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Abstract. A control problem for micropolar fluids is considered. The purpose of this paper is to determine a viscosity coefficient which gives a desired field of microrotation velocity. The existence of an optimal control is obtained; then, the first order necessary conditions of optimality are derived.

1. INTRODUCTION

The theory of microfluids was introduced by Eringen in [1]. A subclass of these fluids is the micropolar fluids. Animal blood, liquid cristals, fluids containing certain additives may be represented by the mathematical model of micropolar fluids. This model can be found in [2]. From the physical point of view, micropolar fluids are characterized by the following property: fluid points contained in a small volume element, in addition to its usual rigid motion, can rotate about the centroid of the volume element in an average sense, described by the gyration tensor, ω. Since for a micropolar fluid the gyration tensor is skew-symmetric ($\omega_{kl} = -\omega_{lk}$), it is possible to replace the gyration tensor by a vector function $\vec{\omega}$ for a 3D flow and by a scalar function

ω for a 2D case, function called microrotation velocity. We shall study in this paper the 2D case. Let $\Omega \subset \mathbf{R}^2$ be an open, bounded, connected set, with $\partial\Omega$ of class C^2 and T a given positive constant. Taking into account the constitutive equations for micropolar fluids given in [2], the non stationnary flow of such a fluid is described by the following coupled system

$$(1.1) \qquad \begin{cases} \vec{v}' + (\vec{v}\cdot\nabla)\vec{v} - (\mu+\chi)\,\Delta\,\vec{v} + \nabla p - \chi\mathrm{rot}\omega = \vec{f} \ \text{ in } \Omega\times(0,T), \\ j\omega' + j\vec{v}\cdot\nabla\omega - \gamma\,\Delta\,\omega + 2\chi\omega - \chi\mathrm{rot}\vec{v} = g \ \text{ in } \Omega\times(0,T), \\ \mathrm{div}\vec{v} = 0 \ \text{ in } \Omega\times(0,T), \\ \vec{v} = \vec{0},\ \omega = 0 \ \text{ on } \partial\Omega\times(0,T), \\ \vec{v}(x,0) = \vec{0},\ \omega(x,0) = 0 \ \text{ in } \Omega, \end{cases}$$

where χ, μ, j, γ are positive given constants associated to the properties of the material, $\vec{f}$, g are given external fields and $\vec{v}$, ω, p are the unknown of the problem: the velocity, the microrotation and the pressure of the micropolar fluid, respectively. In section 2 we give the variational formulation of (1.1) and we establish existence and uniqueness results. Section 3 deals with the control problem. Since the viscosity coefficient χ has a special semnification for the system (1.1) (for $\chi = 0$ this system decouples), we took as control variable this coefficient. We want to determine the coefficient χ which realises a desired field of the microrotation velocity. The necessary conditions of optimality are deduced in the last section.

2. THE VARIATIONAL FORMULATION OF THE PROBLEM

The proof of the existence and uniqueness of weak solutions is, for micropolar fluids, similar to that for Navier-Stokes equations. Some results concerning existence and uniqueness of the solutions for micropolar fluids can be found in [3], [4].

For obtaining the variational formulation of the problem (1.1) we shall need the following spaces (for their properties see, e. g. [5])

$$(2.1)\begin{cases} V=\{\vec{u}\in (H_0^1(\Omega))^2\,/\,\mathrm{div}\vec{u}=0\},\\ H=\{\vec{u}\in (L^2(\Omega))^2\,/\,\mathrm{div}\vec{u}=0,\ \vec{u}\cdot\vec{n}/_{\partial\Omega}=0\},\\ W(0,T;X,X')=\{u\in L^2(0,T;X)/u'\in L^2(0,T;X')\},\ X\text{-Hilbert space.}\end{cases}$$

The following notations will be used throughout the paper

$(\cdot,\cdot)$ the scalar product , $|\cdot|$ the norm in $L^2(\Omega)$ or $(L^2(\Omega))^2$,

$((\cdot,\cdot))_0$ the scalar product , $\|\cdot\|$ the norm in $H_0^1(\Omega)$ or $(H_0^1(\Omega))^2$,

$\langle\cdot,\cdot\rangle_{X',X}$ the duality pairing between a space X and its dual X',

$B_1(\vec{u},\vec{v})=(\vec{u}\cdot\nabla)\vec{v},\quad B_2(\vec{u},\varphi)=\vec{u}\cdot\nabla\varphi\quad\forall\vec{u},\vec{v}\in(H_0^1(\Omega))^2,\ \varphi\in H_0^1(\Omega).$

