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# A property of Sobolev spaces and existence in optimal design

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*Abstract* We prove that for bounded open sets $\Omega$ with continuous boundary, Sobolev spaces of type $W_0^{l,p}(\Omega)$ are characterised by the zero extension outside of $\Omega$. Combining this with a compactness result for domains of class $\mathcal{C}$, we obtain a general existence theorem for shape optimization problems governed by nonlinear nonhomogenous Dirichlet boundary value problems of arbitrary order and with general cost functionals.



AMS Classifications : 49D37, 65K10


## 1. Introduction

The literature concerning existence theory for shape optimization problems is very rich. There are several types of results: using regularity assumptions for the boundary of the unknown domains (see Chenais [6], Pironneau [14]), using certain capacitary constraints (Sverak [15], Bucur and Zolesio [3], Zhong [17]) or using the notion of generalized perimeter and constraints or penalty terms constructed with it (Bucur and Zolesio [3], [5]). In the second case, conditions on the dimension of the underlying Euclidean space have to be imposed in order to obtain the compactness of certain families of open sets with respect to the Hausdorff-Pompeiu distance.

In this work, we study bounded open sets of class $\mathcal{C}$, in the sense of Maz'ya [13] or, equivalently, with the segment property, according to Adams [1].

In section 2, we prove a compactness result in this class of open sets. An announcement result may be found in Liu, Neittaanmaki and Tiba [12], in a different context.

Section 3 extends the wellknown property for Sobolev spaces (see Henrot [7]): if $z \in H_0^1(D)$ and $z = 0$ quasieverywhere in $D - \Omega$ (where $\Omega \subset D$ are open sets), then $z \in H_0^1(\Omega)$. Our results just assume that $z = 0$ a.e. in $D - \Omega$ and $\Omega \in \mathcal{C}$.

The last section discusses existence in shape optimization problems governed by arbitrary order nonlinear and nonhomogeneous Dirichlet boundary value problems and with general cost functionals. Notice that the case of linear elliptic operators of order $2l$, $l \in N$, is a special case of our results. Moreover, our general setting allows nonuniqueness for the solution of the nonlinear Dirichlet problem.

It is recognized in the literature that, in establishing the continuity of the mapping between an open set and the solution of some partial differential equation defined on it, the convergence of the associated characteristic functions is a fundamental property. Our final result discusses the necessity of this property, under certain supplementary assumptions.

## 2. Convergence of open sets

Let $\mathcal{O}$ be the family of open sets contained in a prescribed bounded domain $D \subset R^m$, $m \in N$.

The usual topology on $\mathcal{O}$ is given by the Hausdorff - Pompeiu distance between the complementary sets (which are closed):

$$\rho(\Omega_1, \Omega_2) = dist\left(\overline{D}\backslash\Omega_1, \overline{D}\backslash\Omega_2\right), \quad \forall \Omega_1, \Omega_2 \in \mathcal{O}. \tag{2.1}$$

We denote by *Hlim*, the limit in the sense of (2.1) and it is wellknown that $\rho$ has the compactness property: if $\Omega_n \subset D$, $n \in N$, are open bounded sets, there is $\Omega \subset D$, open, such that $\Omega = Hlim\, \Omega_n$, on a subsequence (see Kuratowski [9], Pironneau [14]).

Moreover, for any open subset $K \subset\subset \Omega$ (compactly embedded) there is $n_K = n(K) \in N$ such that $K \subset\subset \Omega_n$ for $n \geq n_K$. This is called the $\Gamma$ property and proofs may be found in Pironneau [14], Liu and Rubio [11].

We say that an open set $\Omega$ is of class $\mathcal{C}$ if there is a family $\mathcal{F}_\Omega$ of continuous functions $g : \overline{S(0, k_\Omega)} \to R$, with $k_\Omega > 0$ and $S(0, k_\Omega) \subset R^{m-1}$ being the open ball of center 0 and radius $k_\Omega$, such that

$$\partial\Omega = \bigcup_{g \in \mathcal{F}_\Omega} \{(s, g(s));\ s \in S(0, k_\Omega)\}. \tag{2.2}$$

Here, we have slightly modified the corresponding definition from Maz'ya [13], by imposing that all the local charts are defined on balls with the same radius, which is always possible. Furthermore, there is $r_\Omega \in ]0, k_\Omega[$ such that the "restricted" local charts defined on $\overline{S(0, r_\Omega)}$ also give a covering of $\partial\Omega$ :

$$(2.3) \qquad \partial\Omega = \bigcup_{g \in \mathcal{F}_\Omega} \left\{ (s, g(s)) ; \ s \in \overline{S(0, r_\Omega)} \right\}.$$

Open sets of class $\mathcal{C}$ have the segment property (interior and exterior), Maz'ya [13], Adams [1]: for any local chart $g \in \mathcal{F}_\Omega$, there are $y_g \in R^m$, vectors of length one, and $a_\Omega > 0$ such that the points

$$(2.4) \qquad (s, g(s) - ty_g) \in \Omega, \quad t \in ]0, a_\Omega[, \quad s \in S(0, k_\Omega)$$

$$(2.5) \qquad (s, g(s) + ty_g) \in R^m - \overline{\Omega}, \quad t \in ]0, a_\Omega[, \quad s \in S(0, k_\Omega).$$

Notice that the vector $y_g$ may be choosen as the "vertical" axis in the local coordinate system corresponding to the local chart $g \in \mathcal{F}_\Omega$.

