INSTITUTUL DE MATEMATICA
AL ACADEMIEI ROMANE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

LINEAR QUADRATIC CONTROL AND TRACKING PROBLEMS FOR TIME–VARYING STOCHASTIC DIFFERENTIAL SYSTEMS PERTURBED BY A MARKOV CHAIN

by

T. MOROZAN

Preprint nr. 20/1999

BUCURESTI

LINEAR QUADRATIC CONTROL AND TRACKING PROBLEMS FOR TIME–VARYING STOCHASTIC DIFFERENTIAL SYSTEMS PERTURBED BY A MARKOV CHAIN

by

T. MOROZAN*

December, 1999

*Institute of Mathematics of the Romanian Academy, P.O.Box 1–764, RO–70700, Bucharest, Romania

LINEAR QUADRATIC CONTROL AND TRACKING PROBLEMS FOR TIME-VARYING STOCHASTIC DIFFERENTIAL SYSTEMS PERTURBED BY A MARKOV CHAIN

T. MOROZAN

Linear quadratic control and tracking problems, under a wide class of admissible controls, for linear time-varying stochastic systems described by differential equations with Markovian jumping and multiplicative white noise are discussed. Some results concerning stochastic observability for such systems are also given.

1. NOTATIONS

The following notations will be used throughout this paper R^n is the real n-dimensional space. $\mathbb{R}_+$ is the set of nonnegative real numbers.

If X is a matrix (or a vector) X^* is the transpose of X; $|A|$ is the operator norm of the matrix A. I_n is the identity matrix in R^n.

$R^{n\times m}$ is the set of all real $n \times m$ matrices.

$H > 0 (H \geq 0)$ means that H is a symmetric positive (semi)definite matrix.

By $\mathcal{S}_n$ we denote the space of all $n \times n$ symmetric matrices.

In this paper $\mathcal{D} = \{1, 2, \dots, d\}$. By $\mathcal{S}_n^d$ we denote the space of all $H = (H(1), \dots, H(d))$ with $H(i) \in \mathcal{S}_n$.

$\mathcal{S}_n^d$ is a real Hilbert space with the inner product $< H, G >= \sum\limits_{i=1}^{d} Tr(H(i)G(i))$, where TrA is the trace of A.

If $H \in \mathcal{S}_n^d, |H| = \max\{|H(i)|; i \in \mathcal{D}\}$ and $H > 0 (H \geq 0)$ means that $H(i) > 0 (H(i) \geq 0)$ for all $i \in \mathcal{D}$. Obviously $|H|^2 \leq < H, H > \leq nd|H|^2, H \in \mathcal{S}_n^d$.

By J we denote the element in $\mathcal{S}_n^d$ with $J(i) = I_n$ for all $i \in \mathcal{D}$. If $T : \mathcal{S}_n^d \to \mathcal{S}_n^d$ is a linear operator, then $\|T\|$ is the operator norm of T induced by the norm $|\cdot|$ on $\mathcal{S}_n^d$. If T is a linear operator on $\mathcal{S}_n^d, T^*$ stands for its adjoint operator. A linear operator $T : \mathcal{S}_n^d \to \mathcal{S}_n^d$ is called positive (and we write $T \geq 0$)if $H \geq 0$ implies $TH \geq 0$. It is easy to see that if T is a linear positive operator then $\|T\| = |TJ|$. If $H \in \mathcal{S}_n^d$ sometimes we shall write H_i for $H(i)$ and if $M : \mathbb{R}_+ \to \mathcal{S}_n^d$ we shall write $M_i(t)$ or $M(t,i)$ for $M(t)(i), i \in \mathcal{D}$.

A function $M : \mathbb{R}_+ \times \mathcal{D} \to R^{n \times m}$ is said to be continuous (bounded) on $\mathbb{R}_+$ if for every $i \in \mathcal{D}$ the function $M(\cdot, i)$ is continuous (bounded) on $\mathbb{R}_+$.

A function $M : \mathbb{R}_+ \times \mathcal{D} \to \mathcal{S}_m$ is said to be uniformly positive definite if there exists $\delta > 0$ such that $M(t,i) \geq \delta I_m$ for all $t \in \mathbb{R}_+$ and $i \in \mathcal{D}$.

A function $M : \mathbb{R}_+ \to \mathcal{S}_m^d$ is said to be uniformly positive definite if the function $\widetilde{M} : \mathbb{R}_+ \times \mathcal{D} \to \mathcal{S}_m$ defined by $\widetilde{M}(t,i) = M_i(t)$ is uniformly positive definite.

Throughout this paper $\{\Omega, \mathcal{F}, \mathcal{P}\}$ is a given probability space; the argument $\omega \in \Omega$ will not be written.

E denotes expectation and $E[x|\eta(t) = i]$ stands for conditional expectation on the event $\eta(t) = i$.

2. PRELIMINARIES

Throughout this paper $w(t) = (w_1(t), \ldots, w_r(t))^*, t \geq 0$ is a standard r-dimensional Wiener process (see [7]), and $\eta(t), t \geq 0$ is a right continuous homogeneous Markov chain with state space the set $\mathcal{D}$ and the probability transition matrix $P(t) = [p_{ij}(t)] = e^{Qt}, t > 0$; here $Q = [q_{ij}]$ with $\sum_{j=1}^{d} q_{ij} = 0, i \in \mathcal{D}$ and $q_{ij} \geq 0$ if $i \neq j$. It is known [2] that $\lim_{t \to \infty} P(t)$ exists and $p_{ii}(t) > 0$ for all $t > 0$ and $i \in \mathcal{D}$. We shall assume that $\pi_i = \mathcal{P}\{\eta(0) = i\} > 0$ for all i. Thus, from the elementary inequality $\mathcal{P}\{\eta(t) = i\} \geq \pi_i p_{ii}(t)$ it follows that $\mathcal{P}\{\eta(t) = i\} > 0$ for all $t > 0$ and $i \in \mathcal{D}$.

For each $t \geq 0$ we denote $\mathcal{F}_t \subset \mathcal{F}$ the smallest σ-algebra containing all $M \in \mathcal{F}$ with $\mathcal{P}(M) = 0$ and with respect to which all functions $w(s), 0 \leq s \leq t$ are measurable.

By $\mathcal{G}_t, t \geq 0$ we denote σ-algebra generated by $\eta(s), 0 \leq s \leq t$.

Throughout this paper we assume that for every $t \geq 0$ the σ-algebra$\mathcal{F}_t$ is independent of the σ-algebra $\mathcal{G}_t$.

$\mathcal{H}_t$ stands for the smallest σ-algebra containing σ-algebras $\mathcal{F}_t$ and $\mathcal{G}_t$.

By $L^2_{\eta,w}([t_0,\infty), R^m), t_0 \geq 0$ we denote the space of all measurable functions $u : [t_0,\infty) \times \Omega \to R^m$ with the properties: $u(t)$ is $\mathcal{H}_t$ measurable ($\mathcal{H}_t$-adapted) for every $t \geq t_0$ and $E\int_{t_0}^{\infty} |u(t)|^2 dt < \infty$.

The space $L^2_{\eta,w}([t_0,T], R^m), 0 \leq t_0 < T$, is defined in a similar way.

By $\mathcal{U}(t_0,m)$ we denote the space of all functions $u : [t_0,\infty) \times \Omega \to R^m$ with the property that $u_T \in L^2_{\eta,w}([t_0,T], R^m)$ for all $T > t_0$, where u_T is the restriction of u to the set $[t_0,T] \times \Omega$.

Further, let $0 \leq t_0 < T$ and $\sigma : [t_0,T] \to R^{n\times r}$ be a matrix valued function with the columns $\sigma_1(t),\ldots,\sigma_r(t), \sigma_k \in L^2_{\eta,w}([t_0,T], R^n), 1 \leq k \leq r$. The stochastic integral $z(t) = \int_{t_0}^{t} \sigma(s)dw(s), t \in [t_0,T]$ is well-defined (see[7]) because the σ-algebras $\mathcal{H}_t, t \geq 0$ have the properties used in the theory of stochastic Itô integral, namely: $\mathcal{H}_{t_1} \subseteq \mathcal{H}_{t_2}$ if $t_1 < t_2, \mathcal{F}_t \subset \mathcal{H}_t$ and $\mathcal{H}_t$ is independent of the σ-algebra generated by $\{w(t+h) - w(t), h > 0\}$ for every $t \geq 0$.

Thus (see [7]), $z(t)$ is a continuous process, $z \in L^2_{\eta,w}([t_0,T], R^n)$ and

$$E[|z(t)|^2|\eta(t_0) = i] = \sum_{j=1}^{r} E[\int_{t_0}^{t} |\sigma_j(s)|^2 ds|\eta(t_0) = i], \tag{1}$$

for all $t \in [t_0,T]$ and $i \in \mathcal{D}$.

Now, let us consider $a \in L^2_{\eta,w}([t_0,T], R^n)$. It will follow that

$$x(t) = x_0 + \int_{t_0}^{t} a(s)ds + \int_{t_0}^{t} \sigma(s)dw(s),\ t \in [t_0,T], \tag{2}$$

(with $x_0 \in R^n$) is a continuous process and $x \in L^2_{\eta,w}([t_0,T], R^n)$. If $x(t)$ verifies (2) we write

$$dx(t) = a(t)dt + \sigma(t)dw(t),\ t \in [t_0,T],\ x(t_0) = x_0$$

The following result proved in [3] will be often used in this paper.

PROPOSITION 1. *(A Itô type formula) Let a and σ be as above and let $v(t,x,i) = x^*K(t,i)x + 2k^*(t,i)x + k_0(t,i), t \in [t_0,T], x \in R^n, i \in \mathcal{D}$, where*

$K : [t_0, T] \times \mathcal{D} \to \mathcal{S}_n, k : [t_0, T] \times \mathcal{D} \to R^n$ and $k_0 : [t_0, T] \times \mathcal{D} \to \mathbb{R}$ are C^1-functions with respect to t.