Taking the regularity $\vec{f}\in L^2(0,T;V')$, $g\in L^2(0,T;H^{-1}(\Omega))$, the variational formulation of the problem (1.1) is given by

$$(2.2)\begin{cases} \vec{v}\in W(0,T;V,V'),\ \omega\in W(0,T;H_0^1(\Omega),H^{-1}(\Omega)),\\ \langle\vec{v}'(t),\vec{z}\rangle_{V',V}+(\mu+\chi)((\vec{v}(t),\vec{z}))_0+\langle B_1(\vec{v}(t),\vec{v}(t)),\vec{z}\rangle_{V',V}\\ \qquad -\chi(\mathrm{rot}\omega(t),\vec{z})=\langle\vec{f}(t),\vec{z}\rangle_{V',V}\ \forall\vec{z}\in V,\\ j\langle\omega'(t),\zeta\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}+\gamma((\omega(t),\zeta))_0+j\langle B_2(\vec{v}(t),\omega(t)),\zeta\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\\ +2\chi(\omega(t),\zeta)-\chi(\mathrm{rot}\vec{v}(t),\zeta)=\langle g(t),\zeta\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\ \forall\zeta\in H_0^1(\Omega),\\ \vec{v}(0)=\vec{0},\ \omega(0)=0.\end{cases}$$

THEOREM 2.1. *The problem* (2.2) *has a unique solution* $(\vec{v},\omega)$. *Moreover, there exists* $p\in\mathcal{D}'(\Omega\times(0,T))$, *unique up to the addition of a distribution in* $(0,T)$, *which satisfies, together with* $(\vec{v},\omega)$, *the system* (1.1).

Proof. The main steps of the proof are similar to those of [5], for Navier-Stokes equations, so we shall skip them. Analogously as in [5], the proof is based on the properties of B_1 and B_2.

3. THE DISTRIBUTED CONTROL PROBLEM

We consider the control problem

$$(3.1)\qquad \begin{cases} \text{Find } \chi* \in [0,r] \text{ such that} \\ J(\chi*) = \min\{J(\chi)\,/\,\chi \in [0,r]\}, \end{cases}$$

$$(3.2)\qquad J(\chi) = \frac{1}{2}\int_{\Omega_T} (\omega_\chi - \omega_d)^2 \mathrm{d}x\mathrm{d}t,\ \ J : [0,\infty) \mapsto \mathbf{R}$$

with $(\vec{v}_\chi, \omega_\chi)$ the unique solution of (2.2), $\Omega_T = \Omega \times (0,T)$, ω_d a given function in $L^2(\Omega_T)$ and r an arbitrarily large constant.

We want to determine the viscosity coefficient χ so that the corresponding microrotation velocity of the fluid, ω_χ, have a desired configuration, ω_d.

THEOREM 3.1. *The control problem* (3.1) *has at least a solution.*

Proof. We shall prove that the function J is continuous; the assertion of the theorem will follow, by using a classical theorem of Weierstrass.

Let $\{\chi_n\}_{n\in\mathbf{N}}$ be a convergent sequence to an element $\chi \in [0,r]$. We denote by $(\vec{v}_n,\ \omega_n)$ the unique solution of (2.2) corresponding to χ_n and by $(\vec{v},\ \omega)$ the solution of (2.2) corresponding to χ. We shall prove that $(\vec{v}_n,\ \omega_n) \to (\vec{v},\ \omega)$ strongly in $L^2(0,T;V) \times L^2(0,T;H_0^1(\Omega))$, when $n \to \infty$.

Subtracting the equations of (2.2) corresponding to χ_n and χ, respectively, taking $\vec{z} = \vec{v}_n(t) - \vec{v}(t)$, $\zeta = \omega_n(t) - \omega(t)$ and adding the equations we get

$$\tfrac{1}{2}\left(|\ \vec{v}_n(t) - \vec{v}(t)\ |^2 + j\ |\ \omega_n(t) - \omega(t)\ |^2\right)' + (\mu+\chi)\,\|\vec{v}_n(t) - \vec{v}(t)\|_0^2 +$$
$$\gamma\|\omega_n(t) - \omega(t)\|_0^2 + 2\chi\ |\ \omega_n(t) - \omega(t)\ |^2 = \chi(\mathrm{rot}(\omega_n(t) - \omega(t)), \vec{v}_n(t) - \vec{v}(t)) +$$
$$\chi(\omega_n(t) - \omega(t), \mathrm{rot}(\vec{v}_n(t) - \vec{v}(t))) - \langle B_1(\vec{v}_n(t) - \vec{v}(t), \vec{v}(t)), \vec{v}_n(t) - \vec{v}(t)\rangle_{V',V}$$
$$-j\langle B_2(\vec{v}_n(t) - \vec{v}(t), \omega(t)), \omega_n(t) - \omega(t)\rangle + (\chi_n - \chi)(-((\vec{v}_n(t), \vec{v}_n(t) - \vec{v}(t)))_0$$
$$+(\mathrm{rot}\omega_n(t), \vec{v}_n(t) - \vec{v}(t)) - 2(\omega_n(t), \vec{v}_n(t) - \vec{v}(t)) + (\omega_n(t) - \omega(t), \mathrm{rot}\vec{v}_n(t))).$$