Let $\Omega_0 = Hlim\ \Omega_n$ and $\Omega_n$, $n \geq 1$ be some open subsets of class $\mathcal{C}$. We denote by $k_n, r_n, a_n > 0$ the corresponding constants from (2.2) – (2.5).

**Theorem 2.1** *Assume that* $k_n \geq k > 0$, $r_n \leq r < k$, $a_n \geq a > 0$, $\forall n \geq 1$ *and that the family* $\mathcal{F} = \bigcup \mathcal{F}_{\Omega_n}$ *is equicontinuous and equibounded. Then* $\Omega_0 = Hlim\Omega_n$ *is of class* $\mathcal{C}$ *with* $k_{\Omega_0} \geq k$, $r_{\Omega_0} \leq r$, $a_{\Omega_0} \geq a$ *and the characteristic functions* $\chi_n$, *associated to* $\Omega_n$ *in* $D, n \geq 0$, *satisfy* :

$$(2.6) \qquad \chi_n \to \chi_0 \quad a.e.\ in\ D.$$

*Proof*

Denote by $d_n : \overline{D} \to R$ the following distance type functions:

$$(2.7) \qquad d_n(x) = \begin{cases} dist(x, \overline{D} - \Omega_n), & x \in \Omega_n, \\ 0, & x \in \partial\Omega_n, \\ -dist(x, \overline{\Omega}_n), & x \in \overline{D} - \overline{\Omega}_n. \end{cases}$$

They are uniformly Lipschitzian in $\overline{D}$ and we may assume that $d_n \to \widehat{d} \in C(\overline{D})$, uniformly. Let $\Lambda = \left\{ x \in \overline{D};\ \widehat{d}(x) \geq 0 \right\}$ be a closed set, clearly nonvoid. Take $\widehat{x} \in \Lambda$ with $\widehat{d}(\widehat{x}) = 0$. Then $d_n(\widehat{x}) \to 0$, by the definition of $\widehat{d}$. By the definition of $d_n$, as a distance function, there is $x_n \in \partial\Omega_n$, i.e. $d_n(x_n) = 0$ and $x_n \to \widehat{x}$.

By (2.3), there is $g_n \in \mathcal{F}_{\Omega_n}$ such that $x_n = (s_n, g_n(s_n))$, $s_n \in \overline{S(0, r_n)}$. Under our assumptions, we may assume that $s_n \to \widehat{s} \in \overline{S(0, r)}$, and $g_n \to \widehat{g}$ uniformly in $S(0, k)$, $\widehat{g}$ being continuous and bounded, with the same modulus of continuity as the family $\mathcal{F}$. We have

$$(2.8) \qquad \widehat{x} = \lim x_n = \lim (s_n, g_n(s_n)) = (\widehat{s}, \widehat{g}(\widehat{s})),$$

$$(2.9) \qquad d_n(s, g_n(s)) \to \widehat{d}(s, \widehat{g}(s)) = 0, \quad \forall s \in S(0,k),$$

by the uniform convergence of $d_n$, $g_n$ and (2.7).

We show the segment property.

Take any $\varepsilon \in ]0, a[$ and consider the point $(s, \widehat{g}(s) - \varepsilon) \in R^m$, $s \in S(0,k)$. We have that $(s, g_n(s) - \varepsilon) \to (s, \widehat{g}(s) - \varepsilon)$ and $(s, g_n(s) - \varepsilon) \in \Omega_n$ by (2.4). Then $d_n(s, g_n(s) - \varepsilon) > 0$ and, consequently, $\widehat{d}(s, \widehat{g}(s) - \varepsilon) \geq 0$ for $s \in S(0,k), \varepsilon \in ]0, a[$, *i.e.* $(s, \widehat{g}(s) - \varepsilon) \in \Lambda$ for such values of the parameters $s, \varepsilon$.

For the outside segment property a sharper estimate is needed. By the equicontinuity of $g_n$, there is $\delta > 0$ (depending only on $\varepsilon$ and independent of $s \in S(0,k)$ and of $n \in N$) such that

$$(2.10) \qquad |g_n(t) - g_n(s)| < \frac{\varepsilon}{2}, \ \forall\, n, \ \forall\, t \in S(s,\delta) \cap S(0,k).$$

Then, for $\varepsilon < \frac{2}{3}a$, we get

$$(2.11) \qquad dist\left[(s, g_n(s) + \varepsilon), \partial\Omega_n\right] \geq min\left\{\frac{\varepsilon}{2}, \delta, a - \frac{3\varepsilon}{2}, dist(s, \partial S(0,k))\right\}$$

Here, we use the uniform outside segment property (2.5), i.e. $(s, g_n(s) + \varepsilon) \in R^m - \overline{\Omega}$ for any $s \in S(0,k)$, $\forall\, \varepsilon \in ]0, a[$. The inequality (2.11) comes from (2.10) which simply says that the cylinder

$$[S(0,k) \cap S(s,\delta)] \times \left[g_n(s) + \frac{\varepsilon}{2}, g_n(s) + a - \frac{\varepsilon}{2}\right]$$

cannot intersect $\partial\Omega_n$, for any $n$. And the right-hand side in (2.11) is an estimate from below of the distance between $(s, g_n(s) + \varepsilon)$ and the boundary of this cylinder. Notice that this point is inside the cylinder if $\varepsilon < \frac{2}{3}a$. It yields

$$(2.12) \qquad d_n(s, g_n(s) + \varepsilon) \leq -min\left\{\frac{\varepsilon}{2}, \delta, a - \frac{3}{2}\varepsilon, dist(s, \partial S(0,k))\right\}.$$