Then we have

$$E[v(t,x(t),\eta(t))|\eta(t_0)=i] - v(t_0,x_0,i)\cdot$$
$$= E[\int_{t_0}^{t}\{x^*(s)K'(s,\eta(s))x(s) + 2[k'(s,\eta(s))]^*x(s) + k_0'(s,\eta(s))$$
$$+2[x^*(s)K(s,\eta(s)) + k^*(s,\eta(s))]a(s) + Tr(\sigma^*(s)K(s,\eta(s))\sigma(s))$$
$$+\textstyle\sum_{j=1}^{d} v(s,x(s),j)q_{\eta(s)j}\}ds|\eta(t_0)=i]$$

for all $t \in [t_0, T], i \in \mathcal{D}$ where $x(t), t \in [t_0, T]$ verifies (2).

3. SOME STABILITY RESULTS

Consider the following linear stochastic systems

$$dx(t) = A_0(t,\eta(t))x(t)dt + \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t), \; t \geq 0 \tag{3}$$

$$\frac{dx(t)}{dt} = A_0(t,\eta(t))x(t) \qquad t \geq 0 \tag{4}$$

where $A_k : R_+ \times \mathcal{D} \to R^{n\times n}, 0 \leq k \leq r$ are continuous on R_+

By $X(t,s), t \geq s$ we denote the fundamental random matrix of solutions associated to system (3) and by $\widetilde{X}(t,s)$ we denote the fundamental (random) matrix of solutions associated to system (4). Obviously if $A_k(t,i) = 0, t \in \mathbb{R}_+, i \in \mathcal{D}, 1 \leq k \leq r$ then $X(t,s)$ becomes $\widetilde{X}(t,s)$.

On the Hilbert space $\mathcal{S}_n^d$ we define the following linear operators $\mathcal{L}_k(t), t \geq 0, 1 \leq k \leq 3$, by

$$(\mathcal{L}_1(t)H)(i) = \sum_{k=1}^{r} A_k(t,i)H(i)A_k^*(t,i),$$

$$(\mathcal{L}_2(t)H)(i) = (A_0(t,i) + \frac{1}{2}q_{ii}I_n)H(i) + H(i)(A_0^*(t,i) + \frac{1}{2}q_{ii}I_n)$$
$$(\mathcal{L}_3(t)H)(i) = \sum_{j\neq i} q_{ji}H(j), \quad i \in \mathcal{D}, \quad H \in \mathcal{S}_n^d$$

Obviously $\mathcal{L}_1(t) \geq 0, \mathcal{L}_3(t) \geq 0, t \geq 0$

Let us define

$$\widehat{\mathcal{L}}(t) = \mathcal{L}_2(t) + \mathcal{L}_3(t), \mathcal{L}(t) = \widehat{\mathcal{L}}(t) + \mathcal{L}_1(t), t \geq 0 \tag{5}$$

It is easy to verify that

$$(\mathcal{L}^*(t)H)(i) = A_0^*(t,i)H(i) + H(i)A_0(t,i) + \sum_{k=1}^r A_k^*(t,i)H(i)A_k(t,i)$$
$$+\sum_{j=1}^d q_{ij}H(j),\ i \in \mathcal{D}, H \in \mathcal{S}_n^d.$$

On the space $\mathcal{S}_n^d$ we consider the linear differential equation

$$\frac{dS(t)}{dt} = \mathcal{L}(t)S(t), t \geq 0 \tag{6}$$

By $S(t, t_0, H)$ we denote the solution of (6) with $S(t_0, t_0, H) = H, \quad H \in \mathcal{S}_n^d$.

By $\mathcal{T}(t, t_0)$ we denote the linear evolution operator on $\mathcal{S}_n^d$ associated with the equation (6), i.e. $\mathcal{T}(t, t_0)H = S(t, t_0, H), H \in \mathcal{S}_n^d$.

By $\widehat{\mathcal{T}}(t, t_0)$ and $\mathcal{T}_2(t, t_0)$ we denote the linear evolution operators on $\mathcal{S}_n^d$ associated to linear operators $\widehat{\mathcal{L}}(t)$ and $\mathcal{L}_2(t)$ respectively.

It is obvious that if $A_k(t, i) = 0,\ t \geq 0, i \in \mathcal{D}$ and $1 \leq k \leq r$ then $\mathcal{T}(t, t_0)$ becomes $\widehat{\mathcal{T}}(t, t_0)$.

It is easy to show that: $\mathcal{T}(t, s)\mathcal{T}(s, t_0) = \mathcal{T}(t, t_0), \mathcal{T}(t, s) = (\mathcal{T}(s, t))^{-1}$, $\mathcal{T}(s, s) = \widetilde{J}(\widetilde{J}$ being the identity operator on $\mathcal{S}_n^d)$, and

$$\begin{aligned} \frac{d}{dt}\mathcal{T}(t, s) = \mathcal{L}(t)\mathcal{T}(t, s), \frac{d}{dt}\mathcal{T}^*(t, s) = \mathcal{T}^*(t, s)\mathcal{L}^*(t), \\ \frac{d}{dt}\mathcal{T}^*(s, t) = -\mathcal{L}^*(t)\mathcal{T}^*(s, t), \quad t \geq 0 \end{aligned} \tag{7}$$

Also we check easily that

$$(\mathcal{T}_2^*(t, t_0)H)(i) = \widetilde{X}_i^*(t, t_0)H(i)\widetilde{X}_i(t, t_0), i \in \mathcal{D}, H \in \mathcal{S}_n^d \tag{8}$$

where $\widetilde{X}_i(t, t_0)$ is the fundamental matrix of solutions associated with the linear deterministic system

$$\frac{dx}{dt} = (A_0(t, i) + \frac{1}{2}q_{ii}I_n)x(t), \quad i \in \mathcal{D}, t \geq 0 \tag{9}$$

and

$$\widetilde{X}_i(t, t_0) = e^{\frac{1}{2}q_{ii}(t-t_0)}X_i(t, t_0) \tag{10}$$

where $X_i(t, t_0)$ is the fundamental matrix of solutions associated with the linear deterministic system

$$\frac{dx(t)}{dt} = A_0(t,i)x(t), \quad i \in \mathcal{D}. \tag{11}$$

From (5) it follows easily that

$$\mathcal{T}(t,t_0) = \widehat{\mathcal{T}}(t,t_0) + \int_{t_0}^{t} \widehat{\mathcal{T}}(t,s)\mathcal{L}_1(s)\mathcal{T}(s,t_0)ds, \quad t \geq t_0$$

$$\widehat{\mathcal{T}}(t,t_0) = \mathcal{T}_2(t,t_0) + \int_{t_0}^{t} \mathcal{T}_2(t,s)\mathcal{L}_3(s)\widehat{\mathcal{T}}(s,t_0)ds, \quad t \geq t_0$$

Hence

$$\mathcal{T}^*(t,t_0) = \widehat{\mathcal{T}}^*(t,t_0) + \int_{t_0}^{t} \mathcal{T}^*(s,t_0)\mathcal{L}_1^*(s)\widehat{\mathcal{T}}^*(t,s)ds \tag{12}$$

$$\widehat{\mathcal{T}}^*(t,t_0) = \mathcal{T}_2^*(t,t_0) + \int_{t_0}^{t} \widehat{\mathcal{T}}^*(s,t_0)\mathcal{L}_3^*(s)\mathcal{T}_2^*(t,s)ds \tag{13}$$

The next result has been proved in [3]

PROPOSITION 2. *We have*

$$(\mathcal{T}^*(t,t_0)H)(i) = E[X^*(t,t_0)H(\eta(t))X(t,t_0)|\eta(t_0) = i]$$

for all $t \geq t_0, i \in \mathcal{D}$ *and* $H \in \mathcal{S}_n^d$.

From Proposition 2 it follows that $\mathcal{T}^*(t,t_0) \geq 0, \widehat{\mathcal{T}}^*(t,t_0) \geq 0$ for all $t \geq t_0$.

Thus, since $\mathcal{L}_1^*(s) \geq 0, \mathcal{L}_3^*(s) \geq 0$, from (8), (12) and (13) we get

$$\mathcal{T}^*(t,t_0) \geq \widehat{\mathcal{T}}^*(t,t_0) \geq \mathcal{T}_2^*(t,t_0) \geq 0, \qquad t \geq t_0 \tag{14}$$

$$\|\mathcal{T}^*(t,t_0)\| \geq \|\widehat{\mathcal{T}}^*(t,t_0)\| \geq \|\mathcal{T}_2^*(t,t_0)\|, \qquad t \geq t_0 \tag{15}$$

In the time-invariant case i.e. $A_k(t,i) = A_k(i), t \geq 0, 0 \leq k \leq r, i \in \mathcal{D}$, the linear operators $\mathcal{L}(t) = \mathcal{L}, \widehat{\mathcal{L}}(t) = \widehat{\mathcal{L}}$ do not depend upon t and therefore

$$\mathcal{T}^*(t,t_0) = e^{\mathcal{L}^*(t-t_0)}, \widehat{\mathcal{T}}^*(t,t_0) = e^{\widehat{\mathcal{L}}^*(t-t_0)} \tag{16}$$

Definition 1. a) We say that the trivial solution of system (3) is exponentially stable in mean square if there exist $\beta \geq 1$ and $\alpha > 0$ such that

$$E[|X(t,t_0)x_0|^2|\eta(t_0) = i] \leq \beta e^{-\alpha(t-t_0)}|x_0|^2$$

for all $t \geq t_0 \geq 0$ and all $i \in \mathcal{D}, x_0 \in R^n$

b) We say that the trivial solution of system (4) is exponentially stable in mean square if there exist $\beta \geq 1$ and $\alpha > 0$ such that

$$E[|\widehat{X}(t,t_0)x_0|^2|\eta(t_0) = i] \leq \beta e^{-\alpha(t-t_0)}|x_0|^2$$

for all $t \geq t_0, x_0 \in R^n$ and $i \in \mathcal{D}$.

