_+$ we say that the system (4.1) is "in the stationary case". In this case the operator $L : \mathcal{S}_n^d \to \mathcal{S}_n^d$ does not depend upon t, and the linear evolution operator defined by the equation (4.2) is $T(t,t_0) = e^{L(t-t_0)}$ where $e^{Lt} = \sum_{k=0}^{\infty} \frac{(LT)^k}{k!}$, the sum is convergent uniformly with respect to t in every compact subsets. It is easy to see that $T^*(t,t_0) = e^{L^*(t-t_0)}$.

Lemma 4.1 *We have*

$$(T^*(t,t_0)H)(i) = E[\Phi^*(t,t_0)H(\eta(t))\Phi(t,t_0)|\eta(t_0) = i]$$

for all $t \geq t_0 \geq 0, H \in \mathcal{S}_n^d, i \in \mathcal{D}$.

Proof: Let $\mathcal{U}(t,t_0) : \mathcal{S}_n^d \to \mathcal{S}_n^d$ be defined by

$$(\mathcal{U}(t,t_0)(H))(i) = E[\Phi^*(t,t_0)H(\eta(t))\Phi(t,t_0)|\eta(t_0) = i],$$

$H \in \mathcal{S}_n^d, i \in \mathcal{D}, t \geq t_0$.

Take $H \in \mathcal{S}_n^d$, we define $v(t,x,i) = x^*H(i)x, x \in \mathbf{R}^n, i \in \mathcal{D}, t \geq 0$.

Applying Theorem 3.1 to function $v(t,x,i)$ and to equation (4.1) we obtain

$$x^*(\mathcal{U}(t,t_0)(H))(i)x - x^*H(i)x = x^* \int_{t_0}^t (\mathcal{U}(s,t_0)(L^*(s)H))(i)dsx.$$

Hence

$$\frac{d}{dt}\mathcal{U}(t,t_0) = \mathcal{U}(t,t_0)L^*(t).$$

Since $\mathcal{U}(t_0,t_0) = T^*(t_0,t_0)$ and using (4.3) it follows that

$$\mathcal{U}(t,s) = T^*(t,s)$$

$t \geq s$ and the proof is complete. $\square$

Combining the results of Lemma 4.1 and Lemma 2.1 we get:

Corollary 4.2 *The evolution operator* $T(t,s)$ *and its adjoint operator* $T^*(t,s)$ *are positive operators on* $\mathcal{S}_n^d$ *for every* $t \geq s \geq 0$.

Proposition 4.3 *Under the considered assumptions, there exists* $\gamma \geq 1$ *such that*

$$e^{\gamma(t-t_0)}J \geq T^*(t,t_0)J \geq e^{-\gamma(t-t_0)}J$$

for all $t \geq t_0 \geq 0$, *where* $J \in \mathcal{S}_n^d, J(i) = I_n, i \in \mathcal{D}$.

Proof: Applying Itô type formula (3.3) for the function $v(t,x,i) = |x|^2$, $t \geq 0, x \in \mathbf{R}^n, i \in \mathcal{D}$ and the equation (4.1) we obtain for all $i \in \mathcal{D}$

$$E[|\Phi(t,t_0)x_0|^2|\eta(t_0) = i] - |x_0|^2 = E[\int_{t_0}^t g(s)ds|\eta(t_0) = i]$$

$t \geq t_0$, where

$$g(t) = x_0^* \Phi^*(t,t_0)[A_0^*(t,\eta(t)) + A_0(t,\eta(t)) + \sum_{k=1}^{r} A_k^*(t,\eta(t))A_k(t,\eta(t))]\Phi(t,t_0)x_0.$$

Since $A_k(t,i)$ are bounded functions it follows that there exists $\gamma > 0$ such that $|g(t)| \leq \gamma|\Phi(t,t_0)x_0|^2$. On the other hand,

$$\frac{d}{dt}E|\Phi(t,t_0)x_0|^2|\eta(t_0) = i] = E[g(t)|\eta(t_0) = i]$$

$t \geq t_0$, $i \in \mathcal{D}$,

$$\begin{aligned} &\gamma E[|\Phi(t,t_0)x_0|^2|\eta(t_0) = i] \\ \geq\ &\frac{d}{dt}E[|\Phi(t,t_0)x_0|^2|\eta(t_0) = i] \\ \geq\ &-\gamma E[|\Phi(t,t_0)x_0|^2|\eta(t_0) = i] \end{aligned}$$

$t \geq t_0$. Hence,

$$e^{\gamma(t-t_0)}|x_0|^2 \geq E[|\Phi(t,t_0)x_0|^2|\eta(t_0) = i] \geq e^{-\gamma(t-t_0)}|x_0|^2, t \geq t_0, i \in \mathcal{D}.$$

The conclusion follows from Lemma 4.1.

C. Let us consider $A_k : I \subseteq \mathbf{R} \to \mathcal{M}_n^d$ (I being an interval) which are supposed to be continuous functions, $k = 0, 1, ..., r$, $Q \in \mathbf{R}^{dxd}$ is a given constant matrix whose elements satisfy the condition $q_{ij} \geq 0$, if $i \neq j$. The system $(A_0, A_1, ..., A_r; Q)$ defines the linear operator $L(t) : \mathcal{S}_n^d \to \mathcal{S}_n^d$ by

$$\begin{aligned} (L(t)H)(i) &= A_0(t,i)H(i) + H(i)A_0^*(t,i) \\ &+ \sum_{k=1}^{r} A_k(t,i)H(i)A_k^*(t,i) + \sum_{j=1}^{d} q_{ji}H(j), i \in \mathcal{D} \end{aligned} \tag{4.4}$$

for all $H = (H(1), H(2), ...H(d)) \in \mathcal{S}_n^d$.

Consider the linear differential equation on $\mathcal{S}_n^d$:

$$\frac{d}{dt}S(t) = L(t)S(t) \tag{4.5}$$

where $L(t)$ is given by (4.4). Let $T(t,t_0)$ be the linear evolution operator on $\mathcal{S}_n^d$ defined by the differential equation (4.5).

We show that the results of Corollary 4.2 and Proposition 4.3 still hold also for the operator $T(t,t_0)$ and its adjoint operator $T^*(t,t_0)$, even if the operator $L(t)$ in (4.4) is not associated with a system of differential stochastic equation with Markovian jumping.

Proposition 4.4 *If $T(t,t_0)$ is a linear evolution operators on $\mathcal{S}_n^d$ defined by the linear differential equation (4.5) then the following hold:*

(i) $T(t,t_0) \geq 0, T^*(t,t_0) \geq 0$ *for all* $t \geq t_0, t, t_0 \in I$.

(ii) *If* $t \to A_k(t)$ *are bounded functions, then there exist* $\delta > 0, \gamma > 0$ *such that:*

$$T(t,t_0)J \geq \delta e^{-\gamma(t-t_0)}J, \quad T^*(t,t_0)J \geq \delta e^{-\gamma(t-t_0)}J$$

for all $t \geq t_0, t, t_0 \in I$.

Proof. To prove (i) we consider the linear operators $L_1(t) : \mathcal{S}_n^d \to \mathcal{S}_n^d, \tilde{L}(t) : \mathcal{S}_n^d \to \mathcal{S}_n^d$ by

$$\begin{aligned}(L_1(t)H)(i) &= [A_0(t,i) + \frac{1}{2}q_{ii}I_n]H(i) + H(i)[A_0(t,i) + \frac{1}{2}q_{ii}I_n]^* \\ (\tilde{L}(t)H)(i) &= \sum_{k=1}^{r} A_k(t,i)H(i)A_k^*(t,i) + \sum_{j=1, j\neq i}^{d} q_{ji}H(j), i \in \mathcal{D}\end{aligned}$$

$H = (H(1),\ H(2), ..., H(d)) \in \mathcal{S}_n^d, t \in I$.

It is easy to see that for each $t \in I$, the operator $\tilde{L}(t)$ is a positive operator on $\mathcal{S}_n^d$. Let us consider the linear differential equation

$$\frac{dS(t)}{dt} = L_1(t)S(t) \tag{4.6}$$

and denote $T_1(t,t_0)$ the linear evolution operator on $\mathcal{S}_n^d$ defined by (4.6). By direct calculation, we obtain that

$$(T_1(t,t_0)H)(i) = \Phi_i(t,t_0)H(i)\Phi_i^*(t,t_0)$$

for all $t \geq t_0, i \in \mathcal{D}, H \in \mathcal{S}_n^d$ where $\Phi_i(t,t_0)$ is a fundamental matrix solution of the deterministic differential equation on $\mathbf{R}^n$

$$\frac{d}{dt}x(t) = [A_0(t,i) + \frac{1}{2}q_{ii}I_n]x(t).$$

It is clear that for each $t \geq t_0, T_1(t,t_0) \geq 0$. Since the linear differential equation (4.5) is written as

$$\frac{d}{dt}S(t) = L_1(t)S(t) + \tilde{L}(t)S(t)$$

we may write the following representation formula

$$T(t,t_0)H = T_1(t,t_0)H + \int_{t_0}^{t} T_1(t,s)\tilde{L}(s)T(s,t_0)Hds$$

for all $H \in \mathcal{S}_n^d, t \geq t_0, t, t_0 \in I$.

Let $H \in \mathcal{S}_n^d, H \geq 0$ be fixed. We define the sequence of Voltera approximations $S_k(t), k \geq 0, t \geq t_0$ by

$$S_0(t) = T_1(t, t_0)H$$
$$S_{k+1}(t) = T_1(t, t_0)H + \int_{t_0}^{t} T_1(t, s)\tilde{L}(s)S_k(s)ds, k = 1, 2, ...$$

Since $T_1(t, t_0)$ is a positive operator on $\mathcal{S}_n^d$, we get inductively, that $S_k(s) \geq 0$ for all $s \geq t_0, k = 1, 2, ...$ Taking into account that $\lim_{k\to\infty} S_k(t) = T(t, t_0)H$ we conclude that $T(t, t_0)H \geq 0$, hence $T(t, t_0) \geq 0$. By using Lemma 2.1 we get that the adjoint operator $T^*(t, t_0) \geq 0$, is positive.

(ii) Firstly, we show that there exist $\delta > 0, \gamma > 0$, such that

$$|T(t, t_0)H| \geq \delta e^{-\gamma(t-t_0)}|H|$$
$$|T^*(t, t_0)H| \geq \delta e^{-\gamma(t-t_0)}|H| \tag{4.7}$$

for all $H \in \mathcal{S}_n^d, t \geq t_0, t, t_0 \in I$.

Let us denote $v(t) = \frac{1}{2}|||T(t, t_0)H|||^2 = \frac{1}{2}\langle T(t, t_0)H, T(t, t_0)H\rangle$. By direct calculation, we obtain

$$\frac{d}{dt}v(t) = \langle L(t)T(t, t_0)H, T(t, t_0)H\rangle, t \geq t_0.$$

Under the considered assumptions there exists $\gamma > 0$ such that

$$|\frac{d}{dt}v(t)| \leq \gamma|||T(t, t_0)H|||^2,$$

$$|\frac{d}{dt}v(t)| \leq 2\gamma v(t), \quad t \geq t_0.$$

Further we have

$$\frac{d}{dt}v(t) \geq -2\gamma v(t), \quad t \geq t_0$$

or equivalently

$$\frac{d}{dt}[v(t)e^{2\gamma(t-t_0)}] \geq 0$$

for all $t \geq t_0$. Hence the function $t \to v(t)e^{2\gamma(t-t_0)}$ is not decreasing and $v(t) \geq e^{-2\gamma(t-t_0)}v(t_0)$. Considering the definition of $v(t)$ and using (2.1) we conclude that there exists $\delta > 0$ such that

$$|T(t, t_0)H| \geq \delta e^{-\gamma(t-t_0)}|H|$$

which is the first inequality in (4.7).

To prove the second inequality (4.7), we consider the function

$$\hat{v}(s) = 1/2|||T^*(t,s)H|||^2, H \in S_n^d, s \leq t, s, t \in I.$$

By direct computation we obtain

$$\frac{d}{ds}\hat{v}(s) = -\langle L^*(s)T^*(t,s)H, T^*(t,s)H\rangle.$$

Further we have

$$|\frac{d}{ds}\hat{v}(s)| \leq 2\gamma\hat{v}(s)$$

and

$$\frac{d}{ds}[\hat{v}(s)e^{2\gamma(t-s)}] \leq 0,$$

thus we obtain that the function $s \to \hat{v}(s)e^{2\gamma(t-s)}$ is not increasing and therefore $\hat{v}(s)e^{2\gamma(t-s)} \geq \hat{v}(t)$ for all $s \leq t$ hence

$$|||T^*(t,s)H|||^2 \geq e^{-\gamma(t-s)}|||H|||^2.$$

Using again (2.1) we obtain the second inequality in (4.7).

Let $x \in \mathbf{R}^n, i \in \mathcal{D}$ be fixed; consider $\tilde{H} \in S_n^d$ defined by $\tilde{H}(j) = 0$ if $j \neq i, \tilde{H}(j) = xx^*$ if $j = i$.

We may write succesively

$$\begin{aligned} x^*[T(t,t_0)J](i)x &= Tr[xx^*(T(t,t_0)J)(i)] = \langle \tilde{H}, T(t,t_0)J\rangle \\ &= \langle T^*(t,t_0)\tilde{H}, J\rangle = \sum_{i=1}^{d} Tr[T^*(t,t_0)\tilde{H}](i) \\ &\geq \sum_{i=1}^{d} |(T^*(t,t_0)H)(i)| \geq \max_{i\in\mathcal{D}} |(T^*(t,t_0)\tilde{H})(i)| \\ &= |T^*(t,t_0)\tilde{H}| \geq \delta e^{-\gamma(t-t_0)}|x|^2 \end{aligned}$$

Since $x \in \mathbf{R}^n$ is arbitrary we get

$$[T(t,t_0)J](i) \geq \delta e^{-\gamma(t-t_0)}I_n, (\forall) i \in \mathcal{D}, t \geq t_0 \geq 0$$

or equivalently $(T(t,t_0)J) \geq \delta e^{-\gamma(t-t_0)}J, \forall t \geq t_0$. The second inequality in (ii) may be proved in the same way.

D. Definition 4.5 *We say that the zero solution of the system (4.1) is exponentially stable in mean square, or that the system (4.1) generates a mean square exponentially stable evolution, if there exist $\beta \geq 1, \alpha > 0$ such that*

$$E[|\Phi(t,t_0)x_0|^2|\eta(t_0) = i] \leq \beta e^{-\alpha(t-t_0)}|x_0|^2$$

for all $t \geq t_0 \geq 0, x_0 \in \mathbf{R}^n, i \in \mathcal{D}$.

Now we prove:

Proposition 4.6 *The following are equivalent:*

(i) The zero solution of the system (4.1) is exponentially stable in mean square.

(ii) There exist $\beta \geq 1, \alpha > 0$ *such that*

$$||T^*(t,t_0)|| \leq \beta e^{-\alpha(t-t_0)}$$

for all $t \geq t_0 \geq 0$.