Inequality (2.12) is independent of $n$ and we can pass to the limit to obtain

$$(2.13) \qquad \widehat{d}(s, \widehat{g}(s) + \varepsilon) \leq -min\left\{\frac{\varepsilon}{2}, \delta, a - \frac{3}{2}\varepsilon, dist(s, \partial S(0,k))\right\},$$

that is $\widehat{d}(s, \widehat{g}(s) + \varepsilon) < 0$, $\forall\, s \in S(0,k)$, $\forall \varepsilon \in \left]0, \frac{2}{3}a\right[$ and, consequently $(s, \widehat{g}(s) + \varepsilon) \notin \Lambda$ for these values of the parameters $s$, $\varepsilon$. By choosing a smaller $\delta > 0$, if necessary, we can replace $\frac{\varepsilon}{2}$ by $\frac{\varepsilon}{l}$, $l \in N$ and $\frac{3}{2}\varepsilon$ by $\frac{l+1}{l}\varepsilon$ in inequalities (2.10) - (2.13). Finally, we get that $(s, \widehat{g}(s) + \varepsilon) \notin \Lambda$ for $s \in S(0,k)$ and $\varepsilon \in ]0, a[$.

Notice that estimates like (2.11), (2.12) can also be obtained for $\widehat{d}(s, \widehat{g}(s) - \varepsilon)$, $s \in S(0,k)$, $\varepsilon \in ]0, a[$ , with the reversed sign. Then:

$$\widehat{\Omega} = \left\{x \in D;\ \widehat{d}(x) > 0\right\} \tag{2.14}$$

is a nonvoid and open subset of $\Lambda$. Relations (2.8), (2.9) show that $\partial\widehat{\Omega} = \left\{x \in D;\ \widehat{d}(x) = 0\right\}$ and it has a local representation by continuous mappings with the same modulus of continuity as the family $\mathcal{F}$. Moreover, the above argument yields that $\widehat{\Omega}$ satisfies (2.2) - (2.5) with constants limited by $k, r, a$ as required. We continue by

**Lemma 2.2** *$\overline{D} - \widehat{\Omega}$ is the Hausdorff – Pompeiu limit of $\overline{D} - \Omega_n$, i.e. $\widehat{\Omega} = \Omega_0$.*

*Proof*

If $x \in \Omega_0 \Rightarrow \lim_{n\to\infty} dist\left(x, \overline{D} - \Omega_n\right) > 0 \Rightarrow \lim_{n\to\infty} d_n(x) > 0 \Rightarrow \widehat{d}(x) > 0 \Rightarrow x \in \widehat{\Omega}$.

Conversely, assuming $x \in \widehat{\Omega}$ and $x \in \overline{D} - \Omega_0$, then $\widehat{d}(x) > 0$ and there is $x_n \in \overline{D} - \Omega_n$, $x_n \to x$. This means $\widehat{d}(x) > 0$ and $d_n(x_n) \leq 0, x_n \to x$. Finally, we get a contradiction: $\widehat{d}(x) > 0$ and $\widehat{d}(x) \leq 0$, by the uniform convergence $d_n \to \widehat{d}$ in $\overline{D}$. We conclude that $\widehat{\Omega} \cap \left(\overline{D} - \Omega_0\right) = \Phi$ and the Lemma is proved.

*Proof of Theorem 2.1 (continued)*

We have to show (2.6). We remark that $meas\ (\partial\Omega_n) = 0,\ n \geq 0$, since $\partial\Omega_n$ can be represented as a finite union of graphs of continuous functions, by the first part of the theorem.

Consider $H \subset R \times R$ to be the maximal monotone extension of the Heaviside mapping, i.e.

$$H(y) = \begin{cases} 0 & y < 0, \\ [0,1] & y = 0, \\ 1 & y > 0. \end{cases} \tag{2.14}$$

Notice that $\chi_n = H(d_n),\ n \geq 1,\ \chi_0 = H(\widehat{d})$, due to (2.14) and to the fact that $meas\,(\partial\Omega_n) = 0,\ n \geq 0$. If $\widehat{d}(x) > 0$, then $d_n(x) > 0$ for $n \geq n_x$ (by $d_n \to \widehat{d}$ ) and $\chi_n(x) = H(d_n(x)) = H\left(\widehat{d}(x)\right) = \chi_0(x) = 1$. If $\widehat{d}(x) < 0$, we get similarly that $\chi_n(x) = \chi_0(x) = 0$ for $n \geq n_x$.

These two situations are valid a.e. in $D$ and the proof is finished.

*Remark* Domains of class $C$ may have cusps and infinitely many oscillations with vanishing amplitude (to preserve equicontinuity) are allowed. However, cracks or oscillations dense in a set of positive measure are not permitted under assumptions of Theorem 2.1.

## 3. A property of Sobolev spaces

We start with a simple situation when the following "global" representation is valid:

$$\Omega = \left\{(s,y) \in \widetilde{D}\ s \in U,\ y < g(s)\right\} \subset R^m, \tag{3.1}$$

where $U \subset R^{m-1}$ is a bounded open set, $\widetilde{D} = U \times ]0,b[$ and $g : U \to R_+$ is continuous such that $b \geq g(s) \geq c > 0$, $\forall s \in U$, with $b$, $c$ some positive constants.