From Proposition 2 the following result follows directly

PROPOSITION 3. *The trivial solution of system (3) is exponentially stable in mean square iff there exist $\beta \geq 1$ and $\alpha > 0$ such that $||\mathcal{T}^*(t,t_0)|| \leq \beta e^{-\alpha(t-t_0)}$ for all $t \geq t_0$.*

The trivial solution of system (4) is exponentially stable in mean square iff there exist $\beta \geq 1$ and $\alpha > 0$ such that $||\widehat{\mathcal{T}}^*(t,t_0)|| \leq \beta e^{-\alpha(t-t_0)}$ for all $t \geq t_0$.

The next result follows directly from (8), (15) and Proposition 3.

PROPOSITION 4. (i) *If the trivial solution of system (3) is exponentially stable in mean square then the trivial solution of system (4) s exponentially stable in mean square.*

(ii) If the trivial solution of system (4) is exponentially stable in mean square then for every $i \in \mathcal{D}$, the trivial solution of system (9) is exponentially stable.

Remark 1. From Proposition 3 it follows that the mean square exponential stability of the trivial solution of system (3) is completely caracterized by the matrices A_k and $Q, 0 \leq k \leq r$. Therefore we shall say that the system $(A_0, A_1, \ldots, A_r, Q)$ is stable instead of "the trivial solution of system (3) is exponentially stable in mean square".

A detailed study of stability of the system $(A_0, A_1, \ldots, A_r, Q)$ is made in [3].

PROPOSITION 5. *Assume that $A_k, 0 \leq k \leq r$ are bounded on $\mathbb{R}_+$. If there exist $\tau > 0$ and $\delta \in (0,1)$ such that*

$$E[X^*(t+\tau,t)G(t+\tau)(\eta(t+\tau))X(t+\tau,t)|\eta(t) = i] \leq \delta G_i(t)$$

for all $i \in \mathcal{D}, t \geq 0$, where $G : \mathbb{R}_+ \to \mathcal{S}_n^d$ is a bounded and uniformly positive definite function, then the system $(A_0, A_1, \ldots, A_r, Q)$ is stable

Proof. From Proposition 2 we get $\mathcal{T}^*(t+\tau, t)G(t+\tau) \leq \delta G(t), t \geq 0$. Let $t_0 \geq 0$. Since $T^*(t, t_0) \geq 0$ for all $t \geq t_0$ we get by induction that $\mathcal{T}^*(t_0 + n\tau, t_0)G(t_0 + n\tau) \leq \delta^n G(t_0) \leq \beta\delta^n J, n \geq 1$, with some $\beta > 0$: Since G is uniformly positive definite we can write

$$\mathcal{T}^*(t_0 + n\tau, t_0)J \leq \beta_1 \delta^n J, \quad \|\mathcal{T}^*(t_0 + n\tau, t_0)\| \leq \beta_1 \delta^m, n \geq 1$$

But $\sup\{\|\mathcal{L}^*(t)\|; t \geq 0\} < \infty$.

Thus, taking into account (7) one gets easily that there exists $\beta_2 > 0$ such that

$$\|\mathcal{T}^*(t, s)\| \leq \beta_2 \text{ if } 0 \leq t - s \leq \tau$$

Hence we obtain that $\|\mathcal{T}^*(t, t_0)\| \leq \beta_3 e^{-\alpha(t-t_0)}$ for all $t \geq t_0$, with $\alpha = -\frac{1}{\tau}\ln \delta$. Thus, by Proposition 3 the proof is complete.

4. STOCHASTIC OBSERVABILITY

Let $C : \mathbb{R}_+ \times \mathcal{D} \to R^{p \times n}$ be a continuous function. $\tilde{C}(t) \in \mathcal{S}_n^d$ is defined by $\tilde{C}_i(t) = C^*(t, i)C(t, i)$. In the time-invariant case $C(t, i) = C(i), t \geq 0, i \in \mathcal{D}$ and $\tilde{C}(i) = C^*(i)C(i); \tilde{C} \in \mathcal{S}_n^d$.

Definition 2. a) The system $(C; A_0, A_1, \ldots, A_r, Q)$ is uniformly observable (in the time-invariant case we say simple that the system $(C; A_0, A_1, \ldots, A_r, Q)$ is observable) if there $\beta > 0$ and $\tau > 0$ such that

$$\int_t^{t+\tau} \mathcal{T}^*(s, t)\tilde{C}(s)ds \geq \beta J \text{ for all } t \geq 0$$

b) The system $(C; A_0, Q)$ is uniformly observable (in the time-invariant case we say that the system $(C; A_0, Q)$ is observable) if there exists $\beta > 0$ and $\tau > 0$ such that $\int_t^{t+\tau} \hat{\mathcal{T}}^*(s, t)\tilde{C}(s)ds \geq \beta J$ for all $t \geq 0$.

It is well known that the concept of uniform observability for time-varying deterministic systems was established by Kalman [13]. If $\mathcal{D} = \{1\}$ the Definition 2 can be find in [15] and if $A_k = 0, 1 \leq k \leq r$ Definition 2 is given in [16]. For time-invariant stochastic discrete-time linear systems the concept of observability has been defined in [19]. The next result follows directly from Proposition 2.

PROPOSITION 6. *The system $(C; A_0, A_1, \ldots, A_r, Q)$ is uniformly observable iff there exist $\beta > 0$ and $\tau > 0$ such that*

$$E[\int_t^{t+\tau} X^*(s,t)C^*(s,\eta(s))C(s,\eta(s))X(s,t)ds|\eta(t)=i] \leq \beta I_n$$

for all $t \geq 0$ and $i \in \mathcal{D}$.

By using (14), (8) and (10) we can conclude that the next result holds.

PROPOSITION 7a). *If for every $i \in \mathcal{D}$ the pair $(C(\cdot,i), A_0(\cdot,i))$ is uniformly observable (see [13]) then the system $(C; A_0, Q)$ is uniformly observable*

(b) *If the system $(C; A_0, Q)$ is uniformly observable then the system $(C; A_0, A_1, \ldots, A_r, Q)$ is uniformly observable.*

PROPOSITION 8. *Suppose that $A_k(t,i) = A_k(i), C(t,i) = C(i)$ for all $t \geq 0, i \in \mathcal{D}, 0 \leq k \leq r$. Then the following assertions are equivalent*

a) *The system $(C; A_0, \ldots, A_r, Q)$ is observable*

b) *There exists $\tau > 0$ such that*

$$\int_0^{\tau} e^{\mathcal{L}^* t}\widetilde{C}dt > 0$$

c) *There exists $\tau > 0$ such that $K_0(\tau) > 0$ where $K_0(t)$ is the solution of the differential equation on $\mathcal{S}_n^d$*

$$K'(t) = \mathcal{L}^* K(t) + \widetilde{C}, t \geq 0$$

with $K_0(0) = 0$

Proof. a) $\Longleftrightarrow$ b) follows from (16)

Since $K_0(t) = \displaystyle\int_0^t e^{\mathcal{L}^*(t-s)}\widetilde{C}ds = \int_0^t e^{\mathcal{L}^* s}\widetilde{C}ds,\ t \geq 0$ it follows that c) $\Longleftrightarrow$ b). The proof is complete.

THEOREM 1. *Under the assumption of Proposition 8 if the system $(C; A_0, \ldots, A_r, Q)$ is not observable then there exist $x_0 \in R^n, x_0 \neq 0$ and $i_0 \in \mathcal{D}$ such that*

(i) $C(i_0)x_0 = 0$

(ii) $q_{i_0 i}C(i)x_0 = 0$ *for all $i \in \mathcal{D}$*

(iii) $C(i_0)(A_0(i_0))^m x_0 = 0$ *for all $m \geq 1$*

(iv) $q_{i_0 i} q_{ij} C(j) x_0 = 0$ *for all* $i \neq i_0, j \in \mathcal{D}$
(v) $C(i_0) A_k(i_0) x_0 = 0,\ 1 \leq k \leq r$

Proof. Suppose that $(C; A_0, \ldots, A_r, Q)$ is not observable. From Proposition 8 it follows that there exist $x_0 \in R^n, x_0 \neq 0$ and $i_0 \in \mathcal{D}$ such that $x_0^* \int_0^1 (e^{\mathcal{L}^* t} \widetilde{C})(i_0) dt x_0 = 0$. Hence $x_0^*(e^{\mathcal{L}^* t}\widetilde{C})(i_0) x_0 = 0$ for all $t \in [0,1]$. Since $e^{\mathcal{L}^* t} \geq e^{\widehat{\mathcal{L}n}^* t} \geq e^{\mathcal{L}_2^* t}$ (see (14)-(16)) one gets $x_0^*(e^{\widehat{\mathcal{L}}^* t}\widetilde{C})(i_0) x_0 = 0, x_0^*(e^{\mathcal{L}_2^* t}\widetilde{C})(i_0) x_0 = 0, t \in [0,1]$. From the last equality we get $C(i_0) e^{A_0(i_0)t} x_0 = 0, t \in [0,1)$.