(iii) There exists $\delta > 0$ *such that*

$$E[\int_{t_0}^{\infty} |\Phi(t,t_0)x_0|^2 dt|\eta(t_0) = i] \leq \delta |x_0|^2$$

for all $t_0 \geq 0, x_0 \in \mathbf{R}^n, i \in \mathcal{D}$.

(iv) The Liapunov type equation on S_n^d

$$K'(t) + L^*(t)K(t) + J = 0 \tag{4.8}$$

has a bounded on $\mathbf{R}_+$ *and uniformly pozitive solution.*

(v) There exists a C^1 *function* $\tilde{K} : \mathbf{R}_+ \to S_n^d$ *uniformly positive and bounded with its derivative bounded which verifies the following linear inequality on* S_n^d

$$\frac{d}{dt}\tilde{K}(t) + L^*(t)\tilde{K}(t) < 0 \tag{4.9}$$

uniformly with respect to $t \in \mathbf{R}_+$.

Proof: From Definition 4.5 and Lemma 4.1 it follows that (i) $\leftrightarrow$ (ii). It is easy to see that (i) $\Rightarrow$ (iii). We show now that (iii) $\Rightarrow$(iv). Let

$$\hat{K}(t) = \int_t^{\infty} T^*(s,t)Jds.$$

From (iii) and Lemma 4.1 it follows that $0 \leq \hat{K}(t) \leq \delta J, t \geq 0$. Using Proposition 4.3 we obtain that $\hat{K}(t) \geq \frac{1}{\gamma}J, t \geq 0$.

From (4.3) it follows that $\hat{K}$ is a solution of the differential equation (4.8).

(iv) $\Rightarrow$ (v) follows immediately since a uniformly positive and bounded solution of the equation (4.8) is also a solution of the inequality (4.9).

We prove (v) $\Rightarrow$(iv). If $\tilde{K}(t) = (\tilde{K}(t,1), \tilde{K}(t,2), ..., \tilde{K}(t,d))$ is a uniformely positive and bounded C^1-function with bounded derivative which solves (4.9) uniformely with respect to $t \in \mathbf{R}_+$, we define

$$\tilde{M}(t) = -\frac{d}{dt}\tilde{K}(t) - L^*(t)\tilde{K}(t). \tag{4.10}$$

$\tilde{M}(t) = (\tilde{M}(t,1), \tilde{M}(t,2), ..., \tilde{M}(t,d))$. It follows that there exist $\mu_k > 0, \nu_k > 0, k = 1,2$, such that

$$\mu_1 J \le \tilde{M}(t) \le \mu_2 J \quad \nu_1 J \le \tilde{K}(t) \le \nu_2 J \quad \forall t \in \mathbf{R}_+. \tag{4.11}$$

By using (4.3) it follows that

$$\tilde{K}(t) = T^*(s,t)\tilde{K}(s) + \int_t^s T^*(u,t)\tilde{M}(u)du, \quad s \ge t. \tag{4.12}$$

Hence

$$\mu_1 \int_t^s T^*(u,t)Jdu \le \int_t^s T^*(u,t)\tilde{M}(u)du \le \tilde{K}(t) \le \nu_2 J, \quad s \ge t.$$

Hence

$$\hat{K}(t) = \int_t^\infty T^*(u,t)Jdu \le \nu_3 J$$

and solves the equation (4.8). Thus $(v) \Rightarrow (iv)$.

$(iv) \Rightarrow (ii)$.

Let $\tilde{K}(t)$ be a uniformly positive and bounded solution of the equation (4.8). Using (4.3) we may write the representation

$$\tilde{K}(t) = T^*(s,t)\tilde{K}(s) + \int_t^s T^*(\tau,t)Jd\tau.$$

Also there exist $\tilde{\mu}_1, \tilde{\mu}_2 > 0$ such that $\tilde{\mu}_1 J \le \tilde{K}(t \le \tilde{\mu}_2 J$ for all $t \ge 0$. Since $T^*(s,t)$ is positive, we have $T^*(s,t)J \ge 0$, and $T^*(s,t)\tilde{K}(s) \ge 0$ hence $0 \le \int_t^s T^*(\tau,t)Jd\tau \le \tilde{K}(t) \le \tilde{\mu}_2 J$ for all $s \ge t \ge 0$. Therefore is well defined the function $t \to K_0(t) = \int_t^\infty T^*(\tau,t)Jd\tau : \mathbf{R}_+ \to S_n^d$.

Applying Proposition 4.3 we deduce that there exist $\tilde{\mu}_3 > 0$ such that $\tilde{\mu}_3 J \le K_0(t) \le \tilde{\mu}_2 J$ for all $t \ge 0$.

Let $t \ge t_0 \ge 0$; define $G(t) = T^*(t,t_0)K_0(t)$. Since $T^*(t,t_0)$ is positive we get $\tilde{\mu}_3 T^*(t.t_0)J \le G(t) \le \tilde{\mu}_2 T^*(t,t_0)J$.

Taking into account that $T^*(\tau,t_0)T^*(\tau,t) = T^*(\tau,t_0)$ we deduce that

$$G(t) = \int_t^\infty T^*(\tau,t_0)Jd\tau.$$

Hence $G'(t) = -T^*(t,t_0)J$ and $G'(t) \leq -\frac{1}{\tilde{\mu}_2}G(t)$ which leads to $G(t) \leq e^{-\alpha(t-t_0)}G(t_0)$ for all $t \geq t_0 \geq 0$, $(\alpha = \frac{1}{\tilde{\mu}_2})$.

Further we write

$$T^*(t,t_0)J \leq \frac{1}{\tilde{\mu}_3}G(t) \leq \frac{\tilde{\mu}_2}{\tilde{\mu}_3}e^{-\alpha(t-t_0)}J,$$

$$\|T^*(t,t_0)\| = |T^*(t,t_0)J| \leq \frac{\tilde{\mu}_2}{\tilde{\mu}_3}e^{-\alpha(t-t_0)}$$

and hence (ii) holds and the proof is complete.

Remark. a) From Proposition 4.6 $(i) \leftrightarrow (ii) \leftrightarrow (iv) \leftrightarrow (v)$ follows that the exponential stability in mean square of the zero solution of the system (4.1) is completely caracterized by the matrices $A_k(t,i), i \in \mathcal{D}, k = 0,1,2,...$ and Q. Therefore we will say that the system $(A_0, A_1, ..., A_r; Q)$ is stable instead of "the zero solution of system (4.1) is exponentially stable in mean square".

b) The equivalence $(i) \leftrightarrow (iii)$ in Proposition 4.6 is a Datko type condition for exponential stability for the stochastic differential equation by type (4.1).

In the case when the functions A_k are defined on the whole real axis, we may introduce the following definition.

Definition 4.7 *We say that the system $(A_0, A_1, ..., A_r; Q)$ is stable if there exist $\beta \geq 1, \alpha > 0$ such that $\|T(t,t_0)\| \leq \beta e^{-\alpha(t-t_0)}, (\forall)\ t \geq t_0, t, t_0 \in \mathbf{R}$.*

The results of Proposition 4.6 show that when the coefficients of the system (4.1) are defined only on $\mathbf{R}_+$, then the type of "stability" introduced by Definition 4.5 is equivalent to the "stability" stated in Definition 4.7.

Proposition 4.8 *The following are equivalent:*

(i) The system $(A_0, A_1, ..., A_r; Q)$ is stable.

(ii) For each $H : \mathbf{R}_+ \to S_n^d$ continuous, bounded and uniform positive, the linear differential equation on S_n^d:

$$\frac{d}{dt}K(t) + L^*(t)K(t) + H(t) = 0 \tag{4.13}$$

has a bounded and uniform positive solution $\tilde{K} : \mathbf{R}_+ \to S_n^d$.

(iii) For each $H : \mathbf{R}_+ \to S_n^d$ continuous and bounded, with $H(t) \geq 0, t \in \mathbf{R}_+$ there exists a C^1-function $\tilde{K} : \mathbf{R}_+ \to S_n^d$ uniform positive bounded with bounded derivative which verifies the linear inequality on S_n^d:

$$\frac{d}{dt}\tilde{K}(t) + L^*(t)\tilde{K}(t) + H(t) < 0 \tag{4.14}$$

uniformly with respect to $t \in \mathbf{R}_+$.

Proof: (i) $\Rightarrow$(ii).

From the proof of the implication (iii) $\Rightarrow$(iv) of Proposition 4.6 it follows that

$$\tilde{K}(t) = \int_t^\infty T^*(s,t)H(s)ds \tag{4.15}$$

is well defined and we have $\nu_1 J \leq \tilde{K}(t) \leq \nu_2 J$ for all $t \geq 0$, for some positive constants ν_1, ν_2. Using (4.3) we deduce that $\tilde{K}(t)$ is a solution of the equation (4.13).

(ii) $\Rightarrow$ (iii) follows easily, since a bounded and uniform positive solution of the differential equation

$$\frac{d}{dt}K(t) + L^*(t)K(t) + H(t) + J = 0$$

is a solution of the inequation (4.14).

The proof of (iii) $\Rightarrow$ (i) is similar to the one of (v) $\Rightarrow$ (iv) of Proposition 4.6. The proof is complete.

Remark The proof of Proposition 4.6 (see $(v) \Rightarrow (iv) \Rightarrow (i)$ and the equality (4.13)) shows that:

a) If the linear differential equation (4.13) with H satisfying (ii) has a bounded solution

$$K(t) = (K(t,1), K(t,2), ..., K(t,d))$$

where $K(t,i) \geq 0, i \in \mathcal{D}$, we may consider the function

$$v : \mathbf{R}_+ \times \mathbf{R}^n \times \mathcal{D} \to \mathbf{R}, v(t,x,i) = x^* K(t,i)x.$$

Applying the Itô type formula (3.3) to the function v and to the system (4.1) and taking into account the equation (4.13) we deduce:

$$\begin{aligned} &E[x^*(t)K(t,\eta(t))x(t)|\eta(t_0) = i] - x_0^* K(t_0,i)x_0 \\ &= -E[\int_{t_0}^t x^*(s)H(s,\eta(s))x(s)ds|\eta(t_0) = i] \end{aligned} \tag{4.16}$$

$x(t) = \Phi(t,t_0)x_0, t \geq t_0 \geq 0$. Hence

$$E[\int_{t_0}^t |x(s)|^2 ds|\eta(t_0) = i] \leq \frac{1}{\nu_1} x_0^* K(t_0,i)x_0$$

$\nu_1 > 0$ being such that $\nu_1 J \leq H(t)$.

It follows that $E[\int_{t_0}^{\infty} |x(s)|^2 ds | \eta(t_0) = i] \leq \delta |x_0|^2$ for all $t_0 \geq 0, x_0 \in \mathbf{R}^n$.

By using (iii) $\Rightarrow$ (i) of the Proposition 4.6 and using Lemma 4.1 we conclude that

$$K(t_0, i) = \int_{t_0}^{\infty} (T^*(s, t_0) H(s))(i) ds.$$

Hence $K(t) = \tilde{K}(t)$, where $\tilde{K}(t)$ is given by (4.15). Therefore if the equation (4.13) with H verifying (ii) has a bounded solution $K(t) \geq 0$, then $(A_0, A_1, ..., A_r; Q)$ is stable, the solution $K(t)$ is uniform positive and it is the unique bounded solution of the equation (4.13) with this property.

b) In the same way we can prove that if the system $(A_0, A_1, ..., A_r; Q)$ is stable and if $H : \mathbf{R}_+ \to \mathcal{S}_n^d$ is continuous and bounded, then the linear differential equation (4.13) has a unique bounded solution $\tilde{K} : \mathbf{R}_+ \to \mathcal{S}_n^d$ which is given by (4.15).

Corollary 4.9 *Suppose that $A_k(t) : \mathbf{R}_+ \to \mathcal{M}^d, k = 0, 1, ..., r, H : \mathbf{R}_+ \to \mathcal{S}_n^d$ are continuous and θ-periodic functions. If the system $(A_0, A_1, ..., A_r; Q)$ is stable, then the unique bounded and symmetric solution of the equation (4.13) is θ-periodic function.*

Further we have:

Proposition 4.10 *The following are equivalent:*

(i) The system $(A_0, A_1, ..., A_r; Q)$ is stable.

(ii) There exist $\alpha > 0, \beta \geq 1$, such that $||T(t, t_0)|| \leq \beta e^{-\alpha(t-t_0)}$ for all $t \geq t_0 \geq 0$.

(iii) There exists $\delta > 0$ such that

$$0 \leq \int_{t_0}^{t} T(t, s) J ds \leq \delta J,$$

for all $t \geq t_0 \geq 0$.

Proof: (i) $\Leftrightarrow$ (ii). If the system $(A_0, A_1, ..., A_r; Q)$ is stable, then from the Proposition (4.6) ((i) $\Rightarrow$ (ii)) we deduce that there exist $\beta_1 \geq 1, \alpha_1 > 0$ such that $||T^*(t, t_0)|| \leq \beta_1 e^{-\alpha_1(t-t_0)}, \forall t \geq t_0 \geq 0$.

On the other hand from (2.1) we get

$$||T(t, t_0)|| \leq c_1 |||T(t, t_0)||| = c_1 |||T^*(t, t_0)||| \leq c_2 ||T^*(t, t_0)||$$

and finally we obtain

$$||T(t, t_0)|| \leq c_2 \beta_1 e^{-\alpha_1(t-t_0)},$$

for all $t \geq t_0 \geq 0$ and some $c_1, c_2 > 0$.

(ii) $\Rightarrow$ (iii). It follows immediatelly from the inequality $0 \leq T(t,s)J \leq ||T(t,s)||J, t \geq s \geq 0$.

Let $H : \mathbf{R}_+ \to \mathcal{S}_n^d$ be continuous and bounded function. It follows that there exist the real constants δ_1, δ_2 such that $\delta_1 J \leq H(s) \leq \delta_2 J$ for all $s \in \mathbf{R}_+$.

Since $T(t,s)$ is a positive operator defined on $\mathcal{S}_n^d$, we deduce $\delta_1 T(t,s)J \leq T(t,s)H(s) \leq \delta_2 T(t,s)J$ for all $t \geq s \geq 0$. Hence

$$\delta_1 \int_0^t T(t,s)Jds \leq \int_0^t T(t,s)H(s)ds \leq \delta_2 \int_0^t T(t,s)Jds$$

for all $t \geq 0$. Thus, if (iii) holds we deduce that there exist the real constants $\tilde{\delta}_1, \tilde{\delta}_2$ such that

$$\tilde{\delta}_1 J \leq \int_0^t T(t,s)H(s)ds \leq \tilde{\delta}_2 J$$

for all $t \geq 0$ which shows that $t \to \int_0^t T(t,s)H(s)ds$ is bounded on $\mathbf{R}_+$ for all continuos and bounded function $H(s)$.