Using the property of B_1 (see [5], Lemma 3.4., p. 292), which can be obtained also for B_2, and majorizing the right-hand side of the above equality, we get

$$(\mid \vec{v}_n(t)-\vec{v}(t)\mid^2+j\mid \omega_n(t)-\omega(t)\mid^2)'+(\mu+\chi)\|\vec{v}_n(t)-\vec{v}(t)\|_0^2+$$
$$\gamma\|\omega_n(t)-\omega(t)\|_0^2\leq A(t)(\mid \vec{v}_n(t)-\vec{v}(t)\mid^2+j\mid\omega_n(t)-\omega(t)\mid^2)+(\chi_n-\chi)E_n(t),$$

with $A(t)=2\max(\frac{12}{\mu+\chi}\|\vec{v}(t)\|_0^2+\frac{12j^2}{\mu+\chi}\|\omega(t)\|_0^2+\frac{4\chi^2}{\gamma};\frac{8j}{\gamma}\|\omega(t)\|_0^2+\frac{24\chi^2}{j(\mu+\chi)})$, $E_n(t)=2\mid-((\vec{v}_n(t),\vec{v}_n(t)-\vec{v}(t)))_0+(\mathrm{rot}\omega_n(t),\vec{v}_n(t)-\vec{v}(t))-2(\omega_n(t),\vec{v}_n(t)-\vec{v}(t))+(\omega_n(t)-\omega(t),\mathrm{rot}\vec{v}_n(t))\mid$. Integrating the inequality from 0 to T it follows

$$\text{(3.3)}\quad \begin{aligned}&(\mid\vec{v}_n(T)-\vec{v}(T)\mid^2+j\mid\omega_n(T)-\omega(T)\mid^2)\exp(-\int_0^T A(t)dt)+(\mu+\chi)\|\vec{v}_n-\vec{v}\|^2_{L^2(0,T;V)}\\&+\gamma\|\omega_n-\omega\|^2_{L^2(0,T;H_0^1(\Omega))}\leq 2(\chi_n-\chi)\exp(-\int_0^T A(t)dt)\int_0^T E_n(t)dt\end{aligned}$$

The system (2.2) written for $\chi=\chi_n$ with $\vec{z}=\vec{v}_n(t)$, $\zeta=\omega_n(t)$ gives the boundedness of the sequence $\{(\vec{v}_n,\ \omega_n)\}_{n\in\mathbf{N}}$ in $L^2(0,T;V)\times L^2(0,T;H_0^1(\Omega))$; hence, from the definition of $E_n(t)$ we obtain the boundedness of the sequence $\{\int_0^T E_n(t)dt\}_{n\in\mathbf{N}}$ and the proof of the theorem is achieved.

PROPOSITION 3.2. *The function J is differentiable on $[0,r]$ and*

$$\text{(3.4)}\qquad J'(\chi_0)(\chi-\chi_0)=\int_{\Omega_T}(\omega^*-\omega_0)(\omega_0-\omega_d)dxdt,\ \forall\chi_0,\chi\in[0,r]$$