Hence differentiating successively we have

$$x_0^*((\mathcal{L}^*)^m \widetilde{C})(i_0) x_0 = 0, \quad m \geq 0 \tag{17}$$

$$C(i_0)(A_0(i_0))^m x_0 = 0, \quad m \geq 0 \tag{18}$$

$$x_0^*((\widehat{\mathcal{L}}^*)^m \widetilde{C})(i_0) x_0 = 0, x_0^*((\mathcal{L}_2^*)^m \widetilde{C})(i_0) x_0 = 0 \tag{19}$$

for all $m \geq 0$

Thus (i) and (iii) follow from (18)

Now, from (17) and (19) we have

$$0 = x_0^*(\mathcal{L}^* \widetilde{C})(i_0) x_0 = x_0^*(\mathcal{L}_1^* \widetilde{C})(i_0) x_0 + x_0^*(\widehat{\mathcal{L}}^* \widetilde{C})(i_0) x_0$$

$$= x_0(\mathcal{L}_1^* \widetilde{C})(i_0) x_0 = x_0^* \sum_{k=1}^{r} A_k^*(i_0) C^*(i_0) C(i_0) A_k(i_0) x_0$$

and thus (v) follows

Further, by (19) we can write

$$\begin{aligned} 0 &= x_0^*(\widehat{\mathcal{L}}^* \widetilde{C})(i_0) x_0 = x_0^*(\mathcal{L}_2^* \widetilde{C})(i_0) x_0 + x_0^*(\mathcal{L}_3^* \widetilde{C})(i_0) x_0 = \\ &= x_0^*(\mathcal{L}_3^* \widetilde{C})(i_0) x_0 = x_0^* \sum_{j \neq i_0} q_{i_0 j} C^*(j) C(j) x_0 \end{aligned}$$

and since $q_{ij} \geq 0$ if $i \neq j$ one gets (ii).

Also from (19) it follows that

$$\begin{aligned} 0 &= x_0^*((\widehat{\mathcal{L}}^*)^2 \widetilde{C})(i_0) x_0 = \\ &= x_0^*\{([(\mathcal{L}_2^*)^2 + \mathcal{L}_2^*\mathcal{L}_3^* + \mathcal{L}_3^*\mathcal{L}_2^* + (\mathcal{L}_3^*)^2]\widetilde{C}](i_0)\} x_0 = \\ &= x_0^*[(\mathcal{L}_2^*\mathcal{L}_3^*\widetilde{C})(i_0) + (\mathcal{L}_3^*\mathcal{L}_2^*\widetilde{C})(i_0) + ((\mathcal{L}_3^*)^2\widetilde{C})(i_0)] x_0 \end{aligned}$$

But, by using (ii) we can write

$$x_0^*(\mathcal{L}_2^*\mathcal{L}_3^*\widetilde{C})(i_0)x_0 = 2x_0^*[A_0^*(i_0) + \frac{1}{2}q_{i_0i_0}I_n] \sum_{i\neq i_0} q_{i_0i}C^*(i)C(i)x_0 = 0$$

$$x_0^*(\mathcal{L}_3^*\mathcal{L}_2^*\widetilde{C})(i_0)x_0 = 2x_0^* \sum_{i\neq i_0} q_{i_0i}(A_0^*(i) + \tfrac{1}{2}q_{ii}I_n)C^*(i)C(i)x_0 = 0$$

Hence one gets

$$0 = x_0^*((\mathcal{L}_3^*)^2\widetilde{C})(i_0)x_0 = x_0^* \sum_{i\neq i_0} \sum_{j\neq i} q_{i_0i}q_{ij}C^*(j)C(j)x_0$$

and since $q_{i_0i}q_{ij} \geq 0$ for $i \neq i_0, j \neq i$ one obtains $q_{i_0i}q_{ij}C(j)x_0 = 0$ for all $i \neq i_0$ and $j \neq i$ and thus by (ii) it follows that (iv) holds and hence the proof is complete.

COROLLARY 1. *Under the assumption of Proposition 8 if for every $i \in \mathcal{D}$, rank $M(i) = n$, where*

$$M(i) = [C^*(i), A_0^*(i)C^*(i), \ldots, (A_0^*(i_0))^{n-1}C^*(i_0), q_{i1}C^*(1), \ldots, q_{id}C^*(d),$$
$$A_1^*(i)C^*(i), \ldots, A_r^*(i)C^*(i)]$$

then the system $(C; A_0, A_1, \ldots, A_r, Q)$ is observable.

THEOREM 2. *Assume that A_k, $0 \leq k \leq r$ are bounded on $\mathbb{R}_+$. If the system $(C; A_0, \ldots, A_r, Q)$ is uniformly observable and if the following differential equation on $\mathcal{S}_n^d$.*

$$K'(t) + \mathcal{L}^*(t)K(t) + \widetilde{C}(t) = 0, \quad t \geq 0 \tag{20}$$

has a bounded and positive semidefinite solution $\widetilde{K}(t)$ then:

(i) the system $(A_0, A_1, \ldots, A_n, Q)$ is stable

(ii) $\widetilde{K}$ is uniformly positive definite

(iii) The equation (20) has a unique positive semidefinite and bounded on $\mathbb{R}_+$ solution

Proof. From (7) it follows that

$$\widetilde{K}(t) = \mathcal{T}^*(s,t)\widetilde{K}(s) + \int_t^s \mathcal{T}^*(u,t)\widetilde{C}(u)du, \quad s \geq t \tag{21}$$

Since $0 \leq \widetilde{K}(s) \leq \beta_0 J$ with some $\beta_0 > 0$ and $\mathcal{T}(s,t) \geq 0$ one gets $0 \leq \int_t^s T^*(u,t)\tilde{C}(u)du \leq \widetilde{K}(t) \leq \beta_0 J$ for all $s \geq t \geq 0$

Hence the integral $\widehat{K}(t) = \int_t^\infty \mathcal{T}^*(s,t)\tilde{C}(s)ds$ is convergent and $0 \leq \widehat{K}(t) \leq \beta_0 J, \quad t \geq 0$

By (7) it follows directly that $\widehat{K}$ is a solution of the equation (20)

Since $(C; A_0, \ldots, A_r, Q)$ is uniformly observable it follows that $\widehat{K}$ is uniformly observable. Since $\mathcal{T}^*(t+\tau,t)\mathcal{T}^*(s,t+\tau) = \mathcal{T}^*(s,t)$ we have

$$\mathcal{T}^*(t+\tau,t)\widehat{K}(t+\tau) = \int_{t+\tau}^{\infty} \mathcal{T}^*(s,t)\tilde{C}(s)ds = \widehat{K}(t) - \int_t^{t+\tau} \mathcal{T}^*(s,t)\tilde{C}(s)ds$$

Hence $\mathcal{T}^*(t+\tau)\widehat{K}(t+\tau) \leq \widehat{K}(t) - \beta J \leq (1 - \frac{\beta}{\beta_0})\widehat{K}(t), t \geq 0$. Thus by Propositions 2 and 5 it follows that the system $(A_0, \ldots, A_r, Q)$ is stable. Hence by Proposition 3, $\|T^*(s,t)\| \leq \gamma e^{-\alpha(s-t)}, s \geq t$.

Taking $s \to \infty$ in (21) one gets $\widetilde{K}(t) = \widehat{K}(t), t \geq 0$ and thus the proof is complete.

COROLLARY 2. *Suppose that $A_k(t,i) = A_k(i), C(t,i) = C(i), t \geq 0, i \in \mathcal{D}, 0 \leq k \leq r$*

Assume that $(C; A_0, \ldots, A_r, Q)$ is observable and the algebraic equation on $\mathcal{S}_n^d$

$$\mathcal{L}^* K + \tilde{C} = 0 \tag{22}$$

has a solution $\widetilde{K} \geq 0$.

Then :

(i) The system $(A_0, A_1, \ldots, A_r, Q)$ is stable

(ii) $\widetilde{K} > 0$

(iii) The equation (22) has a unique positive semidefinite solution.

5. LINEAR QUADRATIC CONTROL PROBLEM ON FINITE HORIZON

Consider the following linear control system

$$\begin{aligned} dx(t) &= [A_0(t,\eta(t))x(t) + B(t,\eta(t))u(t)]dt + \\ &+ \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t), \qquad t \geq 0 \end{aligned} \tag{23}$$

and the output $y(t) = C(t, \eta(t))x(t)$ where A_k, B and C are continuous functions on $\mathbb{R}_+$.

If $t_0 \geq 0$ and $x_0 \in R^n$ and $u \in \mathcal{U}(t_0, m)$, by the standard procedure of succesive approximations and by using the properties of the stochastic integral, it is easy to see that there exists a unique solution $x_u(t, t_0, x_0)$ of (23) defined for all $t \geq t_0$, with the properties: $x_u(t_0, t_0, x_0) = x_0, x(\cdot, t_0, x_0) \in \mathcal{U}(t_0, n); x_u(t, t_0, x_0)$ is a continuous (with probability one) process.