**Proposition 3.1** *If $z \in H_0^1\left(\widetilde{D}\right)$ has compact support in $\widetilde{D}$ and $z = 0$ a.e. in $\widetilde{D} - \Omega$, then $z \in H_0^1(\Omega)$.*

*Proof*

We denote by $\Gamma$ the part of $\partial\Omega$ represented by g. According to (3.1), the segment property (inside and outside $\Omega$) is valid on $\Gamma$, with "vertical" segments of length at least $c > 0$.

We define the "translated" functions

$$z_t(s,y) = \widetilde{z}(s, y+t),\ y \in ]0,b[,\ t > 0,\ s \in U, \tag{3.2}$$

where $\widetilde{z}$ is the extension by 0 of $z$ to $\mathrm{R}^m$. If $t < min\left\{\frac{1}{2}dist\left(supp\ z, \partial\widetilde{D}\right), c\right\}$, then $z_t$ is well defined and $z_t \in H_0^1\left(\widetilde{D}\right)$ with $supp\ z_t \subset \Omega$. This follows by the observation that $z_t = 0$ a.e. in the interior band (to $\Omega$)

$$\{(s,y) \in \Omega;\ g(s) - t < y < g(s)\},$$

a translation of the exterior (to $\Omega$) band

$$\{(s,y) \in R^m;\ s \in U,\ g(s) + t > y > g(s)\},$$

both given by the segment property. Moreover, the interior band is neighbourhood of $\Gamma_t = \left\{w \in \Gamma;\ dist\left(w, \partial\widetilde{D}\right) > t\right\}$, in $\overline{\Omega}$, again by the segment property.

Concerning $\partial\Omega - \Gamma$, there is a neighbourhood of it such that $z_t$ is null a.e. in this neighbourhood, due to $t < \frac{1}{2}dist\left(supp\ z, \partial\widetilde{D}\right)$.

We conclude that $z_t \in H_0^1(\Omega)$, $\forall\ t > 0$ sufficiently small. By the continuity, in the mean of the translation, we get that $\lim_{t\to 0} z_t = z$ in $H^1(\Omega)$, i.e. $z \in H_0^1(\Omega)$. This ends the proof.

**Theorem 3.2** *Let $\Omega$ be a bounded domain of class $C$. If $z \in H^1(R^m)$ and $z = 0$ a.e. in $R^m \backslash \Omega$, then $z \in H_0^1(\Omega)$.*

*Proof*

We may assume that $\partial\Omega$ is covered by a finite number of local maps, denoted by $\mathcal{O}_j$:

$$\bigcup_{j=1}^{k} \mathcal{O}_j \supset \partial\Omega. \tag{3.3}$$

Assume that $\lambda > 0$ is the minimum length of the segments given by the segment property in all the local charts, inside and outside $\Omega$. Let $U_j \subset R^{m-1}$ be open subsets and $g_j : U_j \to R_+$, continuous, be the local representation, in each $\mathcal{O}_j$, of $\partial\Omega$, such that the local "vertical" axis is given by the segments provided by the segment property.

We may take $\mathcal{O}_j$, $j = \overline{1,k}$, to be given by the union of inside and outside segments from the segment property, which generate a neighbourhood of $\partial\Omega \cap \mathcal{O}_j$. By restricting $U_j$ if necessary, we may assume that

$$\max_{x\in U_j} g_j(x) - \min_{x\in U_j} g_j(x) \le \frac{\lambda}{4}, \tag{3.4}$$

due to the continuity of $g_j$. Then, the system of local axes may be translated in the "vertical" direction such that the cylinder:

$$V_j = \left\{ (s,y) \in R^m;\ s \in U_j,\ 0 = \min g_j(s) - \frac{\lambda}{4} < y < \max g_j(x) + \frac{\lambda}{4} \right\}$$

satisfies $V_j \subset \mathcal{O}_j$ and

$$\bigcup_{j=1}^{k} V_j \supset \partial\Omega. \tag{3.3$'$}$$

There is an open set $V_0$ with $\overline{V}_0 \subset \Omega$ such that $\bigcup_{j=0}^{k} V_j \supset \Omega$. We consider a partition of unity $\{\Psi_j\}_{j=\overline{0,k}}$, subordinated to the covering $\{V_j\}_{j=\overline{0,k}}$, such that $\Psi_j \in C_o^\infty(V_j)$, $\Psi_j \ge 0$, $j = \overline{0,k}$, and

$$\sum_{j=0}^{k} \Psi_j(x) = 1,\ x \in \overline{\Omega}. \tag{3.5}$$

We denote by $z_j = z\Psi_j \in H_0^1(V_j)$. Then, (3.5) gives

$$z(x) = \sum_{j=0}^{k} \widetilde{z}_j(x), \quad x \in R^m \tag{3.6}$$

where $\widetilde{z}_j$ are the extensions by zero of $z_j$ to $R^m$, $j = \overline{0,k}$ and we use that $z$ is zero a.e. in $R^m \backslash \Omega$. Clearly, $\widetilde{z}_0 \in H_0^1(\Omega)$ and we show the same property for $\widetilde{z}_j$, $j = \overline{1,k}$. This follows by Proposition 3.1, applied in each $V_j$ as $\widetilde{D}$ and with the obvious conclusion that $z_j \in H_0^1(Q_j)$, $Q_j = \{(s,y) \in V_j;\ 0 < y < g_j(s)\}$. As $Q_j \subset \Omega$, the proof is finished by (3.6).