Applying Perron's theorem (see [14]) we deduce that there exist the constants $\beta \geq 1, \alpha > 0$ such that

$$\|T(t,s)\| \leq \beta e^{-\alpha(t-s)}, \forall t \geq s \geq 0$$

and thus the proof is complete.

Proposition 4.11 *Assume that the system (4.1) is in the "stationary case". Then the following are equivalent:*

(i) The system $(A_0, A_1, ..., A_r; Q)$ is stable.

(ii) For all $H = (H(1), H(2), ...H(d)) \in \mathcal{S}_n^d, H(i) > 0, i \in \mathcal{D}$ the algebraic linear equation on $\mathcal{S}_n^d$.

$$L^* K + H = 0 \tag{4.17}$$

has a unique solution $K = (K(1), K(2), ...K(d)) \in \mathcal{S}_n^d, K(i) > 0, i \in \mathcal{D}$.

(iii) For each $H = (H(1),\ H(2),\ ...H(d)) \in \mathcal{S}_n^d, H(i) \geq 0, i \in \mathcal{D}$ the linear inequality

$$L^* K + H < 0 \tag{4.18}$$

has a solution $K = (K(1),\ K(2),\ ...K(d)), K(i) > 0, i \in \mathcal{D}$.

(iv) There exists $K \geq 0$ satisfying $L^* K < 0$.

(v) For each $H \in \mathcal{S}_n^d, H > 0$, the linear equation on $\mathcal{S}_n^d$

$$LK + H = 0 \tag{4.19}$$

has a unique positive solution $K = (K(1), K(2), ...K(d))$.

(vi) For each $H \in S_n^d, H \geq 0$ *the linear inequality*

$$LK + H < 0 \tag{4.20}$$

has a solution $K > 0$.

(vii) There exists $K \geq 0$ *satisfying* $LK < 0$.

Proof (i) $\Rightarrow$ (ii). From the equivalence (i) $\leftrightarrow$ (ii) in Proposition 4.8 we get that the equation

$$\frac{d}{dt}K(t) + L^*K(t) + H = 0$$

has a unique bounded and uniform positive solution $\tilde{K}(t)$. Moreover $\tilde{K}(t)$ is given by

$$\tilde{K}(t) = \int_t^\infty e^{L^*(s-t)}Hds.$$

We have $\tilde{K}(t) = \int_0^\infty e^{L^*s}Hds = \tilde{K}(0)$, for all $t \geq 0$. Hence $\tilde{K}(t)$ is constant and it verifies the equation (4.17).

(ii) $\Rightarrow$ (iii).

(ii) implies that the equation $L^*K + H + J = 0$ has a solution $\hat{K} > 0$. Hence $\hat{K}$ verifies (4.18).

(iii) $\Rightarrow$ (iv) follows immediately (taking $H = J$).

(iv) $\Rightarrow$ (i) follows from Remark a).

(i) $\Rightarrow$ (v).

Let $H > 0$. Therefore $\beta_2 J \leq H \leq \beta_1 J$ and with $\beta_1 \geq \beta_2 > 0$. Since $\|e^{Lt}\| \leq \beta e^{-\alpha t}, t \geq 0$ for some $\beta \geq 1, \alpha > 0$ the integral $\hat{K} = \int_0^\infty e^{Lt}Hdt$ is convergent and since e^{Lt} is a positive operator we have according Proposition 4.4

$$\beta_3 J \leq \beta_2 \int_0^\infty e^{Lt}Jdt \leq \hat{K} \leq \frac{\beta}{\alpha}\beta_1 J.$$

Further, we can write

$$L\hat{K} = \int_0^\infty \frac{d}{dt}(e^{Lt}H)dt = -H$$

and thus the proof of (i) $\Rightarrow$ (v) is complete.

(v) $\Rightarrow$ (vi) follows by using the same reasoning as in the proof (ii) $\Rightarrow$ (iii).

(vi) $\Rightarrow$ (vii) follows immediately (taking $H = J$).

(vii) $\Rightarrow$ (i).

Let $H = -LK$. Thus $LK + H = 0$ with $H > 0$ and $K \geq 0$. Since K is a constant solution of the equation $K'(t) = LK(t) + H$ we have

$$K = e^{L(t-t_0)}K + \int_{t_0}^{t} e^{L(t-s)}Hds, t \geq t_0.$$

Since e^{Lt} is a positive operator and $H \geq \gamma J$ with some $\gamma > 0$ we can write

$$\gamma \int_{t_0}^{t} e^{L(t-s)}Jds \leq \int_{t_0}^{t} e^{L(t-s)}Hds \leq K \leq \delta J.$$

Thus, by Proposition 4.10 the proof is complete.

Remark From the proof of Proposition 4.11 it follows that:

a) If there exist $H > 0$ and $K \geq 0$ such that $L^*K + H = 0$ then the stationary system $(A_0, A_1, ..., A_r; Q)$ is stable and $K = \hat{K} = \int_0^\infty e^{L^*t}Hdt$.

b) If there exist $H > 0$ and $\tilde{K} \geq 0$ such that $L\tilde{K} + H = 0$, then the stationary system $(A_0, A_1, ..., A_r; Q)$ is stable and

$$\tilde{K} = \bar{K} = \int_0^\infty e^{Lt}Hdt.$$

5 Affine systems

Consider the system

$$dx(t) = [A_0(t, \eta(t))x(t) + f_0(t)]dt + \sum_{k=1}^{r}[A_k(t, \eta(t))x(t) + f_k(t)]dw_k(t) \quad (5.1)$$

where $A_k(t, i), 0 \leq k \leq r$ are bounded on $\mathbf{R}_+$ and continuous matrix valued functions. Denote

$$u(t) = (f_0^*(t),\ f_1^*(t),\ ...,\ f_r^*(t))^*.$$

If $t_0 \geq 0, x_0 \in \mathbf{R}^n$ and $f_k \in L^2_{\eta,w}([t_0, T], \mathbf{R}^n), 0 \leq k \leq r$ for all $T > t_0$ by standard procedure of succesive aproximations and using properties of stochastic integral, it is easy to obtain that there exists a unique solution $x_u(t, t_0, x_0)$ of the system (5.1) with $x_u(t_0, t_0, x_0) = x_0$ and $x_u(\cdot, t_0, x_0) \in L^2_{\eta,w}([t_0, T], \mathbf{R}^n), T > t_0$.

Remark. If we denote

$$z_u(t, t_0, x_0) = \Phi^{-1}(t, t_0)x_u(t, t_0, x_0)$$

$\Phi(t,t_0)$ being the fundamental (random) matrix solution defined by the linear part of the system (5.1), then, using the Itô formula, we deduce that:

$$
\begin{aligned}
z_u(t,t_0,x_0) &= x_0 + \int_{t_0}^{t} \Phi^{-1}(s,t_0)[f_0(s) - \sum_{k=1}^{r} A_k(s,\eta(s))f_k(s)]ds \\
&\quad + \sum_{k=1}^{r} \int_{t_0}^{t} \Phi^{-1}(s,t_0)f_k(s)dw_k(s)
\end{aligned}
$$

for all $t \geq t_0$.

Thus, we obtain the following representation formula of the solution of system (5.1):

$$
x_u(t,t_0,x_0) = \Phi(t,t_0)x_0 + \Phi(t,t_0)\int_{t_0}^{t} \Phi^{-1}(s,t_0)[f_0(s) \tag{5.2}
$$

$$
-\sum_{k=1}^{r} A_k(s,\eta(s))f_k(s)]ds + \sum_{k=1}^{r} \Phi(t,t_0)\int_{t_0}^{t} \Phi^{-1}(s,t_0)f_k(s)dw_k(s)
$$

for all $t \geq t_0$. This is the stochastic version of the well known variation constant formula in the deterministic framework. Unfortunately the above formula cannot be used to obtain some useful estimates for solutions of system (5.1) as in the deterministic case. Such estimations are obtained in an indirect way using some techniques based on Liapunov functions.

Theorem 5.1: *Assume that the system $(A_0, A_1, ..., A_r; Q)$ is stable. Then:*

(i) There exists $c \geq 1, \alpha > 0$ such that

$$
E[|x_u(t,t_0,x_0)|^2|\eta(t_0)=i] \leq c(e^{-\alpha(t-t_0)}|x_0|^2 + \sum_{k=0}^{r} E[\int_{t_0}^{t} e^{-\alpha(t-s)}|f_k(s)|^2 ds|\eta(t_0)=i])
$$

for all $t \geq t_0 \geq 0, x_0 \in \mathbf{R}^n, i \in \mathcal{D}$ and all $f_k \in L^2_{\eta,w}([t_0,\infty),\mathbf{R}^n), 0 \leq k \leq r$.

(ii) There exists $\beta > 0$ such that

$$
E[\int_{t_0}^{\infty} |x_u(t,t_0,x_0)|^2|\eta(t_0)=i] \leq \beta(|x_0|^2 + \sum_{k=0}^{r} E[\int_{t_0}^{\infty} |f_k(s)|^2 ds|\eta(t_0)=i])
$$

for all $t_0 \geq 0, x_0 \in \mathbf{R}^n, f_k \in L^2_{\eta,w}([t_0,\infty),\mathbf{R}^n), 0 \leq k \leq r, i \in \mathcal{D}$.

(iii) $\lim_{t\to\infty} E|x_u(t,t_0,x_0)|^2 = 0$ for all $t_0 \geq 0, x_0 \in \mathbf{R}^n, f_k \in L^2_{\eta,w}([t_0,\infty),\mathbf{R}^n)$, $0 \leq k \leq r$.

Proof: Since $(A_0, A_1, ..., A_r; Q)$ is stable then by Proposition 4.6 the Lyapunov type equation (4.8) has a unique bounded on $\mathbf{R}_+$ and uniformly positive solution $\tilde{K}(t) = (\tilde{K}(t,1), ..., \tilde{K}(t,d))$. Therefore there exist $\alpha_1 > 0, \alpha_2 > 0$ such that

$$\alpha_1 J \leq \tilde{K}(t) \leq \alpha_2 J, \quad t \geq 0.$$

Let $x_u(t) = x_u(t, t_0, 0), t \geq t_0$. Applying the Itô type formula (3.3) to the function $v(t, x, i) = x^* \tilde{K}(t, i)x$ and to the system (5.1), taking into account the equation (4.8) for $\tilde{K}(t)$ we obtain:

$$E[v(t, x_u(t), \eta(t))|\eta(t_0) = i] = E[\int_{t_0}^{t} \{-|x_u(s)|^2 + 2x_u^*(s)[\tilde{K}(s, \eta(s))f_0(s)$$

$$+\sum_{k=1}^{r} A_k^*(s, \eta(s))\tilde{K}(s, \eta(s))f_k(s)] + \sum_{k=1}^{r} f_k^*(s)\tilde{K}(s, \eta(s))f_k(s)\}ds|\eta(t_0) = i].$$

We denote

$$h_i(t) = E[v(t, x_u(t), \eta(t))|\eta(t_0) = i], i \in \mathcal{D}$$

$$m_i(t) = \sqrt{E[|x_u(t)|^2|\eta(t_0) = i]}, \imath \in \mathcal{D}$$

$$g_i(t) = \sqrt{\sum_{k=0}^{r} E[|f_k(t)|^2|\eta(t_0) = i]}, i \in \mathcal{D}.$$

We may write

$$h_i'(t) = E[\{-|x_u(t)|^2 + 2x_u^*(t)[\tilde{K}(t, \eta(t))f_0(t) + \sum_{k=1}^{r} A_k^*(t, \eta(t))\tilde{K}(t, \eta(t))f_k(t)]$$

$$+\sum_{k=1}^{r} f_k^*(t)\tilde{K}(t, \eta(t))f_k(t)\}|\eta(t_0) = i]$$

a.e. $t \geq t_0, i \in \mathcal{D}$.

Since $A_k, \tilde{K}$ are bounded, there exist $\gamma > 0, \delta > 0$ such that

$$h_i'(t) \leq -m_i^2(t) + \gamma[m_i(t)g_i(t) + g_i^2(t)] \leq -\frac{1}{2}m_i^2(t) + \delta g_i^2(t).$$

Since $\alpha_1 I_n \leq \tilde{K}(t, \eta(t)) \leq \alpha_2 I_n$ we have

$$\alpha_1 m_i^2(t) \leq h_i(t) \leq \alpha_2 m_i^2(t).$$

Hence $h_i'(t) \leq -\frac{1}{2\alpha_2}h_i(t) + \delta g_i^2(t)$. Since $h_i(t_0) = 0$ we obtain

$$\alpha_1 m_i^2(t) \leq h_i(t) \leq \delta \int_{t_0}^{t} e^{-\alpha(t-s)} g_i^2(s)ds, \; t \geq t_0, i \in \mathcal{D} \tag{5.3}$$

with $\alpha = \frac{1}{2a_2}$. On the other hand

$$x_u(t, t_0, x_0) = x_u(t, t_0, 0) + \Phi(t, t_0)x_0. \tag{5.4}$$

Combining (5.3) and (5.4), (i) is proved. (ii) follows by (i) and Fubini theorem. We prove now (iii). Since

$$\sum_{i=1}^{d} E[\int_{t_0}^{\infty} \sum_{k=0}^{r} |f_k(t)|^2 dt | \eta(t_0) = i] < \infty,$$

it follows that for every $\varepsilon > 0$ there exists $t_\varepsilon > t_0$ such that

$$\sum_{i=1}^{d} \int_{t_\varepsilon}^{\infty} g_i^2(t)dt < \varepsilon.$$

For each $t \geq t_\varepsilon$ we have

$$\begin{aligned} \int_{t_0}^{t} e^{-\alpha(t-s)} g_i^2(s)ds &= e^{-\alpha(t-t_\varepsilon)} \int_{t_0}^{t_\varepsilon} e^{-\alpha(t_\varepsilon - s)} g_i^2(s)ds + \int_{t_\varepsilon}^{t} e^{-\alpha(t-s)} g_i^2(s)ds \\ &\leq e^{-\alpha(t-t_\varepsilon)} \int_{t_0}^{\infty} g_i^2(s)ds + \varepsilon. \end{aligned}$$

From this inequality and (5.3) we conclude

$$\lim_{t\to\infty} E[|x_u(t, t_0, 0)|^2 | \eta(t_0) = i] = 0.$$

Finally, using (5.4) we obtain

$$\lim_{t\to\infty} E[|x_u(t, t_0, x_0)|^2 | \eta(t_0) = i] = 0$$

and the proof is complete. $\square$

Remark. If we do not know that the system $(A_0, A_1, ..., A_r; Q)$ is stable then the estimation from Theorem 5.1 (i) is not uniform with respect to $t, t_0 \in \mathbf{R}_+$. In general we may prove that for any compact interval $[t_0, t_1]$ there exists a positive constant c depending upon $t_1 - t_0$ such that

$$E[|x_u(t, t_0, x_0)|^2 | \eta(t_0) = i] \leq c(|x_0|^2 + \sum_{k=0}^{r} E[\int_{t_0}^{t_1} |f_k(s)|^2 ds | \eta(t_0) = i]) \tag{5.5}$$

for all $t \in [t_0, t_1], x_0 \in \mathbf{R}^n, i \in \mathcal{D}$ and all $f_k \in L^2_{\eta,w}([t_0, t_1], \mathbf{R}^n), 0 \leq k \leq r$. To this end we remark that since $A_k(t, i), 0 \leq k \leq r, i \in \mathcal{D}$ are bounded on $\mathbf{R}_+$, from (5.1) and (3.1) it follows easily that there exists an absolute constant $\gamma > 1$ such that for all $t \in [t_0, t_1], i \in \mathcal{D}$ we have

$$E[|x_u(t, t_0, x_0)|^2 | \eta(t_0) = i] \leq \gamma\{|x_0|^2 + E[\int_{t_0}^{t} |x_u(s, t_0, x_0)|^2 ds | \eta(t_0) = i]((t_1$$

$$-t_0) + 1) + \sum_{k=0}^{r} E[\int_{t_0}^{t_1} |f_k(s)|^2 ds | \eta(t_0) = i]((t_1 - t_0) + 1)\}.$$

By using the Gronwall Lemma we get

$$\sup_{t_0 \le t \le t_1} E[|x_u(t, t_0, x_0)|^2 \eta(t_0) = i] \le c(|x_0|^2 + \sum_{k=0}^{r} E[\int_{t_0}^{t_1} |f_k(s)|^2 | \eta(t_0) = i]), i \in \mathcal{D}$$

where $c > 0$ depends only on $t_1 - t_0$.