Remark 2. From (1) and (23) one gets easily by the Gronwall Lemma that for every $T > 0$, there exists $\beta(T) > 0$ such that

$$E[|x_u(t, t_0, x_0)|^2 |\eta(t_0) = i] \leq \beta(T) \left\{|x_0|^2 + E[\textstyle\int_{t_0}^T |u(t)|^2 dt| \right.$$
$$\left. \eta(t_0) = i]\right\} \text{ for all } 0 \leq t_0 < T, t \in [t_0, T], u \in L^2_{\eta,w}([t_0, T], R^m),$$

and $x_0 \in R^n, i \in \mathcal{D}$

If $A_k, 0 \leq k \leq r$ and B are bounded on $\mathbb{R}_+$ then there exists a function $\gamma : (0, \infty) \to (0, \infty)$ such that

$$E[|x_u(t, t_0, x_0)|^2 |\eta(t_0) = i] \leq \gamma(T - t_0)\{[|x_0|^2 +$$
$$+ E[\textstyle\int_{t_0}^T |u(t)|^2 dt|\eta(t_0) = i]\} \text{ for all } i \in \mathcal{D},$$

$x_0 \in R^n$ and all $0 \leq t_0 < T, t \in [t_0, T]$ and $u \in L^2_{\eta,w}([t_0, T], R^m)$

We associate the quadratic cost

$$\begin{aligned} V_T(x_0, t_0, u, i) &= E[\int_{t_0}^T \{|C(t, \eta(t))x_u(t, t_0, x_0)|^2 + \\ &+ u^*(t)R(t, \eta(t))u(t)\}dt|\eta(t_0) = i] \end{aligned}$$

where $0 \leq t_0 < T, x_0 \in R^n, i \in \mathcal{D}$ and $u \in L^2_{\eta w}([t_0, T], R^m)$ and $R(t, i) = R^*(t, i)$ is a continuous function with the property that $R(t, i) > 0$ for all $t \geq 0, i \in \mathcal{D}$.

From Remark 2 it follows that $V_T(x_0, t_0, u, i) < \infty$. In this section we solve the problem: Given arbitrary, but fixed $0 \leq t_0 < T$ and $x_0 \in R^n$, find $\hat{u} \in L^2_{\eta,w}([t_0, T), R^m)$ such that for all $i \in \mathcal{D}$ we have $V_T(x_0, t_0, u, i) \geq V_T(x_0, t_0, \hat{u}, i)$ for all $u \in L^2_{\eta,w}([t_0, T], R^m)$. If $\hat{u}$ has the above property we shall write

$$\min\{V_T(x_0, t_0, u, i); u \in L^2_{\eta,w}([t_0, T], R^m)\} = V_T(x_0, t_0, \hat{u}, i), i \in \mathcal{D}.$$

Let us consider the following Riccati type system

$$
(24)\qquad
\begin{aligned}
&\frac{d}{dt}K(t,i) + A_0^*(t,i)K(t,i) + K(t,i)A_0(t,i)+\\
&+\sum_{k=1}^{r} A_k^*(t,i)K(t,i)A_k(t,i)\\
&+\sum_{j=1}^{d} q_{ij}K(t,j) + C^*(t,i)C(t,i)\\
&-K(t,i)B(t,i)(R(t,i))^{-1}B^*(t,i)K(t,i) = 0,\ t \geq 0,\ i \in \mathcal{D}.
\end{aligned}
$$

If $K : R_+ \times \mathcal{D} \to \mathcal{S}_n$ we define $F_K(t,i) = -(R(t,i))^{-1}B^*(t,i)K(t,i)$ and $\Delta_K(t,i) = K(t,i)B(t,i)F_K(t,i)$.

With the above notations the system (24) can be written in the following form on $\mathcal{S}_n^d$.

$$
(25)\qquad K'(t) = G(t, K(t)), \quad t \geq 0
$$

where $K(t) = (K(t,1), \ldots, K(t,d)), G(t,K) \in \mathcal{S}_n^d$,

$$
G(t,K) = -\mathcal{L}^*(t)K - \Delta_K(t), (\Delta_K(t))(i) = \Delta_K(t,i)
$$

If $K(t,i), t \in [t_0, T]$ is a symmetric solution of system (24), then applying Proposition 1 for system (23) and $v(t,x,i) = x^*K(t,i)x,\ x \in R^n, i \in \mathcal{D}, t \in [t_0, T]$ and taking into account the relations (24), $t \in [t_0, T], i \in \mathcal{D}$ one gets

$$
(26)\qquad E[x_u^*(T,t_0,x_0)K(t,\eta(T))x_u(T,t_0,x_0)|\eta(t_0) = i] - x_0^*K(t_0,i)x_0
$$

$$
= -V_T(x_0,t_0,u,i) + E[\int_{t_0}^{T}(u(t) - F_K(t,\eta(t))x_u(t,t_0,x_0))^*R(t,\eta(t))(u(t)-
$$

$$
-F_K(t,\eta(t))x_u(t,t_0,x_0))dt|\eta(t_0) = i]
$$

for all $t \in [t_0, T], i \in \mathcal{D}, x_0 \in R^n, u \in L^2_{\eta,w}([t_0,T], R^m)$

In what follows by $K_T(t,i)$ we denote the solution of system (24) with $K_T(T,i) = 0, i \in \mathcal{D}$.

THEOREM 3. *For every $T > 0$, the solution $K_T(t,i)$ is defined for all $t \in [0,T]$, $i \in \mathcal{D}$ and has the properties:*

(i) $K_T(t,i) \geq 0$

(ii) $\min\{V_T(x_0,t_0,u,i); u \in L^2_{\eta,w}([t_0,T],R^m)\} = V_T(x_0,t_0,\widehat{u}_T,i) = x_0^* K_T(t_0,i)x_0$ *for all* $0 \le t_0 < T$, $i \in \mathcal{D}, x_0 \in R^n$ *where* $\widehat{u}_T(t) = F_K(t,\eta(t))\widehat{x}(t), t \in [t_0,T]$ *and* $\widehat{x}(t)$ *being the solution of system*

$$dx(t) = [A_0(t,\eta(t)) + B(t,\eta(t))F_{K_T}(t,\eta(t))]x(t)dt+$$

$$+\sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t), t \in [t_0,T],$$

with $\widehat{x}(t_0) = x_0$

(iii) $K_{T_1}(t,i) \le K_{T_2}(t,i)$ *for all* $T_1 < T_2$ *and* $t \in [0,T_1], i \in \mathcal{D}$,

Proof. Let $t_0 \in [0,T)$ be such that the solution K_T is defined on $[t_0,T]\times\mathcal{D}$.

From the relations (26) for the solution K_T one gets that (ii) holds and therefore $K_T(t_0,i) \ge 0$. Moreover by Remark 2 we have $x_0^*K_T(t_0,i)x_0 \le V_T(x_0,t_0,0,i) \le \beta(T)|x_0|^2$.

Hence $0 \le K_T(t_0,i) \le \beta(T)I_n, i \in \mathcal{D}$. Since $\beta(T)$ does not depend on t_0, it follows that K_T is defined on $[0,T]\times\mathcal{D}$ and thus it remains only to verify (iii).

Indeed let $0 < T_1 < T_2$ and $t_0 \in [0,T_1]$.

We have by (ii)

$$x_0^*K_{T_2}(t_0,i)x_0 = V_{T_2}(x_0,t_0,\hat{u}_{T_2},i) \ge V_{T_1}(x_0,t_0,u_{T_1},i) \ge x_0^*K_{T_1}(t_0,i)x_0$$

where u_{T_1} is the restriction of $\widehat{u}_{T_2}$ to the interval $[t_0,T_1]$. The proof is complete.

6. LINEAR QUADRATIC CONTROL PROBLEM ON INFINITE HORIZON

Throughout this section we assume that A_k, B, C and $R = R^*, 0 \le k \le r$ are continuous and bounded on $\mathbb{R}_+$ and $R(t,i)$ is uniformly positive definite.

Condsider the quadratic cost

$$V(x_0,t_0,u,i) = E[\int_{t_0}^{\infty}\{|C(t,\eta(t))x_u(t,t_0,x_0)|^2+$$

$$+u^*(t)R(t,\eta(t))u(t)\}dt|\eta(t_0)=i], t_0 \ge 0, i \in \mathcal{D}, x_0 \in R^n, u \in \mathcal{U}(t_0,m)$$

By $\mathcal{U}_m(t_0,x_0)$ we denote the space of all functions $u \in \mathcal{U}(t_0,m)$ with the property that

$$V(x_0,t_0,u,.i) < \infty,\ i \in \mathcal{D}$$

Obviously if $u \in L^2_{\eta,w}([t_0,\infty), R^m)$ has the property that $x_u(\cdot, t_0, x_0) \in L^2_{\eta,w}([t_0,\infty), R^n)$ then $u \in \mathcal{U}_m(t_0, x_0)$.

In this section we solve the problem:

Given $t_0 \geq 0$ and $x_0 \in R^n$ arbitrary, but fixed, find $\tilde{u} \in \mathcal{U}_m(t_0, x_0)$ such that $V(x_0, t_0, u, i) \geq V(x_0, t_0, \tilde{u}, i)$ for all $i \in \mathcal{D}$ and all $u \in \mathcal{U}_m(t_0, x_0)$

If $\tilde{u}$ has the above property, we shall write $\min\{V(x_0, t_0, u, i); u \in \mathcal{U}_m(t_0, x_0)\} = V(x_0, t_0, \tilde{u}, i)$, $i \in \mathcal{D}$.

It is obvious that $V(x_0, t_0, u, i) = \lim\limits_{T\to\infty} V_T(x_0, t_0, u_T, i), i \in \mathcal{D}, u \in \mathcal{U}_m(t_0, x_0), u_T$ being the restriction of u to the set $[t_0, T] \times \Omega$.

Definition 3. A continuous and bounded function $F : R_+ \times \mathcal{D} \to R^{m\times n}$ is said to be a stabilizing feedback gain if the system $(A_0 + BF, A_1, \ldots, A_r, Q)$ is stable.

Definition 4. A bounded solution $K : R_+ \times \mathcal{D} \to \mathcal{S}_n$ of system (24) is said to be stabilizing if F_K is a stabilizing feedback gain.