**Corollary 3.3** *Let $\Omega$ be a bounded open set of class $C$ in $R^m$. If $z \in W^{l,p}(R^m)$, $l \in N$, $1 \le p < \infty$, and $z = 0$ a.e. in $R^m \backslash \Omega$, then $z \in W_0^{l,p}(\Omega)$.*

*Remark* In case $\partial\Omega$ is Lischitzian, trace theorems may be applied and Theorem 3.2 is obvious. Weaker conditions of Hölder type were considered by Pironneau [14], Ladyzenskaya and Uraltseva [10]. If $z = 0$ in $R^m - \Omega$ quasieverywhere, similar results are known and a recent survey with applications is Henrot [7]. Our assumptions allow cusps for $\partial\Omega$ and use the Lebesgue mesure.

## 4. Existence in optimal design problems involving Dirichlet boundary conditions

Let $\mathcal{O}$ be a family of open subsets of class $C$ in the bounded open domain $D \subset R^m$. To each $\Omega \in \mathcal{O}$, we associate $f_\Omega \in L^2(\Omega)$ and $h_\Omega \in W^{l,p}(\Omega)$, $1 < p < \infty$, assumed to be bounded in the norms of their spaces, with respect to all $\Omega$ in $\mathcal{O}$. The functions $f_\Omega$ may be extended by 0 to the whole $D$ or to $R^m$ and we shall preserve the same notation and, clearly, $\{f_\Omega\}$ remains bounded in $L^2(R^m)$. This is not possible for the mappings $h_\Omega$, due to the absence of regularity properties for $\Omega \in \mathcal{O}$.

In $D$, we consider the partial differential operator

(4.1) $$Az = \sum_{|\alpha| \le l} (-1)^{|\alpha|} D^\alpha A_\alpha \left(x, z, \ldots, D^l z\right) \quad , \quad x \in D,$$

where $z \in W^{l,p}(D)$ and $D^\alpha$, $D^l$ denote derivatives in the sense of distributions, $\alpha$ is a multiindex of length $|\alpha| \le l \in N$ and $A_\alpha : D \times R^T \to R$ ($T$ is the number of partial derivatives in $R^m$ from the order 0 up to the order $l$) satisfy:

(4.2) $A_\alpha$ are measurable in $x \in D$ and continuous in the other variables, denoted by $\xi \in R^T$.

(4.3) $$|A_\alpha(x,\xi)| \le c\left(|\xi|_{R^T}^{p-1} + \mu(x)\right), \quad x \in D, \quad \xi \in R^T$$

with $\mu \in L^q(D)$, $q^{-1} + p^{-1} = 1$, $p > 1$.

(4.4) $$\sum_{|\alpha| \le l} \left(A_\alpha(x,\xi) - A_\alpha(x,\eta)\right)(\xi_\alpha - \eta_\alpha) \ge 0,$$

for any $\xi, \eta \in R^T$ and a.e. $x \in D$. The nonlinear operator A, introduced above, is called the generalized divergence operator or the Leray-Lions operator. Linear elliptic operators of order $2l$ are a special case of operator $A$, corresponding to $p = 2$. It is known that $A$ is maximal monotone in $W_0^{l,p}(D) \times W^{-l,q}(D)$, Barbu [2], Tiba [16]. If, moreover, the coercivity assumption:

$$\sum_{|\alpha| \le l} A_\alpha(x,\xi)\xi_\alpha \ge c\left(|\xi'|^p_{R^{T'}} + c_1\right), \; c > 0,\; x \in D,\; \xi \in R^T \tag{4.5}$$

with $\xi'$ denoting the components of $\xi$ corresponding to the highest order derivatives (see (4.1)) and their number being $T'$, is satisfied, then $A$ is coercive in $W_0^{l,p}(D)$ and onto and its realization in $L^2(D)$ with domain

$$dom\left(A_{L^2}\right) = \left\{z \in W_0^{l,p}(D);\; Az \in L^2(D)\right\} \tag{4.6}$$

is maximal monotone and onto.

The assumptions and the definitions (4.1)-(4.6) are directly inherited by any $\Omega \in \mathcal{O}$ for functions in $W_0^{l,p}(\Omega)$. Consequently, for any $\Omega \in \mathcal{O}$, the (nonlinear) Dirichlet boundary value problem:

$$A\widehat{z}_\Omega = f_\Omega \tag{4.7}$$

has at least one solution $\widehat{z}_\Omega \in dom\left(A_{L^2}\right)$, with homogeneous boundary conditions. Uniqueness may be also proved if the inequality (4.4) is strict for $\xi \neq \eta$.