6 STOCHASTIC STABILIZABILITY AND STOCHASTIC DETECTABILITY

Consider the linear controlled system described by

$$\begin{aligned} dx(t) &= [A_0(t, \eta(t))x(t) + B(t, \eta(t))u(t)]dt + \sum_{k=1}^{r} A_k(t, \eta(t))x(t)dw_k(t) \\ y(t) &= C(t, \eta(t))x(t) \end{aligned} \quad (6.1)$$

where $A_k : \mathbf{R}_+ \to \mathcal{M}_n^d$, $B : \mathbf{R}_+ \to \mathcal{M}_{n,m}^d$, $C : \mathbf{R}_+ \to \mathcal{M}_{p,n}^d$ are continuous and bounded functions.

Definition 6.1 *We shall say that the triple $[A, B; Q]$ (where $A = (A_0, A_1, ..., A_r)$) is stochastically stabilizable (or equivalently the system (6.1) is stochastically stabilizable) if there exists a continuous and bounded function $F : \mathbf{R}_+ \to \mathcal{M}_{m,n}^d$ such that the system $(A_0 + BF, A_1, A_2, ..., A_r; Q)$ is stable.*

The function F with the above property will be termed as a stabilizing feedback gain.

Definition 6.2 *We shall say that the triple $(C, A; Q)$ is stochastically detectable if there exists $H : \mathbf{R}_+ \to \mathcal{M}_{n,p}^d$ continuous and bounded function such that the system $(A_0 + HC, A_1, ..., A_r; Q)$ is stable.*

The function H with the above properties will be called "stabilizing injection".

Remark If the system (6.1) is in "stationary case" then the stabilizing feedback gain and the stabilizing injection are supposed to be of the form $F = (F(1), ..., F(d)), H = (H(1), ..., H(d))$.

Based on the result of Proposition 4.11 we get the following result which can be used to verify the stochastic stabilizability and stochastic detectability, respectively.

Corollary 6.3 *If the system (6.1) is in the stationary case the following are equiavalent:*

(i) The triple $(A, B; Q)$ *is stochastically stabilizable.*

(ii) For each $H = (H(1),\ H(2),\ ...,H(d)) \in \mathcal{S}_n^d, H(i) > 0, i \in \mathcal{D}$ *the system of linear equations:*

$$A_0(i)X_i + X_iA_0(i)^* + \sum_{k=1}^{r} A_k(i)X_iA_k^*(i)$$

$$+\sum_{j=1}^{d} q_{ji}X_j + B(i)\Gamma(i) + \Gamma^*(i)B^*(i) + H(i) = 0 \qquad (6.2)$$

$i \in \mathcal{D}$ *has a solution* $(X, \Gamma), X = (X_1, ..., X_d) \in \mathcal{S}_n^d, \Gamma = (\Gamma(1), ..., \Gamma(d)) \in \mathcal{M}_{m,n}^d, X(i) > 0, i \in \mathcal{D}$.

Moreover, $F = (F(1), ..., F(d))$ *with* $F(i) = \Gamma(i)X_i^{-1}, i \in \mathcal{D}$, *is a stabilizing feedback gain.*

(iii) For each $H = (H(1),\ H(2),\ ...,H(d)) \in \mathcal{S}_n^d, H > 0$ *the system of lincar incqualitics*

$$A_0(i)X_i + X_iA_0^*(i) + \sum_{k=1}^{r} A_k(i)X_iA_k^*(i)$$

$$+\sum_{j=1}^{d} q_{ji}X_j + B(i)\Gamma(i) + \Gamma^*(i)B^*(i) + H(i) < 0 \qquad (6.3)$$

$i \in \mathcal{D}$, *has a solution* $(X, \Gamma), X \in \mathcal{S}_n^d, X > 0, \Gamma \in \mathcal{M}_{m,n}^d$. *Moreover if* (X, Γ) *is a solution of the system (6.3) with* $X > 0$, *then* $F = (F(1),\ F(2), ..., F(d))$ *with* $F(i) = \Gamma(i)X_i^{-1}, i \in \mathcal{D}$, *is a stabilizing feedback gain.*

Corollary 6.4 *If the system (6.1) is in the stationary case, then the following are equivalent:*

(i) $(C, A; Q)$ *is stochastically detectable.*

(ii) For each $H = (H(1),\ H(2), ..., H(d)) \in \mathcal{S}_n^d, H > 0$ *the system of linear equations:*

$$A_0^*(i)X_i + X_iA_0(i) + \sum_{k=1}^{r} A_k^*(i)X_iA_k(i) + \sum_{j=1}^{d} q_{ij}X_j + \Gamma_iC_i + C_i^*\Gamma_i^* + H(i) = 0 \ (6.4)$$

$i \in \mathcal{D}$ *has a a solution* $X = (X_1\ X_2 ... X_d) \in \mathcal{S}_n^d, X > 0, \Gamma = (\Gamma_1\ \Gamma_2 ... \Gamma_d) \in \mathcal{M}_{n,p}^d$. *Moreover if* (X, Γ) *is a solution of the system (6.4) with*

$X > 0$, *then* $K = (K(1), K(2), ..., K(d))$ *with* $K(i) = X_i^{-1}\Gamma_i$ *is a stabilizing injection (here* $C_i = C(i)$*).*

(iii) For each $H \in \mathcal{S}_n^d, H > 0$ *the system of linear inequalities*

$$A_0^*(i)X_i+X_iA_0(i)+\sum_{k=1}^{r} A_k^*(i)X_iA_k(i)+\sum_{j-1}^{d} q_{ij}X_j+\Gamma_iC_i+C_i^*\Gamma_i^*+H(i) < 0, \quad (6.5)$$

$i \in \mathcal{D}$ *has a solution* $(X, \Gamma), X \in \mathcal{S}_n^d, X > 0, \Gamma \in \mathcal{M}_{n,p}^d$. *Moreover, if* $(X, \Gamma), X > 0$ *is a solution of the system (6.5) then* $K = (K(1), K(2), ..., K(d)) \in \mathcal{M}_{n,p}^d$ *with* $K(i) = X_i^{-1}\Gamma_i, i \in \mathcal{D}$, *is a stabilizing injection.*

Now we prove the following theorem, which extends a well known result from the deterministic framework:

Theorem 6.5 *Suppose:*

(i) $(C, A; Q)$ *is stochastically detectable.*

(ii) The differential equation

$$\frac{d}{dt}K(t) + L^*(t)K(t) + \tilde{C}(t) = 0 \quad (6.6)$$

has a bounded solution $\tilde{K} : \mathbf{R}_+ \to \mathcal{S}_n^d, \tilde{K}(t) = (\tilde{K}(t,1), ..., \tilde{K}(t,d)), \tilde{K}(t,i) \geq 0, t \geq 0, i \in \mathcal{D}$ *where* $\tilde{C}(t) = (\tilde{C}(t,1), ..., \tilde{C}(t,d)), \tilde{C}(t,i) = C^*(t,i)C(t,i)$.

Then the solution of the system (4.1) is mean square exponentially stable (or equivalently the system $((A_0, A_1, ..., A_r); Q)$ *is stable.*

Proof: Consider $v : \mathbf{R}_+ \times \mathbf{R}^n \times \mathcal{D} \to \mathbf{R}, v(t,x,i) = x^*\tilde{K}(t,i)x$. Let $x(t) = x(t, t_0, x_0)$ be a solution of the system (4.1).

Applying the identity (3.3) to the function v and to the system (4.1) and taking into account the equation (6.6) we get for all $t \geq t_0$ and $i \in \mathcal{D}$

$$E[v(t,x(t),\eta(t))|\eta(t_0) = i] - x_0^*\tilde{K}(t_0,i)x_0 = -E[\int_{t_0}^{t} |C(s,\eta(s))x(s)|^2 ds|\eta(t_0) = i].$$

Hence

$$E[\int_{t_0}^{\infty} |C(t,\eta(t))x(t)|^2 dt|\eta(t_0) = i] \leq x_0^*K(t_0,i)x_0 \leq \gamma|x_0|^2 \quad (6.7)$$

$t_0 \geq 0, x_0 \in \mathbf{R}^n, i \in \mathcal{D}$.

We may write:

$$\begin{aligned} dx(t) &= \{[A_0(t,\eta(t)) + H(t,\eta(t))C(t,\eta(t))]x(t) + f_0(t)\}dt \\ &\quad + \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t) \end{aligned}$$

where $f_0(t) = -H(t,\eta(t))C(t,\eta(t))x(t)$.

Since the system $(A_0 + HC, A_1 ... A_r; Q)$ is stable and $f_0 \in L^2_{\eta,w}([t_0,\infty) \times \mathbf{R}^n)$ (see (6.7)) we may use the Theorem 5.1 (ii) to obtain;

$$E[\int_{t_0}^{\infty} |\Phi(t,t_0)x_0|^2 dt|\eta(t_0) = i] \leq \delta[|x_0|^2 + E[\int_{t_0}^{\infty} |f_0(t)|^2 dt|\eta(t_0) = i] \leq \beta|x_0|^2$$

for all $t_0 \geq 0, x_0 \in \mathbf{R}^n, i \in \mathcal{D}$.

Using Proposition 4.6 we conclude that the system $(A_0, A_1, ... A_r; Q)$ is stable and the proof is complete. ▱

7 STOCHASTIC VERSION FOR BOUNDED REAL LEMMA

7.1 Input-output operators

Let us consider

$$\begin{aligned} dx(t) &= [A_0(t,\eta(t))x(t) + B(t,\eta(t))u(t)]dt + \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t) \\ y(t) &= C(t,\eta(t)x(t) + D(t,\eta(t))u(t) \end{aligned} \qquad (7.1)$$

$t \geq 0$, with the input $u \in R^m$, the output $y \in R^p$ and the states $x \in R^n$. The coefficients $A_k : \mathbf{R}^+ \to \mathcal{M}^d_n, k = 0,1,...,r, B : \mathbf{R}_+ \to \mathcal{M}^d_{n,m}, C : \mathbf{R}_+ \to \mathcal{M}^d_{p,n}, D : \mathbf{R}_+ \to \mathcal{M}^d_{p,m}$ are bounded and continuous functions. For each $u \in L^2_{\eta,w}\{[t_0,T], \mathbf{R}^m\}, 0 \leq t_0 < T$ we denote $x(t,t_0,u), t \in [t_0,T]$ the solution of the system (7.1) which verifies the initial condition $x(t_0,t_0,u) = 0$.

The stochastic process $x(t,t_0,u), t \in [t_0,T]$ depends linearly upon $u \in L^2_{\eta,w}\{[t_0,T], \mathbf{R}^n\}$. Moreover, using (5.5) we obtain that there exists $c > 0$ (depending upon $T - t_0$) such that

$$\sum_{i=1}^{d} E[\int_{t_0}^{T} |x(t,t_0,u)|^2 dt|\eta(t_0) = i] \leq c \sum_{i=1}^{d} E[\int_{t_0}^{T} |u(t)|^2 dt|\eta(t_0) = i].$$

It follows that the system (7.1) defines a bounded linear operator from the space of stochastic processes $u \in L^2_{\eta,w}([t_0,T], \mathbf{R}^m)$ to the space of stochastic processes $y \in L^2_{\eta,w}([t_0,T], \mathbf{R}^p)$ by

$$(\mathbf{T}_{t_0,T}u)(t) = C(t,\eta(t))x(t,t_0,u) + D(t,\eta(t))u(t).$$

The linear operator $\mathbf{T}_{t_0,T}$ will be termed "the input-output operator" defined by the system (7.1) on the interval $[t_0, T]$ and the system (7.1) will be called "a state space realization" of the operator $\mathbf{T}_{t_0,T}$.

If the zero solution of the system (7.1) with $u(t) = 0$ is mean square exponentially stable we can deduce (by using the theorem 5.1) that the system (7.1) defines a bounded linear operator $\mathbf{T}_{t_0} : L^2_{\eta,w}([t_0,\infty), \mathbf{R}^m) \to L^2_{\eta,w}([t_0,\infty), \mathbf{R}^p)$ by

$$(\mathbf{T}_{t_0}u)(t) = C(t,\eta(t))x(t,t_0,u) + D(t,\eta(t))u(t)$$

for all $t \in [t_0,\infty), u \in L^2_{\eta,w}([t_0,\infty), \mathbf{R}^m)$.

Remark It is obvious that the space $L^2_{\eta,w}([t_0,T]; \mathbf{R}^m)$ may be identify as the subspace of $L^2_{\eta,w}([t_0,\infty), \mathbf{R}^m)$ consisting in the processes $u \in L^2_{\eta,w}([t_0,\infty), \mathbf{R}^m)$ with the property $u(t) = 0$ if $t > T$. Under this convention we may write: $L^2_{\eta,w}([t_0,T], \mathbf{R}^m) \subset L^2_{\eta,w}([t_0,\infty), \mathbf{R}^m)$ for all $T > t_0$. It is easy to verify that if the system $(A_0, A_1, ..., A_r; Q)$ is stable, then we have $\mathbf{T}_{t_0,T} = \Pi_{t_0,T}\mathbf{T}_{t_0}\,|_{L^2_{\eta,w}([t_0,T],\mathbf{R}^m)}$ where $\Pi_{t_0,T} : L^2_{\eta,w}([t_0,\infty), \mathbf{R}^p) \to L^2_{\eta,w}([t_0,T], \mathbf{R}^p)$ is the canonical projection.