where $(\vec{v}^*,\omega^*)$ is the unique solution of the system

$$\text{(3.5)}\begin{cases}\vec{v}^*\in W(0,T;V,V'),\ \omega^*\in W(0,T;H_0^1(\Omega),H^{-1}(\Omega)),\\ \langle\vec{v}^{*\prime}(t),\vec{z}\rangle_{V',V}+(\mu+\chi_0)((\vec{v}^*(t),\vec{z}))_0+\langle B_1(\vec{v}^*(t),\vec{v}_0(t)),\vec{z}\rangle_{V',V}\\ +\langle B_1(\vec{v}_0(t),\vec{v}^*(t)),\vec{z}\rangle_{V',V}-\chi_0(\mathrm{rot}\omega^*(t),\vec{z})=\langle\vec{f}(t),\vec{z}\rangle_{V',V}-(\chi-\chi_0)\\ ((\vec{v}_0(t),\vec{z}))_0+\langle B_1(\vec{v}_0(t),\vec{v}_0(t)),\vec{z}\rangle_{V',V}+(\chi-\chi_0)(\mathrm{rot}\omega_0(t),\vec{z})\ \forall\vec{z}\in V,\\ j\langle\omega^{*\prime}(t),\zeta\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}+\gamma((\omega^*(t),\zeta))_0+j\langle B_2(\vec{v}_0(t),\omega^*(t)),\zeta\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\\ +j\langle B_2(\vec{v}^*(t)-\vec{v}_0(t),\omega_0(t)),\zeta\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}+2\chi_0(\omega^*(t),\zeta)-\chi_0(\mathrm{rot}\vec{v}^*(t),\zeta)\\ =\langle g(t),\zeta\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}-(\chi-\chi_0)(2(\omega_0(t),\zeta)-(\mathrm{rot}\vec{v}_0(t),\zeta))\ \forall\zeta\in H_0^1(\Omega),\\ \vec{v}^*(0)=\vec{0},\ \omega^*(0)=0,\end{cases}$$

and $(\vec{v}_0, \omega_0)$ is the unique solution of (2.2) corresponding to $\chi = \chi_0$.

Proof. The existence and uniqueness of $(\vec{v}^*,\ \omega^*)$ follow with similar techinques as those of Theorem 2.1. Let $\alpha \in (0,1)$ and let $(\vec{v}_{\alpha\chi},\ \omega_{\alpha\chi})$ be the unique solution of (2.2) corresponding to $\chi_0 + \alpha(\chi - \chi_0)$. We introduce the functions $\vec{v}_\alpha = (\vec{v}_{\alpha\chi} - \vec{v}_0)/\alpha + \vec{v}_0$; $\omega_\alpha = (\omega_{\alpha\chi} - \omega_0)\alpha + \omega_0$. We shall obtain the problem satisfied by $(\vec{v}_\alpha,\ \omega_\alpha)$ computing ((2.2) for $\chi = \chi_0 + \alpha(\chi - \chi_0) -$ (2.2) for $\chi = \chi_0)/\alpha +$ (2.2) for $\chi = \chi_0$; taking $\vec{z} = \vec{v}_\alpha(t)$, $\zeta = \omega_\alpha(t)$ in the obtained system, adding the two equalities and using the same technique as the one of Theorem 3.1. we obtain the boundedness of $\{(\vec{v}_\alpha,\ \omega_\alpha)\}_{\alpha\in(0,1)}$ in $L^2(0,T;V) \times L^2(0,T;H_0^1(\Omega))$,then the convergence of $\{(\vec{v}_\alpha,\ \omega_\alpha)\}_{\alpha\in(0,1)}$ in $W(0,T;V,V') \times W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$. Finally, using this convergence, we obtain (3.4). Let χ_0 be a solution of the control problem (3.1) and $(\vec{v}_0,\ \omega_0)$ the corresponding solution of (2.2). Then (3.4) yields

$$\int_{\Omega_T} (\omega^* - \omega_0)(\omega_0 - \omega_d)\mathrm{d}x\mathrm{d}t \geq 0. \tag{3.6}$$

4. OPTIMALITY CONDITIONS

Let χ_0 be an optimal control and $(\vec{v}_0,\ \omega_0)$ the unique solution of (2.2) corresponding to $\chi = \chi_0$. We consider the following adjoint problem

$$\tag{4.1} \begin{cases} \vec{u}_0 \in W(0,T;V,V'),\ \ \rho_0 \in W(0,T;H_0^1(\Omega),H^{-1}(\Omega)), \\ -\langle \vec{u}_0'(t), \vec{z}\rangle_{V',V} + (\mu + \chi_0)((\vec{u}_0(t),\vec{z}))_0 + \langle B_1(\vec{z},\vec{v}_0(t)), \vec{u}_0(t)\rangle_{V',V} \\ -\langle B_1(\vec{v}_0(t),\vec{u}_0(t)),\vec{z}\rangle_{V',V} - j\langle B_2(\vec{z},\omega_0(t)),\rho_0(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ +\chi_0(\mathrm{rot}\rho_0(t),\vec{z}) = 0 \ \ \forall \vec{z} \in V, \\ -j\langle \rho_0'(t),\zeta\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \gamma((\rho_0(t),\zeta))_0 - j\langle B_2(\vec{v}_0(t),\rho_0(t)),\zeta\rangle_{H^{-1},H_0^1} \\ +2\chi_0(\rho_0(t),\zeta) + \chi_0(\mathrm{rot}\vec{u}_0(t),\zeta) = (\omega_0(t) - \omega_d(t),\zeta) \ \ \forall\zeta \in H_0^1(\Omega), \\ \vec{u}_0(T) = \vec{0},\ \ \rho_0(T) = 0. \end{cases}$$

PROPOSITION 4.1. *The system (4.1) has a unique solution* $(\vec{u}_0, \rho_0)$.