Definition 5. a) The system (23) is stabilizable (or equivalently the system $(A_0, A_1, \ldots, A_r, Q; B)$ is stabilizable) if there exists a stabilizing feedback gain.

b) The system $(C; A_0, A_1, \ldots, A_r, Q)$ is said to be detectable if there exists $H : R_+ \times \mathcal{D} \to R^{n\times p}$ continuous and bounded such that the system $(A_0 + HC, A_1, \ldots, A_r, Q)$ is stable

Remark 3. From Theorem 5.1. in [3] it follows that if $u(t) = F(t, \eta(t))x(t) + f(t), t \geq t_0$ where F is a stabilizing feedback gain and $f \in L^2_{\eta,w}([t_0,\infty), R^m)$ and $x(t)$ verifies

(27)

$$\begin{aligned} dx(t) &= \{[A_0(t,\eta(t)) + B(t,\eta(t))F(t,\eta(t))]x(t) + B(t,\eta(t))f(t)\}dt + \\ &+ \textstyle\sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t), t \geq t_0, x(t_0) = x_0 \end{aligned}$$

then $x_u \in L^2_{\eta,w}([t_0,\infty), R^n)$ and therefore $u \in \mathcal{U}_m(t_0, x_0)$

THEOREM 4. *Suppose that the system (23) is stabilizable*

Then:

a) There exists $\rho > 0$ such that $K_T(t,i) \leq \rho I_n$ for all $i \in \mathcal{D}, T > 0$ and $t \in [0, T]$

b) $\lim\limits_{T\to\infty} K_T(t,i) = \widehat{K}(t,i)$ exists, and $\widehat{K}$ is a positive semidefinite and bounded on $\mathbb{R}_+$ solution of (24)

c) $\min\{V(x_0, t_0, u, i); u \in \mathcal{U}_m(t_0, x_0)\} = V(x_0, t_0, \tilde{u}, i) = x_0^*\widehat{K}(t_0, i)x_0$ for all $t_0 \geq 0, x_0 \in R^n, i \in \mathcal{D}$, where $\tilde{u}(t) = F_{\widehat{K}}(t, \eta(t))\tilde{x}(t), t \geq t_0$ and $\tilde{x}(t), t \geq t_0$

is the solution of the system

$$dx(t) = [A_0(t,\eta(t)) + B(t,\eta(t))F_{\widehat{K}}(t,\eta(t))]x(t)dt+ \\ +\sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t),\ t \geq t_0, \tilde{x}(t_0) = x_0$$

d) If in addition A_k, B, C and R are θ-periodic functions with respect to $t \in \mathbb{R}_+$ for each $i \in \mathcal{D}$ then $\widehat{K}(t+\theta,i) = \widehat{K}(t,i)$ for all $t \geq 0, i \in \mathcal{D}$.

Proof. Let F be a stabilizing feedback gain and $\overline{u}(t) = F(t,\eta(t))\overline{x}(t), t \geq 0, \overline{x}$ verifying(27) with $f(t) = 0, \overline{x}(t_0) = x_0$

Since $(A_0 + BF, A_1, \ldots, A_r, Q)$ is stable there exists $\rho > 0$ suchthat $V(x_0,t_0,\overline{u},i) \leq \rho|x_0|^2$ for all $t_0 \geq 0, x_0 \in R^n$ and $i \in \mathcal{D}$. Thus, by Theorem 3 one gets $x_0^* K_T(t_0,i)x_0 \leq \rho|x_0|^2$.

Hence $0 \leq K_T(t_0,i) \leq \rho I_n, T > 0, t_0 \in [0,T]$ and $i \in \mathcal{D}$.

Therefore from Theorem 3, (iii) it follows that $\lim_{T\to\infty} K_T(t,i)$ exists and its limit $\widehat{K}(t,i)$ is bounded and positive semidefinite.

Now, let $0 \leq \alpha < \beta$ and $T > \beta$. Since $K_T(t) = (K_T(t,1),\ldots,K_T(t,d))$ is a solution of the equation (25) $0 \leq K_T(t,i) \leq \rho I_n$ and the function G is continuous and locally Lipschitz with respect to the second argument, by the Gronwall Lemma one obtains that there exists $\gamma > 0$ depending only on $\beta - \alpha$ such that

$$|K_{T_2}(t) - K_{T_1}(t)| \leq \gamma |K_{T_2}(\alpha) - K_{T_1}(\alpha)| \text{ for all } t \in [\alpha,\beta] \text{ and all } T_2 > T_1 > \beta.$$

Hence $\lim_{T\to\infty} K_T(t,i) = \widehat{K}(t,i), i \in \mathcal{D}$ uniformly with respect to t in every compact interval $[\alpha,\beta] \subset \mathbb{R}_+$. Therefore $\widehat{K}$ is a solution of system (24) and $0 \leq \widehat{K}(t,i) \leq \rho I_n, t \geq 0, i \in \mathcal{D}$.

From (26) we have

$$V_T(x_0,t_0,\tilde{u},i) = x_0^*\widehat{K}(t_0,i)x_0 - E[\tilde{x}^*(T)\widehat{K}(T,\eta(T))\tilde{x}(T)|\eta(t_0) = i]$$

Hence $x_0^* K_T(t_0,i)x_0 \leq V_T(x_0,t_0,\tilde{u},i) \leq x_0^*\widehat{K}(t_0,i)x_0$

Taking $T \to \infty$ one obtains $V(x_0,t_0,\tilde{u},i) = x_0^*\widehat{K}(t_0,i)x_0$. On the other hand we can write for $u \in \mathcal{U}_m(t_0,x_0), V_T(x_0,t_0,u,i) \geq x_0^* K_T(t_0,i)x_0$

Taking $T \to \infty$ one gets $V(x_0,t_0,u,i) \geq x_0^*\widehat{K}(t_0,i)x_0$ and thus c) follows.

To prove d), let $\widehat{K}_T : [0,T]\times\mathcal{D} \to \mathcal{S}_n$ defined as follows $\widehat{K}(t,i) = K_{T+\theta}(t+\theta,i)$

Obviously $\widehat{K}_T(t,i)$ verif y equations (24) for $t \in [0,T]$ and $i \in \mathcal{D}$ and since $\widehat{K}_T(T,i) = 0 = K_T(T,i), i \in \mathcal{D}$ from uniqueness it follows that $\widehat{K}_T(t,i) =$

$K_T(t,i), t \in [0,T], i \in \mathcal{D}$. Taking $T \to \infty$ in the above equality we get $\widehat{K}(t+\theta,i) = \widehat{K}(t,i), t \geq 0, i \in \mathcal{D}$ and thus the proof is complete.

PROPOSITION 9. *Suppose that the system $(C; A_0, \ldots, A_r, Q)$ is detectable. Then every positive semidefinite and bounded on $\mathbb{R}_+$ solution of system (24) is stabilizing.*

Proof. Let $K(t,i)$ be a positive semidefinite and bounded on $\mathbb{R}_+$ solution of (24).

It is easy to see that

$$\frac{d}{dt}K(t,i) + \widetilde{A}_0^*(t,i)K(t,i) + K(t,i)\widetilde{A}_0(t,i) + M^*(t,i)M(t,i) +$$
$$+ \sum_{k=1} A_k^*(t,i)K(t,i)A_k(t,i) + \sum_{j=1}^{d} q_{ij}K(t,j) = 0, t \geq 0$$

where $\widetilde{A}_0(t,i) = A_0(t,i) + B(t,i)F_K(t,i)$ and

$$M(t,i) = \begin{pmatrix} R^{1/2}(t,i)F_K(t,i) \\ C(t,i) \end{pmatrix}$$

We shall prove that $(M; \widetilde{A}_0, A_1, \ldots, A_r, Q)$ is detectable. Indeed, let $H(t,i)$ be a matrix valued function satisfying the assumption in Definition 5, b).

We take $\widehat{H}(t,i) = [-B(t,i)R^{-1/2}(t,i) \quad H(t,i)]$

We have

$$\widetilde{A}_0(t,i) + \widehat{H}(t,i)M(t,i) = A_0(t,i) + H(t,i)C(t,i).$$

Hence $(\widetilde{A}_0 + \widehat{H}M, A_1, \ldots, A_r, Q)$ is stable. Thus $(M; \widetilde{A}_0, A_1, \ldots, A_r, Q)$ is detectable and therefore by virtue of Theorem 6.4 in [3] we can conclude that $(\widetilde{A}_0, A_1, \ldots, A_r, Q)$ is stable; hence K is a stabilizing solution and the proof is complete.

PROPOSITION 10. *Suppose that $(C; A_0, A_1, \ldots, A_r, Q)$ is uniformly observable. Then if K is a positive semidefinite and bounded on $\mathbb{R}_+$ solution of system (24) we have*

(i) K is uniformly positive definite

(ii) K is a stabilizing solution

Proof. Let K be a positive semidefinite and bounded on $\mathbb{R}_+$ solution of system (24).

Let $\widetilde{A}_0(t,i) = A_0(t,i) + B(t,i)F_K(t,i)$ and $\widetilde{X}(t,t_0)$ be the fundamental matrix solution associated with the linear system $dx(t) = \widetilde{A}_0(t,\eta(t))x(t)dt + \sum_{k=1}^{r} A_k(t,\eta(t))x(t)dw_k(t)$

Let $\tau > 0$ and $\beta > 0$ verifying the inequality in Proposition 6.

Define

$$G(t,i) = E[\int_t^{t+\tau} \widetilde{X}^*(s,t)[C^*(s,\eta(s))C(s,\eta(s)) + F_K^*(s,\eta(s))R(s,\eta(s)).$$

$$F_K(s,\eta(s))]\widetilde{X}(s,t)ds|\eta(t) = i], t \geq 0,\ i \in \mathcal{D}).$$

We shall prove $\inf\{x^*G(t,i)x; |x| = 1, t \geq 0, i \in \mathcal{D}\} > 0$. Suppose on the contrary that for every $\varepsilon > 0$ there exist $x_\varepsilon \in R^n, |x_\varepsilon| = 1, t_\varepsilon \geq 0$ and $i_\varepsilon \in \mathcal{D}$ such that $x_\varepsilon^* G(t_\varepsilon, i_\varepsilon)x_\varepsilon < \varepsilon$.