B. Concerning the product of two input-output operators it is easy to prove:

Proposition 7.1 *Let $\mathbf{T}^1_{t_0,T} : L^2_{\eta,w}([t_0,T], \mathbf{R}^m) \to L^2_{\eta,w}([t_0,T], \mathbf{R}^p)$ and $\mathbf{T}^2_{t_0,T} : L^2_{\eta,w}([t_0,T], \mathbf{R}^q) \to L^2_{\eta,w}([t_0,T], \mathbf{R}^m)$ be the input output operator having the state space realizations:*

$$\begin{aligned} dx^j(t) &= [A^j_0(t,\eta(t))x^j(t) + B^j(t,\eta(t))u^j(t)]dt + \sum_{k=1}^{r} A^j_k(t,\eta(t))x^j(t)dw_k(t) \\ y^j(t) &= C^j(t,\eta(t))x^j(t) + D^j(t,\eta(t))u^j(t), j = 1,2 \end{aligned}$$

then a state space realization of the linear operator $\mathbf{T}^1_{t_0,T}\mathbf{T}^2_{t_0,T}$ is given by

$$d\tilde{x}(t) = [\tilde{A}_0(t,\eta(t))\tilde{x}(t) + \tilde{B}(t,\eta(t))u^2(t)]dt + \sum_{k=1}^{r} \tilde{A}_k(t,\eta(t))\tilde{x}(t)dw_k(t), t \in [t_0,T]$$

$$\tilde{y}(t) = \tilde{C}(t,\eta(t))\tilde{x}(t) + \tilde{D}(t,\eta(t))u^2(t)$$

where

$$\tilde{A}_0(t,i) = \begin{pmatrix} A^1_0(t,i) & B^1(t,i)C^2(t,i) \\ 0 & A^2_0(t,i) \end{pmatrix}; \quad \tilde{A}_k(t,i) = \begin{pmatrix} A^1_k(t,i) & 0 \\ 0 & A^2_k(t,i) \end{pmatrix},$$

$$k = 1,2,...,r, \tilde{B}(t,i) = \begin{pmatrix} B^1(t,i)D^2(t,i) \\ B^2(t,i) \end{pmatrix},$$

$$\tilde{C}(t,i) = (C^1(t,i) \quad D^1(t,i)C^2(t,i)),\ \tilde{D}(t,i) = D^1(t,i)D^2(t,i), \tilde{x} = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}.$$

Concerning the invertibility of a input-output operator we can prove:

Proposition 7.2 *Assume that for the system (7.1) the number of the inputs equal the number of the outputs and* $\inf_{t\in[t_0,t_1]} |detD(t,i)| > 0, i \in \mathcal{D}$, *then the corresponding input-output operator* $\mathbf{T}_{t_0,t_1} : L^2_{\eta,w}([t_0,t_1],\mathbf{R}^m) \to L^2_{\eta,w}([t_0,t_1],\mathbf{R}^m)$ *is invertible with bounded inverse.*

A state space realization of the operator $\mathbf{T}^{-1}_{t_0,t_1}$ *is given by:*

$$\begin{aligned} dx(t) &= \{[A_0(t,\eta(t)) - B(t,\eta(t))D^{-1}(t,\eta(t))C(t,\eta(t))]x(t) \\ &\quad + B(t,\eta(t))D^{-1}(t,\eta(t))y(t)\}dt + \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t) \quad (7.2) \\ u(t) &= -D^{-1}(t,\eta(t))C(t,\eta(t))x(t) + D^{-1}(t,\eta(t)y(t). \end{aligned}$$

Moreover if $\inf_{t\geq t_0} |detD(t,i)| > 0$ *and the system* $(A_0 - BD^{-1}C, A_1, ..., A_r; Q)$ *is stable then the input output operator* $\mathbf{T}_{t_0} : L^2_{\eta,w}([t_0,\infty),\mathbf{R}^m) \to L^2_{\eta,w}([t_0,\infty),\mathbf{R}^m)$ *is invertible with bounded inverse and a state space realization of the operator* $\mathbf{T}^{-1}_{t_0}$ *is given by (7.2).*

C. It is clear that if $0 \leq t_1 < t_2$ and if $u \in L^2_{\eta,w}([t_2,\infty);\mathbf{R}^m)$, then

$$\hat{u}(t) = \begin{cases} u(t) & t \in [t_2,\infty) \\ 0 & t_1 \leq t < t_2 \end{cases} \quad (7.3)$$

belongs to $L^2_{\eta,w}([t_1,\infty);\mathbf{R}^m)$.

Thus, (7.3) defines a cannonical injection $i_{t_2,t_1} : L^2_{\eta,w}([t_2,\infty),\mathbf{R}^m) \to L^2_{\eta,w}([t_1,\infty),\mathbf{R}^m)$. We remark that if $L^2_{\eta,w}([t_1,\infty),\mathbf{R}^m), k = 1,2$ is endowed with the norm generated by the inner product (2.3) then the injection $i_{t_2t_1}$ is not an isometry, as it happens when on $L^2_{\eta,w}([t_k,\infty),\mathbf{R}^m), k = 1,2$ the standard L^2-norm is considered. However, we point out that for the developments in the next sections it is essentialy to consider the norm generated by the inner product (2.3) (see [28]).

Under these remarks it follows that $t_0 \to ||\mathbf{T}_{t_0}||$ is not a decreasing function as in deterministic framework or in the case when the system is subjected only to the white noise perturbations. At the end of this subsection we prove:

Proposition 7.3 *Assume that the zero solution of the system (7.1) for* $u = 0$ *is exponentially stable in mean square and* $\|\mathbf{T}_0\| \leq \gamma$. *Then there exists* $\varepsilon_0 > 0$ *such that:*

$$(\gamma^2 - \varepsilon_0^2)I_m - D^*(t,i)D(t,i) \geq 0$$

for all $(t,i) \in \mathbf{R}^+ \times \mathcal{D}$.

Proof. Our proof is based on some ideas of Hinrichsen and Pritchard in [16]. We choose ε_0 such that $\|\mathbf{T}_0\| \langle (\gamma^2 - \varepsilon_0^2)^{1/2}$ and denote $\hat{\gamma} = (\gamma^2 - \varepsilon_0^2)^{1/2}$.

We show that $\hat{\gamma}^2 I_m - D^*(t,i)D(t,i) \geq 0$ for all $(t,i) \in \mathbf{R}^+ \times \mathcal{D}$. If this is not true, then there exist $t_0 \geq 0, i_0 \in \mathcal{D}, u_0 \in \mathbf{R}^m, |u_0| = 1$ such that

$$u_0^*(\hat{\gamma}^2 I_m - D^*(t_0,i_0)D(t_0,i_0)u_0 = -2\alpha < 0$$

for some $\alpha > 0$.

Since $D(\cdot,i_0)$ is a continuous function then there exist $\delta_0 > 0$, such that

$$u_0^*[\hat{\gamma}^2 I_m - D^*(t,i_0)D(t,i_0)]u_0 < -\alpha$$

for all $t \in [t_0, t_0 + \delta_0], \delta \in [0, \delta_0]$ and define the stochastic process $v_\delta(t)$ defined by

$$V_\delta(t) = \begin{cases} 0 & \text{if} \quad t \in [0,t_0) \cup (t_0 + \delta, \infty) \\ u_0 \chi_{\eta(t)=i_0} & \text{if} \quad t \in [t_0, t_0 + \delta] \end{cases}$$

Obviously $v_\delta \in L^2_{\eta,w}((0,\infty); \mathbf{R}^m)$. Let $x_\delta(t)$ be the solution of the system 97.1) corresponding to the input $v_\delta(0) = 0$.

Let $T > t_0 + \delta_0$ be fixed and $K(t) = (K(t,1), K(t,2), ..., K(t,d))$ be the solution of the equation (6.6) which verifies $K(T,i) = 0, i \in \mathcal{D}$. Applying the Itô type formula (3.3) to the function $v(t,x,i) = x^* K(t,i)x$ and to the system (7.1) and taking into account the equation (6.6) we obtain easily:

$$E[\int_0^T \{|y_\delta(t)|^2 - \gamma^2 |v_\delta(t)|^2\} dt | \eta(0) = i] =$$

$$E[\int_0^T \{2x_\delta^*(t)\mathcal{N}(t,\eta(t))v_\delta(t) - v_\delta^* \Delta(t,\eta(t))v_\delta(t)\} dt | \eta(0) = i$$

$i \in \mathcal{D}$ where

$$\begin{aligned} y_\delta(t) &= C(t,\eta(t))x_P\delta(t) + D(t,\eta(t))v_\delta(t) \\ \mathcal{N}(t,i) &= K(t,i)B(t,i) + C^*(t,i)D(t,i) \\ \Delta(t,i) &= \hat{\gamma}^2 I_m - D^*(t,i)D(t,i). \end{aligned}$$

Further we may write

$$E[\int_0^T \{2x_\delta^*(t)\mathcal{N}(t,\eta(t))v_\delta(t) - v_\delta^*(t)\Delta(t,\eta(t))v_\delta(t)\} dt | \eta(0) = i] =$$

$$E[\int_{t_0}^{t_0+\delta} \{2x_\delta^*(t)\mathcal{N}(t,\eta(t))u_0 - u_0^*\Delta(t,\eta(t))u_0\}\chi_{\eta(t)=i_0} dt | \eta(0) = i] =$$

$$E[\int_{t_0}^{t_0+\delta} \sum_{j=1}^{d} \{2x_\delta(t)\mathcal{N}(t,j)u_0 - u_0^*\Delta(t,j)u_0\}\chi_{\eta(t)=j}\chi_{\eta(t)=i_0} dt | \eta(0) = i] =$$

$$E[\int_{t_0}^{t_0+\delta}\{2x_\delta^*(t)\mathcal{N}(t,i_0)u_0 - u_0^*\Delta(t,i_0)u_0\}\chi_{\eta(t)=i_0}dt|\eta(0)=i] =$$

$$2E[\int_{t_0}^{t_0+\delta} x_\delta^*(t)\mathcal{N}(t,i_0)u_0\chi_{\eta(t)=i_0}dt|\eta(0)=i] - \int_{t_0}^{t_0+\delta} u_0^*\Delta(t,i_0)u_0 p_{i,i_0}(t)dt.$$

Hence

$$E[\int_0^T \{|y_\delta(t)|^2 - \hat{\gamma}^2|v_\delta(t)|^2\}dt|\eta(0)=i] \geq$$

$$\alpha\int_{t_0}^{t_0+\delta} p_{i,i_0}(t)dt - 2E[|\int_{t_0}^{t_0+\delta} x_{delta}^*(t)\mathcal{N}(t,i_0)u_0\chi_{\eta(t)=i_0}dt|\eta(0)=i].$$

On the other hand using (5.5) we deduce that there exist $c_1 > 0$ not depending upon δ such that

$$\sup_{0\leq t\leq T} E[|x_\delta(t)|^2|\eta(0)=i] \leq \tilde{c}_1 E[\int_0^T |v_\delta(t)|^2dt|\eta(0)=i] \leq c_1\delta.$$

Thus we conclude that there exists $c_2 > 0$ not depending upon δsuch that

$$\sum_{i=1}^d E[\int_0^T \{|y_\delta(t)|^2 - \hat{\gamma}^2|v_\delta(t)|^2\}dt|\eta(0)=i] \geq$$

$$\alpha\sum_{i=1}^d\int_{t_0}^{t_0+\delta} p_{i,i_0}(t) - c_2\delta\sqrt{\delta}.$$

Since $p_{i_0,i_0}(t) > 0$ for all $t \geq 0$ and $t \to p_{i_0,i_0}(t)$ is a continuous function (see[6]) we deduce that there exists $\delta \in (0,\delta_0)$ such that

$$p_{i_0,i_0}(t) \geq \frac{1}{2}p_{i_0,i_0}(t_0) > 0, (\forall)t \in [t_0, t_0+\delta].$$

Hence

$$\sum_{i=1}^d p_{i_0,i_0}(t) \geq p_{i_0,i_0}(t) \geq \frac{1}{2}p_{i_0,i_0}(t_0) > 0, (\forall)t \in [t_0, t_0+\delta].$$

Thus we get that for $\delta > 0$ small enough we have

$$\sum_{i=1}^d E[\int_0^T \{|y_\delta(t)|^2 - \hat{\gamma}^2|v_{delta}(t)|^2\}dt|\eta(0)=i]$$

$$\geq \delta[\frac{\alpha}{2}p_{i_0,i_0}(t_0) - c_2\sqrt{\delta}] > 0.$$

On the other hand, we write

$$\begin{aligned}
0 &\leq \|\mathbf{T}_0 v_\delta\|^2 - \hat{\gamma}^2\|v_\delta\|^2 \\
&= \sum_{i=1}^d E[\int_0^\infty |y_\delta(t)|^2dt|\eta(0)=i] - \hat{\gamma}^2\sum_{i=1}^d E[\int_0^\infty |v_\delta(t)|^2dt|\eta(0)=i] \\
&\geq \sum_{i=1}^d[\int_0^T \{|y_\delta(t)|^2 - \hat{\gamma}^2|v_\delta(t)|^2\}dt|\eta(0)=i] > 0
\end{aligned}$$

which is a contradiction and thus the proof is complete.

Remark. The result proved in the Proposition 7.3 allows to prove the stochastic version of the Bounded Real Lemma of the system (7.1). However for the sake of simplicity in the next subsection we'll consider only the case when $D(t,i) = 0, (t,i) \in \mathbf{R}_+ \times \mathcal{D}$.

7.2 Stochastic Version Of Bounded Real Lemma

A. In this subsection we establish a necessary and sufficient condition assuaring that the norm of input-output operator defined by the system

$$\begin{aligned} dx(t) &= [A_0(t,\eta(t))x(t) + B(t,\eta(t))u(t)]dt + \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t) \\ y(t) &= C(t,\eta(t))x(t) \end{aligned} \tag{7.4}$$

is less then a prefixed level γ.

We extend the result of [28] and [29] to the case when the controlled system is subjected both to "white noise" perturbations and Markovian jumping.

We associate the following system of Riccati type differential equations parametrized by γ.

$$\frac{d}{dt}X_i(t) + A_0^*(t,i)X_i(t) + X_i(t)A_0(t,i) + \sum_{k=1}^{r} A_k^*(t,i)X_i(t)A_k(t,i) +$$
$$\sum_{j=1}^{d} q_{ij}X_j(t) + \gamma^{-2}X_i(t)B(t,i)B^*(t,i)X_i(t) + C^*(t,i)C(t,i) = 0, i \in \mathcal{D}. \tag{7.5}$$

When $\mathcal{D} = \{1\}$ (7.5) was considered in [?, 29] for time-varying case and in [16] for time-invariant case. For $A_k(t,i) = 0, k = 1,2,...,r$ system (7.5) was intensively investigated in [28].