In the nonhomogeneous case, a weak solution $z_\Omega \in W^{l,p}(\Omega)$ is defined by

$$\sum_{|\alpha| \le l} \int_\Omega D^\alpha A_\alpha\left(x, z_\Omega, \ldots, D^l z_\Omega\right) D^\alpha v dx = \int_\Omega f_\Omega v dx,\; \forall v \in W_0^{l,p}(\Omega), \tag{4.8}$$

$$z_\Omega - h_\Omega \in W_0^{l,p}(\Omega). \tag{4.9}$$

The existence in (4.8), (4.9) follows by considering the shifted mappings $\widetilde{A}_\alpha(x,\xi) = A_\alpha(x, \xi + [h_\Omega(x)\;, \ldots, D^l h_\Omega(x)])$ for $x \in D$, $\xi \in R^T$. We associate to them the differential operator $\widetilde{A}$ constructed as in (4.1) and acting in $W_0^{l,p}(\Omega) \times W^{-l,q}(\Omega)$. We notice that for any $z \in W_0^{l,p}(\Omega)$:

$$\int_\Omega \left|\widetilde{A}_\alpha\left(x, z(x), \ldots, D^l z(x)\right)\right|^q dx \le c\left(|z|^p_{W_0^{l,p}(\Omega)} + 1\right)$$

$$\sum_{|\alpha|\le l}\int_\Omega \widetilde{A}_\alpha\left(x,z(x),\ldots,D^l z(x)\right)D^\alpha z(x)dx \ge c\left|D^l z\right|^p_{L^p(\Omega)^{T'}}+c_1,\ c>0$$

by (4.3), (4.5) and the Clarkson inequalities, Hewitt and Stronberg [8]. Here $c$, $c_1$ are some generic constants, $c>0$, which may change from one relation to another. We conclude that $\widetilde{A}$ is well defined in $W_0^{l,p}(\Omega)$, it is maximal monotone and coercive. The corresponding equation (4.7) has at least one solution $\widetilde{z}_\Omega \in W_0^{l,p}(\Omega)$ and one can easily check that $z_\Omega=\widetilde{z}_\Omega+h_\Omega$ satisfies (4.8), (4.9).

Notice that, in general, $\widehat{z}_\Omega,\widetilde{z}_\Omega$ have higher regularity properties expressed by (4.6) and this remains valid for $z_\Omega$ if $h_\Omega$ satisfies extra regularity assumptions. Such results are known in the literature, both in the linear and in the nonlinear cases.

To maintain a general setting, we assume

$$|h_\Omega|_{W^{l+\varepsilon,p}(\Omega)}\le c, \tag{4.10}$$

$$|z_\Omega|_{W^{l+\varepsilon,p}(\Omega)}\le c\left(|h_\Omega|_{W^{l+\varepsilon,p}(\Omega)}+|f_\Omega|_{L^2(\Omega)}\right), \tag{4.11}$$

for some given $\varepsilon>0$ and with $z_\Omega$ satisfying (4.8), (4.9).

It is our aim to study the shape optimization problem

$$\mathrm{Min}_{\Omega\in\mathcal{O}}\int_\Omega L\left(x,z_\Omega,\ldots,D^l z_\Omega\right)dx \tag{4.12}$$

subject to (4.8), (4.9), where $L$ satisfies (4.2) and

$$0\le L(x,\xi)\le c\left(|\xi|^p_{R^T}+\xi(x)\right)\ \text{with}\ \xi\in L^1(D). \tag{4.3$'$}$$

**Theorem 4.1** *Assume that $\mathcal{O}$ is as in Theorem 2.1 and conditions (4.2) – (4.5), (4.10), (4.11) are fulfilled. Then, the shape optimization problem (4.12), (4.8), (4.9) has at least one optimal domain $\Omega^*\in\mathcal{O}$.*

*Proof*

Let $\Omega_n\in\mathcal{O}$ be a minimizing sequence for the problem (4.12). We may find by Theorem 2.1 some $\Omega^*\in\mathcal{O}$, such that $\Omega^*=Hlim\Omega_n$ and its characteristic function, $\chi^*$ satisfies $\chi_n\to\chi^*$ a.e. in $D$ and $\chi_n\to\chi^*$ strongly in $L^\beta(D)$, $\forall\,\beta>1$.

By the $\Gamma-$ property, for any $K\subset\subset\Omega^*$, there is $n(K)\in N$, such that $K\subset\subset\Omega_n$ for $n\ge n(K)$.

We denote shortly by $h_n$, $f_n$, $z_n$ the corresponding data and solutions of (4.8), (4.9), in $\Omega_n$. Hypotheses (4.10), (4.11) yield $h_n\to h^*$ strongly in $W^{l,p}(K)$, $z_n\to z^*$ strongly in $W^{l,p}(K)$, $f_n\to f^*$ weakly in $L^2(D)$, on a subsequence again denoted by $n$.

The mappings $h^*$, $z^*$, $f^*$ satisfy $h^* \in W^{l,p}(\Omega^*)$, $z^* \in W^{l,p}(\Omega^*)$, $f^* \in L^2(\Omega^*)$ and are constructed by taking an increasing sequence of open subsets of $\Omega^*$, compactly embedded in $\Omega^*$, such that their union gives $\Omega^*$. In each subset, such limit functions may be constructed as above and it can be extended to larger subsets by taking further subsequence. The regularity of $h^*$, $z^*$, $f^*$ is a simple consequence of a distributions argument in $\Omega^*$, Liu and Rubio [11].

We show that $z^*$ is the solution of (4.8), (4.9) associated to $h^*$, $f^*$.