Let $x_\varepsilon(t) = \widetilde{X}(t, t_\varepsilon)x_\varepsilon$ and $u_\varepsilon(t) = F_K(t,\eta(t))x_\varepsilon(t)$

We can write

$$\varepsilon > x_\varepsilon^* G(t_\varepsilon, i_\varepsilon) \geq E[\int_{t_\varepsilon}^{t_\varepsilon+\tau} u_\varepsilon^*(t)R(t,\eta(t))u_\varepsilon(t)dt|\eta(t_\varepsilon) = i_\varepsilon]$$
$$\geq \delta E[\int_{t_\varepsilon}^{t_\varepsilon+\tau} |u_\varepsilon(t)|^2 dt|\eta(t_\varepsilon) = i_\varepsilon]$$

with some $\delta > 0$. But $x_\varepsilon(t) = X(t,t_\varepsilon)x_\varepsilon + \hat{x}_\varepsilon(t), \quad t \geq t_\varepsilon$ where $\widehat{x}_\varepsilon(t_\varepsilon) = 0$ and

$$d\widehat{x}_\varepsilon(t) = (A_0(t,\eta(t))\widehat{x}_\varepsilon(t) + B(t,\eta(t))u_\varepsilon(t))dt + \sum_{k=1}^{r} A_k(t,\eta(t))\widehat{x}_\varepsilon(t)dw_k(t)$$

Hence, by Remark 2 there exists $\gamma_0 > 0$ such that

$$E[|\widehat{x}_\varepsilon(t)|^2|\eta(t_\varepsilon) = i_\varepsilon] \leq \gamma_0 E[\int_{t_\varepsilon}^{t_\varepsilon+\tau} |u_\varepsilon(t)|^2 dt|\eta(t_\varepsilon) = i_\varepsilon] \leq \delta_1\varepsilon$$

Further, we can write

$$\varepsilon > x_\varepsilon^* G(t_\varepsilon, i_\varepsilon)x_\varepsilon \geq E[\int_{t_\varepsilon}^{t_\varepsilon+\tau} |C(t,\eta(t))x_\varepsilon(t)|^2 dt|\eta(t_\varepsilon) = i_\varepsilon]$$
$$= E[\int_{t_\varepsilon}^{t_\varepsilon+\tau} |C(t,\eta(t))X(t,t_\varepsilon)x_\varepsilon + C(t,\eta(t))\widehat{x}_\varepsilon(t)|^2 dt|\eta(t_\varepsilon) = i_\varepsilon]$$
$$\geq \tfrac{1}{2}E[\int_{t_\varepsilon}^{t_\varepsilon+\tau} |C(t,\eta(t))X(t,t_\varepsilon)x_\varepsilon|^2 dt|\eta(t_\varepsilon) = i_\varepsilon]$$
$$-E[\int_{t_\varepsilon}^{t_\varepsilon+\tau} |C(t,\eta(t))\widehat{x}_\varepsilon(t)|^2 dt|\eta(t_\varepsilon) = i_\varepsilon]$$
$$\geq \tfrac{1}{2}\beta - \delta_2\varepsilon, \quad \varepsilon > 0$$

and thus we get a contradiction, since $\beta > 0$. Hence, there exists $\beta_1 > 0$ such that $G(t,i) \geq \beta_1 I_n, \ t \geq 0, i \in \mathcal{D}$. Now we take in (26) $u(s) = F_K(s,\eta(s))\widetilde{X}(s,t)x_0$ and we get

$$x_0^* E[\widetilde{X}^*(t+\tau,t)K(t+\tau,\eta(t+\tau))\widetilde{X}(t+\tau,t)|\eta(t) = i]x_0 - x_0^* K(t,i)x_0 =$$
$$-x_0^* G(t,i)x_0, \quad t \geq 0,\ x_0 \in R^n, i \in \mathcal{D}$$

Therefore

$$\beta_1|x_0|^2 \leq x_0^* K(t,i) x_0 \leq \beta_2 |x_0|^2, \quad t \geq 0, i \in \mathcal{D}, x_0 \in R^n$$

Thus K is uniformly positive definite and

$$E[\widetilde{X}^*(t+\tau,t)K(t+\tau,\eta(t+\tau))\widetilde{X}(t+\tau,t)|\eta(t)=i] \leq (1-\frac{\beta_1}{\beta_2})K(t,i)$$

By virtue of Proposition 5, the proof is complete.

THEOREM 5. *Assume:*

(i) the system (23) is stabilizable

(ii) the system $(C; A_0, A_1, \ldots, A_r, Q)$ is either detectable or uniformly observable

Then the Riccati type system (24) has a unique positive semidefinite and bounded on $\mathbb{R}_+$ solution. Moreover this solution is stabilizing

Proof. It is know (see [4]) that the Riccati type system (24) has at most one symmetric stabilizing one bounded on $\mathbb{R}_+$ solution. Thus by virtue of Theorem 4 and Propositions 9 and 10, the proof is complete.

Remark 4. From Remark 3 and Theorem 5 it follows that under the assumptions of Theorem 5,the optimal control $\tilde{u}$, defined in Theorem 4 has the property that $\tilde{u} \in L^2_{\eta,w}([t_0,\infty), R^m)$.

In the case $A_k(t,i) = 0, 1 \leq k \leq r$ the results in this section have been proved in [16]; in [16] one considers only admissible controls of the form $u(t) = \varphi(t, x(t), \eta(t))$, where $\varphi(t,x,i)$ are continuous functions and Lipschitz with respect to x.

For time-varying stochastic Itô systems, the linear quadratic control problem on infinite horizon has been discussed in [1] and [15].

For time-invariant linear differential systems with jump Markov perturbations the linear quadratic control problem has been investigated in many papers (see [11], [12], [18] and the references therein).

7. QUADRATIC TRACKING PROBLEM

Throughout this section, we also assume that A_k, B, C an $R, 0 \leq k \leq r$ are continuous and bounded on $\mathbb{R}_+$ matrix valued functions and $R(t,i)$ is uniformly positive definite.

Given a continuous and bounded on $\mathbb{R}_+$ signal $r(t,i)$ we want to minimize the cost

$$\overline{\lim}_{T\to\infty}\frac{1}{T-t_0}E[\int_{t_0}^{t}[|C(t,\eta(t))[x(t)-r(t,\eta(t))]|^2+$$
$$+u^*(t)R(t,\eta(t))u(t)\}dt|\eta(t_0)=i]$$

in a suitable class of controls $u(t)$.

For fixed $t_0 \geq 0$ and $x_0 \in R^n$ we denote by $U_m(t_0,x_0)$ the set of all $u \in \mathcal{U}(t_0,m)$ with the property that $\sup\limits_{t\geq t_0} E|x_u(t,t_0,x_0)|^2 < \infty$

Remark 5. From Theorem 5.1 in [3] it follows that if $u(t) = F(t,\eta(t))x(t)+f(t), t \geq t_0$ where F is a stabilizing feedback gain and $f \in \mathcal{U}(t_0,m)$ with $\sup\limits_{t\geq t_0} E|f(t)|^2 < \infty$ and $x(t)$ verifies (27) then $\sup\limits_{t\geq t_0} E|x(t)|^2 < \infty$ and therefore $u \in U_m(t_0,x_0)$.

The quadratic tracking problem is to minimize in the class $U_m(t_0,x_0)$

$$\overline{\lim_{T\to\infty}}\frac{1}{T-t_0}W_T(x_0,t_0,u,i),$$

$$W_T(x_0,t_0,u,i) = E[\int_{t_0}^{T}\{|C(t,\eta(t))[x_u(t,t_0,x_0)-r(t,\eta(t))]|^2$$
$$+u^*(t)R(t,\eta(t))u(t)\}dt|\eta(t_0)=i]$$

We shall assume the conditions in Theorem 5 are satisfied, hence the system (24) has a unique positive semidefinite and bounded on $\mathbb{R}_+$ solution $\widetilde{K}(t,i)$. Since $\widetilde{K}$ is stabilizing, from Proposition 4 it follows that the trivial solution of the system $\frac{dx}{dt} = \widetilde{A}(t,\eta(t))x(t)$ is exponentially stable in mean square, where $\widetilde{A}(t,i) = A_0(t,i) - B(t,i)R^{-1}(t,i)B^*(t,i)\widetilde{K}(t,i)$

Therefore by virtue of Proposition 4 in [16] the following system

$$\frac{d}{dt}g(t,i) + \widetilde{A}^*(t,i)g(t,i) + \sum_{j=1}^{d} g(t,j)q_{ij} - C^*(t,i)C(t,i)r(t,i) = 0 \tag{28}$$

has a unique bounded on $\mathbb{R}_+$ solution. We denote this solution by $\widetilde{g}(t,i)$.