A C^1-function $X : \mathbf{R}_+ \to S_n^d, X(t) = (X_1(t), \quad X_2(t), ..., X_d(t))$ is called stabilizing solution of the system (7.5) if it verifies (7.5) and additionally, the zero solution of the linear system

$$dx(t) = (A_0(t,\eta(t)) + B(t,\eta(t))F(t,\eta(t)))X(t)dt + \sum_{k=1}^{r} A_k(t,\eta(t))X(t)dw_k(t)$$

is exponentially stable in mean-square, where

$$F(t,i) = \gamma^{-2}B^*(t,i)X_i(t). \tag{7.6}$$

Let $\gamma > 0, 0 \leq t_0 < t_1, x_0 \in \mathbf{R}^n, i \in \mathcal{D}$ be fixed. Consider the cost functions

$$\mathcal{V}_\gamma(t_0, t_1, x_0, i, \cdot) : L^2_{\eta,w}([t_0, t_1], \mathbf{R}^m) \to \mathbf{R}$$

$$\mathcal{V}_\gamma(t_0, \infty, x_0, i, \cdot) : L^2_{\eta,w}([t_0, \infty), \mathbf{R}^m) \to \mathbf{R}$$

defined by:

$$\mathcal{V}_\gamma(t_0, t_1, x_0, i, u) = E[\int_{t_0}^{t_1} (|y_u(t, t_0, x_0)|^2 - \gamma^2 |u(t)|^2) dt | \eta(t_0) = i]$$

$$\mathcal{V}_\gamma(t_0, \infty, x_0, i, u) = E[\int_{t_0}^{\infty} (|y_u(t, t_0, x_0)|^2 - \gamma^2 |u(t)|^2) dt | \eta(t_0) = i]$$

where

$$y_u(t, t_0, x_0) = C(t, \eta(t)) x_u(t, t_0, x_0)$$

being the solution of the system (7.4) determined by the input u.

Directly, by Theorem 3.1 we obtain:

Proposition 7.4

If $X : [t_0, t_1] \to \mathcal{S}_n^d$ is a solution of the system (7.5),

$$X(t) = (X_1(t), ..., X_d(t))$$

then we have

$$\mathcal{V}_\gamma(t_0, t_1, x_0, i, u) = x_0^* X_i(t_0) x_0 - E[x_u^*(t_1) X_{\eta(t_1)}(t_1) x_u(t_1) | \eta(t_0) = i]$$

$$-\gamma^2 E[\int_{t_0}^{t_1} |u(t) - F(t, \eta(t)) x_u(t)|^2 dt | \eta(t_0) = i]$$

for all $i \in \mathcal{D}, x_0 \in \mathbf{R}^n, u \in L^2_{\eta,w}([t_0, t_1], \mathbf{R}^n), x_u(t) = x_u(t, t_0, x_0)$ and $X(t, i) = X_i(t), F(t, i)$ defined as in (7.6).

With the same techniques as in the Proof of Proposition 3 in [28] we obtain:

Proposition 7.5

a) Assume that

$$sup_{\tau \in [t_0, t_1]} ||\mathbf{T}_\tau|| < \gamma.$$

Then there exists a positive constant ρ depending upon $t_0, t_1, t_1 - t_0$ such that

$$\mathcal{V}_\gamma(t_0, t_1, x_0, i, u) \leq \rho |x_0|^2$$

for all $u \in L^2_{\eta,w}([t_0, t_1], \mathbf{R}^m), x_0 \in \mathbf{R}^n$.

b) If the system $(A_0, A_1, ..., A_r; Q)$ is stable and

$$\sup_{\tau \geq 0} ||\mathbf{T}_\tau|| < \gamma$$

then there exists $\rho > 0$ not depending upon t_0,

$$\mathcal{V}_\gamma(t_0, \infty, x_0, i, u) \leq \rho |x_0|^2$$

for all $u \in L^2_{\eta,w}([t_0, \infty), \mathbf{R}^m), x_0 \in \mathbf{R}^n, t_0 \geq 0, i \in \mathcal{D}$.

B. The next result is a stochastic version of the Bounded Real Lemma for a linear system described by stochastic differential equation with Markovian jumping on a compact interval $[t_0, t_1]$.

Theorem 7.6 *The following are equivalent:*

(i)

$$\sup_{\tau \in [t_0, t_1]} ||\mathbf{T}_\tau|| < \gamma$$

(ii) the solution $X_{t_1}(t) = (X(\cdot, 1), ..., X(\cdot, d))$ of the system (7.5) which verifies the conditon $X_{t_1}(t_1, i) = 0, i \in \mathcal{D}$ is defined on the whole interval $[t_0, t_1]$.

Proof: Follows directly applying Proposition 7.4 and Proposition 7.5 (a).

Remark:

The statement of (i) of Theorem 7.6 differs by the corresponding result in deterministic framework since $\tau \rightarrow ||\mathbf{T}_\tau||$ is not a decreasing function.

C. In the same way as in [28] we can prove

Proposition 7.7

Assume that:

(i) The system $(A_0, A_1, ..., A_r; Q)$ is stable

(ii)

$$\sup_{\tau \geq 0} ||\mathbf{T}_\tau|| < \gamma$$

Let $X_T(t) = (X_T(t, 1) \quad X_T(t, 2)...X_T(t, d))$ be the solution of system (7.5) which verifies the condition $X_T(T, i) = 0, i \in \mathcal{D}$.

Then we have:

a) The solution $X_T(\cdot)$ is defined on the whole interval $[0, T]$.

b) There exists $\rho > 0$ independent of T, t such that $0 \leq X_T(t, i) \leq \rho I_n$, for all $0 \leq t \leq T, i \in \mathcal{D}$.

c) For each $t_0 \in [0, T]$ we have:

$$max\{\mathcal{V}_\gamma(t_0, T, x_0, i, u) | u \in L^2_{\eta,w}([t_0, T], \mathbf{R}^m)\} = x_0^* X_T(t_0, i) x_0 = \mathcal{V}_\gamma(t_0, T, x_0, i, u_T)$$

where

$$u_T(t) = F_T(t, \eta(t)) x_T(t),$$

$x_T(\cdot)$ is the solution of the problem

$$dx(t) = [A_0(t, \eta(t)) + B(t, \eta(t)) F_T(t, \eta(t))] x(t) dt + \sum_{k=1}^{r} A_k(t, \eta(t)) x(t) dw_k(t) \quad (7.7)$$

$t \geq t_0, x_T(t_0) = x_0$.

$$F_T(t, i) = \gamma^{-2} B^*(t, i) X_T(t, i). \quad (7.8)$$

d)

$$0 \leq X_{T_1}(t, i) \leq X_{T_2}(t, i)$$

for all $0 \leq t \leq T_1 < T_2, i \in \mathcal{D}$.

The main result of this subsection is:

Theorem 7.8 *The following are equivalent:*

(i) The system $(A_0, A_1, ..., A_r; Q)$ is stable and

$$\sup_{\tau \geq 0} ||\mathbf{T}_\tau|| < \gamma. \quad (7.9)$$

(ii) The system (7.5) has a unique bounded on $\mathbf{R}_+$ and stabilizing solution $\tilde{X}(t) \geq 0$. Moreover if $A_k(\cdot), k = 0, 1, ...r, B(\cdot), C(\cdot)$ are θ-periodic functions then the stabilizing solution of system (7.5) is also a θ-periodic function.

Proof: $(i) \Longrightarrow (ii)$. From the Proposition 7.7 it follows that the function $\tilde{X}(t) = (\tilde{X}(t, 1), \tilde{X}(t, 2), ..., \tilde{X}(t, d))$ is well defined by

$$\tilde{X}(t, i) = \lim_{T \to \infty} X_T(t, i) \quad (7.10)$$

$t \geq 0, i \in \mathcal{D}$.

By standard argument we get that the function $\tilde{X}(t)$ defined by (7.10) is a solution of the system (7.5). Applying again the Proposition 7.7 (c), we deduce that $0 \leq \tilde{X}(t) \leq \rho J$.

The fact that $\tilde{X}(t)$ is the unique bounded stabilizing solution of the system (7.5) may be obtained in the same way as in the case when the system is only subjected to Markovian jumping (see proof of Theorem 1 in [28]) and we ommited it for shortness.

Assume now that $A_k(\cdot), B(\cdot), C(\cdot)$ are θ-periodic functions. Let $\hat{X}_T(t) = (\hat{X}_T(t,1), ..., \hat{X}_T(t,d))$ defined by $\hat{X}_T(t,i) = X_{T+\theta}(t+\theta, i)$, $(\hat{X}_T(T,i) = 0, i \in \mathcal{D})$.

By uniqueness arguments of the solution of the system (7.5) we deduce that $\hat{X}_T(t,i) = X_T(t,i)$ for all $t \in [0,T], i \in \mathcal{D}$.

Hence we have

$$\tilde{X}(t) = \lim_{T\to\infty} X_T(t) = \lim_{T\to\infty} X_{T+\theta}(t+\theta) = \tilde{X}(t+\theta), t \in \mathbf{R}_+, i \in \mathcal{D}.$$

This shows that if the coefficients of the system (7.5) are θ-periodic functions and if (i) in the statement holds, then the unique stabilizing solution of (7.5) is θ-periodic function.

(ii)$\Longrightarrow$(i). To obtain that the system $(A_0, A_1, ..., A_r; Q)$ is stable we use Theorem 6.5. To this end we remark that the system (7.5) can be written into a compact form:

$$\frac{d}{dt}X(t) + L^*(t)X(t) + \tilde{C}(t) = 0 \tag{7.11}$$

where

$$\tilde{C}(t) = (\tilde{C}(t,1), ..., \tilde{C}(t,d))$$

with

$$\tilde{C}(t,i) = \hat{C}^*(t,i)\hat{C}(t,i)$$

where

$$\hat{C}(t,i) = \begin{pmatrix} \gamma^{-1}B^*(t,i)X(t,i) \\ C(t,i) \end{pmatrix}, t \geq 0, i \in \mathcal{D}.$$

We have to check that the triple $(\hat{C}, A, Q)$ is stochastically detectable, where $A = (A_0, A_1, ..., A_r)$. To this end we take $\hat{H}(t) = (\hat{H}(t,1), \hat{H}(t,2), ..., \hat{H}(t,d))$ with $\hat{H}(t,i) = (\gamma^{-1}B(t,i) \quad 0), t \geq 0, i \in \mathcal{D}$.

We have

$$(A_0 + \hat{H}\hat{C}, A_1, ...A_r; Q) = (A_0 + B\tilde{F}, A_1, ..., A_r; Q)$$

which is stable.

Applying Theorem 6.4 to the equation (7.11) we deduce that the system $(A_0, A_1, ...A_r; Q)$ is stable.

The inequality (7.9) is obtained in the same way as in the case when the system is subjected only to Markovian jumping (see proof of Theorem 2 in [28]). Thus the proof is complete.

Corollary 7.9 *The following are equivalent:*

(i) The system $(A_0, A_1, ...A_r; Q)$ *is stable and* $\sup_{\tau\geq 0}||\mathbf{T}_\tau|| < \gamma$.

(ii) There exists a bounded C^1*-function* $X : \mathbf{R}_+ \to S_n^d$ *uniform positive which is bounded with bounded derivative and verifies the system of matrix differential inequalities.*

$$\frac{d}{dt}X(t,i) + A_0^*(t,i)X(t,i) + X(t,i)A_0(t,i) + \sum_{k=1}^{r} A_k^*(t,i)X(t,i)A_k(t,i)$$
$$+\sum_{j=1}^{d} q_{ij}X(t,j) + \gamma^{-2}X(t,i)B(t,i)B^*(t,i)X(t,i) + C^*(t,i)C(t,i) < 0 \quad (7.12)$$

uniformly with respect to $t \geq 0, i \in \mathcal{D}$.

Moreover if the coefficients of the system are θ*-periodic functions and if (i) holds then there exists a uniform positive* C^1*-function* $\tilde{X} : \mathbf{R}_+ \to S_n^d$, *which is* θ*-periodic and solves (7.12).*

Proof: $(i) \Rightarrow (ii)$ It follows applying Theorem 7.8 to the augmented system described by

$$dx(t) = [A_0(t,\eta(t))x(t) + B(t,\eta(t))u(t)]dt + \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t)$$

$$y_\delta(t) = \begin{pmatrix} C(t,\eta(t) \\ \delta I_n \end{pmatrix} x(t)$$

where $\delta > 0$ is sufficiently small.

To prove $(ii) \Rightarrow (i)$, firstly we observe that from (7.12) and Theorem 4.6 it follows that $(A_0, A_1, ..., A_r; Q)$ is stable. Further by using again (7.12) and reasoning as in the proof of Theorem 2 in [28] one concludes that $\sup_{\tau\geq 0}||T_\tau|| < \infty$ and thus the proof is complete.

D. For each $t_0 \geq 0$ we denote $\Gamma(t_0) = \{\gamma > 0 |$ the system (7.5) has a bounded and stabilizing solution $\tilde{X} : [t_0, \infty) \to S_n^d\}$,

$$\tilde{\gamma}(t_0) = \sup_{\tau\geq t_0} ||\mathbf{T}_\tau||.$$

The next result can be proved in the same way as in the case of the system containing only Markovian jumping (see [28]).

Proposition 7.10 *a) For each* $t_0 \geq 0$, *we have* $\Gamma(t_0) = (\tilde{\gamma}(t_0), \infty)$.

b) If the coefficients of the system (7.4) are θ*-periodic functions, then* $\Gamma(t_0 + \theta) = \Gamma(t_0)$ *for all* $t_0 \geq 0$ *and* $t_0 \to \tilde{\gamma}(t_0)$ *is constant.*

c) If the system (7.4) is in "the stationary case " then,

$$||\mathbf{T}_\tau|| = ||\mathbf{T}_0||$$

for all $\tau \geq 0$.

The next result is the stochastic version of the bounded real Lemma for the system (7.4) in the stationary case.

Corollary 7.11 *Assume that system (7.4) is in "the stationary case". Then the following are equivalent:*

(i) the system $(A_0, A_1, .., A_r; Q)$ *is stable and* $||\mathbf{T}_0|| < \gamma$.

(ii) The system of algebraic Riccati type equations

$$A_0(i)X_i + X_i A_0(i) + \sum_{k=1}^{r} A_k^*(i) X_i A_k(i) + \sum_{j=1}^{d} q_{ij} X_j + \gamma^{-2} X_i B(i) B^*(i) X_i$$
$$+ C^*(i)C(i) = 0, \quad i \in \mathcal{D}$$

has a unique stabilizing solution $\tilde{X} = (\tilde{X}_1 \quad \tilde{X}_2 ... \tilde{X}_d) \in \mathcal{S}_n^d, \tilde{X}_i \geq 0, i \in \mathcal{D}$.

(iii) The system of matrix inequality

$$\begin{pmatrix} L_i(X) & X_i B(i) \\ B^*(i) X_i & -\gamma^2 I_m \end{pmatrix} < 0, \quad i \in \mathcal{D}$$

has a positive solution $\tilde{X} = (\tilde{X}_1, \quad \tilde{X}_2, ..., \tilde{X}_d) \in \mathcal{S}_n^d$, *where*
$L_i(X) = A_0^*(i)X_i + X_i A_0(i) + \sum_{k=1}^{r} A_k^*(i) X_i A_k(i) + \sum_{j=1}^{d} q_{ij} X_j + C^*(i)C(i)$.