We denote by $y_n = z_n - h_n \in W_0^{l,p}(\Omega_n)$ and we extend it by 0 to the domain $D$, $\widetilde{y}_n \in W_0^{l,p}(D)$ and $\widetilde{y}_n|_{\Omega_n} = y_n$. Clearly $\{\widetilde{y}_n\}$ is bounded in $W_0^{l,p}(D)$ and we may assume that $\widetilde{y}_n \to \widetilde{y} \in W_0^{l,p}(D)$, weakly in $W_0^{l,p}(D)$, on a subsequence. Notice that

$$0 = \int_{D-\Omega^*} |\widetilde{y}|\, dx = \int_D (1-\chi^*)\, |\widetilde{y}|\, dx = \lim_{n\to\infty} \int_D (1-\chi_n)\, |\widetilde{y}_n|\, dx$$

by the strong convergences $\chi_n \to \chi^*$ and $\widetilde{y}_n \to \widetilde{y}$ in $L^q(D)$, $L^p(D)$, $q^{-1}+p^{-1}=1$. Then, $\widetilde{y}=0$ a.e. in $D-\Omega^*$ and Corollary 3.3 gives that $y = \widetilde{y}|_{\Omega^*} \in W_0^{l,p}(\Omega^*)$. On the other side, $\widetilde{y} = z^* - h^*$ in $\Omega^*$, by taking the restriction to any $K \subset\subset \Omega^*$ by $y_n = z_n - h_n \to z^* - h^*$ in $W^{l,p}(K)$. We conclude that $z^*$, $h^*$ satisfy (4.9).

To pass to the limit in (4.8), we use that $\{A_\alpha(x, z_n, \dots, D^l z_n)\}$ are bounded in $L^q(\Omega_n)$ for any $n \in N$ and $|\alpha| \le l$. This is also valid in $L^q(K)$, any $K \subset\subset \Omega^*$, for $n \ge n(K)$. We denote by $a_\alpha \in L^q(\Omega^*)$ the mapping constructed such that $A_\alpha(\cdot, z_n(\cdot), \dots, D^l z_n(\cdot)) \to a_\alpha$ weakly in $L^q(K)$, any $K \subset\subset \Omega^*$.

This is again possible by taking an increasing sequence of open subsets of $\Omega^*$, compactly embedded in $\Omega^*$ and with their union giving $\Omega^*$, as in the construction of $z^*$.

By (4.11), (4.10), $\{z_n\}$ is strongly convergent in $W^{l,p}(K)$, for any $K \subset\subset \Omega$. Then, we get the a.e. convergence in $K$ of $z_n$ and all its derivates up to the order $l$ to $z^*, D^\alpha z^*, D^l z^*$, ..., respectively. The Caratheodory assumption (4.2) shows that $A_\alpha(x, z_n(x), \dots, D^l z_n(x)) \to A_\alpha(x, z^*(x), \dots, D^l z^*(x))$ a.e. in $K$.

Consequently, we have $a_\alpha(x) = A_\alpha(x, z^*(x), \dots, D^l z^*(x))$ a.e. in $\Omega^*$, which identifies the limits of the nonlinear terms in (4.8). Therefore we can pass to the limit in (4.8) and $z^*$ is the solution of (4.8), (4.9) associated to the domain $\Omega^* \in \mathcal{O}$ and to the nonhomogeneous data $h^*, f^*$. At the last step of the proof, we show that $\Omega^*$ is optimal for the problem (4.12). The same argument as above gives that

$$L(x, z_n(x), \dots, D^l z_n(x)) \to L(x, z^*(x), \dots, D^l z^*(x)) \tag{4.13}$$

a.e. in any $K \subset\subset \Omega^*$, due to (4.2) (valid also for $L$) and to the strong convergence of $\{z_n\}$ in $W^{l,p}(K)$. By assumption (4.3)' and the Vitali theorem, we get that the convergence in (4.13) is true in $L^1(K)$, $\forall K \subset\subset \Omega^*$.

We have:

$$\inf_{\Omega\in\mathcal{O}} \int_\Omega L(x, z_\Omega, \dots, D^l z_\Omega)\, dx = \lim_{n\to\infty} \int_{\Omega_n} L(x, z_n, \dots, D^l z_n)\, dx \ge$$

$$\geq \liminf_{j\to\infty} \lim_{n\to\infty} \int_{G_j} L\left(x, z_n, \ldots, D^l z_n\right) dx$$

$$= \liminf_{j\to\infty} \int_{\Omega^*} \chi_{G_j} L(x, z^*, \ldots, D^l z^*) dx \geq$$

$$\geq \int_{\Omega^*} \liminf_{j\to\infty} \chi_{G_j} L\left(x, z^*, \ldots, D^l z^*\right) dx =$$

$$= \int_{\Omega^*} L\left(x, z^*, \ldots, D^l z^*\right) dx,$$

where we have used the positivity of $L$ and the Fatou lemma and $G_j \subset\subset \Omega^*$ are open subsets such that $\bigcup_{j=1}^{\infty} G_j = \Omega^*$.

The above inequality proves the optimality of $\Omega^*$ in the open sets family $\mathcal{O}$ and the proof is finished.

*Remark* In proving the continuity of the application $\Omega \to z_\Omega$ an essential step was the convergence of the characteristic functions, associated to the corresponding open sets. The next result shows that this property is also necessary, in certain situations.

We consider the simple case of the Laplace operator:

$$-\Delta y_\Omega = 1 \; in \; \Omega, \tag{4.14}$$

$$y_\Omega = 0 \; on \; \partial\Omega, \tag{4.15}$$

where $\Omega \in \mathcal{O}$ and $y_\Omega \in H_0^1(\Omega)$ is the unique weak solution of (4.14), (4.15). Clearly, the assumptions (4.2) - (4.5) are valid in this setting with $p = 2$ and $l = 1$, in arbitrary space dimension $m$. If a cost functional is associated to (4.14), (4.15) satisfying (4.2), (4.3)', then Theorem 4.1 gives an existence result for this shape optimization problem which extends the works of Sverak [15], Zhong [17].