Let $h_T(t,i)$ be the solution of the following system

$$\frac{d}{dt}h(t,i) + \sum_{j=1}^{d} q_{ij}h(t,j) + \widetilde{m}_i(t) = 0, \quad t \geq 0,\ i \in \mathcal{D} \tag{29}$$

with $h_T(T,i)=0,\ i\in\mathcal{D}$ where

$$\begin{aligned}\widetilde{m}_i(t)\ &= r^*(t,i)C^*(t,i)C(t,i)r(t,i)-\\ &-\widetilde{g}^*(t,i)B(t,i)R^{-1}(t,i)B^*(t,i)\widetilde{g}(t,i).\end{aligned}$$

THEOREM 6. *Under the assumptions of Theorem 5 we have*

$$\min_{u\in U_m(t_0,x_0)}\ \overline{\lim}_{T\to\infty}\frac{1}{T-t_0}W_T(x_0,t_0,u,i)=$$

$$\overline{\lim}_{T\to\infty}\frac{1}{T-t_0}W_T(t_0,x_0,\bar{u},i)$$

$$=\overline{\lim}_{T\to\infty}\frac{1}{T}\int_0^T\sum_{j=1}^d\widetilde{p}_{ij}\widetilde{m}_j(t)dt,\quad \textit{for all}\ t_0\geq 0,\ x_0\in R^n,\ i\in\mathcal{D}$$

where $\widetilde{P}=[\widetilde{p}_{ij}]$, $\widetilde{P}=\lim_{t\to\infty}P(t)$ *and* $\overline{u}(t)=F_{\widetilde{K}}(t,\eta(t))\overline{x}(t)+\widetilde{f}(t),\widetilde{f}(t)=-R^{-1}(t,\eta(t))B^*(t,\eta(t))\widetilde{g}(t,\eta(t))$ *and* $\overline{x}$ *verifies (27) corresponding to* $F_{\widetilde{K}}$ *and* $\widetilde{f}\overline{x}(t_0)=x_0$.

Proof. Firstly we remark that by virtue of Remark 5, $\overline{u}\in U_m(t_0,x_0)$ and therefore $\sup\limits_{t\geq t_0}E|\overline{x}(t)|^2<\infty$.

Let $t_0\geq 0, x_0\in R^n, T>t_0$, and $u\in U_m(t_0,x_0)$.

Consider the function $v_T(t,x,i)=x^*\widetilde{K}(t,i)x+2x^*\widetilde{g}(t,i)+h_T(t,i), t\geq 0, x\in R^n, i\in D$.

Applying Proposition 1 to the above function v_T and to system (23) and taking into account the equations for $\widetilde{K},\widetilde{g}$ and h_T one gets

$$\begin{aligned}&E[v_T(T,x_u(T),\eta(T))|\eta(t_0)=i]-v_T(t_0,x_0,i)=\\ &-W_T(x_0,t_0,u,i)+E[\textstyle\int_{t_0}^T\{(u(t)-\\ &F_{\widetilde{K}}(t,\eta(t))x_u(t)-\widetilde{f}(t))^*R(t,\eta(t))(u(t)\\ &-F_{\widetilde{K}}(t,\eta(t))x_u(t)-\widetilde{f}(t))\}dt|\eta(t_0)=0],\quad i\in\mathcal{D}\end{aligned}\tag{30}$$

where $x_u(t)=x_u(t,t_0,x_0),\ t\geq t_0$

From (30) we get

$$W_T(x_0,t_0,\overline{u},i)=v_T(t_0,x_0,i)-E[v_T(T,\overline{x}(T),\eta(T))|\eta(t_o)=i]$$

$$W_T(x_0,t_0,u,i)\geq v_T(t_0,x_0,i)-E[v_T(T,x_u(T),\eta(T))|\eta(t_0)=i]$$

Since $\sup_{t\geq t_0} E|x_u(t)|^2 < \infty$ we have

$$\begin{aligned}&\lim_{T\to\infty} \tfrac{1}{T-t_0} E[v_T(T,\overline{x}(T),\eta(T))|\eta(t_0)] = i] = 0 = \\ &= \lim_{T\to\infty} \tfrac{1}{T-t_0} E[v_T(T,x_u(T),\eta(T))|\eta(t_0) = i]\end{aligned}$$

On the other hand $\overline{\lim}_{T\to\infty}\dfrac{1}{T-t_0}v_T(t_0,x_0,i) = \overline{\lim}_{T\to\infty}\dfrac{1}{T-t_0}h_T(t_0,i)$

Therefore we have to prove that

$$\overline{\lim_{T\to\infty}}\frac{1}{T-t_0}h_T(t_0,i) = \overline{\lim_{T\to\infty}}\frac{1}{T}\int_0^T \sum_{j=1}^d \widetilde{p}_{ij}\widetilde{m}_j(t)dt$$

Indeed, let $h_T(t) = (h_T(t,1),\ldots,h_T(t,d))^*, \widetilde{m}(t) = (\widetilde{m}(t,1),\ldots,\widetilde{m}(t,d))^*$.

From (29) one gets

$$h_T(t) = \int_t^T e^{Q(s-t)}\widetilde{m}(s)ds, \quad T \geq t$$

Hence

$$h_T(t_0) = \int_{t_0}^T P(s-t_0)\widetilde{m}(s)ds = \int_{t_0}^T (P(s-t_0)-\widetilde{P})\widetilde{m}(s)ds + \int_{t_0}^T \widetilde{P}\widetilde{m}(s)ds$$

Since $\lim_{t\to\infty}(P(t-t_0)-\widetilde{P}) = 0$ and $\widetilde{m}(t)$ is bounded we have

$$\overline{\lim_{T\to\infty}}\frac{1}{T-t_0}h_T(t_0) = \overline{\lim_{T\to\infty}}\frac{1}{T-t_0}\int_{t_0}^T \widetilde{P}\widetilde{m}(t)dt.$$

But $\widetilde{m}(t)$ is a continuous and bounded on $\mathbb{R}_+$ function. Thus it is easy to verify that $\overline{\lim}_{T\to\infty}\frac{1}{T-t_0}\int_{t_0}^T \widetilde{P}\widetilde{m}(t)dt = \overline{\lim}_{T\to\infty}\frac{1}{T}\int_0^T \widetilde{P}\widetilde{m}(t)dt$ and the proof is complete.

Remark 6. From Theorem 6 it follows that under the assumptions of Theorem 5, the optimal value of the quadratic tracking problem does not depend upon t_0 and x_0.

Remark 7. If A_k, B, C, R and $r(t,i)$, $0 \leq k \leq r$ are asymptotically almost periodic functions (see [6]) then under the assumptions of Theorem 5 it follows from [4] that $\widetilde{K}$ and $\widetilde{g}$ are asymptoticaly almost periodic functions, hence $\widetilde{m}(t)$ is asymptotically almost periodic and therefore from [6] it follows that

$$\lim_{t\to\infty}\frac{1}{T}\int_0^T \sum_j \widetilde{p}_{ij}\widetilde{m}_j dt \text{ exists for every } i \in \mathcal{D}.$$

If $A_k(t,i) = 0$, $1 \leq k \leq r$, $t \geq 0, i \in \mathcal{D}$, Theorem 6 has been proved in [16], for $u \in U_m(t_0, x_0)$ only of the form $u(t) = \varphi(t, x(t), \eta(t))$, with φ continuous functions and Lipschitz in the second argument. For stochastic differential equations different tracking type problems are discussed in [1], [5], [8]-[10]. [14], [17].

REFERENCES

1. G. Da Prato and A. Ichikawa, *Quadratic control for linear time-varying systems*, SIAM J. Control Optim, **28**, (1990), 359-381.

2. J. L. Doob, *Stochastic processes*, Wiley, New-York, 1967.

3. V. Dragan and T. Morozan, *Stability and robust stabilization to linear stochastic systems described by differential equations with Markovian jumping and multiplicative white noise*, Stochastic Analysis Appl. (to appear).

4. V. Dragan and T. Morozan, *General coupled Riccati type equations*, Submitted for publication.

5. V. Dragan, A. Halanay and T. Morozan, *Performance estimates in tracking under white noise perturbations*, Int. J. Control, **50**, *6*, (1989), 2269-2283.

6. A. M. Fink, *Almost periodic differential equations*, Lecture Notes in Math., **337**, Springer Verlag 1974.

7. A. Friedman, *Stochastic differential equations and Applications*, vol. 1, Academic Press, 1975.

8. A. Haurie and A. Leizarowitz, *Overtaking optimal regulation and trocking of picewise diffusion linear systems* SIAM J. Control Optim. **30**, *4*, (1992), 816-837.

9. A. Halanay, T. Morozan and C. Tudor, *Tracking almost periodic signals under white noise perturbations*, Stochastics, **21**, (1987), 287-301.

10. A. Halanay and T. Morozan, *Adaptive stabilization tracking under white noise perturbations*, Int. J. Control, **49**,*2*, (1989), 533-544.

11. Y. Ji and H. J. Chizeck, *Jump linear quadratic Gaussian control: Steady state solution and testable conditions*, Contr. Theory and Adv. Technol. **6**,*3*, (1990), 389-319.

12. Y. Ji and H.J. Chizeck, *Controllability, stabilizability and continuous-time Markovian jump linear quadratic control*, IEEE Trans. Automat. Contr., **35**, *7*, (1990), 277-288.

13. R. E. Kalman, *Contributions to the theory of optimal control*, Bull. Soc. Math. Mexicana, **5**, (1960), 102-119.

14. A. Leizarowitz, *Infinite horizon stochastic regulation and tracking with overtaking criterion*, Stochastics, **22**, (1987), 117-150.

15. T. Morozan, *Stochastic uniform observability and Riccati equations of stochastic control*, Rev. Roumaine Math. Pures Appl., **38**, *9*, (1993), 771-781.

16. T. Morozan, *Stability and control for linear systems with jump Markov perturbations*, Stochastic Analysis Appl., **13**, *1* (1995), 91-110.

17. T. Morozan, *Optimal stationary control for dynamic systems with Markov perturbations*, Stochastic Analysis Appl., **1**, *3*, (1983), 299-325.

18. W.M. Wonham, *Random differential equations in control theory*, Probabilistic methods in Appl. Math., vol. 2, Academic Press, 1970.

19. J. Zabczyk, *On optimal control of discrete-time systems in Hilbert space*, SIAM J. Contr. **13**, (1975), 1217-1234.

Institute of Mathematics
of the Romanian Academy
P.O. Box 1-764
Bucharest, Romania