8 ROBUST STABILIZATION OF STOCHASTIC SYSTEMS WITH MARKOVIAN JUMPING

In this section we shall study the problem of stabilization by state feedback of a linear stochastic system with Markovian jumping with parametric structured uncertainties.

8.1 The Stochastic Version Of The Small Gain Theorem

Using the notation of the previous subsection, we prove:

Theorem 8.1 *Assume that the zero solution of the system (7.4) (for $u = 0$) is exponentially stable in mean square, the number of inputs equals the number of outputs, and*

$$\sup_{\tau>0} ||\mathbf{T}_\tau|| < 1. \tag{8.1}$$

Under these conditions the zero solution of the system

$$dx(t) = \tilde{A}_0(t,\eta(t))x(t)dt + \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t) \tag{8.2}$$

is exponentially stable in mean square with

$$\tilde{A}_0(t,i) = A_0(t,i) + B(t,i)C(t,i).$$

Proof: Using Corollary 7.9 $(i) \Longrightarrow (ii)$ for $\gamma = 1$, we deduce that there exists a C^1 uniform positive function $\tilde{X} : \mathbf{R}_+ \to \mathcal{S}_n^d, \tilde{X}(t) = (\tilde{X}_1(t),...,\tilde{X}_d(t)), \tilde{X}_i(t) > 0$ bounded with is derivatives bounded, which verify the linear differential matrix inequalities. (7.12)

By direct calculus we get

$$\frac{d}{dt}\tilde{X}_i(t) + \tilde{A}_0^*(t,i)\tilde{X}_i(t) + \tilde{X}_i(t)\tilde{A}_0(t,i) + \sum_{k=1}^{r} A_k^*(t,i)\tilde{X}_i(t)A_k(t,i)$$

$$+\sum_{j=1}^{d} q_{ij}\tilde{X}_j(t) + [F(t,i) - C(t,i)]^*[F(t,i) - C(t,i)] < 0, i \in \mathcal{D}, t \geq 0$$

where we denote $\tilde{A}_0(t,i)$ as before, and $F(t,i) = B^*(t,i)X_i(t)$.

Hence

$$\frac{d}{dt}\tilde{X}_i(t)+\tilde{A}_0^*(t,i)\tilde{X}_i(t)+\tilde{X}_i(t)\tilde{A}_0(t,i)+\sum_{k=1}^{r} A_k^*(t,i)\tilde{X}_i(t)A_k(t,i)+\sum_{j=1}^{d} q_{ij}\tilde{X}_j(t) < 0$$

uniformly with respect to $t \geq 0, i \in \mathcal{D}$.

Applying Proposition 4.6 $(v) \Longrightarrow (i)$ we deduce that the zero solution of the system (8.2) is exponentially stable in mean square and the proof is complete.

Remark. Combining Theorem 8.1 and Proposition 7.2 we obtain that the operator $I - \mathbf{T}_{t_0}$ is invertible with bounded inverse $\forall t_0 \geq 0$ and a state space realization of its inverse is:

$$\begin{aligned} dx(t) &= [\tilde{A}_0(t,\eta(t))x(t) + B(t,\eta(t))y(t)]dt \\ &\quad + \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t) \\ u(t) &= C(t,\eta(t))x(t) + y(t). \end{aligned}$$

Let us consider the controlled systems described by:

$$\begin{aligned} dx(t) &= [A_0(t,\eta(t))x(t) + B(t,\eta(t))u(t)]dt + \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t) \\ y(t) &= C(t,\eta(t))x(t) \end{aligned} \tag{8.3}$$

$$\begin{aligned} dx^c(t) &= [A_0^c(t,\eta(t))x^c(t) + B^c(t,\eta(t))u^c(t)]dt + \sum_{k=1}^{r} A_k^c(t,\eta(t))x^c(t)dw_k(t) \\ y^c(t) &= C^c(t,\eta(t))x^c(t) + D^c(t,\eta(t))u^c(t) \end{aligned} \tag{8.4}$$

$x \in \mathbf{R}^n, x^c \in \mathbf{R}^{n_c}$ (are the states), $u \in \mathbf{R}^m, u^c \in \mathbf{R}^p$ (are the inputs), $y \in \mathbf{R}^p, y^c \in \mathbf{R}^m$ (the outputs) and matrix coefficients. When coupling the system (8.3) with the system (8.4) by taking $u(t) = y^c$ and $u^c = y(t)$ we obtain the following closed-loop system:

$$dx_{cl}(t) = A_{0,cl}(t,\eta(t))x_{cl}(t)dt + \sum_{k=1}^{r} A_{k,cl}(t,\eta(t))x_{cl}(t)dw_k(t) \tag{8.5}$$

where

$$A_{0,cl}(t,i) = \begin{pmatrix} A_0(t,i) + B(t,i)D^c(t,i)C(t,i) & B(t,i)C^c(t,i) \\ B^c(t,i)C(t,i) & A^c(t,i) \end{pmatrix}$$

$$A_{k,cl}(t,i) = \begin{pmatrix} A_K(t,i) & 0 \\ 0 & A_k^c(t,i) \end{pmatrix}.$$

The following result is a stochastic version of the Small Gain Theorem [10].

Theorem 8.2 *Assume that*

a) the zero solution of the system (8.3) and the system (8.4) for $u = 0, u^c = 0$ respectively are exponentially stable in mean square.

b) If $\{\mathbf{T}_\tau\}_{\tau\geq 0}, \{\mathbf{T}_\tau^c\}$ are families of input-output operators associated to systems (8.3), (8.4) respectively, we assume

$$\sup_{\tau\geq 0} ||\mathbf{T}_\tau|| < \gamma,$$

$$\sup_{\tau\geq 0} ||\mathbf{T}_\tau^c|| < \gamma^{-1}$$

for some $\gamma > 0$.

Under these assumptions the zero solution of the closed-loop system (8.5) is exponentially stable in mean square.

Proof: Let us consider the controlled system with the input $\hat{u}$ and the output $\hat{y}$:

$$\begin{aligned} d\hat{x}(t) &= [\hat{A}_0(t,\eta(t))\hat{x}(t) + \hat{B}(t,\eta(t))\hat{u}(t)]dt + \sum_{k=1}^{r} \hat{A}_k(t,\eta(t))\hat{x}(t)dw_k(t) \\ \hat{y}(t) &= \hat{C}(t,\eta(t))\hat{x}(t) \end{aligned} \tag{8.6}$$

$$\hat{A}_0(t,i) = \begin{pmatrix} A_0(t,i) & B(t,i)C^c(t,i) \\ 0 & A^c(t,i) \end{pmatrix}, \hat{B}(t,i) = \begin{pmatrix} B(t,i)D^c(t,i) \\ B^c(t,i) \end{pmatrix},$$

$$\hat{C}(t,i) = (C(t,i) \quad 0), \hat{x} = \begin{pmatrix} x \\ x^c \end{pmatrix}. \hat{A}_k(t,i) = A_{k,d}(t,i).$$

It is easy to see by using Theorem 5.1 that the zero solution of the system (8.6) for $\hat{u} = 0$ is exponentially stable in mean square. On the other hand if

$$\hat{\mathbf{T}}_\tau : L^2_{\eta,w}\{[\tau,\infty), \mathbf{R}^p\} \to L^2_{\eta,w}\{[\tau,\infty), \mathbf{R}^p\}$$

is the input-output operator associated to system (8.6) then

$$\hat{\mathbf{T}}_\tau = \mathbf{T}_\tau \mathbf{T}^c_\tau.$$

Hence, from b) we have $\sup_{\tau\geq 0} ||\hat{\mathbf{T}}_\tau|| < 1$.

The conclusion follows applying Theorem 8.1 to system (8.6) and the proof is complete.

If in (8.4) $C^c(t,i) = 0$ for all $t \geq 0, i \in \mathcal{D}$, then the operator $\mathbf{T}^c_\tau$ becomes $(\mathbf{T}^c_\tau u)(t) = D^c(t,\eta(t))u(t)$. In this case the corresponding closed-loop system becomes

$$\begin{aligned} dx(t) = [A_0(t,\eta(t)) + B(t,\eta(t))D^c(t,\eta(t))C(t,\eta(t))]x(t)dt \\ + \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t). \end{aligned} \tag{8.7}$$

Corollary 8.3

(i) *Assume that the system* $(A_0, A_1, ..., A_r; Q)$ *is stable and* $\sup_{\tau\geq 0} ||\mathbf{T}_\tau|| <$ γ

(ii)

$$\sup_{\tau\geq 0} \max_{i\in\mathcal{D}} |D^c(\tau,i)| < \gamma^{-1}. \tag{8.8}$$

Then the zero solution of the system (8.7) is exponentially stable in mean square.

Proof: It is easy to verify that

$$||\mathbf{T}^c_{t_0}|| \leq \sup_{\tau\geq t_0} \max_{i\in\mathcal{D}} |D^c(\tau,i)| \leq \sup_{\tau\geq 0} \max_{i\in\mathcal{D}} |D^c(t,i)| < \gamma^{-1}.$$

The conclusion will follow from Theorem 8.2.

Remark: Since $D^c : \mathbf{R}_+ \to \mathcal{M}^d_{m,p}$, the inequality (8.8) may be written as: $\sup_{\tau\geq 0} |D^c(\tau)| < \gamma^{-1}, |\cdot|$ being the norm in $\mathcal{M}^d_{m,p}$.

8.2 Estimates For Stability Radius

Let us consider the following stochastic linear system with Markovian jumping:

$$dx(t) = [A_0(t,\eta(t)) + B(t,\eta(t))\Delta(t,\eta(t))C(t,\eta(t))]x(t)dt + \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t) \qquad (8.9)$$

where $A_k : \mathbf{R}_+ \to \mathcal{M}^d_n, k = 0,1,...,r, B : \mathbf{R}_+ \to \mathcal{M}^d_{n,m}, C : \mathbf{R}_+ \to \mathcal{M}^d_{p,n}$ are continuous and bounded functions which are supposed to be known, $\Delta : \mathbf{R}_+ \to \mathcal{M}^d_{m,p}$ are continuous functions which are unknown and modelled the uncertainties in the system.

The system (8.9) will be called "the perturbed system" of the following nominal system:

$$dx(t) = A_0(t,\eta(t)x(t)dt + \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t) \qquad (8.10)$$

and the pair $(B(t,i),C(t,i))$ describe the structure of the uncertainties. Let us denote $\mathbf{\Delta}_{m,p}$ the set of the functions $\Delta : \mathbf{R}^+ \to \mathcal{M}^d_{m,p}$ which are countinuous and $\sup_{t\geq 0} |\Delta(t)| < \infty, |\cdot|$ being the norm on $\mathcal{M}^d_{m,p}$.

On $\mathbf{\Delta}_{mp}$ we define the norm $||\Delta|| = \sup_{t\geq 0} |\Delta(t)|$.

If the trivial solution of the nominal system (8.10) is exponentially stable in mean square, it is natural to ask if the trivial solution of the perturbed system (8.9) is still exponentially stable in mean square. Thus we can define the stability radius for the perturbed system (8.9) $\rho(A,B,C;Q) = \inf\{\rho > 0 | \exists\Delta \in \mathbf{\Delta}_{m,p}$ with $||\Delta|| = \rho$ such that the zero solution of the system (8.9) is not exponentially stable in mean square.}

Based on the small gain theorem we shall obtain a lower bound of the stability radius $\rho(A,B,C;Q)$. In [9, 28] respectively, were provide estimations of stability radius for the case when the nominal system is subject to the white noise type perturbations and Markov perturbation respectively.

To the perturbed system (8.9) we associate the following fictitions controlled system:

$$\begin{aligned} dx(t) &= [A_0(t,\eta(t))x(t) + B(t,\eta(t))u(t)]dt + \\ &+\sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t) \\ y(t) &= C(t,\eta(t))x(t) \end{aligned} \tag{8.11}$$

and denote $\{\tilde{\mathbf{T}}_\tau\}_{\tau\geq 0}$ the family of the input-output operators corresponding to the system (8.11).

Theorem 8.4 *Assume that the zero solution of the nominal system (8.10) is exponentially stable in mean square. Then the stability radius of the perturbed system (8.9) satisfies*

$$\rho(A,B,C;Q) \geq (\sup_{\tau\geq 0} ||\tilde{\mathbf{T}}_\tau||)^{-1}.$$

Proof: We denote $\gamma_0 = \sup_{\tau\geq 0}||\tilde{\mathbf{T}}_\tau||$. Let $\Delta \in \mathbf{\Delta}_{m,p}$ be arbitrary with $||\Delta|| < \gamma_0^{-1}$.

Consider the linear bounded operator

$$\mathbf{T}_{\Delta,\tau} : L^2_{\eta,w}([\tau,\infty),\mathbf{R}^p) \to L^2_{\eta,w}([\tau,\infty),\mathbf{R}^m)$$

$$(\mathbf{T}_{\Delta,\tau}v)(t) = \Delta(t,\eta(t))v(t), t \geq \tau.$$

We have

$$||\mathbf{T}_{\Delta,\tau}|| \leq ||\Delta||$$

. Consider the auxiliary system

$$\begin{aligned} dx(t) &= [A_0(t,\eta(t))x(t) + B(t,\eta(t))\Delta(t,\eta(t))v(t)]dt \\ &+\sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t) \\ y(t) &= C(t,\eta(t))x(t) \end{aligned} \tag{8.12}$$

having the inputs $v \in \mathbf{R}^p$ and outputs $y \in \mathbf{R}^p$.

It is easy to see that the input output operator of the system (8.12) on the interval $[\tau,\infty)$ is $\mathbf{T}_\tau = \tilde{\mathbf{T}}_\tau \cdot \mathbf{T}_{\delta,\tau}$.

We have

$$\sup_{\tau\geq 0}||\mathbf{T}_\tau|| \leq \sup_{\tau\geq 0}||\tilde{\mathbf{T}}_\tau|| \sup_{\tau\geq 0}||\mathbf{T}_{\Delta,\tau}|| < 1.$$

Applying Theorem 8.1 to the system (8.12) we deduce that the zero solution of the perturbed system

$$\begin{aligned} dx(t) &= [A_0(t,\eta(t)) + B(t,\eta(t))\Delta(t,\eta(t))C(t,\eta(t))]x(t)dt \\ &+ \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t) \end{aligned}$$

is exponentially stable in mean square.

Thus we conclude that

$$||\Delta|| < \rho(A, B, C; Q).$$

Since Δ is arbitrary with $||\Delta < \gamma_0^{-1}$ it will follow that $\rho(A, B, C; Q) \geq \gamma_0^{-1}$ and the proof is complete.