Let us also denote by

$$\widehat{y}_\Omega(x) = \begin{cases} y_\Omega(x), & x \in \Omega, \\ -y_{D-\Omega}(x), & x \in D - \overline{\Omega}. \end{cases} \tag{4.16}$$

We have $\widehat{y}_\Omega \in H_0^1(D)$ since relation (4.16) may be reexpressed by $\widehat{y}_\Omega(x) = \widetilde{y}_\Omega(x) - \widetilde{y}_{D-\Omega}(x)$, where $\widetilde{y}_\Omega, \widetilde{y}_{D-\Omega}$ are the extensions by 0 of $y_\Omega \in H_0^1(\Omega)$, respectively $y_{D-\Omega} \in H_0^1\left(D - \overline{\Omega}\right)$.

**Proposition 4.2** *If the strong maximum principle is valid in $\Omega_n \in \mathcal{O}$, $D - \Omega_n$ and $\widehat{y}_{\Omega_n} \to \widehat{y}_\Omega$ a.e. in , then $\chi_{\Omega_n} \to \chi_\Omega$ a.e. $\in D$.*

*Proof*

We remark that, by (2.14), we get:

$$\chi_{\Omega}(x) = H\left(\widehat{y}_{\Omega}\left(x\right)\right) \text{ a.e. } D, \tag{4.17}$$

for any $\Omega \in \mathcal{O}$, open subset. Here, we use that, by the strong maximum principle and by relation (4.16), $\widehat{y}_{\Omega}(x) > 0$ a.e. in $\Omega$, $\widehat{y}_{\Omega}(x) < 0$ a.e. in $D - \overline{\Omega}$. Notice as well that *meas* $(\partial\Omega) = 0$ since $\partial\Omega$ can be written as a finite union of graphs of continuous functions by (2.2). This gives that (4.17) is valid a.e. in $D$.

Moreover, if $\widehat{y}_{\Omega}(x) > 0$, than $\widehat{y}_{\Omega_n}(x) > 0$ for $n \geq n_x \in N$ and, consequently $H\left(\widehat{y}_{\Omega_n}(x)\right) = H\left(\widehat{y}_{\Omega}(x)\right) = 1$ for $n \geq n_x$. If $\widehat{y}_{\Omega}(x) < 0$, then $\widehat{y}_{\Omega_n}(x) < 0$ for $n \geq n_x$ and $H\left(\widehat{y}_{\Omega_n}(x)\right) = H\left(\widehat{y}_{\Omega}(x)\right) = 0$. These two situations are valid a.e. in $D$ as *meas* $(\partial\Omega) = 0$ and the proof is finished.

*Remark* It is possible to extend Proposition 4.1 to the case of unbounded domains of class $\mathcal{C}$ since we use just that the boundary has zero measure. In the bounded domains case, Theorem 2.1 gives the Hausdorff - Pompeiu convergence and the convergence of characteristic functions, on a subsequence. The significance of Proposition 4.1 is that the limit domain may be identified from the a.e. convergence of the solutions to differential equations defined in these domains.

## References


1. R. Adams - "Sobolev spaces", Academic Press, New York (1975).

2. V. Barbu - "Nonlinear semigroups and differential equations in Banach spaces", Noordhoff, Leyden (1976).

3. D. Bucur and J.P. Zolesio - "Optimisation de forme sous contrainte capacitaire", CRAS Paris, ser. I, 318 (1994), p. 795-800.

4. D. Bucur and J.P. Zolesio - "Free boundary problems and density perimeter", J. Diff. Eq. 126 (1996), p. 224-243.

5. D. Bucur and J.P. Zolesio - "N-dimensional shape optimization under capacitary contraints", J. Diff. Eq. 123 (1995), p. 504-522.

6. D. Chenais - "On the existence of a solution in a domain identification problem", JMAA 52 (1975), p. 189-219.

7. A. Henrot - "Continuity with respect to the domain for the Laplacian: a survey", Control and Cybernetics 23 (1994), p. 427-443.

8. E. Hewitt and K. Stronberg - "Real and abstract analysis", Springer-Verlag, Berlin (1965).

9. K. Kuratowski - "Introduction to set theory and topology", Pergamon Press, Oxford (1961).

10. O. Ladyzenskaya and N. Uraltseva - "Linear and quasilinear elliptic equations", Academic Press, New York (1968).

11. W.B. Liu and J.E. Rubio - "Local convergence and optimal shape design", SIAM J. Control Optimiz. 30 (1992), p. 49-62.

12. W.B. Liu, P. Neittaanmaki and D. Tiba - "Sur les problemes d'optimisation structurelle", CRAS Paris (submitted).

13. V. Maz'ya - "Sobolev spaces", Springer-Verlag, Berlin (1985).

14. O. Pironneau - "Optimal shape design for elliptic systems", Springer- Verlag, Berlin (1984).

15. V. Sverak - "On optimal shape design", J. Math. Pures Appl. 72 (1993), p. 537-551.

16. D. Tiba - "Optimal control of nonsmooth distributed parameter systems", LNM 1459, Springer Verlag, Berlin (1990).

17. Xiao Zhong - "On nonhomogeneous elliptic equations", Annales Academiae Scientiarum Fennicae 117, Helsinki (1998).