Proof. For obtaining the existence, the uniqueness and the regularity of $(\vec{u}_0, \rho_0)$, we use the same remark as the one of the proof of Theorem 2.1.

The last result ov this paper states the optimality conditions associated to a solution χ_0 of (3.1).

THEOREM 4.2. *let* χ_0 *be an optimal control. Then there exist the unique elements* $(\vec{v}_0,\ \omega_0)$, $(\vec{u}_0,\ \rho_0) \in W(0,T;V,V') \times W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ *and the distributions* $p_0,\ \pi_0 \in \mathcal{D}'(\Omega_T)$, *unique up to an additive distribution in* $(0,T)$, *satisfying the following problem*

$$
(4.2)\quad \begin{cases}
\text{system (1.1) for } \chi = \chi_0 \text{ and the unknowns } \vec{v} = \vec{v}_0,\ \omega = \omega_0,\ p = p_0 \\
\begin{cases}
-\dfrac{\partial \vec{u}_0}{\partial t} - (\mu + \chi_0) \triangle \vec{u}_0 - j\rho_0 \nabla \omega_0 + (\nabla \vec{v}_0)^T \vec{u}_0 - B_1(\vec{v}_0,\ \vec{u}_0) + \nabla \pi_0 \\
-\chi_0 \mathrm{rot} \rho_0 = 0 \ \text{ in } \Omega_T, \\
-j\dfrac{\partial \rho_0}{\partial t} - \gamma \triangle \rho_0 - jB_2(\vec{v}_0,\ \rho_0) + 2\chi_0\rho_0 + \chi_0 \mathrm{rot} \vec{u}_0 = \omega_0 - \omega_d \text{ in } \Omega_T, \\
\mathrm{div} \vec{u}_0 = 0 \ \text{ in } \Omega_T, \\
\vec{u}_0 = \vec{0},\ \rho_0 = 0 \ \text{ on } \partial\Omega \times (0,T), \\
\vec{u}_0(x,T) = \vec{0},\ \rho(x,T) = 0 \ \text{ in } \Omega,
\end{cases} \\
(\chi_0 - \chi)\displaystyle\int_{\Omega_T} (\vec{u}_0 \cdot \mathrm{rot}\omega_0 - \rho_0 \mathrm{rot} \vec{v}_0 + 2\omega_0\rho_0 - \nabla \vec{v}_0 \cdot \nabla \vec{u}_0) dxdt \geq 0\ \forall \chi \in [0,r].
\end{cases}
$$

Proof. The first assertion of the theorem has been already proved. The existence of a distribution π_0 is obtained as in [5], for Navier-Stokes equations.

We have to prove next the inequality of the system (4.2). This inequality without contraints replaces the inequality (3.6).

Taking adequate test functions in (4.1) and in (3.5)-(2.2) and using the equality

$$(\mathrm{rot}\omega, \vec{v}) = (\mathrm{rot}\vec{v}, \omega)\ \forall \omega \in H_0^1(\Omega),\ \vec{v} \in (H_0^1(\Omega))^2,$$

we get

$$(\omega^*(t)-\omega_0(t),\omega_0(t)-\omega_d(t))=(\chi_0-\chi)(2(\omega_0(t),\rho_0(t))-(\mathrm{rot}\vec{v}_0(t),\rho_0(t)))$$
$$-j\langle B_2(\vec{v}^*(t)-\vec{v}_0(t),\omega_0(t)),\rho_0(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}+\chi_0(\mathrm{rot}(\vec{v}^*(t)-\vec{v}_0(t)),\rho_0(t))$$
$$+\chi_0(\mathrm{rot}(\vec{u}_0(t),\omega^*(t)-\omega_0(t))=(\chi_0-\chi)(\vec{u}_0(t),\mathrm{rot}\omega_0(t))-(\rho_0(t),\mathrm{rot}\vec{v}_0(t))$$
$$+2(\omega_0(t),\rho_0(t))-((\vec{v}_0(t),\vec{u}_0(t)))_0)$$

Integrating the previous equality from 0 to T and using (3.6) we obtain the inequality $(4.2)_3$ and hence, the proof is achieved.

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