8.3 Robust Stabilization By State Feedback Of a Linear Stochastic System With Markovian Jumping

Let us consider the perturbed system:

$$\begin{aligned} dx(t) &= \{[A_0(t,\eta(t)) + B_1(t,\eta(t))\Delta(t,\eta(t))C_1(t,\eta(t))]x(t) \\ &+ B_2(t,\eta(t))u(t)\}dt + \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t) \end{aligned} \quad (8.13)$$

where $u \in \mathbf{R}^{m_2}$ is a control and $x \in \mathbf{R}^n$ is the state, $A_k : \mathbf{R}_+ \to \mathcal{M}^d_n, k = 0, 1, ...r, B_j : \mathbf{R}_+ \to \mathcal{M}^d_{n,m_j}, j = 1, 2, C_1 : \mathbf{R}_+ \to \mathcal{M}^d_{p_1,n}$ are bounded and continuous functions which are supposed to be known and $\Delta : \mathbf{R}_+ \to \mathcal{M}^d_{m_1,p_1}$ are continuous functions which are unknown. The function Δ modelled the parametric uncertainties of the nominal system:

$$dx(t) = [A_0(t,\eta(t))x(t) + B_2(t,\eta(t))u(t)]dt + \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t).$$

The problem which we wish to solve consists in finding a stabilizing feedback gain $F : \mathbf{R}_+ \to \mathcal{M}^d_{m_2,n}$ which is bounded and continuous function such that the control $u(t) = F(t,\eta(t))x(t)$ stabilizes the perturbed system (8.13) for all perturbation Δ with $||\Delta|| < \rho$ for a prescribed level $\rho > 0$.

The corresponding closed loop system is:

$$dx(t) = [A_0(t,\eta(t)) + B_2(t,\eta(t))F(t,\eta(t)) + B_1(t,\eta(t)\Delta(t,\eta(t))C_1(t,\eta(t))]x(t)dt$$

$$+\sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t).$$

To this system we associate the following auxiliary system:

$$\begin{aligned} dx(t) &= \{[A_0(t,\eta(t)) + B_2(t,\eta(t))F(t,\eta(t)]x(t) + B_1(t,\eta(t))u_1(t)\}dt \\ &\quad + \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t) \\ y(t) &= \begin{pmatrix} C_1(t,\eta(t)) \\ F(t,\eta(t)) \end{pmatrix} x(t) \end{aligned} \tag{8.14}$$

and denote

$$\mathbf{T}_\tau^F : L^2_{\eta,w}([\tau,\infty),\mathbf{R}^{m_1}) \to L^2_{\eta,w}([\tau,\infty),\mathbf{R}^{m_2+p_1})$$

the input-output operator on the interval $[\tau,\infty)$ defined by the auxiliary system (8.14).

Directly, by Theorem 8.4 (taking instead of Δ, $\hat{\Delta} = (\Delta \ \ 0)$, we obtain:

Proposition 8.5 *Suppose that $(A_0 + B_2F, A_1, ..., A_r; Q)$ is stable. If*

$$\sup_{\tau\geq 0} ||\mathbf{T}_\tau^F|| < \rho^{-1} \tag{8.15}$$

then the control $u(t) = F(t,\eta(t))x(t)$ stabilizes the perturbed system (8.13) for all Δ with $||\Delta|| < \rho$.

The next result provides a necessary and sufficient condition which assures the existence of a stabilizing feedback gain which verifies the inequality (8.15).

Theorem 8.6 *The following are equivalent:*

(i) There exists a stabilizing feedback gain $F : \mathbf{R}_+ \to \mathcal{M}^d_{m_2,n}$ bounded and continuous which satisfies (8.15).

(ii) There exists a C^1-function $X : \mathbf{R}_+ \to \mathcal{S}^d_n$ uniform positive which is bounded with bounded derivative and solves the following system of differential inequalities:

$$\begin{aligned} &\frac{d}{dt}X(t,i) + A_0^*(t,i)X(t,i) + X(t,i)A_0(t,i) \\ &\quad + \sum_{k=1}^{r} A_k^*(t,i)X(t,i)A_k(t,i) + \sum_{j=1}^{d} q_{ij}X(t,j) \\ &+X(t,i)[\rho^2 B_1(t,i)B_1^*(t,i) - B_2(t,i)B_2^*(t,i)]X(t,i) + C_1^*(t,i)C_1(t,i) < 0 \end{aligned} \tag{8.16}$$

uniformly with respect to $t \geq 0, i \in \mathcal{D}$.

Proof: (i)$\Longrightarrow$(ii)

Applying Corollary 7.9 to the fictious system (8.14) and the parameter $\gamma = \rho^{-1}$ we deduce that there exists a C_1 function $X : \mathbf{R}_+ \to S_n^d$ uniform positive which is bounded with bounded derivative, verifying the system of differential inequalities:

$$\frac{d}{dt}X(t,i) + [A_0(t,i) + B_2(t,i)F(t,i)]^* X(t,i) + X(t,i)[A_0(t,i) + B_2(t,i)F(t,i)]$$

$$+\sum_{k=1}^{r} A_k^*(t,i)X(t,i)A_k(t,i) + \sum_{j-1}^{d} q_{ij}X(t,j) + \rho^2 X(t,i)B_1(t,i)B_1^*(t,i)X(t,i)$$

$$+C_1^*(t,i)C_1(t,i) + F^*(t,i)F(t,i) < 0$$

uniformly with respect to $t \geq 0, i \in \mathcal{D}$.

Further we may write:

$$\frac{d}{dt}X(t,i) + A_0^*(t,i)X(t,i) + X(t,i)A_0(t,i) + \sum_{k=1}^{r} A_k^*(t,i)X(t,i)A_k(t,i)$$

$$+\sum_{j=1}^{d} q_{ij}X(t,j) + X(t,i)[\rho^2 B_1(t,i)B_1^*(t,i) - B_2(t,i)B_2^*(t,i)]X(t,i)$$

$$+C_1^*(t,i)C_1(t,i) + (F(t,i) + B_2^*(t,i)X(t,i))^*(F(t,i) + B_2^*(t,i)X(t,i)) < 0$$

which shows that the functions $X(t,i)$ solve the system (8.16).

$(ii) \Longrightarrow (i)$

Let $\tilde{X}(t) = (\tilde{X}(t,1), \tilde{X}(t,2), ..., \tilde{X}(t,d))$ a uniform positive solution of the system (8.16) which is bounded with its derivative. Let $\tilde{F}(t,i) = -B_2^*(t,i)X(t,i)$, $t \geq 0, i \in \mathcal{D}$.

It is easy to see that we have:

$$\frac{d}{dt}\tilde{X}(t,i) + [A_0(t,i) + B_2(t,i)\tilde{F}(t,i)]^* \tilde{X}(t,i) + \tilde{X}(t,i)[A_0(t,i) + B_2(t,i)\tilde{F}(t,i)]$$

$$+\sum_{k=1}^{r} A_k^*(t,i)\tilde{X}(t,i)A_k(t,i) + \sum_{j=1}^{d} q_{ij}\tilde{X}(t,j) + \rho^2 \tilde{X}(t,i)B_1(t,i)B_1^*(t,i)\tilde{X}(t,i)$$

$$+C_1^*(t,i)C_1(t,i) + \tilde{F}^*(t,i)\tilde{F}(t,i) < 0$$

uniformly with respect to $t \geq 0, i \in \mathcal{D}$.

Applying Corollary 7.9 (ii)$\Longrightarrow$(i) we deduce that $\tilde{F}(t) = (\tilde{F}(t,1), \tilde{F}(t,2), ...,$ $\tilde{F}(t,d))$ is a stabilizing feedback gain and additionally (8.15) holds. Then the proof is complete.

Corollary 8.7 *If $A_k(t+\theta) = A_k(t), k = 0,1,...,r, B_j(t+\theta) = B_j(t), j = 1,2, C_1(t+\theta) = C_1(t)$ for all $t \geq 0$ then the following are equivalent:*

(i) There exists a stabilizing feedback gain $F(t) = (F(t,1), F(t,2),..., F(t,d))$ which is a θ-periodic function and in addition (8.15) be fulfilled.

(ii) There exists a C^1 function $X : \mathbf{R}_+ \to S_n^d$ which is uniform positive and solves the system of differential inequalities (8.16).

Corollary 8.8 *Assume that : $A_k(t,i) = A_k(i), k = 0,1,...,r$, $B_j(t,i) = B_j(i), C_1(t,i) = C_1(i)$ for all $t \geq 0, i \in \mathcal{D}$.*

Under these conditions the following are equivalent:

(i)There exists a stabilizing feedback gain $F = (F_1, F_2, ..., F_d) \in \mathcal{M}^d_{m_2,n}$ such that

$$||\mathbf{T}_0^F|| < \rho^{-1}.$$

(ii) The system of algebraic nonlinear inequlities

$$A_0^*(i)X(i) + X(i)A_0(i) + \sum_{k=1}^{r} A_k^*(i)X(i)A_k(i) + \sum_{j=1}^{d} q_{ij}X(j) + X(i)[\rho^2 B_1(i)B_1^*(i)$$

$$-B_2(i)B_2(i)^*]X(i) + C_1^*(i)C_1(i) < 0$$

$i \in \mathcal{D}$ has a positive solution $X = (X(1), \quad X(2), \quad ..., X(d))$.

References

[1] L. Arnold, Stochastic differential equations; theory and applications, Wiley, New-York, 1974

[2] M. Athans, Command and control (C2) theory: A challenge to control science, *IEEE Trans. Automat. Control*, **32**, (2), (1987), 286-293.

[3] E. K. Boukas, Q. Zhang, G. Yin, Robust production and maintenance planning in stochastic manufacturing systems, *IEEE Trans. Automat. Control*, **40**, (6), (1995), 1098-1102.

[4] H. Bunke, Gewöhnliche Differential-gleichungen mit zufälligen Parametern, Academie Verlag, Berlin, 1972

[5] R.F. Curtain, Stability of stochastic dynamical systems, *Lecture Notes in Math.*, vol 294, Springer Verlag, 1972

[6] J.L.Doob, Stochastic processes, Wiley, New-York, 1967

[7] V.Dragan, T. Morozan, Global solutions to a game-theoretic Riccati equation of stochastic control, *Jornal of Diff. Eq.*, **138**, no.2, (1997), 328-350

[8] V.Dragan, T. Morozan, Game-theoretic coupled Riccati equations associated to controlled linear differential systems with jump Markov perturbations, *Stochastic Analysis Appl.*, (to appear).

[9] V. Dragan, A. Halanay, T. Morozan, Optimal stabilizing compensator for linear systems with stable dependent noise, *Stochastic Analysis and Applications*, **10**, 5, (1992), 557-572.

[10] V.Dragan, A. Halanay, A.Stoica, A small gain theorem for linear stochastic systems, *System and Control Letters*, 30, (1997), 243-251.

[11] V. Dragan, A. Halanay, A. Stoica, The γ-attenuation problem for systems with state dependent noise - *Stochastic Analysis and Applications*, **17**, 3, (1999), 395-404.

[12] A. Friedman, Stochastic differential equations and applications, Academic Press, vol. I, 1975.

[13] A. Halanay, Differential equations, stability, oscillations, time lag, Academic Press, New York, 1966.

[14] A. Haurie, A. Leizarowitz, Overtaking optimal regulation and tracking of piecewise diffusion linear systems, *SIAM J. Control and Optimiz.*, **30**, 4, (1992), 816-837.

[15] U.G.Hausmann, Optimal stationary control with state and control dependent noise, *SIAM J. Control and Optimization*, **9**, (1971), 184-198.

[16] D. Hinrichsen, A.J. Pritchard, Stochastic H^{∞}, *SIAM J. Control and Optimization*, **36**, (1998), 1504-1538.

[17] D. Hinrichsen, A.J. Pritchard, Stability radii of systems with stochastic uncertainty and their optimization by output feedback, *SIAM J. Control and Optimization*, 34, (1996), 1972-1998.

[18] A. Ichikawa, Dynamic programming approach to stochastic evolution equations, *SIAM Journal Control and Optimization*, **17**, 1, (1979), 153-174.

[19] Y. Ji, H. J. Chizeck, Controllability, stabilizability and continuous time Markovian jump linear quadratic control, *IEEE Transactions on Automatic Control*, **35**, 7, (1990), 777-788.

[20] R.Z. Khasminskii, Stochastic stability of differential equations, Sÿthoff and Noordhoff Alpen aan den Ryn, 1980

[21] H. Kushner, Stochastic stability and control, Academic Press, New-York, 1967

[22] G. S. Ladde, V. Lakshmikantham, Random differential inequalities, Academic Press, New-York, 1980

[23] M. Lewin, On the boundedness recurrence and stability of solutions of an Itô equation perturbed by a Markov chain, *Stochastic Analysis and Appl.*, **4**, 4, (1986), 431-487.

[24] K. A. Loparo, Stochastic stability of coupled linear systems: a survey of methods and results, *Stochastic Analysis Appl.*, **2**, (1984), 193-228.

[25] D.D. Moerder, N. Halyo, J.R. Braussard, A.K. Caglayan, Application of precomputed control laws in a reconfigurable aircraft flight control system, *J. of Guidance, Control and Dynamics*, **12**, (3), (1989), 325-333.

[26] T. Morozan, Optimal stationary control for dynamic systems with MArkov perturbations, *Stochastic Analisys and Applications*, **1**, 3, (1983), 299-323.

[27] T. Morozan, Stability and control for linear system with jump Markov perturbation, *Stochastic Analisys and Applications*, **13**, 1, (1995), 91-110.

[28] T. Morozan, Parametrized Riccati equations for controlled linear differential systems with jump Markov perturbations, *Stochastic Analisys and Applications*, **16**, 4, (1998), 661-682.

[29] T. Morozan, Parametrized Riccati equation and input-output operators for time-varying stochastic differential equations with state dependent noise, *Studii si Cercetari Matematice*, nr.1, 1999.

[30] Z. Pan, T. Basar, H^{∞}-control of Markovian jump systems and solutions to associated piece-wise deterministic differential games, *Annals of the International Society of Dynamic Games, Birkäuser*, **3**, (1995), 61-94

[31] D. Petkovski, Multivariable control system design: a case study of robust control of nuclear power plants, *Fault Detection and Reliability*, **9**, (1987), 239-246.

[32] D.D. Sworder, R.O. Rogers, An LQ- solution to a control problem associated with a solar thermal central receiver,*IEEE Trans. Automat. Control*, **28**(8), (1983), 971-978.

[33] W.H. Wonham, Random diferential equations in control theory, *Probabilistic Methods in Applied Math.*,**2**, (A.T.Barucha-Reid, Ed.) Academic Press, New-York, (1970), 131-212.

[34] J.L. Willems, J.C. Willems, Feedback stabilizability for stochastic systems with state and control dependent noise, *Automatica*, **12**, (1976), 277-283.

[35] A.S. Willsky, A survey of design methods for failure detection in dynamic systems, *Automatica*, **12**(5), (1976), 601-611.